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Period-Doubling of Kink-Antikink Patterns, Quasiperiodicity in Antiferro-Like Structures and Spatial Intermittency in Coupled Logistic Lattice*)

— *Towards a Prelude of a "Field Theory of Chaos"* —

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Qualitative features of a one-dimensional lattice of coupled-logistic maps are investigated. First, kink-antikink patterns of 2^n-periodic cycles with their period-doubling bifurcations are found. Secondly, antiferro-like structures with some kinks are observed, which show the transition from torus to chaos. Lastly, spatial intermittent structures are investigated, with the emphasis on the propagation of bursts.

§ 1. Introduction and models

Recent studies on low-dimensional dynamical systems have made great advances, which elucidate various aspects of chaos and the mechanism of its onset.[1] The success of the low-dimensional theory is, however, limited to systems with a few number of excited modes, which are relevant near the onset of turbulence owing to Ruelle and Takens' picture[2),3)] and abundance of phase lockings.[4)~6)] Then a question arises; what happens in a system with a large number of excited modes? Such a question is important for the study of fully-developed turbulence, chemical turbulence,[7] optical turbulence,[8] nonlinear field theory[9] and a pattern formation theory. The main topics are as follows: characterization of patterns, bifurcations of a solution with a spatial structure and transitions to chaos, characterization of spatial complexity, estimation of the fractal dimensions and Lyapunov spectra,[10),11)] validity of a low-dimensional theory, stability of a direct product state, and phase transitions in spatial structures.

In the present paper we report some preliminary results on such questions by making use of coupled maps.[12)~15)] The models we study in the present paper are given by

$$\text{(I)}\quad x_{n+1}(i)=f(x_n(i))+D((x_n(i+1)+x_n(i-1))/2-x_n(i)),^{12)}$$

$$\text{(II)}\quad x_{n+1}(i)=f(x_n(i))+D'((f(x_n(i+1))+f(x_n(i-1)))/2-f(x_n(i))),^{14)}$$

where $i=1, 2,\cdots, N$ denote one-dimensional lattice sites with periodic boundary condition $(x_n(N+1)=x_n(1))$. The function $f(x)$ is chosen to be $1-Ax^2$ (i.e., coupled logistic map). The number N is chosen to be 100 in most cases. See Ref. 12) for the case with $N=2$. (For Model II, we use the notation $D=D'A$.)

§ 2. Period-doublings of kink-antikink patterns

One characteristic pattern of the coupled logistic lattice (CLL) (I and II) appears for

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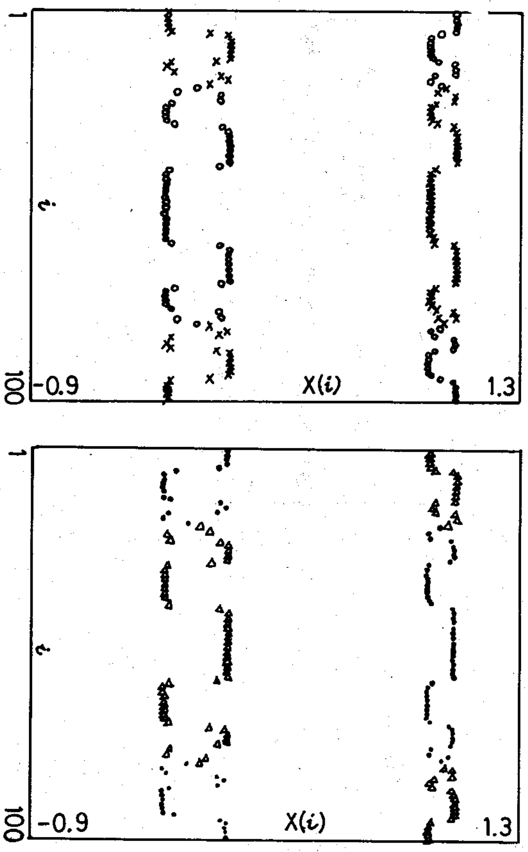

Fig. 1. Pattern of a 4-cycle. Plotted are $x_n(i)$'s for $n=400$ (○), $n=401$ (×), $n=402$ (△), and $n=403$ (●) for Map (II) with $A=1.30$, $D=0.05$ and the initial condition $x(i)=\sin(2\pi i/N)$, after the transients have already decayed out.

the parameter region A where the period-doubling bifurcations proceed for the logistic map. The pattern is characterized by flat regions and domain boundaries (kinks or antikinks). The phase of the periodic oscillation (with a period 2^n) varies among flat regions but is the same within a flat region. An example of the pattern is shown in Fig. 1, where the initial condition is given by $x_0(i)=\sin(2\pi i/N)$ and the period is 4. As A is increased, period-doubling bifurcations proceed, which lead to chaos*) (see Fig. 2(a) for a 2^4-periodic cycle and Fig. 2(b) for chaos). We note the following features:

i) Doubling occurs only a finite number of times in general.[17] The number can depend on initial conditions, coupling D, and the size N.

ii) The width of a kink is rather small (5 or 6 sites in maximum). It increases as the coupling is increased for Model II.

iii) Period-doubling brings about the spatial structure with more complexity. Structures with smaller wavelengths appear owing to the doubling. Small spatial structures are feasible to disappear as the increase of the coupling (i.e., a structure with a fewer kinks appears).

iv) Even after the transition to chaos occurs, patterns of kink-antikinks are conserved and chaos is localized in each domain. As the nonlinearity A is further increased, the structure collapses.

§ 3. Transition from torus to chaos in antiferro-like structures

Another interesting pattern in CLL is an "antiferro-like" structure, which is shown in Fig. 3 (the figure shows a cycle with period two). It is characterized by an alternate structure with a wavelength two. This structure is remarkably seen in Model I though it can be seen in a narrower parameter region for Model II. We note that there can be kinks in the antiferro-like structure as can be seen in Fig. 3, which is rather analogous to solitons in polyacetylene.[18] As the nonlinearity A is increased, there occurs a Hopf bifurcation and a torus appears (see Fig. 4(a)) for the attractor in $x(1)$-$x(2)$ space). The shape of the projected torus (Fig.4(a) for example) differs by the projected space (see, i.e., Fig. 4(b) for the projection into $x(1)$-$x(50)$ space). As the nonlinearity is increased further, the torus is modulated (3-torus is expected[19]) and finally chaos appears (see Fig. 4(c)).[12),20)]

*) In the present paper, we regard the attractor as chaos if the pattern is irregular and the time series of $x(i)$ does neither have a cycle with the period less than 1000, nor does it show a quasiperiodic behavior as in §3.

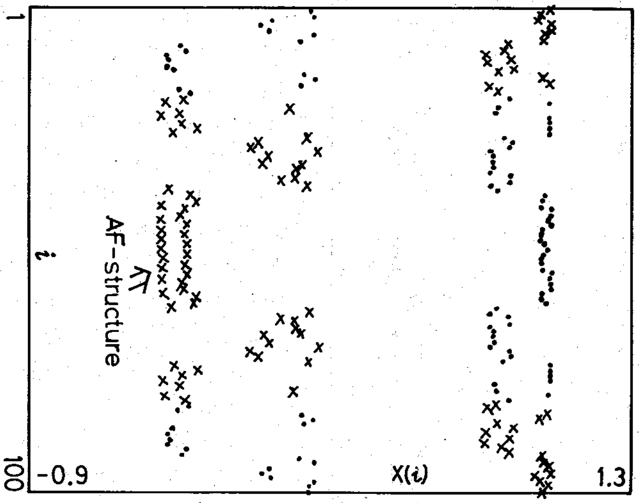

Fig. 2. Snapshot of Map (II) with the initial condition $x(i)=\sin(2\pi i/N)$ after the decay of the transients.

(a) $A=1.4$ and $D=0.005$(2^4-cycle), shown by ●.

(b) $A=1.42$ and $D=0.005$ (chaos), shown by ×.

We note the following features:

i) We can construct various types of structures in which the number of kinks differs, by choosing suitable initial conditions. Kinks cause the modulation of the phase of the torus, which bring about the transition to chaos.

ii) In some parameter regions, two types of patterns can coexist, i.e., the structures of §2 and the present section. Small irregularities in the pattern in §2 can cause the transition to the antiferro-like pattern (e.g., the pattern in §2 with $x(i)=0$ for $i=1, 2, \cdots, 10$ or the initial condition with some singularities (e.g., $x(i)=\sin(\pi i/N)$) show this type of behavior).

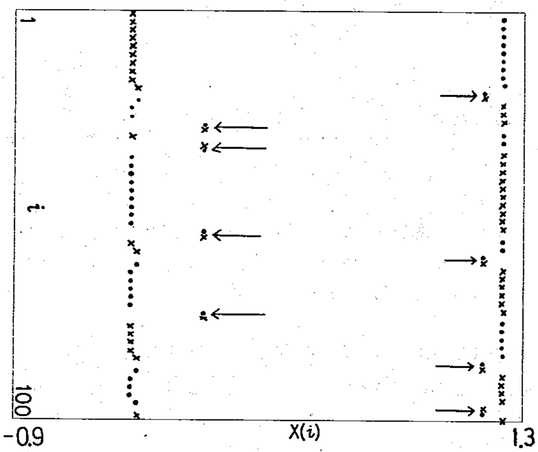

Fig. 3. Snapshot of Map (I) with the initial condition $x(i)=\sin(\pi i/N)$ and $A=0.74$ and $D=0.2$. The symbols × and ● denote the odd and even sites respectively. We note the existence of some kinks (shown by arrows). The attractor is a 2-cycle.

iii) Even after the transition from torus to chaos, the kinks can exist, which do not change their positions by the iteration of the map. As the nonlinearity is increased further, the antiferro-like structure collapses and the chaotic structure with more spatial complexity appears.

iv) The antiferro-like structure is also observed in other parameter regions of A, where basic dynamics shows not a 2-cycle but a 4-cycle or chaos. (See for example Fig. 2.)

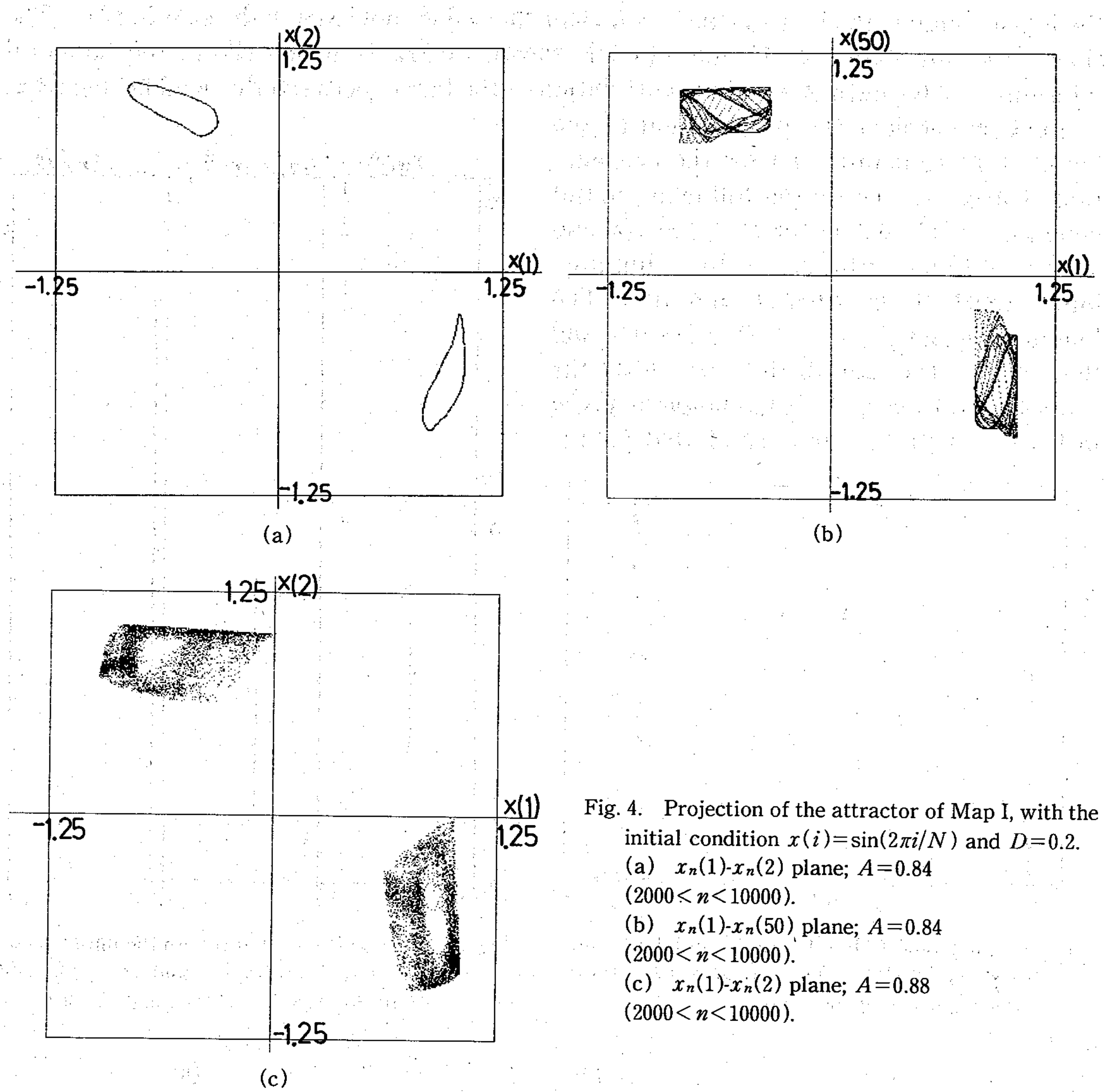

Fig. 4. Projection of the attractor of Map I, with the initial condition $x(i)=\sin(2\pi i/N)$ and $D=0.2$.
(a) $x_n(1)$-$x_n(2)$ plane; $A=0.84$ $(2000<n<10000)$.
(b) $x_n(1)$-$x_n(50)$ plane; $A=0.84$ $(2000<n<10000)$.
(c) $x_n(1)$-$x_n(2)$ plane; $A=0.88$ $(2000<n<10000)$.

§ 4. Spatial intermittency

The last structure which is treated in this paper is seen near the parameter value A where the intermittent transition[21] occurs for the logistic map.[22] Let us consider the system with $A\sim 1.75$ (as A is decreased from 1.75, the intermittent transition from a 3-cycle occurs for the logistic map). A snapshot of the pattern of "spatial intermittency" is given in Fig. 5, where bursts between laminar regions with $x\sim -0.75$, 1.0, and $0.0\sim 0.003$ can be remarkably seen. (In the present section Map II is studied, though similar behavior is observed also for Map I.) For a very small coupling D, a structure only with a laminar region (consequently a cycle with period-three) exists, which is analogous to the structure in §2. The structure is, however, stable only for a much smaller coupling than the case in §2 and a kink with a finite width cannot be observed in the present case. This instability of the kink structures will be due to the existence of the topological chaos for the logistic map with $A\sim1.75$ (the topological chaos does not exist in the case in §2). The time series for $x(10)$, $x(11)$ and $x(50)$ is shown in Fig. 6, which shows the temporal intermittency for each site and the correlation of the bursts between the neighboring sites.

Let us consider the propagation of the bursts in more detail. To see the propagation clearly, we choose the following initial condition; $x(i)=0.0013$ for $i=1, 2, \cdots, 50$ and $x(i)=-0.75$ for $i=51, 52, \cdots, 100$. Initially, bursts exist at the sites 50 and 100. The bursts propagate for $D>D_c\sim0.00185$ and they expand through all the space after the propagation time t_p.*) An example is given in Fig. 7, where the dots are plotted for the

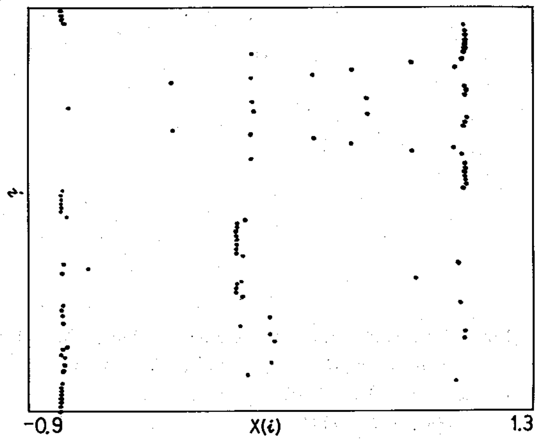

Fig. 5. Snapshot of Map II with the initial condition $x(i)=0.2\sin(\pi i/N)$ and $A=1.752$ and $D=0.0018$, after the decay of the transients.

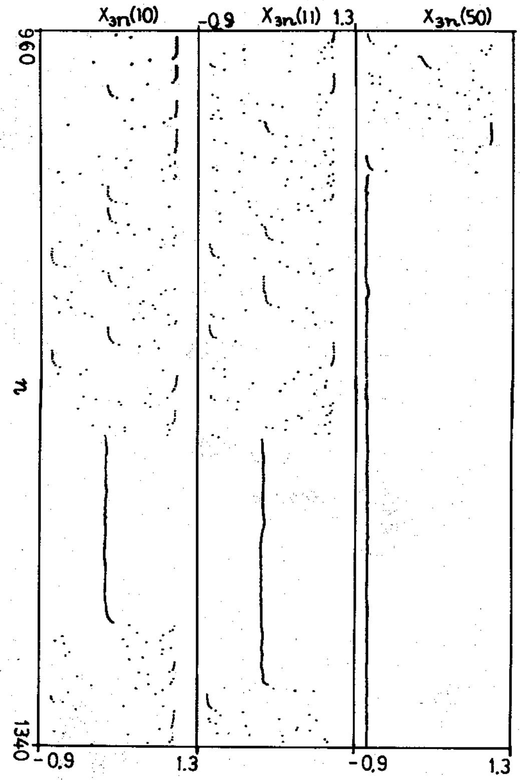

Fig. 6. Time series of Map II with the initial condition $x(i)=0.2\sin(2\pi i/N)$ and $A=1.752$ and $D=0.0018$ for $x_{3n}(10)$, $x_{3n}(11)$, and $x_{3n}(50)$.

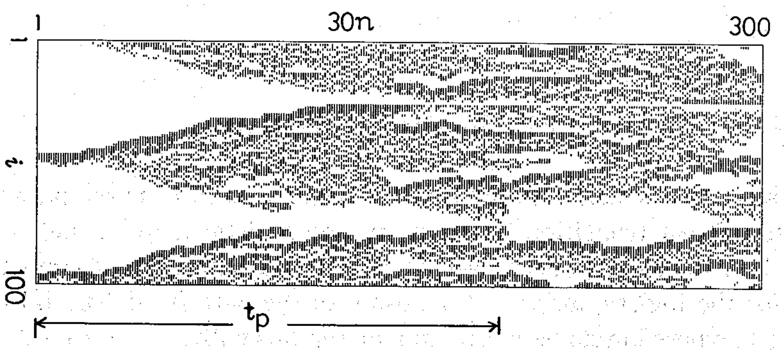

Fig. 7. Propagation of the bursts for Map II with the initial condition $x(i)=0.0013$ $(1\leq i\leq50)$, -0.75 $(51\leq i\leq100)$ and $A=1.752$ and $D=0.002$. $x_{30n}(i)$'s for $0<n<300$ are plotted. See the text for the notations.

*) The propagation time t is defined as the time step when the bursts from $i=0$ and $i=50$ make the first collision (see Fig. 7).

sites which satisfy $|x(i+1)-x(i)|>0.1$. We note the similarity between the figure and the propagation of the perturbation in the experiment by Reynolds.[23),24)] The propagation time obeys the relation $t_p \propto (D-D_c)^{-0.75(\pm 0.05)}$, from numerical data.

§ 5. Summary and discussion

The present paper shows some qualitative results on three typical behaviors in the coupled logistic lattice. In connection with the coupled map lattice systems, the following problems will be important for the future study:

1) It will be useful to introduce a reduction to a symbolic system with a finite number of states. Let us introduce, for example, $y(i)=x(i+1)-x(i)$ and assign a value 1 or 0 to each site according to the sign of $y(i)$. The pattern obtained in this way is rather analogous to the patterns in cellular automata.[25)] The method in §4 (i.e., 1 for burst and 0 for otherwise) also gives a pattern similar to those observed in cellular automaton theory. Especially, we observed a self-similar pattern at $D \sim D_c$ which looks quite like the interesting pattern reported in Ref. 25). Detailed results with the computation of the fractal dimension will be given elsewhere.

2) It may be also useful to introduce "pattern entropy" to characterize the complex structure of the system. Using the reduction to a sequence of [0, 1] in 1), an entropy in a spatial sequence can be defined in a manner similar to the entropy in a time series. It is an open problem to study the meaning of this quantity and its convergence for a large system.

3) As a coupling is increased the correlation among the sites is increased and some sort of "order" appears even if the system is chaotic. The order can be seen in the behavior of spatial correlation function. Thus, it will be important to study a "phase transition" in CLL with various dimensionalities.

4) Some states for our CLL clearly breaks the translational invariance (i.e., the long-time average for $x(i)$ differs sites by sites). The search for the transition accompanied by the breaking of the translational symmetry will be also a fascinating problem.

5) Coupled circle maps are ($f(x)=x+A\sin(2\pi x)+D$) also relevant models for the turbulence, especially in connection with the stability of a high-dimensional torus.[20)]

There are a lot of problems to be solved in future. Some answers to the above problems will be hopefully reported in the forthcoming papers in addition to the detailed study of the three features in the present paper.

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Notes added:

1) The width of a kink increases near the onset of a 2^n-cycle. It is proportional to $(A-A_n)^{-1/2}$(A_n is the parameter value for the onset of the 2^n-cycle), which can be obtained by analytic calculations.
2) To be precise, the value D_c in §4 is not a single parameter. There exists a fine stripe structure between the phases with and without bursts near $D=D_c$.