PERFORMANCE AND ROBUSTNESS ANALYSIS FOR STRUCTURED UNCERTAINTY*

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ABSTRACT

This paper introduces a nonconservative measure of performance for linear feedback systems in the face of structured uncertainty. This measure is based on a new matrix function, which we call the Structured Singular Value.

1. Introduction

The basic requirement of feedback systems is to achieve certain desired levels of performance and also to be tolerant of uncertainties. Performance levels concern such things as command following, disturbance rejection, sensitivity, etc., while uncertainty tolerances deal with the inevitable differences which exist between a physical plant and its mathematical design/analysis model. As discussed in various textbooks and references [eg. 1,2], these two aspects of the feedback problem lead to fundamental tradeoffs and compromises which motivate the entire body of feedback theory.

An essential difficulty in the theory has been to capture both the performance and uncertainty aspects of feedback in a single problem statement. Thus we have optimization theories which emphasize performance, robustness theories which emphasize uncertainties, and a host of ad hoc tools which attempt to compromise the two.

In this paper, we propose a problem formulation which captures both aspects of feedback under the umbrella of what we will call the "block-diagonal bounded perturbation (BDBP) problem." The solution to this problem, introduced in [3], involves a generalization of the ordinary singular value decomposition (SVD). It provides a reliable, nonconservative measure to determine whether both the performance and robustness requirements of a feedback loop are satisfied. This measure, which we will call the structured singular value (SSV) and denote by the symbol μ, serves as an essential analysis tool. Synthesis tools based on the structured singular value are under development.

The paper is organized into seven major sections. Section 2 defines nomenclature. Section 3 formulates the robustness and performance aspects of feedback as a block-diagonal bounded perturbation problem. This problem is solved in Section 4 using the new structured singular value concept. Performance implications of these results are then examined in Section 5, which includes a fundamental theorem relating performance in the face of uncertainty to the SSV. Section 6 provides an example, and Section 7 contains concluding comments.

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2. Notation

$\mathcal{M}(k)$	=	Algebra of complex kxk matrices
$\mathcal{U}(k)$	=	Unitary matrices in $\mathcal{M}(k)$
$\bar{\sigma}(M)$	=	Maximum Singular value of M
$\rho(M)$	=	Spectral radius
	=	Magnitude of largest eigenvalue
M^*	=	Conjugate transpose of M
diag $(M_1, M_2, M_3, \ldots, M_n)$	=	Block diagonal matrix with M_j (not necessarily square) on the diagonal.
$\lvert M \rvert$	=	magnitude of det(M)
$\mathcal{R}(k)$	=	kxk rational matrices

3. Feedback Analysis as a Block-Diagonal Bounded Perturbation Problem

This section formulates the basic feedback problem of achieving performance in the face of uncertainties as a stability problem in the presence of block-diagonal bounded perturbations. The formulation involves cone-bounded transfer functions as basic building blocks, in terms of which both the robustness and performance aspects of feedback can be characterized.

3.1 Basic Building Blocks: Cone-Bounded Transfer Functions

Throughout this paper, we will deal with multi-variable feedback systems whose models are linear, time invariant, and finite dimensional. Hence, they can be represented by transfer function matrices with rational elements. The robustness and performance properties of these systems will be expressed in terms of a collection of transfer matrices, $\Delta_i(s)$, i=1,2, ..., m, which each satisfy

$$\Delta_i(s) = L_i^{-1}(s)\ \Theta(s) R_i(s) \tag{1}$$

where $L_i(s)$ and $R_i(s)$ are constant transfer matrices and $\Theta(s)$ is any stable transfer matrix from a set satisfying

$$\bar{\sigma}[\Theta\ (j\omega)] \leq 1 \quad \forall\, \omega \geq 0$$

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We will also require that L_i and R_i have no poles or zeros in the open right half plane. These assumptions assure that Δ_i has no rhp poles. Reasons for this restriction are discussed later. Note that the functions which satisfy (1) belong to conic sectors, as initially defined by Zames ([4], [5]) and generalized by Safonov ([6], [7]). Their sector centers are zero, and their sector radii are characterized by L(s) and R(s). We will use such cone-bounded transfer functions as basic building blocks in a combined robustness/performance characterization of feedback systems.

3.2 Robustness Characterization

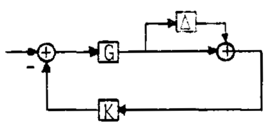
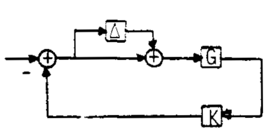
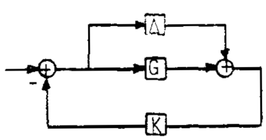
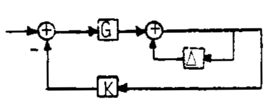
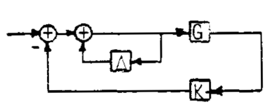
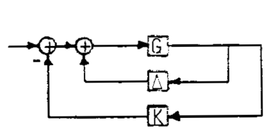
The use of cone-bounded transfer functions to characterize robustness has been a central theme in many recent references, including [8] where such transfer functions were inserted at the inputs or outputs of a plant model in order to represent so called unstructured uncertainties (modelling errors with no assumed structure except for known magnitude bounds on their transfer functions). Necessary and sufficient conditions were then derived for stability robustness in the face of such uncertainties. For example, a stable feedback loop with plant G(s) and compensator K(s) will remain stable in the face of all possible perturbed plants $G'(s) = [I + \Delta_i]G(s)$, with Δ_i given by (1), if and only if

$$\bar{\sigma}[R_i GK(I + GK)^{-1} L_i^{-1}] < 1 \ \forall\ \omega \geq 0 \qquad (2)$$

Note that with R_i and L_i specified, this inequality imposes conditions on the shape of the closed loop frequency response, $GK(I + GK)^{-1}$, which must be satisfied in order to assure robust stability. These conditions are unique to the assumed form of plant perturbations (eg. $G' = (I + \Delta)G$ in the present case). Each such assumed form corresponds to a specific location where Δ is inserted in the nominal feedback loop. The location for our present case is shown in Row 1 of Table 1. Other locations correspond to other assumed forms for G' and produce different necessary and sufficient stability robustness conditions. A representative set of possibilities is summarized in the remaining rows of Table 1. (Most of these cases can be found in [9]).

Table 1 also indicates representative types of physical uncertainties which can be usefully represented by cone bounded perturbations inserted at the indicated locations. For example, the representation $G' = (I + \Delta)G$ in Row 1 is useful for output errors at high frequencies, covering such things as unmodelled high frequency dynamics of sensors or plant, including diffusion processes, transport lags, electro-mechanical resonances, etc. The representation $G' = G(I + \Delta)$ in Row 2 covers similar types of errors occurring at the inputs. Both cases should be contrasted with Rows 4 and 5 which treat $G' = (I + \Delta)^{-1}G$ and $G' = G(I + \Delta)^{-1}$. These representations are more useful for variations in modelled dynamics, such as low frequency errors produced by parameter variations with operating conditions, with aging, or across production copies

TABLE 1 REPRESENTATIVE ROBUSTNESS/PERFORMANCE CONDITIONS

LOCATION OF Δ_i	CONDITIONS IMPOSED ON NOMINAL FEEDBACK LOOP SHAPES	REPRESENTATIVE TYPES OF UNCERTAINTY CHARACTERIZED	REPRESENTATIVE TYPES OF PERFORMANCE SPECS
	$\bar{\sigma}[R\ GK(I + GK)^{-1}L^{-1}] < 1$	- output (sensor) error - neglected HF dynamics - changing numbers of rhp zero $G' = (I + \Delta)G$	- sensor noise attenuation - output response to output commands
	$\bar{\sigma}[R\ KG(I + KG)^{-1}L^{-1}] < 1$	- input (actuator) errors - neglected HF dynamics - changing numbers of rhp zeros $G' = G(I + \Delta)$	- input response to input commands
	$\bar{\sigma}[R\ K(I + GK)^{-1}L^{-1}] < 1$	- additive plant errors - uncertain rhp zeros $G' = G + \Delta$	- input response to output commands
	$\bar{\sigma}[R\ (I + GK)^{-1}L^{-1}] < 1$	- LF plant parameter errors - changing numbers of rhp poles $G' = (I + \Delta)^{-1}G$	- output sensitivity - output errors to output commands and disturbances
	$\bar{\sigma}[R\ (I + KG)^{-1}L^{-1}] < 1$	- LF plant parameter errors - changing numbers of rhp poles $G' = G(I + \Delta)$	- input sensitivity - input errors to input commands and disturbances
	$\bar{\sigma}[R\ (I + GK)^{-1}G\ L^{-1}] < 1$	- LF plant parameter errors - uncertain rhp poles $G' = (G^{-1} + \Delta)^{-1}$	- output errors to input commands and disturbances

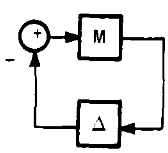
of the same plant. Discussion of still other cases is left to the table. Note from the table that the stability requirements on Δ do not limit our ability to represent variations in either the number or location of rhp singularities.

The most significant thing to understand about Table 1 is that the stability robustness conditions shown are sufficient to assure stability only if all the uncertainties occur at the indicated locations and none occur elsewhere. In order to use the conditions directly, therefore, designers are obliged to reflect all known sources of uncertainty from their known point of occurance to a single reference location in the loop. Such reflected uncertainties invariably have a great deal of structure which must then be "covered up" with a larger, arbitrarily more conservative perturbation in order to maintain a simple cone bounded representation at the reference location.*

Alternatively, designers could choose to treat uncertainties occurring at several different locations in the feedback loop as a single uncertainty occurring at one location in a larger feedback loop. To be specific about this alternative, let Δ_i, i=1, 2, ..., m, denote a collection of such uncertainties positioned at location ℓ_i, i=1, 2, ..., m. Note that at each ℓ_i, the feedback loop has an input, where it receives the signals from Δ_i, and also an output, where it supplies signals to Δ_i. Let M_{ii} be the transfer function matrix between these two sets of signals. Further, let M_{ij} denote the transfer matrix between the inputs at location ℓ_j and the outputs at location ℓ_i. Then the block-structured matrix

$$M \triangleq \{ M_{ij} \} \tag{3}$$

represents all interactions of the feedback loop with its uncertainties, and indeed, the block-diagonal bounded perturbation diagram in Figure 1 is an equivalent representation of the loop.



$$\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \ldots, \Delta_m)$$

Figure 1. Feedback loop as a BDBP Problem

Note that the feedback elements in this larger loop are zero in the absence of uncertainties. Hence, M will be a stable "plant" whenever the original nominal loop is stable. As an example of this representation, consider the system in Figure 2. This system, with two uncertainties present simultaneously, the first from Row 2 and the second from Row 4 of Table 1, is described by the following M matrix.

$$M = \begin{bmatrix} (I + KG)^{-1}KG & (I + KG)^{-1}K \\ (I + GK)^{-1}G & (I + GK)^{-1} \end{bmatrix} \tag{4}$$

* By "arbitrarily more conservative," we mean that examples can be constructed where the degree of conservatism is arbitrarily large. Of course, other examples exist where it is quite reasonable.

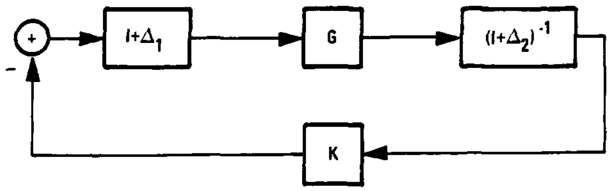


Figure 2. Feedback loop with Two Uncertainties

Given the equivalent system in Figure 1 with Δ_i's characterized by equation (1), it follows from the Small Gain Theorem [10] that the loop remains stable in the presence of these uncertainties if

$$\bar{\sigma}\,[R(j\omega)M(j\omega)L(j\omega)^{-1}] < 1 \;\forall\, \omega \geq 0 \tag{5}$$

where $R = \mathrm{diag}(R_1\ R_2\ \ldots\ R_m)$
and $L = \mathrm{diag}(L_1\ L_2\ \ldots\ L_m)$

This condition provides an alternate test for stability robustness. Like the procedure of reflecting all uncertainties to one reference location, however, the new test can be arbitrarily more conservative because it ignores the known block-diagonal structure of the uncertainties in Figure 1.

The objective of our results in this paper is precisely to reduce the conservatism of robustness and performance tests for block diagonal structures such as Figure 1. We do this by introducing a generalized notion of the maximum singular value for block-diagonal structures. This generalization is developed in Section 4. It is called the structured singular value (SSV) and is denoted by the symbol μ. It yields the following necessary and *sufficient conditions for robust stability of the* BDBP problem:

$$\mu[R(j\omega)M(j\omega)L^{-1}(j\omega)] < 1 \quad \forall\, \omega \geq 0 \tag{6}$$

This represents an extension of the Small Gain Theorem which we call the Small μ Theorem.

Since all simultaneous uncertainties can be put into block-diagonal form by merely constructing the associated matrix M, the SSV allows us to *nonconservatively analyze simultaneous occurances of* uncertainties anywhere in a feedback system. The uncertainties may be cone-bounded errors of individual components of the system (SISO or MIMO), they may be individual parameter variations in the model, or even polynomial approximations of parameters entering nonlinearly. In fact, the only restriction which remain is that all variations must be allowed to be complex. Pure real variations or pure imaginary variations cannot be separated into individual blocks.

3.3 Performance Characterization

The ability to treat simultaneous, structured uncertainties also offers, almost as a free byproduct, the ability to deal simultaneously with the performance and robustness aspects of feedback. This is made evident in Column 4 of Table 1, where each of the conditions imposed on feedback loop shapes by perturbation Δ_i at location ℓ_i is given a performance interpretation. For example, the perturbations in Row 4 impose requirements (through L and R) on the shape of the function $(I + KG)^{-1}$. This function is, of course, the classical (output) sensitivity function of the feedback loop. Small values over some frequency range guarantee low closed loop sensitivity to open loop variations and low command following errors to output commands over that range. A particular specification on these performance parameters can thus be imposed on a design by introducing a "fictitious uncertainty" at the location in Row 4 with cone bounds R and L selected to meet the performance requirement.

To illustrate how such fictitious uncertainties actually enforce performance specs, consider the simple case where a single true uncertainty, say Δ_r from Row 2, and a single fictitious (performance) uncertainty, say Δ_p from Row 4, are specified for our feedback system. Let the structured singular value condition (6) be satisfied for the corresponding M matrix (equation 4). Then the system remains stable in the face of Δ_r and Δ_p occurring simultaneously. Obviously, it will also remain stable for Δ_p with $\Delta_r = 0$. This means that the nominal system must satisfy the performance condition

$$\bar{\sigma}[R_p(I + KG)^{-1}L_p^{-1}] < 1 \qquad \forall \omega \geq 0 , \tag{7}$$

because the latter is also a necessary and sufficient condition for robust stability with Δ_p only. This much is straightforward. What is not so evident but much more important is that Condition (7) is also satisfied for all perturbed feedback loops. That is, for all true plants $G' = G(I + \Delta_r)$ we have

$$\bar{\sigma}[R_p(I + KG')^{-1}L_p^{-1}] < 1 \qquad \forall \omega \geq 0. \tag{8}$$

Hence, the performance spec is satisfied in the face of all possible true uncertainties. A proof of this consequence of the structured singular value condition is left to Section 5.

4. The Structured Singular Value and the Small μ Theorem

We have discussed how the problem of analyzing performance in the face of structured uncertainty can be expressed as a BDBP problem. We noted that standard singular value tests applied to the BDBP can be excessively conservative because they ignore the block diagonal structure. A more general nonconservative test (the Small μ Theorem) is developed in this section which removes this limitation. By nonconservative we mean providing a necessary and sufficient condition. The test is expressed in terms of a new measure, the structured singular value μ. This section begins with review of the results in [3], where μ was introduced.

To provide a more precise description of block diagonal perturbations, let $\mathcal{K} = (m_1, m_2, \ldots, m_n, k_1, k_2, \ldots, k_n)$ be a 2n-tuple of positive integers. All the definitions that follow depend on $\mathcal{K}$, but to simplify notation this dependency will not be explicitly represented. Let

$$k = \sum_{j=1}^{n} m_j k_j \quad \text{and} \quad m = \sum_{j=1}^{n} m_j .$$

For each $\delta \varepsilon [0,\infty)$, let $X_\delta \in \mathcal{M}(k)$ be a set of δ-norm-bounded block-diagonal matrices defined by

$$X_\delta = \{\mathrm{diag}(\underbrace{\Delta_1, \Delta_1, \cdots; \Delta_1}_{m_1}, \underbrace{\Delta_2, \Delta_2, \cdots; \Delta_2}_{m_2}, \Delta_3, \; \Delta_{n-1}, \; \underbrace{\Delta_n, \Delta_n, \cdots; \Delta_n}_{m_n}) \tag{9}$$

$$\Big| \Delta_j \varepsilon \mathcal{M}(k_j) \text{ and } \bar{\sigma}(\Delta_j) \leq \delta \text{ for each } j=1,2,..n\}$$

Let $X_\infty = \bigcup_{j=1}^{\infty} X_j$ be the set of all such matrices with no restriction on the norm, $\mathcal{U}$ be the set of block diagonal unitary matrices,

$$\mathcal{U} = U(k) \cap X_1$$

and $\mathcal{D}$ the set of real diagonal matrices such that

$$\mathcal{D} = \{\mathrm{diag}(d_1 I_k, d_2 I_k, \ldots d_{m_1} I_{k_1}, d_{m+1} I_{k_2}, \ldots, d_m I_{k_m}) \Big| \; d_i \varepsilon R^+ = (0, \infty)\} \tag{10}$$

What is desired is a function (depending on $\mathcal{K}$)

$$\mu : \mathcal{M}(k) \rightarrow [0, \infty) \tag{11}$$

with the property that for $\forall M \varepsilon \mathcal{M}(k)$

$$\det(I+M\Delta) \neq 0 \quad \forall \Delta \varepsilon X_\delta \qquad \text{iff} \quad \delta\mu(M) < 1 \tag{12}$$

This could be taken as a definition of μ. Alternatively, μ could be defined as

$$\mu(M) = \begin{cases} 0 & \text{if no } \Delta \varepsilon X_\infty \text{ solves } \det(I+M\Delta) = 0 \\ \left(\min\limits_{\Delta \varepsilon X_\infty} \{\bar{\sigma}(\Delta) \mid \det(I+M\Delta) = 0\} \right)^{-1} & \text{otherwise} \end{cases} \tag{13}$$

This definition shows that a well-defined function satisfies (12). It probably has little additional value since the optimization problem involved does not appear to have useful properties.

Using these definitions, the following useful properties of μ are easily proven.

1) $\mu(\alpha M) = |\alpha| \mu(M) \qquad \forall M \varepsilon \mathcal{M}(k)$

2) $\mu(I) = 1$

3) $\mu(AB) \leq \bar{\sigma}(A)\mu(B) \qquad \forall A, B \varepsilon \mathcal{M}(k)$

4) $\mu(\Delta) = \bar{\sigma}(\Delta) \qquad \forall \Delta \varepsilon X_\delta$

5) If $n=1$ and $m_1=1$ then $\mu(M) = \bar{\sigma}(M) \qquad \forall M \varepsilon \mathcal{M}(k)$

6) If $n=1$, $k_1=1$, then $k=m_1$,
$X_\delta = \{\lambda I \mid \lambda \varepsilon \;\; , |\lambda| \leq \delta\}$ and
$\mu(M) = \rho(M) \qquad \forall M\varepsilon \; (k)$

7) If $\Delta \varepsilon X_\delta$, $U \varepsilon \mathcal{U}$ then $U\Delta \varepsilon X_\delta$ and $\Delta U \varepsilon X_\delta$

8) For $\forall \Delta \varepsilon X_\infty$ and $\forall D \varepsilon \mathcal{D} \qquad D\Delta D^{-1} = \Delta$

9) For $\forall U \varepsilon \mathcal{U}$ and $M \varepsilon \mathcal{M}(k)$
$\mu(MU) = \mu(UM) = \mu(M)$

10) For $\forall D \varepsilon \mathcal{D}$ and $M \varepsilon \mathcal{M}(k)$
$\mu(DMD^{-1}) = \mu(M)$

11) $\max\limits_{U \varepsilon \mathcal{U}} \rho(UM) \leq \mu(M) \leq \inf\limits_{D \varepsilon \mathcal{D}} \bar{\sigma}(DMD^{-1}) \qquad$ for $\forall M \varepsilon \mathcal{M}(k)$

Properties 5) and 6) show that the structured singular value has as special cases both the spectral radius and the maximum singular value. Property 9) means that μ is $\mathcal{U}$-invariant.

The most important results from [3] are the following, which deal with the bounds in property 11):

a) The left-hand-side inequality in 11) is always an equality. This expresses μ in familiar linear algebraic terms, but the optimization problem involved may have multiple local maxima.

b) The right-hand-side inequality in 11) is an equality when there are three or fewer blocks, and the blocks are not repeated. The blocks themselves, and therefore M, may be of arbitrarily large dimension. A tedious but straightforward computation shows that the optimization problem involved is always convex [11]. Furthermore, the minimization is over only n-1 parameters for n blocks, independent of block size, making this an attractive alternative to a).

Note that the transformation DMD^{-1} is simply a rescaling of the inputs and outputs of M. The SSV is invariant with respect to such rescaling (property 10), while singular values do, of course, vary with rescaling. This implies, for example, that the ad hoc method of performing a change of units can reduce the conservatism associated with singular values. For some time we have been using Osborne's technique [12], which minimizes the Frobenius norm of DMD^{-1} to compute frequency-dependent D matrices. We now have new algorithms which compute D to directly minimize $\bar{\sigma}(DMD^{-1})$.

Numerical software for computing μ has been developed using algorithms based on these results. In addition to using this software to analyze some simple feedback designs, test runs have been made on a large number of psuedo-random matrices. It appears that the global maximum in a) is often easily found, although a simple gradient search is inadequate. Also, the bound obtained in b) appears to be quite good (to within 15%) for cases of more than 3 blocks. These observations are most encouraging, especially considering the experimental and preliminary nature of the software.

There are essentially two direct applications of singular values to the BDBP problem, which provide bounds for μ:

1) Ignore the block diagonal structure and compute $\bar{\sigma}(M)$. This gives an upper bound for μ.

2) Treat each perturbation one at a time. Compute the largest maximum singular value for each of the corresponding diagonal blocks. This gives a lower bound for μ.

The gap between these two bounds may be arbitrarily large.

An extension to 1) was proposed by Lehtomaki ([9], [13]), who uses the singular vectors for $\bar{\sigma}(M)$ to sharpen the bound. Lehtomaki's method checks for structure but not in the BDBP form. The optimism of 2) can be reduced by using a method suggested by Freudenberg, et. al. [14], who evaluate the differential sensitivity of the singular values at one point with respect to perturbations at another. Although this method does not apply to simultaneous, large perturbations, it can be quite useful in indicating when the lower bound for μ obtained by method 2) is optimistic. It should be mentioned that Lehtomaki and Freudenberg did not present their techniques in the context of the BDBP problem.

The preceding discussion of μ and the BDBP problem has dealt with determining the size of the minimum structured perturbation Δ that causes I + MΔ to be nonsingular. We are interested in using the structured singular value to answer robustness, sensitivity, and performance questions for multivariable feedback systems. The connection between μ and these essential feedback properties is provided by the Small μ Theorem, which characterizes the stability robustness properties of a feedback system with respect to block diagonal perturbations. In order to state the Small μ Theorem we need the following additional definitions (all depending on $\mathcal{X}$):

Let L, R $\in \mathcal{R} \cap \{X_\infty \times \mathbb{C}\}$ be such that L and R have no poles or zeros in the open right-half-plane. Then let

$$\mathcal{H} = \{\Theta \in \mathcal{R} \cap (X_\infty \times \mathbb{C}) \mid \bar{\sigma}(\Theta(s)) \leq 1 \quad \forall \operatorname{Re}(s) \geq 0\} \qquad (16)$$

$$\text{and } \mathcal{X} = \{L^{-1}\Theta R \mid \Theta \in \mathcal{H}\}. \qquad (17)$$

For the BDBP problem in Figure 1, $M \in \mathcal{R}(k)$, $\mathcal{X}$ is the set of allowable block diagonal perturbations, and L and R are the weightings for the Δ such that $\bar{\sigma}(L\Delta R^{-1}) \leq 1$. We will say the canonical system in Figure 1 is stable iff I+MΔ is nonsingular in the closed right-half-plane. Although this definition does not distinguish between ill-posedness and instability, it is adequate for our purposes. We can now state and prove the following:

<u>Theorem</u> (Small μ): The canonical system is closed loop stable for all $\Delta \in \mathcal{X}$ iff

$$\mu_c = \sup_\omega \mu(RML^{-1}) < 1 \qquad (18)$$

<u>Proof</u>: To prove the if part, suppose $\mu_c < 1$ and let $\Delta \in \mathcal{X}$. Then using Properties 3) and 11) and the definition of $\mathcal{X}$

$$\sup_{\operatorname{Re} s \geq 0} \rho(M\Delta) = \sup_{s=j\omega} \rho(M\Delta) \leq \sup_{s=j\omega} \mu(RML^{-1}) = \mu_c < 1.$$

Thus I + MΔ is nonsingular for all Res > 0. Since Δ was arbitrary, the canonical system is stable for all $\Delta \in \mathcal{X}$.

Conversely, suppose $\mu(RML^{-1})|_{\omega=\omega_0} \geq 1$ (ω_0 may be ∞).

Then $\exists\, \Theta \in X_1 \ni \det(I + RML^{-1}\Theta)|_{\omega=\omega_0} = 0$.

Thus, $\exists\, \Delta \in \mathcal{X} \ni \det(I + M\Delta)|_{\omega=\omega_0} = 0$ and the canonical system is not stable for all Δ ε H. □

This theorem guarantees that if $\mu(RML^{-1})$ is less than 1 at every frequency, then the closed-loop system is stable for all structured perturbations $\Delta \in \mathcal{X}$. Conversely, if $\mu(RML^{-1})$ is greater than or equal to 1 at some frequency, then there exists a structured perturbation $\Delta \in \mathcal{X}$ that results in closed-loop instability. Note that a destabilizing Δ can be expressed as $L\Theta R^{-1}$ for some constant Θ.

As noted in Section 2, the Small μ Theorem can also guarantee a pre-specified performance level by including a performance block in the BDBP problem. Furthermore, this performance level is guaranteed for all structured perturbations $\Delta \in \mathcal{X}$. These claims are made precise in the next section by a corollary to the Small μ Theorem that treats performance.

5. <u>Performance Implications</u>

Suppose that the plant perturbations are given by

$$\Delta_r = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_m) \in \mathcal{X}_r$$

with corresponding weighting matrices L_r, R_r, interconnection matrix M_r, and 2n-tuple $\mathcal{K}_r = (m_1, \ldots, m_n, k_1, \ldots, k_n)$. Suppose that a performance specification is given as

$$\bar{\sigma}(R_p M_p'(\Delta_r) L_p^{-1}) < 1 \qquad \forall \omega \quad \forall \Delta_r \in X_r \qquad (19)$$

Here M_p' is a $k_p x k_p$ performance matrix which we desire to be small (as weighted by R_p and L_p). Examples include $M_p'=(I + G'K)^{-1}$, $M_p=G'K(I + G'K)^{-1}$, etc., as discussed in Section 3. Note that M_p' depends on the perturbation Δ_r, indicating that this performance should be met for all uncertainties.

Let M_{pr} and M_{rp} denote the transfer function matrices between performance outputs and perturbation outputs and between perturbation inputs and performance inputs, respectively. In terms of these matrices, it can be shown that

$$M_p'(\Delta_r) = M_p + M_{pr}\Delta_r(I + M_r\Delta_r)^{-1} M_{rp}$$

where M_p is the performance matrix in the absence of uncertainties.

Define

$$M_T = \begin{bmatrix} M_p & M_{pr} \\ M_{rp} & M_r \end{bmatrix}$$

$$\mathcal{K}_T = (1, m_1, \ldots, m_n, k_p, k_1, \ldots, k_n)$$

$$L_T = \mathrm{diag}(L_p, L_r)$$

$$R_T = \mathrm{diag}(R_p, R_r)$$

$$\mathcal{H}_r = \mathcal{H}(\mathcal{K}_r)$$

$$\mathcal{H}_p = \mathcal{H}((1, k_r))$$

$$\mathcal{H}_T = \mathcal{H}(\mathcal{K}_T) \simeq \mathcal{H}_p \times \mathcal{H}_r$$

We have noted the dependence on $\mathcal{K}$ here to avoid confusion. Of course, the nominal interconnection matrix M is assumed to be stable.

For these definitions, the following relationship exists between performance, stability robustness, and the SSV.

<u>Theorem</u>: (Robust Performance)

$$M_p'(\Delta_r) \text{ stable and } \bar{\sigma}(R_p M_p' (\Delta_r) L_p^{-1}) < 1 \quad \forall \omega \text{ and } \forall \Delta_r \in \mathcal{K}_r \qquad (20)$$

$$\text{iff } \mu(R_T M_T L_T^{-1}) < 1 \quad \forall \omega$$

<u>Proof</u>: It follows from the Small μ Theorem that $\mu(R_T M_T L_T^{-1}) < 1 \quad \forall \omega$

iff $\left|I + R_T M_T L_T^{-1}\theta\right| > 0, \quad \forall \operatorname{Re} s \geq 0, \quad \forall \theta \in \mathcal{H}_T$

iff $\forall \operatorname{Re} s \geq 0, \; \left|I + R_r M_r L_r^{-1}\theta_r\right| > 0, \quad \forall \theta_r \in \mathcal{H}_r$

and $\left|I + R_p M_p' (L_r^{-1}\theta_r R_r) L_p^{-1} \theta_p\right| > 0, \quad \forall \theta_p \in \mathcal{H}_p$

iff $\forall \operatorname{Re} s \geq 0, \left|I + M_r \Delta_r\right| > 0$ and

$$\left|I + R_p M_p' (\Delta_r) L_p^{-1}\theta_p\right| > 0 \quad \forall \Delta_r \in \mathcal{K}_r, \forall \theta_p \in \mathcal{H}_p$$

iff $M_p' (\Delta_r)$ stable and $\bar{\sigma}(R_p M_p' (\Delta_r) L_p^{-1}) < 1$

for $\forall \omega$ and $\forall \Delta_r \in \mathcal{K}_r$ □

We note that this theorem extends the Small μ Theorem's robust stability results to a composite, simultaneous result on robust stability and performance. Thus, given an uncertain plant model with structured perturbations and a performance specification, we have a necessary and sufficient condition in terms of μ for satisfaction of the performance spec in the face of the uncertainty. If the condition μ < 1 is met, then the desired performance is achieved for all perturbed plants. If $\mu \geq 1$, then there exists a structured perturbation which causes the performance spec to be violated. The robust performance condition may be thought of as arising from an equivalent "fictitious uncertainty," although this interpretation is not necessary.

6. <u>Example</u>

In this section the ideas we have presented will be illustrated by a simple "textbook" example. An example similar to this was discussed in [8].

The configuration for this example is shown in Figure 2 and was discussed briefly in Section 3. The plant transfer function is

$$G(s) = \frac{1}{s} \begin{bmatrix} 10 & 9 \\ 9 & 8 \end{bmatrix} \qquad (21)$$

the uncertainty weightings are $L_1=L_2=I$, $R_1=r_1 I$, and $R_2 = r_2 I$, where

$$r_1(s) = 1 + s \quad \text{and } r_2(s) = 1 + \frac{1}{s} \qquad (22)$$

Interpreted in terms of uncertainty levels, these weightings mean that Δ_1 is large at high frequency and Δ_2 is large at low frequency. A design that yields $\mu < \alpha \;\; \forall \omega$ has the property that the closed loop system would remain stable for all simultaneous perturbations such that

$$\bar{\sigma}(\Delta_i) \leq \frac{r_i}{\alpha} \qquad \forall \omega \qquad (23)$$

Larger perturbations would destabilize the closed loop system. An alternative interpretation in terms of performance would be that low the closed loop system has the following output sensitivity

$$\bar{\sigma}[(I + KG')^{-1}] \leq \frac{\alpha}{r_2} \qquad \forall \omega \qquad (24)$$

in the face of all input perturbations which satisfy

$$\bar{\sigma}[\Delta_1] \leq \frac{r_1}{\alpha} \qquad \forall \omega \qquad (25)$$

The weighting r_2 emphasizes sensitivity at low frequency.

This example is not motivated by a physical design problem, and either interpretation is possible. We will simply compute μ for 3 different designs and compare them with each other rather than with a performance/robustness specification. We will also compute some singular value bounds and μ for the case of $r_1 = r_2 = 1$. No claims are made about the quality of these designs. They are intended merely to be illustrative.

The M matrix for this example is given in equation (4) and

$$\mu(RML^{-1}) = \inf_{d} \bar{\sigma} \begin{bmatrix} r_1T_1 & d\, r_1T_1G^{-1} \\ \frac{1}{d} r_2S_2G & r_2S_2 \end{bmatrix} \qquad (26)$$

where $T_1 = (I + KG)^{-1}KG$ and $S_2 = (I + GK)^{-1}$. The compensator matrices for the designs are

$$1)\ K_1 = \begin{bmatrix} -8 & 9 \\ 9 & -10 \end{bmatrix}$$

$$2)\ K_2 = Q_1Q_2 \qquad (25)$$

$$3)\ K_3 = Q_1 \begin{bmatrix} \frac{10(s+1)}{3s(s+16)} & 0 \\ 0 & \frac{9(16s+1)}{32s(s+1)} \end{bmatrix} Q_2$$

$$\text{where } Q_1 = \begin{bmatrix} 3/4 & -2/3 \\ 2/3 & 3/4 \end{bmatrix},\quad Q_2 = \begin{bmatrix} 3/4 & 2/3 \\ 2/3 & -3/4 \end{bmatrix}$$

In the first design, K_1 basically inverts the plant, making $KG = GK = 1/s\ I$. The unweighted singular values for the two uncertainties considered individually ($\bar{\sigma}(T_1)$ and $\bar{\sigma}(S_2)$) are plotted as the lower two curves in Figure 3. The maximum of these give a lower bound for the unweighted $\mu(M)$. The singular values of M are the upper two curves in Figure 3. The maximum of these give an upper bound for μ. The unweighted SSV itself is shown as the plots labelled with a 1 in Figure 4.

This example illustrates that μ can equal the upper bound $\bar{\sigma}(M)$ or lower bounds $\bar{\sigma}(T_1)$ and $\bar{\sigma}(S_2)$ or be anywhere in between. Note that this design can tolerate either Δ_1 or Δ_2 separately as large as 1, but simultaneous variations of less than 0.1 can be destabilizing. Interpreted in terms of performance for the weightings in (22), we can see that the design has good output sensitivity ($\bar{\sigma}(r_2S_2) \leq 1$) for the unperturbed plant. Small input perturbations, however, can lead to very poor output sensitivity since $\mu(RML^{-1}) >> 1$ implies that $\bar{\sigma}(r_2S_2') >> 1$ for some perturbed S_2'.

Designs 2 and 3 have substantially improved unweighted SSV's, as seen by the plots labelled 2 and 3 in Figure 4. It appears from these plots that Designs 2 and 3 are uniformly better than Design 1, with Design 2 best in the midfrequency range and Design 3 best at the high and low frequency extremes. The significance of these differences, however, can only be interpreted against the given performance and/or robustness specs. This is done with the corresponding weighted SSV's in Figure 5. We see that

$$\alpha = \max_{\omega} \mu[RML^{-1}] = \begin{cases} 18 & \text{for Design 1} \\ 18.5 & \text{for Design 2} \\ 5.5 & \text{for Design 3} \end{cases}$$

Thus, Design 3 offers the smallest α-value for equations (23) - (25) and is the best system when judged against the spec.

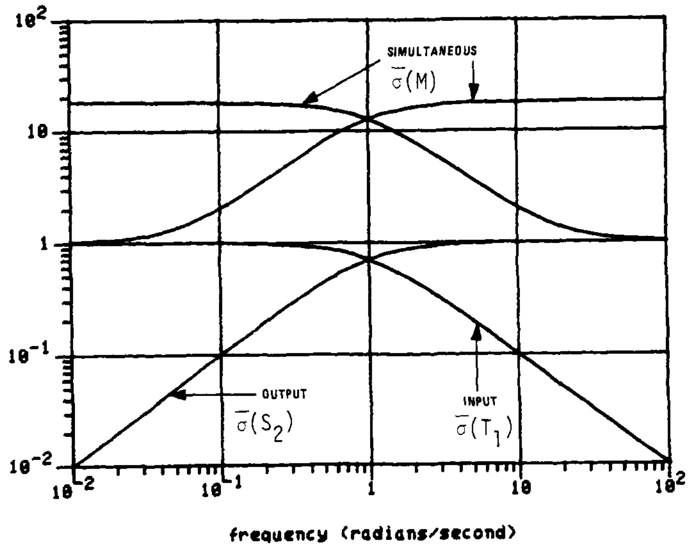


Figure 3. Singular value bounds for Design 1

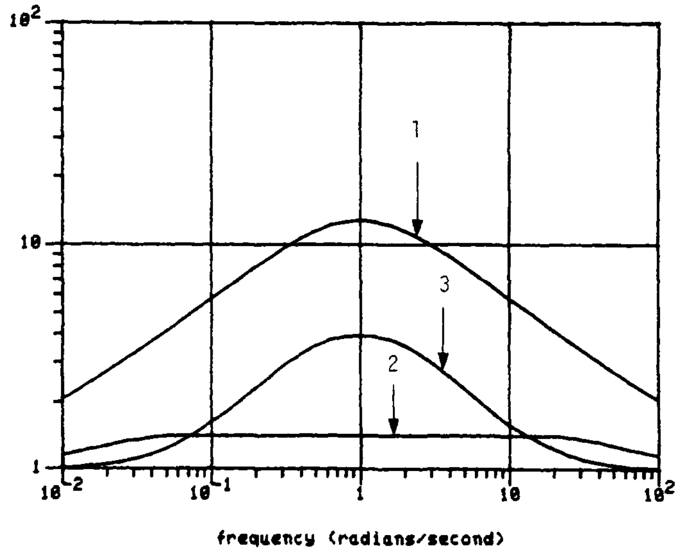


Figure 4. Unweighted μ for three designs

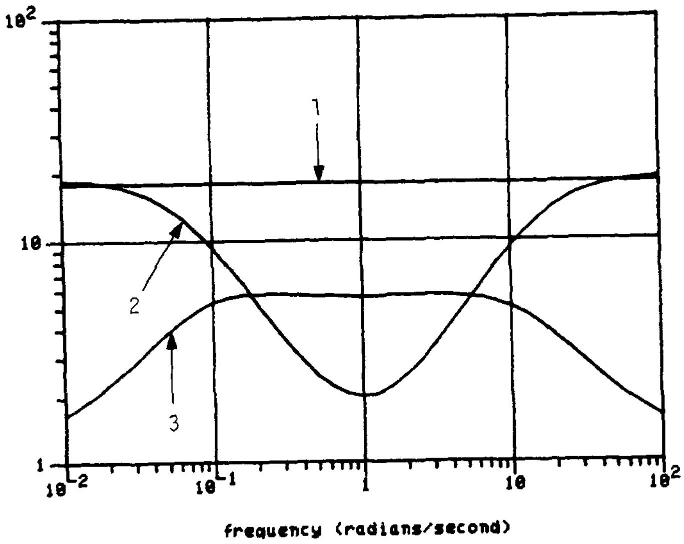


Figure 5. Weighted μ for three designs

This example has illustrated several important points:

1) Unscaled singular value bounds provide, in general, poor estimates for μ.

2) Stability/performance evaluations with perturbations one at a time can be highly optimistic.

3) Designs can be meaningfully compared only with respect to some performance and robustness specification.

7. Concluding Comments

This paper introduced an analysis technique based on the Structured Singular Value μ for linear feedback systems that provide a reliable, nonconservative measure of performance in the face of structured uncertainty. The Small μ Theorem gives a necessary and sufficient condition in terms of μ for stability of a linear system with multiple, simultaneous, norm-bounded perturbations of fixed but arbitrary structure. The Robust Performance Theorem provides a similar condition for the satisfaction of performance specifications in the presence of structured perturbations. Some simple feedback designs were presented to illustrate the theory.

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References

[1] I.M. Horowitz, Synthesis of Feedback Systems, New York: Academic, 1963.

[2] Special Issue on Linear Multi-Variable Control Systems, IEEE Trans. Automatic Control, February 1981.

[3] J.C. Doyle, "Analysis of Feedback Systems with Structured Uncertainties," Proc. IEE, November 1982.

[4] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems - Part I," IEEE Trans. Automatic Control, Volume AC-11, no. 2, pp. 228-238, April 1966.

[5] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems - Part II," IEEE Trans. Automatic Control, Volume AC-11, no. 3, pp. 465-476, July 1966.

[6] M.G. Safonov, Stability and Robustness of Multivarialbe Systems, Cambridge, MA: MIT Press, 1980.

[7] M.G. Safonov, "Tight Bounds on the Response of Multivariable Systems with Component Uncertainty," 16th. Allerton Conference, October, 1978.

[8] J.C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," IEEE Trans. on Automatic Control, Volume AC-26, No. 1, pp. 4-16, February 1981.

[9] N.A. Lehtomaki, "Practical Robustness Measures in Multivariable Control System Analysis," Ph.D. Dissertation, MIT, May 1981.

[10] C.A. Desoer, and M. Vidyasagar, Feedback Systems: Input-Output Properties. New York: Academic, 1975.

[11] J.C. Doyle and M.G. Safonov, "Convexity of the Block Diagonal Scaling Problem," Honeywell Internal Memo, August 1982.

[12] E.E. Osborne, "On pre-conditioning of matrices," I. Assoc. Comput., Mach, 7, 338-345, 1960.

[13] N.A. Lehtomaki, et. al., "Robustness Tests Utilizing the Structure of Modeling Error," Proc. 1981 CDC, San Diego, CA, December 1981.

[14] J.S. Freudenberg, D.P. Looze, and J.B. Cruz, "Robustness Analysis Using Singular Value Sensitivities," Int. J. Control, Vol. 35, No. 1, pp. 95-116, 1982.