Performance Bounds for Coded Free-Space Optical Communications Through Atmospheric Turbulence Channels

Xiaoming Zhu and Joseph M. Kahn, *Fellow, IEEE*

***Abstract*—Error-control codes can help to mitigate atmospheric turbulence-induced signal fading in free-space optical communication links using intensity modulation/direct detection (IM/DD). Error performance bound analysis can yield simple analytical upper bounds or approximations to the bit-error probability. In this letter, we first derive an upper bound on the pairwise codeword-error probability for transmission through channels with correlated turbulence-induced fading, which involves complicated multidimensional integration. To simplify the computations, we derive an approximate upper bound under the assumption of weak turbulence. The accuracy of this approximation under weak turbulence is verified by numerical simulation. Its invalidity when applied to strong turbulence is also shown. This simple approximate upper bound to the pairwise codeword-error probability is then applied to derive an upper bound to the bit-error probability for block codes, convolutional codes, and turbo codes for free-space optical communication through weak atmospheric turbulence channels. We also discuss the choice of interleaver length in block codes and turbo codes based on numerical evaluation of our performance bounds.**

***Index Terms*—Atmospheric turbulence, error-control coding, free-space optical communication, interleaver length, pairwise error probability (PEP), performance-bound analysis.**

I. Introduction

FREE-SPACE optical links using intensity modulation and direct detection (IM/DD) are useful in a variety of applications. However, atmospheric turbulence can greatly degrade the performance of free-space optical links, particularly over ranges of the order of 1 km or longer. Error-control coding can be applied to improve the error performance on such channels [1].

The theoretical error performance of coded systems over time-varying channels has been under research for many years [2]–[4]. In most wireless communication systems, the channel is not memoryless. The error performance of such continuous fading channels with memory often requires lengthy computer simulations. Performance-bound analysis has been widely adopted to study the error performance of communication systems. In this approach, we find an expression for the pairwise codeword-error probability and upper bound the codeword-error probability P_{block} by

$$P_{\text{block}} \leq \sum_{j, C_j \in S_C} P(C_j) \left[\sum_{\substack{k, C_k \in S_C \\ j \neq k}} P_e(Cj, C_k) \right] \tag{1}$$

where S_C is the set of all codewords and $P(C_j)$ is the probability that codeword C_j is transmitted. The pairwise error probability (PEP) $P_e(C_j, C_k)$ is the probability that when codeword C_j is transmitted, the decoder favors selection of an incorrect codeword C_k over C_j. With the knowledge of the weight enumerating function (WEF), we can further simplify the calculation of (1) and extend it to accurately estimate the error bound of constituent codes where the number of codewords is infinite, such as convolutional codes and turbo codes [5], [6]. Many error bounds have been introduced for radio-frequency channels, which often can be well-modeled as Rayleigh or Rician fading channels [3], [4].

In this letter, we will derive an error performance bound for coded on–off keying (OOK) free-space optical communication through atmospheric turbulence channels, where the fading is described by a joint log-normal distribution. The letter is organized as follows. In Section II, we review the joint log-normal distribution for turbulence-induced fading in OOK systems. In Section III, we derive the PEP assuming perfect knowledge of the channel state information. We also present simulations to verify the error bounds and define the limits of their applicability. In Section IV, we apply the PEP upper bound to study the error performance through weak atmospheric turbulence channels of various coding schemes including block coding, convolutional coding, and turbo coding. In Section V, we present simulation results of the approximate bit-error probability upper bound for some practical coding schemes, making use of the error performance analysis of Section IV. We also discuss the use of interleaving which, in conjunction with error-control codes, can help to mitigate correlated channel fading. We present our conclusions in Section VI.

II. System Model

In this letter, we consider IM/DD links using OOK. In most practical systems, the receiver signal-to-noise ratio (SNR) is limited by shot noise caused by ambient light which is much stronger than the desired signal and/or by thermal noise in the electronics following the photodetector [7]. In this case, the noise can usually be modeled to high accuracy as additive, white Gaussian noise (AWGN) that is statistically independent

Paper approved by R. D. Wesel, the Editor for Coding and Communication Theory of the IEEE Communications Society. Manuscript received October 18, 2000; revised July 29, 2002. This work has been supported by the Defense Advanced Research Projects Agency (DARPA) STAB Program under Contract DAAH01-00-C-0089 and by the DARPA MTO MEMS Program under Contract DABT63-98-1-0018. This paper was presented in part at the IEEE International Conference on Communications, Helsinki, Finland, June 11–15, 2001.

X. Zhu was with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720-1772 USA. He is now with QUALCOMM, Inc., Campbell, CA 95008 USA (e-mail: xzhu@qualcomm.com).

J. M. Kahn was with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720-1772 USA. He is now with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305-9515 USA (e-mail: jmk@ee.stanford.edu).

Digital Object Identifier 10.1109/TCOMM.2003.815052

of the desired signal. Let the bit interval be T, and assume that the receiver integrates the received photocurrent for an interval $T_0 \leq T$ during each bit interval. We further assume that $T_0 << \tau_0$, where τ_0 denotes the coherence time for atmospheric turbulence. Therefore, the light intensity can be viewed constant during each exposure interval. At the end of the integration interval, the resulting electrical signal can be expressed as

$$r_e = \eta(I_s + I_b) + n_W \tag{2}$$

where I_s is the received signal light intensity and I_b is the ambient light intensity. The optical-to-electrical conversion efficiency is given by

$$\eta = \gamma_e T_0 \cdot \frac{e\lambda}{hc} \tag{3}$$

where γ_e is the quantum efficiency of the photodetector, e is the electron charge, λ is the signal wavelength, h is Plank's constant, and c is the speed of light. The additive noise n_w is white and Gaussian, and has zero mean and variance $N_0/2$, independent of whether the received bit is off or on. In this letter, we assume that the receiver has knowledge of the ambient light bias ηI_b, and we denote the electrical signal to be $r = r_e - \eta I_b$ after subtraction of the ambient light bias.

For an OOK free-space optical communication system, we assume a n-bit sequence $\vec{s} = [s_1, s_2, \ldots, s_n]$ transmitted. Define the index subset of on-state symbol $S_{\mathrm{On}} = \{n_i \in \{1, 2, \ldots, n\}, s_{n_i} = 1\}_{i=1}^{m}$. We also have the index subset of off-state symbols $S_{\mathrm{Off}} = \{l_j \in \{1, 2, \ldots, n\}, s_{l_j} = 0\}_{j=1}^{n-m}$. Ignoring intersymbol interference (ISI), the receiver would only receive signal light through turbulence when on-state is transmitted. The ith on-state symbol intensity can be expressed as

$$I_{n_i} = I_0 \exp\left(2X_{n_i} - 2E[X]\right) \tag{4}$$

where X_{n_i} is the so-called log-amplitude of the optical signal and can be modeled as a Gaussian random variable with the ensemble average $E[X]$ and covariance σ_X^2. I^0 is the signal light intensity without turbulence. The joint probability density function (PDF) of a log-amplitude sequence $\vec{X} = [X_{n_1} - E[X], X_{n_2} - E[X], \ldots, X_{n_m} - E[X]]$ is jointly Gaussian [1]

$$f(\vec{X}) = \frac{1}{(2\pi)^{\frac{m}{2}} \left|C_X^{\mathrm{On}}\right|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\vec{X} \cdot \left(C_X^{\mathrm{On}}\right)^{-1} \cdot \vec{X}^T\right] \tag{5}$$

where C_X^{On} is the covariance matrix of the on-state bit sequence, as shown in (6) at the bottom of the page. As explained in [1], $b_X(d)$ is the normalized log-amplitude covariance function and the coherence time is

$$\tau_0 = \frac{d_0}{u_\perp} \tag{7}$$

where d_0 is the coherence length of turbulence-induced fading and $u_\perp$ is the wind velocity perpendicular to the light propagation direction. Here we assume $u_\perp$ to be constant and ignore its fluctuations.

The joint distribution of the signal intensity of on-state symbols is, therefore, joint log-normal [1]

$$f(I_{n_1}, I_{n_2}, \ldots, I_{n_m}) = \frac{1}{2^m \Pi_{i=1}^{m} I_{n_i}} \frac{1}{(2\pi)^{\frac{m}{2}} \left|C_X^{\mathrm{On}}\right|^{\frac{1}{2}}} \times \exp\left\{-\frac{1}{8}\left[\ln\left(\frac{I_{n_1}}{I_0}\right) \ldots \ln\left(\frac{I_{n_m}}{I_0}\right)\right] \left(C_X^{\mathrm{On}}\right)^{-1} \begin{bmatrix} \ln\left(\frac{I_{n_1}}{I_0}\right) \\ \cdots \\ \ln\left(\frac{I_{n_m}}{I_0}\right) \end{bmatrix}\right\}. \tag{8}$$

III. Pairwise Codeword-Error Probability Bound

In this analysis, we assume the turbulence-induced fading to be piecewise-constant during each bit interval and known to the receiver, i.e., receiver has perfect state information (SI) about the channel. The receiver utilizes maximum-likelihood (ML) soft decoding.

Consider two n-bit codewords, $C_j = [c_1^j, c_2^j, \ldots, c_n^j]$ and $C_k = [c_1^k, c_2^k, \ldots, c_n^k]$. Define the symmetric difference of their on-state symbol index subsets, i.e., the set consists of all those points that belong to one or the other of the two sets but not to both

$$S_{\mathrm{d}}^{C_j, C_k} = \left\{n_i \in \{1, 2, \ldots, n\}, c_{n_i}^j \neq c_{n_i}^k\right\}_{i=1}^{m_{j,k}} \tag{9}$$

and define its complement set

$$S_{\mathrm{c}}^{C_j, C_k} = \left\{n_i \in \{1, 2, \ldots, n\}, c_{n_i}^j = c_{n_i}^k\right\}_{i=1}^{n-m_{j,k}}. \tag{10}$$

The energy over the symmetric difference set of two codewords can be defined as

$$\varepsilon_{C_j, C_k} = \sum_{\substack{i=1 \\ n_i \in S_{\mathrm{d}}^{C_j, C_k}}}^{m_{j,k}} (\eta I_{n_i})^2. \tag{11}$$

Due to the linearity of the code, the PEP between C_j and C_k can be denoted by the PEP between a codeword whose index

$$C_X^{\mathrm{On}} = \begin{bmatrix} \sigma_X^2 & \sigma_X^2 b_X\left[\frac{|n_1 - n_2|T}{\tau_0} d_0\right] & \cdots & \sigma_X^2 b_X\left[\frac{|n_1 - n_m|T}{\tau_0} d_0\right] \\ \sigma_X^2 b_X\left[\frac{|n_2 - n_1|T}{\tau_0} d_0\right] & \sigma_X^2 & \cdots & \sigma_X^2 b_X\left[\frac{|n_2 - n_m|T}{\tau_0} d_0\right] \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_X^2 b_X\left[\frac{|n_m - n_1|T}{\tau_0} d_0\right] & \sigma_X^2 b_X\left[\frac{|n_m - n_2|T}{\tau_0} d_0\right] & \cdots & \sigma_X^2 \end{bmatrix}_{m \times m} \tag{6}$$

subset of on-state bits is the same as the symmetric difference set $S_{\mathrm{d}}^{C_j,C_k}$ and the all-zero codeword

$$E[P_e(C_j,C_k)] = E\left[Q\left(\sqrt{\frac{\varepsilon_{C_j,C_k}}{2N_0}}\right)\right] = \int_{\vec{X}} f(\vec{X}) \cdot Q\left(\sqrt{\frac{\varepsilon_{C_j,C_k}}{2N_0}}\right) d\vec{X}. \quad (12)$$

Since $Q(\sqrt{x}) \le 1/2\exp(-x/2)$, an upper bound on the PEP is

$$E\left[P_e(C_j,C_k)\right] \le \frac{1}{2} E\left[\exp\left(-\frac{\varepsilon_{C_j,C_k}}{4N_0}\right)\right]. \quad (13)$$

Defining the average SNR $\gamma = (\eta I_0)^2/N_0$, we can express (13) as

$$E\left[P_e(C_j,C_k)\right] \le \frac{1}{2} \cdot \int_{\vec{X}} f(\vec{X}) \cdot \exp\left\{-\frac{\gamma}{4} \sum_{\substack{i=1 \\ n_i \in S_{\mathrm{d}}^{C_j,C_k}}}^{m_{j,k}} [\exp(2X_{n_i})]^2\right\} d\vec{X}$$

$$\cong \frac{1}{2} \cdot \int_{\vec{X}} f(\vec{X}) \cdot \exp\left\{-\frac{\gamma}{4} \sum_{\substack{i=1 \\ n_i \in S_{\mathrm{d}}^{C_j,C_k}}}^{m_{j,k}} \left[\sum_{l=0}^{\infty} \frac{(4X_{n_i})^l}{l!}\right]\right\} d\vec{X}. \quad (14)$$

Under the assumption of weak turbulence ($\sigma_X << 1$), we can approximate the upper bound (14) as

$$E\left[P_e(C_j,C_0)\right] \le \frac{1}{2} \cdot \int_{\vec{X}} f(\vec{X}) \cdot \exp\left[-\frac{\gamma}{4} \sum_{l=0}^{2} \sum_{\substack{i=1 \\ n_i \in S_{\mathrm{d}}^{C_j,C_0}}}^{m_{j,0}} \frac{(4X_{n_i})^l}{l!}\right] d\vec{X} \quad (15)$$

$$E[P_e(C_j,C_0)] \le \frac{1}{2} \cdot \exp\left(-\frac{\gamma m_{j,0}}{4}\right) \cdot \left(\prod_{i=1}^{m_{j,0}} \frac{1}{\sqrt{1+4\gamma\lambda_i}}\right) \cdot \exp\left[\sum_{i=1}^{m_{j,0}} \frac{\gamma^2 u_i^2 \lambda_i}{2(1+4\gamma\lambda_i)}\right] \quad (16)$$

where λ_i is the ith eigenvalue of the covariance matrix $C_X^{j,0}$ and u_i is the sum of the elements in the corresponding eigenvector.

To verify the accuracy of the approximate upper bound (16), we calculate the PEP of two codewords whose difference set is $S_{\mathrm{d}}^{C_j,C_k} = \{1,3,4,6\}$ versus the average SNR, choosing $T/\tau_0 = 0.04$. The approximate upper bounds for log-amplitude variances $\sigma_X = 0.05$, 0.15, and 0.25 are indicated by the solid lines in Fig. 1. The pairwise codeword-error probability calculated using (12) is indicated by the dashed lines in Fig. 1 for comparison. We can see that the upper bound (16) is accurate under the assumption of small σ_X in weak turbulence region.

To demonstrate the limits of applicability of (16), we also consider larger values of σ_X in Fig. 2. In Fig. 2, we see when σ_X increases from 0.25 to 0.65, which should obviously result in higher error probability, the approximate PEP computed using (16) decreases instead. This is because terms of higher order in the log-amplitude, which we ignored in deriving (15), are no longer negligible when σ_X is large, even though the weak turbulence approximation made in (4) is still valid. Therefore, (16) is no longer valid for large σ_X.

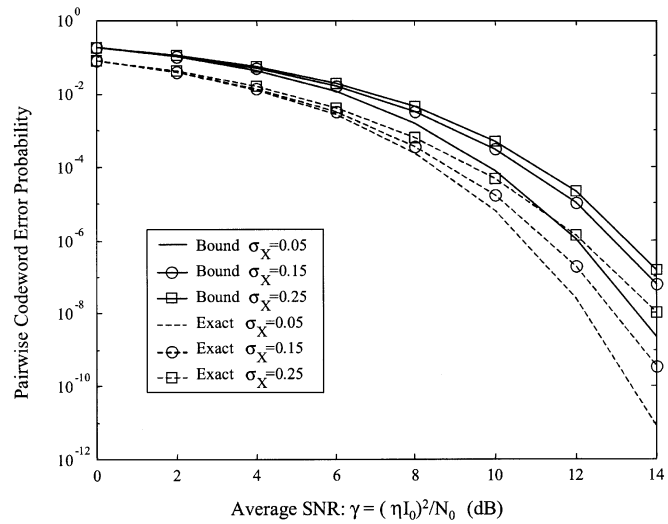

Fig. 1. Pairwise codeword-error probability versus average SNR using exact integration (dashed lines) and using approximate upper bounds (solid lines) for various log-amplitude variances ($\sigma_X = 0.05, 0.15, 0.25$).

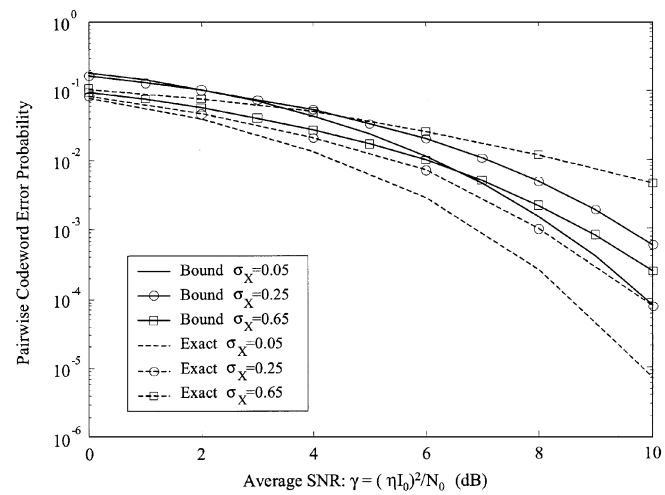

Fig. 2. Pairwise codeword-error probability versus average SNR using exact integration (dashed lines) and using approximate upper bounds (solid lines) for larger values of the log-amplitude variances ($\sigma_X = 0.05, 0.25, 0.65$).

Since most free-space optical communication systems will operate only when turbulence is weak, the approximate upper bound (16) should be widely applicable to estimate the PEP for long block codes and constituent codes, such as convolutional codes and turbo codes, which can be useful in optimizing the design of codes for free-space optical turbulence channels.

IV. Error-Probability Bounds for Various Coding Schemes

Using the approximate pairwise codeword-error upper bound that we have derived, we can compute upper bounds on the error probability for various coding schemes. In this section, we still

consider linear codes, and assume that all codewords are selected with equal probability. We derive upper bounds on the bit-error probability of block codes, convolutional codes, and turbo codes.

A. Block Codes

For a binary linear (n, k) block code with a set of codewords $\{C_0, C_1, \ldots, C_{2^k-1}\}$, where C_0 denotes the all-zero codeword, the average block-error probability with ML decoding is

$$E[P_{\text{block}}] \leq \sum_{j=1}^{2^k-1} E\left[P_e(C_j, C_0)\right] \tag{17}$$

and the average bit-error probability upper bound is

$$E[P_{\text{b}}] \leq \frac{1}{k} \cdot \sum_{j=1}^{2^k-1} B_j E\left[P_e(C_j, C_0)\right] \tag{18}$$

where B_j is the Hamming weight of the information sequence corresponding to codeword C_j.

It is straightforward to apply (16) in (17) and (18) to obtain performance bounds for block codes.

B. Convolutional Codes

Consider a rate $R = k/n$ linear convolutional code, where C_0 is the all-zero codeword and $\{C_j, j > 0\}$ is the set of nonzero codewords whose initial state is the all-zero state, which first remerge with the all-zero state at their final state. With ML decoding, the average bit-error probability can be upper bounded by

$$E[P_{\text{b}}] \leq \frac{1}{k} \cdot \sum_{j=1}^{\infty} B_j E\left[P_e(C_j, C_0)\right] \tag{19}$$

To estimate (19), we have to reorder the pairwise error elements by sorting the codewords according to their Hamming weights. Define $w_{\text{H}}(C_i, C_j)$ to be the Hamming distance between two codewords. Let S_w be the subset of codewords with Hamming weight w: $S_w = \{C_{j_k} | \, w_{\text{H}}(C_{j_k}, C_0) = w\}_{k=1}^{n_w}$, where n_w is the number of codewords in S_w. Equation (19) can be expressed as

$$E[P_{\text{b}}] \leq \frac{1}{k} \cdot \sum_{w=w_{\text{free}}}^{\infty} \left\{ \sum_{\substack{l=1 \\ C_{j_I} \in S_w}}^{n_w} B_{j_l} E\left[P_e(C_{j_l}, C_0)\right] \right\} \tag{20}$$

where w_{free} is the free distance of the code, which is the minimum Hamming weight of any codeword except C_0. From [2], at high SNR, we can simplify the sum in (20) by ignoring negligible terms with large Hamming weights. Our simulations results in Section V also show such an approximation is quite accurate at high SNR. We can approximate the average bit-error probability by

$$E[P_{\text{b}}] \cong \frac{1}{k} \cdot \sum_{\substack{l=1 \\ C_{j_l} \in S_{w_{\text{free}}}}}^{n_{w_{\text{free}}}} B_{j_l} E\left[P_e(C_{j_l}, C_0)\right]. \tag{21}$$

Applying (16) to (21), we obtain the approximate bit-error probability

$$\begin{aligned} E[P_{\text{b}}] \cong \frac{1}{2k} \cdot \exp\left(-\frac{\gamma w_{\text{free}}}{4}\right) \\ \cdot \sum_{\substack{l=1 \\ C_{j_l} \in S_{w_{\text{free}}}}}^{n_{w_{\text{free}}}} \left\{ B_{j_l} \cdot \left(\prod_{i=1}^{w_{\text{free}}} \frac{1}{\sqrt{1 + 4\gamma\lambda_i^{j_l}}} \right) \right. \\ \left. \cdot \exp\left[\sum_{i=1}^{w_{\text{free}}} \frac{\lambda_i^{j_l} \left(\gamma u_i^{j_l}\right)^2}{2\left(1 + 4\gamma\lambda_i^{j_l}\right)} \right] \right\}. \end{aligned} \tag{22}$$

We can also apply (16) to (20) to find an upper bound to the bit-error probability

$$\begin{aligned} E[P_{\text{b}}] \lesssim \frac{1}{2k} \cdot \sum_{w=w_{\text{free}}}^{\infty} \left\{ \exp\left(-\frac{\gamma w}{4}\right) \cdot \sum_{\substack{l=1 \\ C_{j_l} \in S_w}}^{n_w} \right. \\ \left. \times \left[B_{j_l} \cdot \left(\prod_{i=1}^{w} \frac{1}{\sqrt{1+4\gamma\lambda_i^{j_l}}} \right) \cdot \exp\left(\sum_{i=1}^{w} \frac{\lambda_i^{j_l}\left(\gamma u_i^{j_l}\right)^2}{2\left(1+4\gamma\lambda_i^{j_l}\right)} \right) \right] \right\}. \end{aligned} \tag{23}$$

In (22) and (23), $\lambda_i^{j_l}$ is the ith eigenvalue of the covariance matrix $C_X^{j_l,0}$ and $u_i^{j_l}$ is the sum of the elements in the corresponding eigenvector.

To simplify (23), we can define the fading-induced degradation factors

$$\alpha_w = \max_{C_{j_l} \in S_w, 1 \leq l \leq n_w} \prod_{i=1}^{w} \frac{1}{\sqrt{1 + 4\gamma\lambda_i^{j_l}}} \tag{24}$$

and

$$\beta_w = \max_{C_{j_l} \in S_w, 1 \leq l \leq n_w} \sum_{i=1}^{w} \frac{\lambda_i^{j_l} \left(u_i^{j_l}\right)^2}{2\left(1 + 4\gamma\lambda_i^{j_l}\right)}. \tag{25}$$

Note that (24) and (25) are functions of average SNR γ. The upper bound (23) can then be more simply expressed using

$$E[P_{\text{b}}] \leq \frac{1}{2k} \cdot \sum_{w=w_{\text{free}}}^{\infty} B^w \alpha_w \exp\left(-\frac{\gamma w}{4} + \beta_w \gamma^2\right) \tag{26}$$

where B^w is the sum of all B_{j_l} for which codeword C_{j_l} has Hamming weight w. Note that B^w can be obtained using the transfer function of the code [5], [6]. The upper bound (26) can also be applied to long block codes, for which (18) would involve a prohibitive amount of computation.

C. Turbo Codes

Turbo codes offer excellent performance in a variety of applications, including optical communications, and have aroused significant recent interest in the coding community. Turbo codes are parallel concatenated convolutional codes (PCCC) in which the encoder is formed by two or more constituent systematic recursive convolutional encoders joined through an interleaver [5], [6], [8], [9]. The information sequence is divided into blocks

of length equal to the interleaver length K. The input to the first encoder is the original information bit sequence, and the inputs to the other encoders are interleaved versions of the information block. The encoded sequence consists of the information sequence and the parity check bits from all encoders. Many decoding schemes and error-performance analyzes for turbo codes have been documented in the literature. An abstract uniform interleaver approach has been widely used to derive the average of the upper bounds obtainable for the whole class of deterministic interleavers [8].

We start with the definition of the input-redundancy weight enumerating function (IRWEF) for systematic convolutional code

$$A_{i,j}(W,Z) \equiv \sum_{w,z} A_{i,j}^{w,z} W^w Z^z \tag{27}$$

where i and j denote the initial and final states of the codewords. $A_{i,j}^{w,z}$ denotes the number of codewords generated by an input information word having Hamming weight w and having parity check bits of weight z, so that the overall Hamming weight of the systematic codeword is $w+z$. We can also define the conditional WEF

$$A_{i,j}^{w}(w,Z) \equiv W^w A_{i,j}^{C}(Z) = W^w \sum_{z} A_{i,j}^{w,z} Z^z. \tag{28}$$

Making use of the properties of a uniform interleaver [8], we obtain the average conditional WEF of all possible turbo codes with respect to the whole class of interleavers

$$A_{ip,jq}^{w}(w,Z) = W^w \cdot \frac{A_{i,j}^{C_1}(Z) \cdot A_{p,q}^{C_2}(Z)}{\binom{K}{w}} \tag{29}$$

where $A_{i,j}^{C_1}(Z)$ and $A_{p,q}^{C_2}(Z)$ are the conditional WEFs of two encoders, respectively, and K is the interleaver length. It has been shown when K is sufficiently large, we can accurately approximate the error performance with the paths which diverge from the off-states of both constituent encoders and remerge into off-states after K steps, which have WEF $A_{00,00}(W,Z) = \sum_{w,z} A_{00,00}^{w,z} W^w Z^z$. A tight bound for the pairwise codeword-error in correlated turbulence-induced fading channel requires knowledge of the positions of differing symbols, as we discussed above. For simplicity, we can loosen the bound by making the pessimistic assumption that all $d = w+z$ differing symbols are adjacent [8]. Therefore, the average bit-error probability is upper bounded as

$$\begin{aligned} E[P_{\rm b}] &\le \sum_{w,z} \frac{w}{K} \cdot A_{00,00}^{w,z} P_e(w+z) \\ &= \sum_{w=1}^{K} \frac{w}{K} \cdot \left[\sum_{d \ge w} A_{00,00}^{w,d-w} P_e(d) \right] \end{aligned} \tag{30}$$

where $P_e(d)$ is the pairwise codeword-error bound of the all-zero codeword and the codeword with d adjacent on-state bits. Let us define

$$\left[\frac{W}{K} \cdot \frac{\partial A_{00,00}(W,Z)}{\partial W} \right] \Bigg|_{W=Z=D} = \sum_{d} A_d D^d \tag{31}$$

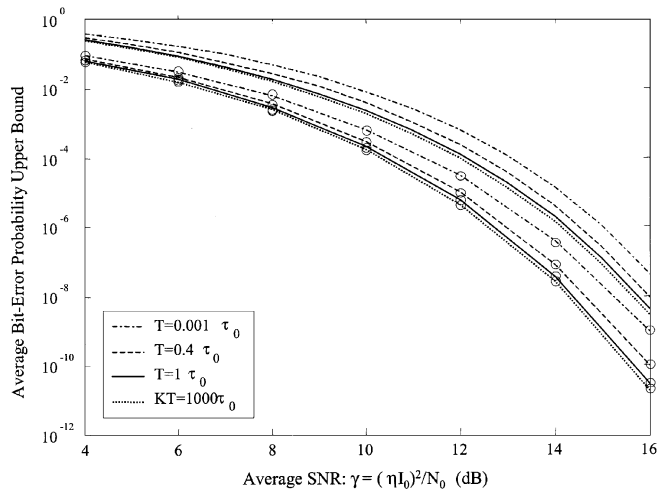

Fig. 3. The simulated bit-error probability (lines with circles) and the approximate upper bound (lines without circles) for (7, 4) Hamming codes versus average SNR. The log-amplitude variance $\sigma_X = 0.2$ and the ratio of the adjacent codeword bit interval T to the coherence time of turbulence-induced fading (T/τ_0) is chosen to be 0.001, 0.4, and 1, respectively. We also show the simulation results where a block interleaver is used with $KT/\tau_0 = 1000$.

which allows us to express (30) as

$$E[P_{\rm b}] \le \sum_{d} A_d P_e(d). \tag{32}$$

We can simplify (32) by ignoring the negligible higher order terms with larger Hamming weights as discussed in Section IV-B.

V. Results

In this section, we will use the approximate error probability upper bounds derived above to numerically evaluate the performance of some practical codes.

We first study a Hamming (7, 4) code. In Fig. 3, we plot the approximate bit-error probability upper bound (lines without circles) versus average SNR with log-amplitude variance $\sigma_X = 0.2$. T/τ_0 takes on the values 0.001, 0.4, and 1, respectively. We also present the bit-error probability estimated using Monte–Carlo simulation (lines with circles) for comparison. Fig. 3 shows that when T/τ_0 increases, the average bit-error probability for (7, 4) Hamming codes will decrease. Therefore, we can implement interleaving to compensate for the coding gain penalty due to the memory of channel. Consider the normal block interleaver of degree K [2], where the codewords are interleaved so that the adjacent bits of coded bit sequence are transmitted at intervals of KT. Fig. 3 shows that interleaving will improve the bit-error performance when KT/τ_0 is large. However, when $KT > \tau_0$, further increase of the interleaver depth will not significantly improve the bit-error performance. As shown in Fig. 3, when KT/τ_0 increases from 1 to 1000, the corresponding average upper bounds to the bit-error probability are very close. Therefore, it is not necessary to further increase the interleaver depth when $KT > \tau_0$.

For convolutional codes, we consider the example of a rate-1/3 code, whose encoder diagram is shown in Fig. 4. We choose the log-amplitude variance $\sigma_X = 0.2$. T/τ_0 takes on

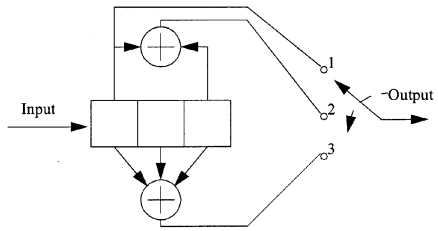

Fig. 4. Rate 1/3 convolutional encoder with three-stage shift registers.

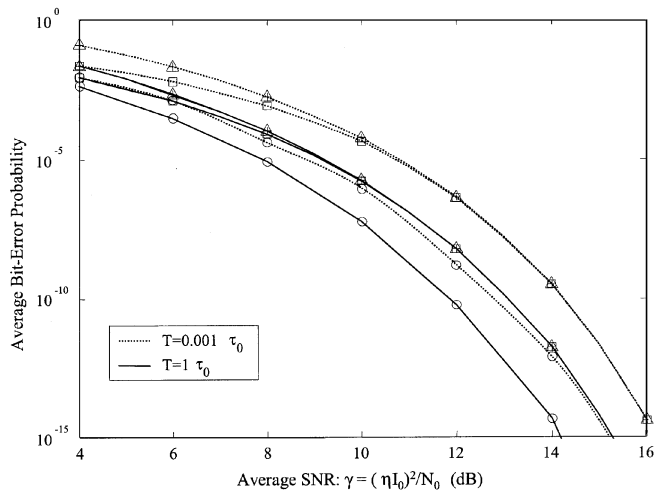

Fig. 5. Average bit-error probability for a three-stage, rate-1/3 convolutional code versus average SNR with log-amplitude variance $\sigma_X = 0.2$ and for $T/\tau_0 = 0.001$ and 1. The estimates are provided by (22) (lines with squares), (23) (lines with triangles) and (26) (lines with crosses), respectively. The simulated average bit-error probability (lines with circles) is also shown for comparison.

the values 0.001 and 1, respectively. In Fig. 5, the approximate bit-error probability versus average SNR, computed using (22) (lines with squares), (23) (lines with triangles) and (26) (lines with crosses), respectively. The simulated bit-error probability is also shown (lines with circles) in Fig. 5 for comparison. We see that (26) yields a very accurate estimate of the approximate upper bound (23). Also, when the SNR is high, the higher Hamming weight terms in (23) are negligible, and the approximate bit-error probability (22) merges with the approximate upper bound (23). Interleaving can help to improve the system performance for convolutional codes as well. Similar to the block-coding case, the interleaver depth K is sufficient when it satisfies $KT > \tau_0$.

Finally, we present simulations for a rate-1/3 turbo-coded system whose encoder structure is shown in Fig. 6. The approximate bit-error probability upper bound is calculated using (32) with log-amplitude variance $\sigma_X = 0.2$ and $T/\tau_0 = 0.001$. The interleaver length K of the uniform interleaver takes on the values 100, 1000, and 10000, respectively. In Fig. 7, we plot the approximate bit-error probability upper bound (lines without circles) and the simulated bit-error probability (lines with circles) versus the average SNR for comparison. Comparing the simulation results shown in Fig. 7 to those of the rate-1/3 convolutional code in Fig. 5, we see turbo coding can achieve better bit-error probability performance through atmospheric turbulence-induced fading channels if the interleaver length is sufficiently long. In Fig. 7, we see that the error-probability performance of turbo codes continues to improve with increasing interleaver length even when $KT > \tau_0$, especially at low average SNR [8]. This favors the use of longer interleavers with turbo codes. However, system complexity and delays in coding and decoding will limit the length of interleavers in practical systems using turbo codes.

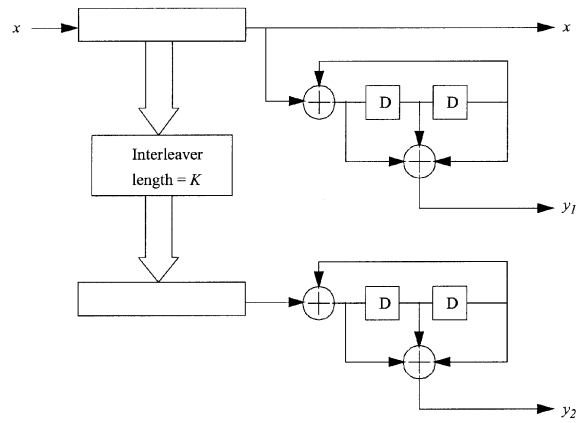

Fig. 6. Encoder structure of a rate-1/3 turbo code with uniform interleaver of length K.

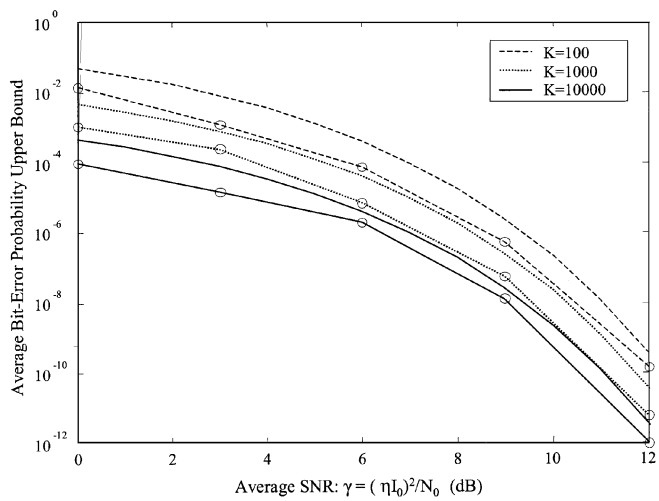

Fig. 7. The simulated bit-error probability (lines with circles) and the approximate upper bound (lines without circles) of a rate-1/3 turbo code versus average SNR with log-amplitude variance $\sigma_X = 0.2$ and $T/\tau_0 = 0.001$. A uniform interleaver with different interleaver depths $(K = 100, 1000, 1000)$ is considered.

VI. Conclusions

Error-control codes can help to mitigate turbulence-induced signal fading in wireless optical communication through atmosphere turbulence. To study the efficiency of various coding schemes, performance bound analysis has been used for its simplicity.

In this letter, we first derived an upper bound on the pairwise codeword-error probability for correlated atmospheric turbulence channels. To avoid complicated multidimensional integration, we have also derived an approximation for this upper bound under the assumption of weak turbulence. The accuracy and the limits of applicability of this approximation

have been demonstrated using numerical simulations. We then applied this approximate upper bound to derive the error performance bounds and their approximations for various coding schemes through atmospheric turbulence channels, including block coding, convolutional coding, and turbo coding. The analytical upper bounds were then applied to compare the performance of a few specific example codes. The effect of varying the interleaver depth was studied through numerical evaluation of the performance bounds.

REFERENCES

[1] X. Zhu and J. M. Kahn, "Free-space optical communication through atmospheric turbulence channels," *IEEE Trans. Commun.*, vol. 50, pp. 1293–1300, Aug. 2002.

[2] J. G. Proakis, *Digital Communication*, 3rd ed. New York: McGraw-Hill, 1995.

[3] F. Gagnon and D. Haccoun, "Bounds on the error performance of coding for nonindependent Rician-fading channels," *IEEE Trans. Commun.*, vol. 40, pp. 351–360, Feb. 1992.

[4] G. Kaplan and S. Shamai, "Achievable performance over the correlated Rician channel," *IEEE Trans. Commun.*, vol. 42, pp. 2967–2978, Nov. 1994.

[5] T. M. Duman and M. Salehi, "Performance bounds for turbo-coded modulation systems," *IEEE Trans.Commun.*, vol. 47, pp. 511–521, Apr. 1999.

[6] K. Kiasaleh, "Turbo-coded optical PPM communication systems," *J. Lightwave Technol.*, vol. 16, pp. 18–26, Jan. 1998.

[7] V. Hsu, J. M. Kahn, and K. S. J. Pister, "Wireless Communications for Smart Dust," Univ. of California, Berkeley, CA, Electron. Res. Lab. Tech. Memo. M98/2, Feb. 1998.

[8] S. Benedetto and G. Montorsi, "Unveiling turbo codes: some results on parallel concatenated coding schemes," *IEEE Trans. Commun.*, vol. 42, pp. 409–428, Mar. 1996.

[9] E. K. Hall and S. G. Wilson, "Design and analysis of turbo codes on Rayleigh fading channels," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 160–174, Feb. 1998.