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Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator

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A driven anharmonic oscillator is described which exhibits period doubling and chaotic behavior. The measured behavior of the oscillator under successive period doublings is in quantitative agreement with a recent theory which describes the behavior of nonlinear systems. Both the scaling and the convergence rate predicted by the theory are verified by the experiment. The oscillator also exhibits period tripling and quintupling.

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A recent theory by Feigenbaum of nonlinear systems which exhibit period doubling[1,2] predicts that these systems should behave in a universal manner independent of the precise equations which govern their dynamics. Consider a system described by the coupled differential equations[3]

$$dx_i/dt = F_i(x_1, x_2, \ldots, x_N, \lambda), \quad i = 1, 2, \ldots, N,$$

for which the $x_i(t)$ are periodic with period $T_n = 2^n T_0$ at $\lambda = \lambda_n$. For example, in the nonlinear electrical network described below, the x_i correspond to the charge and current flowing in the circuit and λ is the amplitude of the driving voltage. The theory predicts that the modulation parameter, λ, should asymptotically satisfy the recurrence relation

$$(\lambda_{n+1} - \lambda_n)/(\lambda_{n+2} - \lambda_{n+1}) = \delta, \qquad (1)$$

where δ is a universal convergence rate that depends only on the nature of the F_i near an extremum. For a quadratic extremum $\delta = 4.669\ldots$. When n is large, the odd frequency components of the Fourier spectrum are related by[3,4]

$$x_{(2k+1)}{}^{(n+1)} \simeq \frac{1}{2\alpha}\left[1 - i(-1)^k\right]\left(1 - i\frac{(-1)^k}{\alpha}\right) \times \sum_{k'} \frac{1}{\pi i} \frac{x_{(2k'+1)}{}^{(n)}}{(2k'+1) - \frac{1}{2}(2k+1)}.$$

Here the quantity $x_{(2k+1)}{}^{(n)}$ is the complex Fourier amplitude for the frequency $(2k+1)/2^n T_0$ and $\alpha = 2.5029\ldots$ is the universal rescaling factor. If the phase of the Fourier amplitudes is not available, the relation can be approximated by interpolating between the components $x_{(2k+1)}{}^{(n)}$ and then rescaling by[4]

$$\mu = 4\alpha/[2(1+\alpha^{-2})]^{1/2} = 6.57,$$

or

$$10\log_{10}\mu = 8.2 \text{ dB},$$

to predict the new Fourier components. Once a Fourier component appears at a particular frequency it is predicted to remain essentially constant through any succeeding period doublings. The rescaling of the Fourier amplitudes has been seen previously in an experiment on turbulent flow.[5,6] This experiment is the first to measure the convergence rate δ.

The anharmonic oscillator built for this experiment was an *RLC* circuit (Fig. 1) driven by a sinusoidal voltage. The nonlinear element was a capacitor whose capacitance varied as a function of the voltage across it. The device used was a varactor diode, IN5470A,[7] with a capacitance which varied as

$$C(V) = \frac{C_0}{(1 + V/\varphi)^\gamma}.$$

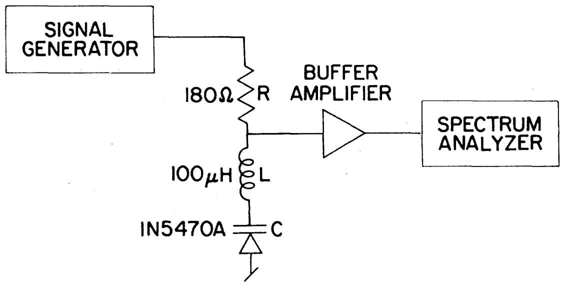

FIG. 1. Experimental apparatus for subharmonic generation.

Here V is the voltage across the diode and is positive for reverse bias. The measured values of capacity were in excellent agreement with this formula for the parameter values $C_0 = 81.8$ pF, $\varphi = 0.6$ V, and $\gamma = 0.44$. The circuit was driven sinusoidally from a signal generator with a 50-Ω source impedance and the output was analyzed by a spectrum analyzer which measured the power contained in each spectral peak.

At very low drive voltages the circuit behaved as a linear RLC circuit with a resonance at $f_1 = 1.78$ MHz. The nonlinear behavior was observed by driving the circuit at this frequency and gradually increasing the drive voltage. At low voltages the circuit displayed the frequency multiplication typical of all nonlinear circuits. The first subharmonic appeared at a drive voltage of 1.9 V. The saturated subharmonic peaks which appear at successive period doublings are shown in Figs. 2(a)–2(d). It is clear from the data that once a spectral peak has appeared and reached full magnitude it remains essentially unchanged through any further period doublings. In Fig. 2(d) the fundamental spectral peak is labeled 0, and the subharmonic peaks are labeled 1 through 4 according to their order of appearance after successive period doublings. The three lines in the figure are a straight line interpolating between the $n = 2$ peaks and two lines, respectively, 8.2 and 16.4 dB below it. The third and fourth generation spectral peaks are in remarkable agreement with the theoretical prediction. Noise in

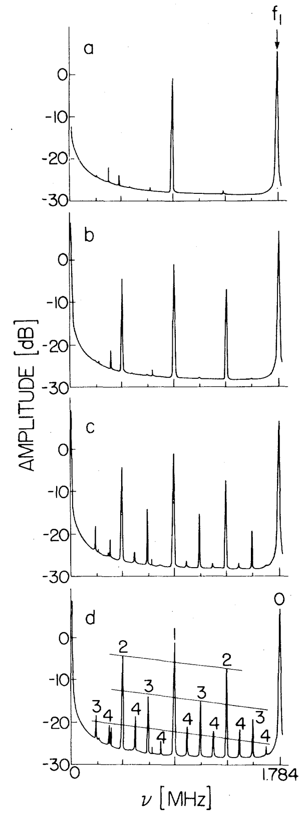

FIG. 2. (a) – (c) Subharmonic spectrum for successive period doublings; (d) final period doubling and comparison with theory.

TABLE I. Measured value of the convergence rate δ.

Subharmonic	$\Delta V_{\text{threshold}}$ (V)	δ_n
$f_1/2$		
	3.2 ± 0.02	
$f_1/4$		4.4 ± 0.1
	0.72 ± 0.02	
$f_1/8$		4.5 ± 0.6
	0.16 ± 0.02	
$f_1/16$		

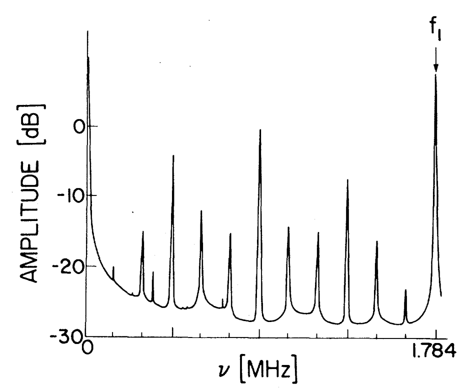

FIG. 3. Subharmonic spectrum prior to the onset of the chaotic spectrum.

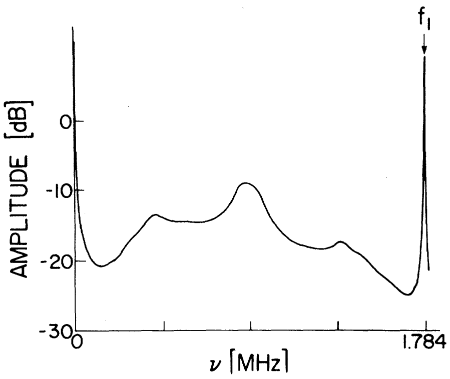

FIG. 4. Fully developed chaotic spectrum.

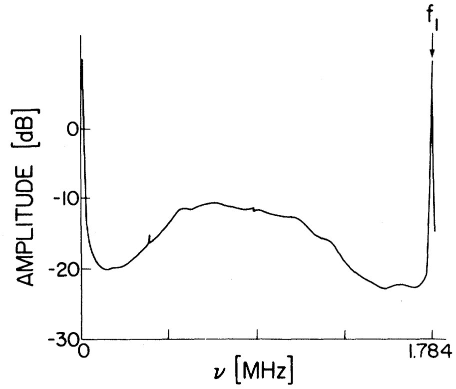

FIG. 6. Secondary chaotic spectrum.

the electronics prevented testing this prediction for fifth generation and higher period doublings.

The convergence rate was obtained by measuring the difference in threshold voltages between period doublings and then using Eq. (1) to calculate δ. Table I lists the subharmonics, the measured voltage differences between the appearance of successive subharmonics, and the calculated value of δ. The predicted value of 4.67 is in good agreement with the measured values.

Increasing the drive voltage further produced chaotic behavior. From the voltage differences in Table I the theory would predict that after a voltage increase of 44 mV the subharmonic spectrum should exhibit an infinite sequence of period doublings with geometrically decreasing Fourier amplitudes. This is unobservable because of the finite sensitivity of the experiment. Raising the voltage first produced noise between the $f_1/4$ subharmonics which was then replaced by subharmonics at multiples of $f_1/12$ (Fig. 3). Beyond this point the subharmonic spectrum became chaotic (Fig. 4). The noise spectrum is definitely not white noise and displays strong enhancements at $f_1/4$, $f_1/2$, and $3f_1/4$.

Further increases in drive voltage produced period quintupling (Fig. 5), noise (Fig. 6), and period tripling (Fig. 7). (The subharmonic at $f_1/2$ is due to subharmonic generation in the spectrum analyzer input amplifier.) Period doubling then produced subharmonics between the $f_1/3$ spectral peaks [Figs. 8(a) and 8(b)]. Unfortunately, Feigenbaum's theory could not be tested here because the spectrum analyzer did not have sufficient sensitivity to detect any further spectral peaks. The approach to chaos of this system is closely paralleled by the band mergings of iterated maps[8] and the order of appearance of new subharmonics is consistent with the U sequences of Metropolis, Stein, and Stein.[9] Other sequences of subharmonics can be generated by varying the value of the resistance.

A separate experiment was carried out in order to determine the nonlinearity which generated the

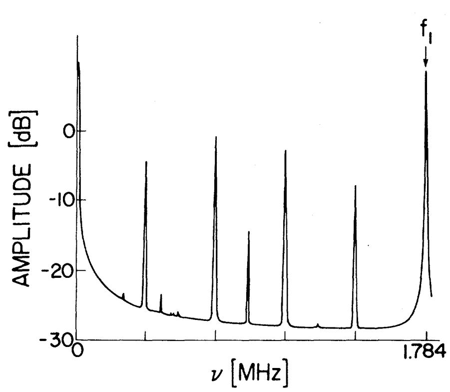

FIG. 5. Period quintupling.

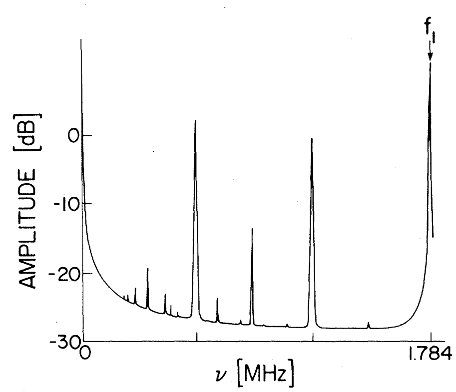

FIG. 7. Period tripling.

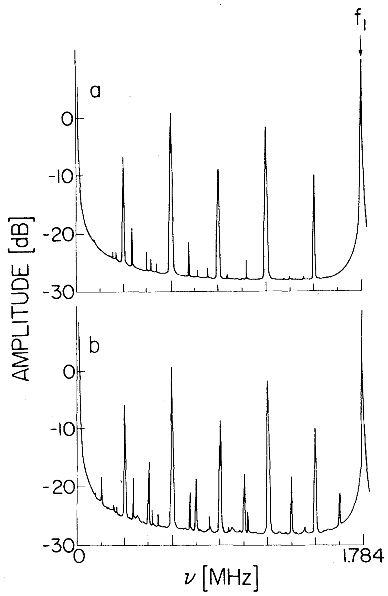

FIG. 8. (a) and (b) Period tripling with additional period doubling.

subharmonics. The varactor was replaced by an 80-pF capacitor in parallel with a 1N4154 diode. This circuit had essentially the same resonant frequency as before but showed no period doubling up to the largest drive voltages available (about 15 V peak to peak). Thus the subharmonic generation was due to the nonlinear capacity and can be described by the coupled equations

$$V_c = \frac{Q}{C(V_c)} = \left(1 + \frac{V_c}{\varphi}\right)^{\gamma} \frac{Q}{C_0} ,$$

$$dQ/dt = I,$$

$$L\,dI/dt = V_0 \sin(2\pi f_1 t) - V_c - RI,$$

where Q is the charge and I is the current flowing in the circuit. The solutions of these equations exhibit behavior similar to the data.

In conclusion, Feigenbaum's predictions of universal convergence and rescaling are in excellent agreement with the behavior of a driven anharmonic oscillator. The simplicity of the system described here and the richness of its behavior allow straightforward tests of other theories of subharmonic generation and chaotic behavior in nonlinear systems.

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Note added.—Since this work was submitted, Giglio, Musazzi, and Perini[10] have also reported the convergence rate δ.

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