Performance Analysis of Maximum Ratio Transmission with Imperfect Channel Estimation

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***Abstract*— Maximal ratio transmission (MRT) is designed assuming the availability of perfect channel state information (CSI) at both the transmitter and the receiver. However, perfect CSI is not available in practice. This paper investigates the impact of Gaussian estimation errors on the MRT performance in independently and identically distributed (i.i.d.) Rayleigh fading channels. We derive the cumulative distribution function (cdf), the probability density function (pdf) and the moment generating function (mgf) of the MRT output signal-to-noise ratio (SNR) with imperfect CSI, enabling the evaluation of some useful performance metrics such as the average error rate and the outage performance. Numerical and simulation results are provided to show the impact of imperfect CSI on the MRT performance.**

***Index Terms*— Channel state information, diversity, Gaussian estimation error, maximal ratio transmission, MIMO systems.**

I. Introduction

ANTENNA diversity has long been employed to combat the impact of multi-path fading on wireless communication systems. While classical research in this area has focused on single-input and multiple-output (SIMO) antenna systems, multiple-input and multiple-output (MIMO) antenna systems have captured considerable attention recently. In MRT [1], the signal is transmitted along the strongest eigenmode and the received signals are combined using maximal ratio combining. The performance of MIMO MRT has been studied for various fading channels [2]–[5]. However, most analytical results are derived under the assumption that perfect CSI is available at both the transmitter and the receiver, an assumption that is not true in practice. For example, when pilot symbols are used for channel estimation, Gaussian errors can arise due to time or frequency separation between the pilot and the signal [6]. Thus, from a theoretical and practical viewpoint, it is important to quantify how the MRT performance degrades due to imperfect CSI.

We denote a MIMO system by the pair (N_r, N_t) where N_r and N_t denote the number of receive and transmit antennas, respectively. The MRT performance with imperfect CSI for $(1, N_t)$ systems (i.e., multiple-input and single-output (MISO) antenna systems) has been analyzed in the literature [7], [8]. However, no analytical results have been published for general (N_r, N_t) MIMO MRT systems with imperfect CSI. This letter analyzes the performance of such systems with imperfect CSI in i.i.d. Rayleigh fading channels. We derive the cdf, the pdf and the mgf of the MIMO MRT output SNR, enabling the evaluation of the MRT performance as a function of the quality of CSI.

This letter is organized as follows. Section II describes the system and channel models. Section III derives the output SNR of MRT with Gaussian channel estimation errors in i.i.d. Rayleigh fading. Section IV derives statistical distributions of the MRT output SNR with Gaussian estimation errors. Section V presents several numerical and simulation results and concludes this letter. The following notations are used. All the matrices and the vectors are denoted by boldfaced capital letters and boldfaced small letters, respectively. The determinant, conjugate transpose, transpose, conjugate of $\mathbf{X}$ are denoted by $\det(\mathbf{X})$, $\mathbf{X}^H$, $\mathbf{X}^T$ and $\mathbf{X}^*$, respectively. The (i, j)-th element of the matrix $\mathbf{X}$ is denoted by the corresponding small letter with subscripts x_{ij}. The average and the absolute value of x are denoted by $E(x)$ and $|x|$, respectively. The $n \times n$ identity matrix is $\mathbf{I}_n$. The Kronecker delta is defined as $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

II. System and Channel Models

Consider a (N_r, N_t) MIMO MRT system. The channel experiences slow and frequency non-selective Rayleigh fading, which is characterized by an $N_r \times N_t$ matrix $\mathbf{H} = [h_{ij}]$ of i.i.d. complex Gaussian random variables (CGRVs) with zero-mean and unit variance, i.e., $E(|h_{ij}|^2) = 1$, where h_{ij} denotes the channel gain from the j-th transmit antenna to the i-th receive antenna. Thus, the $N_r \times 1$ received signal vector $\mathbf{r}$ can be written as

$$\mathbf{r} = \mathbf{H}\mathbf{v_t}s + \mathbf{n} \tag{1}$$

where s is the transmitted data symbol with energy $E(|s|^2) = E_s$, $\mathbf{v_t}$ denotes the $N_t \times 1$ normalized transmit weighting vector and $\mathbf{n}$ is an $N_r \times 1$ additive white Gaussian noise (AWGN) vector with zero-mean and covariance $E(\mathbf{n}\mathbf{n}^H) = \sigma_n^2 \mathbf{I}_{N_r}$. The SNR of data symbols is $\frac{E_s}{\sigma_n^2}$. In practice, the transmit weighting factor $\mathbf{v_t}$ is calculated at the receiver and sent to the transmitter through a feedback channel. This feedback channel is assumed to be perfect. Hence, the transmitter uses exactly the same weighting factor as calculated at the receiver.

Assuming the use of orthogonal pilot sequences, the pilot symbol assisted channel estimate $\hat{\mathbf{H}}$ differs from the actual channel $\mathbf{H}$ by an independent complex Gaussian error $\mathbf{\Delta H}$ which is an $N_r \times N_t$ matrix of i.i.d. CGRVs with zero-mean and variance σ_e^2, i.e., $\hat{\mathbf{H}} = \mathbf{H} + \mathbf{\Delta H}$. The variance of the Gaussian estimation errors σ_e^2 depends on the SNR of the pilot symbols $\sigma_e^2 \propto \left(\frac{E_p}{N_0}\right)^{-1}$ [9] where E_p is the pilot symbol energy. In general, E_p is not always equal to the data symbol energy E_s. Thus, the estimate $\hat{\mathbf{H}}$ is an $N_r \times N_t$ matrix of i.i.d.

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CGRVs with zero-mean and variance $\hat{\sigma}^2 = 1+\sigma_e^2$. It can be shown that $\hat{h}_{ij}$ and h_{ij} are joint complex Gaussian distributed with the normalized correlation coefficient $\frac{1}{\sqrt{1+\sigma_e^2}}$. The actual channel matrix $\mathbf{H}$ can be written in terms of $\hat{\mathbf{H}}$ as [10],

$$\mathbf{H} = \rho\hat{\mathbf{H}} + \tilde{\mathbf{H}} \quad (2)$$

where $\rho = \frac{1}{1+\sigma_e^2}$, $\tilde{\mathbf{H}} = [\tilde{h}_{ij}]$ and $\tilde{h}_{ij}$'s are i.i.d. CGRVs with zero-mean and variance $\tilde{\sigma^2} = E(|\tilde{h}_{ij}|^2) = \frac{\sigma_e^2}{1+\sigma_e^2}$ and $E(\tilde{h}_{ij}\hat{h}_{lk}^*) = 0$, $E(\tilde{h}_{ij}n_k^*) = 0$ for any i, j, k, l.

III. The MRT Output SNR with Gaussian Estimation Error

When perfect CSI is available at both the transmitter and the receiver, in order to maximize the SNR at the receiver, the transmit weighting vector $\mathbf{v_t}$ is selected to be the eigenvector $\mathbf{u}$ of the largest eigenvalue λ_{max} of the Wishart matrix $\mathbf{H}^H\mathbf{H}$. Thus, the coherently combined received signal is given by $\hat{s} = \mathbf{u}^H\mathbf{H}^H(\mathbf{H}\mathbf{u}s+\mathbf{n})$. The output SNR of MRT with perfect CSI is then given by [2]

$$\gamma_{mrt} = \lambda_{max}\frac{E_s}{\sigma_n^2}. \quad (3)$$

However, when the channel estimate $\hat{\mathbf{H}}$ differs from the true channel $\mathbf{H}$, the combined received signal is given by

$$\hat{s} = \hat{\mathbf{u}}^H\hat{\mathbf{H}}^H(\mathbf{H}\hat{\mathbf{u}}s + \mathbf{n}) \quad (4)$$

where $\hat{\mathbf{u}}$ is the eigenvector of the largest eigenvalue $\hat{\lambda}_{max}$ of the Wishart matrix $\hat{\mathbf{H}}^H\hat{\mathbf{H}}$.

Substituting (2) into (4), we obtain the output of the combiner as

$$\hat{s} = \hat{\mathbf{u}}^H\hat{\mathbf{H}}^H\hat{\mathbf{H}}\hat{\mathbf{u}}\rho s + \hat{\mathbf{u}}^H\hat{\mathbf{H}}^H\tilde{\mathbf{H}}\hat{\mathbf{u}}s + \hat{\mathbf{u}}^H\hat{\mathbf{H}}^H\mathbf{n}. \quad (5)$$

Given the estimate $\hat{\mathbf{H}}$ and the transmitted signal s, the second term in (5) is i.i.d. Gaussian distributed. Hence, $\hat{s}$ is a complex Gaussian distributed RV with mean $E(\hat{s}) = \hat{\mathbf{u}}^H\hat{\mathbf{H}}^H\hat{\mathbf{H}}\hat{\mathbf{u}}\rho s = \hat{\lambda}_{max}\rho s$ and variance $\mathrm{VAR}(\hat{s}) = \hat{\lambda}_{max}(E_s\tilde{\sigma}^2+\sigma_n^2)$. Therefore, the effective output SNR can be written as

$$\gamma_{mrt} = \frac{\hat{\lambda}_{max}}{(1+\sigma_e^2)[\sigma_e^2 + (1+\sigma_e^2)\sigma_n^2/E_s]}. \quad (6)$$

As expected, when $\sigma_e^2 = 0$, (6) reduces to (3) for the perfect CSI case.

IV. Output Statistics

To derive the statistics of the MRT output with Gaussian estimation errors (6), we require the statistical distribution of $\hat{\lambda}_{max}$. Fortunately, the distribution of the largest eigenvalue of a Wishart matrix has have already been developed in the mathematics community [11], [12]. In this section, we apply the available results to derive the cdf, the pdf and the mgf of the MRT output SNR with Gaussian estimation errors. These results can be readily used to evaluate the MRT performance.

Using Khatri's results [12, Eq. (6)], we can readily obtain the cdf of the MRT output SNR as

$$F_{mrt}(x) = \Pr(\gamma_{mrt} \le x) = \frac{\det[\mathbf{S}(gx)]}{\prod_{k=1}^{a}(a-k)!(b-k)!} \quad (7)$$

where $a = \min(N_r, N_t)$, $b = \max(N_r, N_t)$, $g = \sigma_e^2 + (1+\sigma_e^2)\sigma_n^2/E_s$ and $\mathbf{S}(x)$ is an $a\times a$ Hankel matrix with elements given by $s_{ij}(x) = \gamma(b-a+i+j-1, x)$ where $\gamma(a, x)$ is the incomplete gamma function defined as [13, 8.350.1]. These notations will be used throughout this letter. Note that the output cdf (7) only depends on the minimum and the maximum values of N_r and N_t. Thus, when the signal energy E_s is fixed, the performance of (L, M) MRT system is the same as that of (M, L) MRT system with Gaussian estimation errors. The same fact has been observed for MRT performance with perfect CSI [2], [4].

Applying (7) and [13, 8.352.1], we can readily obtain the pdf and the mgf of the MRT output SNR with imperfect CSI respectively as [2]

$$p_{mrt}(x) = \frac{dF_{mrt}(x)}{dx} = D\frac{d}{dx}\det[\mathbf{S}(gx)] = gD\sum_{i=1}^{a}\sum_{j=b-a}^{(a+b)i-2i^2} c_{ij}e^{-igx}(gx)^j \text{ and} \quad (8)$$

$$\phi_{mrt}(s) = E(e^{-s\gamma_{mrt}}) = D\sum_{i=1}^{a}\sum_{j=b-a}^{(a+b)i-2i^2} j!\, c_{ij}\left[\frac{g}{s+ig}\right]^{j+1} \quad (9)$$

where $D = \left[\prod_{k=1}^{a}(a-k)!(b-k)!\right]^{-1}$, c_{ij} is the coefficient of the term $e^{-ix}x^j$ in the expansion of $\frac{d}{dx}\det[\mathbf{S}(x)]$. The coefficients c_{ij}'s can be readily determined using mathematical softwares such as MAPLE.

Let us consider two special cases of the mgf (9). For MISO or SIMO systems ($a = 1$), the Hankel matrix $\mathbf{S}(gx)$ only has one element $\gamma(b, gx)$. Noticing that $\frac{d}{dx}\gamma(b, x) = x^{b-1}e^x$, we can readily obtain $c_{1,b-1} = 1$ and the mgf (9) reduces to $\phi_{mrt}(s) = \left(\frac{g}{g+s}\right)^b$, which is equivalent to the previous result for SIMO MRT systems [8].

For two receive antennas and more transmit antennas (or two transmit antennas and more receive antennas) system ($a = 2$), the Hankel matrix $\mathbf{S}(gx)$ is given by

$$\mathbf{S}(gx) = \begin{bmatrix}\gamma(b-1, gx) & \gamma(b, gx)\\ \gamma(b, gx) & \gamma(b+1, gx)\end{bmatrix}. \quad (10)$$

Thus, the mgf (9) can be reduced to closed-form

$$\phi_{mrt}(s) = \left(\frac{g}{g+s}\right)^b\left[\frac{bg}{g+s} + \frac{b(g+s)}{g} + 2(b-1)\right] - \sum_{k=0}^{b-2}\frac{(b+k)!\eta^{b+k+1}}{k!\,(b-1)!} - \sum_{k=0}^{b}\frac{(b+k-2)!\eta^{b+k-1}}{k!\,(b-2)!} - 2\sum_{k=0}^{b-1}\frac{(b+k-1)!\eta^{b+k}}{k!\,(b-2)!} \quad (11)$$

where $\eta = \frac{g}{2g+s}$. Applying the mgf approach with (9), we can obtain the average BER of a wide class of digital modulations.

V. Numerical Results

Numerical results are provided to show how Gaussian estimation errors impact the MRT performance in i.i.d. Rayleigh

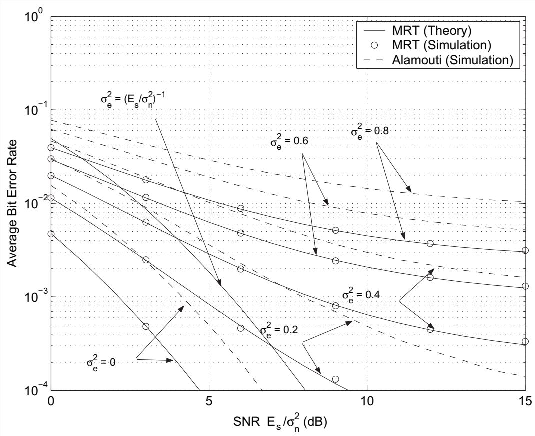

Fig. 1. The BER of BPSK with imperfect MRT and Alamouti's scheme in i.i.d. Rayleigh fading channel. $N_r = 3$, $N_t = 2$.

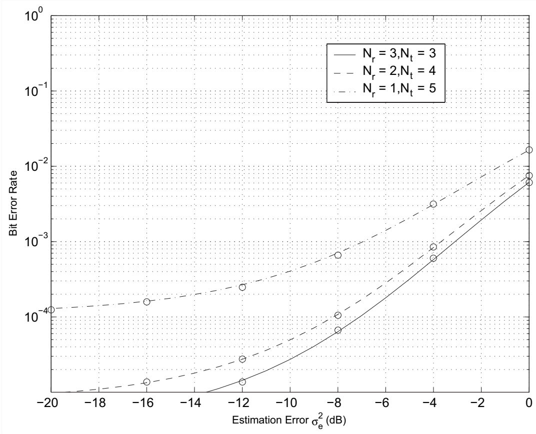

Fig. 2. Comparison of effects of imperfect CSI on different MRT systems in i.i.d Rayleigh fading channel. $N_r + N_t = 6$, $E_s/\sigma_n^2 = 5$dB.

fading channels. Simulation results are provided as an independent check of our analytical results. For brevity, we provide only BPSK results.

Fig. 1 compares the average BERs of MRT$(3, 2)$ and Alamouti $(3, 2)$ [15] employing BPSK modulation versus SNR (E_s/σ_n^2) for different levels of estimation error variance σ_e^2. For a fair comparison, we allocate $\frac{E_s}{2}$ to each symbol in Alamouti's scheme so that the total power from two transmit antennas is E_s, which is the same as that in the MRT system. Alamouti's scheme does not require CSI at the transmitter while MRT requires the feedback of CSI from the receiver to the transmitter. However, Alamouti's scheme performs worse than MRT. As σ_e^2 increases, both MRT and Alamouti's scheme perform worse and the performance loss of Alamouti's scheme compared to MRT increases. For example, to achieve an average BER of 10^{-3}, the performance loss of Alamouti's scheme to MRT is 2dB when $\sigma_e^2 = 0$ and more than 3dB when $\sigma_e^2 = 0.2$. Observe that the imperfect CSI results in degradation of the diversity order (the negative slope of the average BER curves) and the coding gain (the gap between the imperfect curves and the perfect curves) of both MRT and Alamouti's scheme. Error floors can be clearly seen when σ_e^2 is fixed. Observe that no error floor exist when $\sigma_e^2 = \left(\frac{E_s}{N_0}\right)^{-1}$, i.e., the variance of the Gaussian estimation errors decreases as the SNR of the data symbols increases (the pilot symbols have the same energy as the data symbols).

Fig. 2 compares the BER performance of MRT with BPSK versus the variance of Gaussian estimation error σ_e^2 at $E_s/\sigma_n^2 = 5$dB. We consider three MRT systems each with a total of six antennas. The MRT$(3, 3)$ system outperforms the MRT$(2, 4)$ and the MRT$(1, 5)$ systems. Hence, for a fixed number of antennas, better performance is obtained by evenly distributing the antennas between the transmitter and the receiver. This observation agrees with those in [2], [4]. However, as σ_e^2 increases, the performance gain of MRT$(3, 3)$ system over the other two systems diminishes. The MRT$(3, 3)$ system is more sensitive to the estimation error.

In conclusion, this letter analyzed the general (N_r, N_t) MIMO MRT performance with Gaussian channel estimation errors in i.i.d Rayleigh fading channels. We derived the effective output SNR and the related statistics. It is instructive to compare the performance of MRT and that of the Alamouti's scheme, which is optimal when CSI is not available at the transmitter. The performance gain of MRT over Alamouti's scheme increases as channel estimation becomes worse. Although the MRT system with evenly distributed transmitter antennas and receiver antennas provides better performance, such a system is more sensitive to imperfect CSI.

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