PERIOD MAPPINGS AND PERIOD DOMAINS

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Introductory Examples

The basic idea of Hodge theory is that the cohomology of an algebraic variety has more structure than one sees when viewing the same object as a "bare" topological space. This extra structure helps us understand the geometry of the underlying variety, and it is also an interesting object of study in its own right. Because of the technical complexity of the subject, in this chapter, we look at some motivating examples which illuminate and guide our study of the complete theory. We shall be able to understand, in terms of specific and historically important examples, the notions of Hodge structure, period map, and period domain. We begin with elliptic curves, which are the simplest interesting Riemann surfaces.

1.1 Elliptic Curves

The simplest algebraic variety is the Riemann sphere, the complex projective space $\mathbb{P}^1$. The next simplest examples are the branched double covers of the Riemann sphere, given in affine coordinates by the equation

$$y^2 = p(x),$$

where $p(x)$ is a polynomial of degree d. If the roots of p are distinct, which we assume they are for now, the double cover C is a one-dimensional complex manifold, or a Riemann surface. As a differentiable manifold it is characterized by its genus. To compute the genus, consider two cases. If d is even, all the branch points are in the complex plane, and if d is odd, there is one branch point at infinity. Thus the genus g of such a branched cover C is $d/2$ when d is even and $(d-1)/2$ when d is odd. These facts follow from Hurwitz's formula, which in turn follows from a computation of Euler characteristics (see Problem 1.1.2). Riemann surfaces of genus 0, 1, and 2 are illustrated in Fig. 1. Note that if $d = 1$ or $d = 2$, then C is topologically a sphere. It is not hard to prove that it is also isomorphic to the Riemann sphere as a complex manifold.

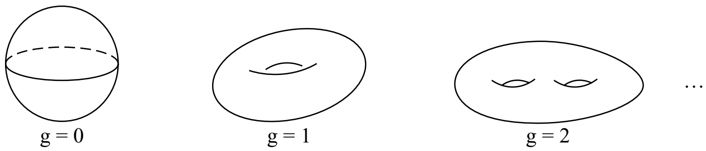

Figure 1. Riemann surfaces.

Now consider the case $d = 3$, so that the genus of C is 1. By a suitable change of variables, we may assume the three roots of $p(x)$ to be 0, 1, and λ, where $\lambda \neq 0, 1$:

$$y^2 = x(x-1)(x-\lambda). \tag{1.1.1}$$

We shall denote the Riemann surface defined by this equation (1.1.1) by $\mathcal{E}_\lambda$, and we call the resulting family the Legendre family. As topological spaces, and even as differentiable manifolds, the various $\mathcal{E}_\lambda$, are all isomorphic, as long as $\lambda \neq 0, 1$, a condition which we assume to be now in force. However, we shall prove the following:

Theorem 1.1.2 *Suppose that $\lambda \neq 0, 1$. Then there is an $\epsilon > 0$ such that for all λ' within distance ϵ from λ, the Riemann surfaces $\mathcal{E}_\lambda$ and $\mathcal{E}_{\lambda'}$ are not isomorphic as complex manifolds.*

Our proof of this result, which guarantees an infinite supply of essentially distinct elliptic curves, will lead us directly to the notions of period map and period domain and to the main ideas of Hodge theory.

The first order of business is to recall some basic notions of Riemann surface theory so as to have a detailed understanding of the topology of $\mathcal{E}_\lambda$, which for now we write simply as $\mathcal{E}$. Consider the multiple-valued holomorphic function

$$y = \sqrt{x(x-1)(x-\lambda)}.$$

On any simply connected open set which does not contain the branch points $x = 0,\ 1,\ \lambda,\ \infty$, it has two single-valued determinations. Therefore, we cut the Riemann sphere from 0 to 1 and from λ to infinity, as in Fig. 2. Then analytic continuation of y in the complement of the cuts defines a single-valued function. We call its graph a "sheet" of the Riemann surface. Note that analytic continuation of y around δ returns y to its original determination, so δ lies in a single sheet of $\mathcal{E}$. We can view it as lying in the Riemann sphere

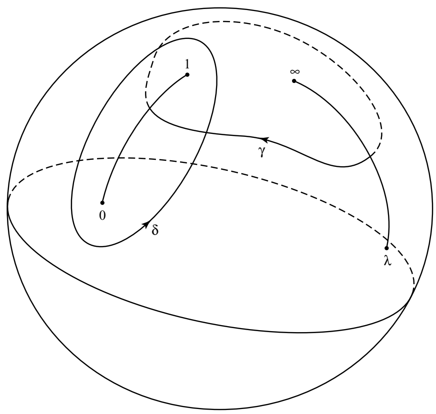

Figure 2. Cuts in the Riemann sphere.

itself. But when we analytically continue along γ, we pass from one sheet to the other as we pass the branch cut. That path is therefore made of two pieces, one in one sheet and one in the other sheet.

Thus the Riemann surface of y consists of two copies of the Riemann sphere minus the cuts, which are then "cross-pasted": we glue one copy to the other along the cuts but with opposite orientations. This assembly process is illustrated in Fig. 3. The two cuts are opened up into two ovals, the opened-up Riemann sphere is stretched to look like the lower object in the middle,

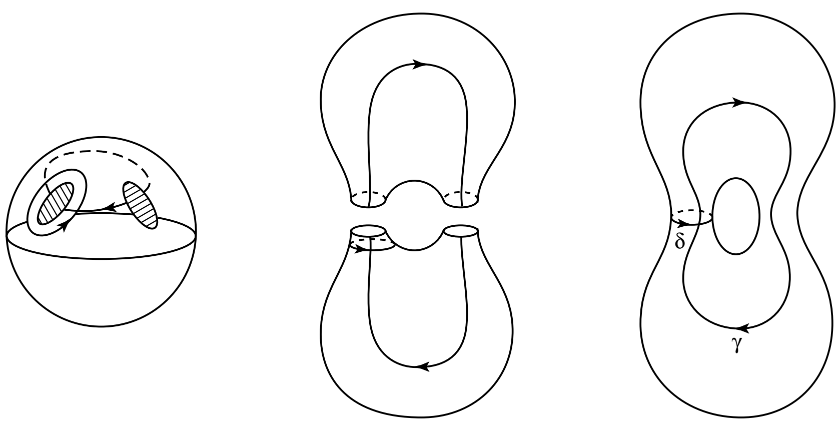

Figure 3. Assembling a Riemann surface.

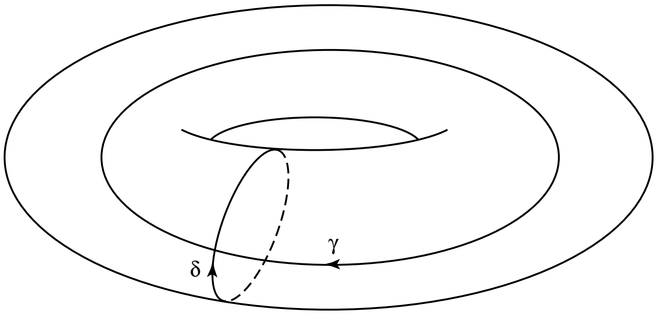

Figure 4. Torus.

a second copy is set above it to represent the other sheet, and the two sheets are cross-pasted to obtain the final object.

The result of our assembly is shown in Fig. 4.

The oriented path δ indicated in Fig. 4 can be thought of as lying in the Riemann sphere, as in Fig. 2, where it encircles one branch cut and is given parametrically by

$$\delta(\theta) = 1/2 + (1/2 + k)e^{i\theta}$$

for some small k. The two cycles δ and γ are oriented oppositely to the x and y axes in the complex plane, and so the intersection number of the two cycles is

$$\delta \cdot \gamma = 1.$$

We can read this information off either Fig. 3 or Fig. 2. Note that the two cycles form a basis for the first homology of $\mathcal{E}$ and that their intersection matrix is the standard unimodular skew form,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With this explanation of the homology of our elliptic curve, we turn to the cohomology. Recall that cohomology classes are given by linear functionals on homology classes, and so they are given by integration against a differential form. (This is de Rham's theorem – see Theorem 2.1.1). In order for the line integral to be independent of the path chosen to represent the homology class, the form must be closed. For the elliptic curve $\mathcal{E}$ there is a naturally given differential one-form that plays a central role in the story we are recounting. It is defined by

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}. \quad (1.1.3)$$

As discussed in Problem 1.1.1, this form is holomorphic, that is, it can be written locally as

$$\omega = f(z)dz,$$

where z is a local coordinate and $f(z)$ is a holomorphic function. In fact, away from the branch points, x is a local coordinate, so this representation follows from the fact that $y(x)$ has single-valued holomorphic determinations. Because f is holomorphic, ω is closed (see Problem 1.1.7). Thus it has a well-defined cohomology class.

Now let δ^* and γ^* denote the basis for $H^1(\mathcal{E}; \mathbb{Z})$ which is dual to the given basis of $H_1(\mathcal{E}; \mathbb{Z})$. The cohomology class of ω can be written in terms of this basis as

$$[\omega] = \delta^* \int_\delta \omega + \gamma^* \int_\gamma \omega.$$

In other words, the coordinates of $[\omega]$ with respect to this basis are given by the indicated integrals. These are called the *periods* of ω. In the case at hand, they are sometimes denoted A and B, so that

$$[\omega] = A\delta^* + B\gamma^*. \tag{1.1.4}$$

The expression (A, B) is called the *period vector* of $\mathcal{E}$.

From the periods of ω we are going to construct an invariant that can detect changes in the complex structure of $\mathcal{E}$. In the best of all possible worlds this invariant would have different values for elliptic curves that have different complex structures. The first step toward constructing it is to prove the following.

Theorem 1.1.5 *Let $H^{1,0}$ be the subspace of $H^1(\mathcal{E}; \mathbb{C})$ spanned by ω, and let $H^{0,1}$ be the complex conjugate of this subspace. Then*

$$H^1(\mathcal{E}; \mathbb{C}) = H^{1,0} \oplus H^{0,1}.$$

The decomposition asserted by this theorem is the *Hodge decomposition* and it is fundamental to all that follows. Now there is no difficulty in defining the $(1, 0)$ and $(0, 1)$ subspaces of cohomology: indeed, we have already done this. The difficulty is in showing that the defined subspaces span the cohomology, and that (equivalently) their intersection is zero. In the case of elliptic curves, however, there is a quite elementary proof of this fact. Take the cup product of (1.1.4) with its conjugate to obtain

$$[\omega] \cup [\bar{\omega}] = (A\bar{B} - B\bar{A})\delta^* \cup \gamma^*.$$

Multiply the previous relation by $\mathrm{i} = \sqrt{-1}$ and use the fact that $\delta^* \cup \gamma^*$ is the fundamental class of $\mathcal{E}$ to rewrite the preceding equation as

$$\mathrm{i} \int_{\mathcal{E}} \omega \wedge \bar{\omega} = 2\,\mathrm{Im}(B\bar{A}).$$

Now consider the integral above. Because the form ω is given locally by $f\,dz$, the integrand is locally given by

$$\mathrm{i}|f|^2 dz \wedge d\bar{z} = 2|f|^2 dx \wedge dy,$$

where $dx \wedge dy$ is the natural orientation defined by the holomorphic coordinate, that is, by the complex structure. Thus the integrand is locally a positive function times the volume element, and so the integral is positive. We conclude that

$$\mathrm{Im}(B\bar{A}) > 0.$$

We also conclude that neither A nor B can be 0 and, therefore, that the cohomology class of ω cannot be 0. Consequently the subspace $H^{1,0}(\mathcal{E})$ is nonzero.

Because neither A nor B can be 0 we can rescale ω and assume that $A = 1$. For such "normalized" differentials, we conclude that *the imaginary part of the normalized B-period is positive*:

$$\mathrm{Im}\, B > 0. \tag{1.1.6}$$

Now suppose that $H^{1,0}$ and $H^{0,1}$ do not give a direct sum decomposition of $H^1(\mathcal{E};\mathbb{C})$. Then $H^{1,0} = H^{0,1}$, and so $[\bar{\omega}] = \lambda[\omega]$ for some complex number λ. Therefore

$$\delta^* + \bar{B}\gamma^* = \lambda(\delta^* + B\gamma^*).$$

Comparing coefficients, we find that $\lambda = 1$ and then that $B = \bar{B}$, in contradiction with the fact that B has a positive imaginary part. This completes the proof of the Hodge theorem for elliptic curves, Theorem 1.1.5.

An Invariant of Framed Elliptic Curves

Now suppose that $f : \mathcal{E}_\mu \longrightarrow \mathcal{E}_\lambda$ is an isomorphism of complex manifolds. Let ω_μ and ω_λ be the given holomorphic forms. Then we claim that

$$f^*\omega_\lambda = c\,\omega_\mu \tag{1.1.7}$$

for some nonzero complex number c. This equation is certainly true on the level of cohomology classes, although we do not yet know that c is nonzero.

However, on the one hand,

$$\int_{[\mathcal{E}_\mu]} f^*\omega_\lambda \wedge f^*\bar{\omega}_\lambda = |c|^2 \int_{[\mathcal{E}_\mu]} \omega_\mu \wedge \bar{\omega}_\mu,$$

and on the other,

$$\int_{[\mathcal{E}_\mu]} f^*\omega_\lambda \wedge f^*\bar{\omega}_\lambda = \int_{f_*[\mathcal{E}_\mu]} \omega_\lambda \wedge \bar{\omega}_\lambda = \int_{[\mathcal{E}_\lambda]} \omega_\lambda \wedge \bar{\omega}_\lambda.$$

The last equality uses the fact that an isomorphism of complex manifolds is a degree-one map. Because $\mathrm{i}\,\omega_\lambda \wedge \bar{\omega}_\lambda$ is a positive multiple of the volume form, the integral is positive and therefore

$$c \neq 0. \tag{1.1.8}$$

We can now give a preliminary version of the invariant alluded to above. It is the ratio of periods B/A, which we write more formally as

$$\tau(\mathcal{E}, \delta, \gamma) = \frac{\int_\gamma \omega}{\int_\delta \omega}.$$

From Eq. (1.1.6) we know that τ has a positive imaginary part. From the just-proved proportionality results (1.1.7) and (1.1.8), we conclude the following.

Theorem 1.1.9 *If $f : \mathcal{E} \longrightarrow \mathcal{E}'$ is an isomorphism of complex manifolds, then $\tau(\mathcal{E}, \delta, \gamma) = \tau(\mathcal{E}', \delta', \gamma')$, where $\delta' = f_*\delta$ and $\gamma' = f_*\gamma$.*

To interpret this result, let us define a *framed elliptic curve* $(\mathcal{E}, \delta, \gamma)$ to consist of an elliptic curve and an integral basis for the first homology such that $\delta \cdot \gamma = 1$. Then we can say that "if framed elliptic curves are isomorphic, then their τ-invariants are the same."

Holomorphicity of the Period Mapping

Consider once again the Legendre family (1.1.1) and choose a complex number $a \neq 0,\ 1$ and an $\epsilon > 0$ which is smaller than both the distance from a to 0 and the distance from a to 1. Then the Legendre family, restricted to λ in the disk of radius ϵ centered at a, is trivial as a family of differentiable manifolds. This means that it is possible to choose two families of integral homology cycles δ_λ and γ_λ on $\mathcal{E}_\lambda$ such that $\delta_\lambda \cdot \gamma_\lambda = 1$. We can "see" these cycles by modifying Fig. 2 as indicated in Fig. 5. A close look at Fig. 5 shows that we

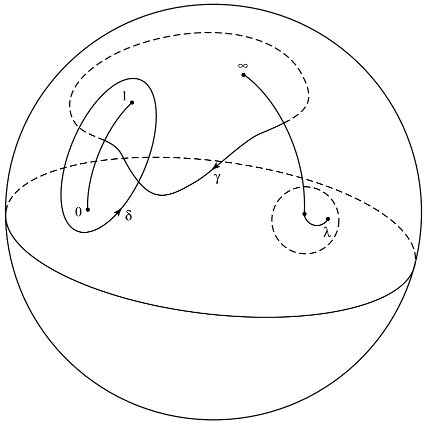

Figure 5. Modified cuts in the Riemann sphere.

can move λ within a small disk Δ without changing either δ_λ or γ_λ. Thus we can view the integrals defining the periods A and B as having constant domains of integration but variable integrands.

Let us study these periods more closely, writing them as

$$A(\lambda) = \int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \qquad B(\lambda) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

We have suppressed the subscript on the homology cycles in view of the remarks made at the end of the previous paragraph. The first observation is the following.

Proposition 1.1.10 *On any disk Δ in the complement of the set $\{0, 1, \infty\}$, the periods of the Legendre family are single-valued holomorphic functions of λ.*

The proof is straightforward. Since the domain of integration is constant, we can compute $\partial A/\partial\bar{\lambda}$ by differentiating under the integral sign. But the integrand is a holomorphic expression in λ, and so that derivative is 0. We conclude that the period function $A(\lambda)$ is holomorphic, and the same argument applies to $B(\lambda)$.

Notice that the definitions of the period functions A and B on a disk Δ depend on the choice of a symplectic homology basis $\{\delta,\ \gamma\}$. Each choice of basis gives a different determination of the periods. However, if δ' and γ' give a different basis, then

$$\delta' = a\delta + b\gamma$$
$$\gamma' = c\delta + d\gamma,$$

where the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has determinant 1. The periods with respect to the new basis are related to those with respect to the old one as follows:

$$A' = aA + bB$$
$$B' = cA + dB.$$

Thus the new period vector (A', B') is the product of the matrix T and the old period vector (A, B). The τ-invariants are related by the corresponding fractional linear transformation:

$$\tau' = \frac{d\tau + c}{b\tau + a}.$$

The ambiguity in the definition of the periods and of the τ invariant is due to the ambiguity in the choice of a homology basis. Now consider a simply connected open set U of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ and a point λ_0 and λ of U. The choice of homology basis for $\mathcal{E}_{\lambda_0}$ determines a choice of homology basis for all other fibers $\mathcal{E}_\lambda$. Thus the periods $A(\lambda)$ and $B(\lambda)$ as well as the ratio $\tau(\lambda)$ are single-valued holomorphic functions on U. On the full domain $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, however, these functions are multivalued.

We can now state a weak form of Theorem 1.1.2.

Theorem 1.1.11 *The function τ is nonconstant.*

If τ, defined on a simply connected open set U, is a nonconstant holomorphic function then its derivative is not identically zero. Therefore its derivative has at most isolated zeroes. For a randomly chosen point, different from one of these zeroes, τ is a locally injective function.

There are at least two ways to prove that τ is nonconstant. One is to compute the derivative directly and to show that it is nonzero. The other is to show that τ tends to infinity as λ approaches infinity along a suitable ray in the complex plane. We give both arguments, beginning with an analysis of τ along a ray.

Asymptotics of the Period Map

Let us show that τ is a nonconstant function of λ by showing that τ approaches infinity along the ray $\lambda > 2$ of the real axis. Indeed, we will show that $\tau(\lambda)$ is asymptotically proportional to $\log \lambda$. To see this, assume $\lambda \gg 2$, and observe that

$$\int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \sim \int_\delta \frac{dx}{x\sqrt{-\lambda}} = \frac{2\pi}{\sqrt{\lambda}}.$$

By deforming the path of integration, we find that

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = -2\int_1^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

The difference between the last integrand and $1/x(\sqrt{x-\lambda})$ is $1/(2x^2) +$ higher powers of $1/x$, an expression with asymptotically negligible integral $-\lambda^{-1}/4 +$ higher powers of λ^{-1}. The residual integral,

$$-2\int_1^\lambda \frac{dx}{x\sqrt{(x-\lambda)}},$$

can be computed exactly:

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}} = \frac{4}{\sqrt{\lambda}} \arctan \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}} \sim \frac{2\mathrm{i}}{\sqrt{\lambda}} \log \lambda.$$

Thus one finds

$$\tau(\lambda) \sim \frac{\mathrm{i}}{\pi} \log \lambda, \tag{1.1.12}$$

as claimed. Note also that $\tau(\lambda)$ has a positive imaginary part, as asserted in (1.1.6).

Derivative of the Period Map

We now prove the strong form of Theorem 1.1.2 by showing that $\tau'(\lambda) \neq 0$ for $\lambda \neq 0,\ 1$ for any determination of τ. To this end, we write the holomorphic differential ω_λ in terms of the dual cohomology basis $\{\delta^*,\ \gamma^*\}$:

$$\omega_\lambda = A(\lambda)\delta^* + B(\lambda)\gamma^*.$$

The periods are coefficients that express ω_λ in this basis, and the invariant $\tau(\lambda)$ is an invariant of the line spanned by the vector ω_λ. The expression

$$\omega'_\lambda = A'(\lambda)\delta^* + B'(\lambda)\gamma^*$$

is the derivative of the cohomology class ω_λ with respect to the "Gauss–Manin connection." This is by definition the connection on the bundle of cohomology vector spaces

$$\bigcup_{\lambda\in\mathbb{P}^1\setminus\{0,1,\infty\}} H^1(\mathcal{E}_\lambda)$$

with respect to which the classes δ^*, γ^* are (locally) constant. Then

$$[\omega_\lambda]\cup[\omega'_\lambda] = (AB' - A'B)\,\delta^*\cup\gamma^*.$$

However, $\tau' = 0$ if and only if $AB' - A'B = 0$. Thus, to establish that $\tau'(\lambda) \neq 0$, it suffices to establish that $[\omega_\lambda]\cup[\omega'_\lambda] \neq 0$.

Now the derivative of ω_λ is represented by the meromorphic form

$$\omega'_\lambda = \frac{1}{2}\frac{dx}{\sqrt{x(x-1)(x-\lambda)^3}}. \tag{1.1.13}$$

This form has a pole of multiplicity two at $p = (\lambda, 0)$. To see this, note that at the point p, the function y is a local coordinate. Therefore the relation $y^2 = x(x-1)(x-\lambda)$ can be written as $y^2 = u(y)\lambda(\lambda-1)(x-\lambda)$ where $u(y)$ is a holomorphic function of y satisfying $u(0) = 1$. Solving for x, we obtain $x = \lambda +$ terms of order ≥ 2 in y. Then setting $p(x) = x(x-1)(x-\lambda)$, we have

$$\omega = \frac{dx}{y} = \frac{2dy}{p'(x)} \sim \frac{2dy}{\lambda(\lambda-1)},$$

where $a \sim b$ means that a and b agree up to lower-order terms in y. Using (1.1.13), the previous expression, and the expansion of x in terms of y, we find

$$\omega' = \frac{1}{2}\frac{dx}{y(x-\lambda)} \sim \frac{dy}{\lambda(\lambda-1)(x-\lambda)} \sim \frac{dy}{y^2} + \text{a regular form}. \tag{1.1.14}$$

To explain why such a form represents a cohomology class on $\mathcal{E}$, not just on $\mathcal{E}\setminus\{p\}$, we first note that its residue vanishes. Recall that the residue of ϕ is defined as

$$\frac{1}{2\pi \mathrm{i}}\int_{C_p}\phi = \mathrm{res}(\phi)(p),$$

where C_p is a small positively counterclockwise–oriented circle on S centered at p. Next, note that the residue map in fact is defined on the level of cohomology classes (just apply Stokes' theorem). In fact the resulting map "res" is the coboundary map from the exact sequence of the pair $(\mathcal{E}, \mathcal{E}\backslash\{p\})$,

$$0 \longrightarrow H^1(\mathcal{E}) \longrightarrow H^1(\mathcal{E}\backslash\{p\}) \xrightarrow{\text{res}} H^2(\mathcal{E}, \mathcal{E}\backslash\{p\}),$$

provided we identify the third vector space with $\mathbb{C}$ using the isomorphism

$$H^2(\mathcal{E}, \mathcal{E}\backslash\{p\}) \cong H^0(\{p\}) \cong \mathbb{C}.$$

See Problem 1.1.10, where this sequence is discussed in more detail.

Observe that the sequence above is the simplest instance of the so-called Gysin sequence for a smooth hypersurface (here just the point p) inside a smooth variety (here the curve $\mathcal{E}$). The Gysin sequence is at the heart of many calculations and is treated in detail in Section 3.2.

From the Gysin sequence we see that ω'_λ represents a cohomology class on $\mathcal{E}$, not just on $\mathcal{E}\backslash\{p\}$. We now claim that

$$\int_{\mathcal{E}} [\omega] \cup [\omega'] = \frac{4\pi\mathrm{i}}{\lambda(\lambda - 1)}. \tag{1.1.15}$$

By establishing this formula we will complete the proof that $\tau'(\lambda) \neq 0$ for $\lambda \neq 0,\ 1$. To do this, first observe that the formula (1.1.14) implies that $\omega' + d(1/y)$ has no pole at p. To globalize this computation, let U be a coordinate neighborhood of p on which $|y| < \epsilon$, and let $\rho(z)$ be a smooth function of $|z|$ alone which vanishes for $|z| > \epsilon/2$, which is identically one for $|z| < \epsilon/4$, and which decreases monotonically in $|z|$ on the region $\epsilon/4 < |z| < \epsilon/2$. Then the form

$$\tilde{\omega}' = \omega' + d(\rho(y)/y)$$

lies in the same cohomology class on $\mathcal{E}\backslash\{p\}$ as does ω'. By construction, it extends to a form on $\mathcal{E}$ and represents the cohomology class of ω' there. Because ω and $\tilde{\omega}'$ are both holomorphic one-forms on the complement of U,

$$\int_{\mathcal{E}} \omega \wedge \tilde{\omega}' = \int_U \omega \wedge d(\rho/y).$$

Because $\omega \wedge d(\rho/y) = -d(\rho\omega/y)$, Stokes' theorem yields

$$\int_U d\frac{\rho\omega}{y} = -\int_{|y|=\frac{\epsilon}{4}} \frac{\rho\omega}{y} = -\int_{|y|=\frac{\epsilon}{4}} \frac{\omega}{y}.$$

A standard residue calculation of the line integral then yields

$$\int_{|y|=\epsilon/4} \frac{\omega}{y} = \frac{4\pi\mathrm{i}}{\lambda(\lambda - 1)}.$$

This completes the proof.

Picard–Fuchs Equation

In computing the derivative of the period map, we proved that the form ω and its derivative ω' define linearly independent cohomology classes. Therefore the class of the second derivative must be expressible as a linear combination of the first two classes. Consequently there is a relation

$$a(\lambda)\omega'' + b(\lambda)\omega' + c(\lambda)\omega = 0 \tag{1.1.16}$$

in cohomology. The coefficients are meromorphic functions of λ, and on the level of forms the assertion is that the left-hand side is exact on $\mathcal{E}_\lambda$. Let ξ be a one-cycle and set

$$\pi(\lambda) = \int_\xi \omega.$$

Then (1.1.16) can be read as a differential equation for the period function

$$a\pi'' + b\pi' + c\pi = 0.$$

One can determine the coefficients in this expression. The result is a differential equation with regular singular points at 0, 1, and ∞:

$$\lambda(\lambda - 1)\pi'' + (2\lambda - 1)\pi' + \frac{1}{4}\pi = 0. \tag{1.1.17}$$

Solutions are given by hypergeometric functions (see [45], Section 2.11). To find the coefficients a, b, c above, we seek a rational function f on $\mathcal{E}_\lambda$ such that df is a linear combination of ω, ω', and ω'' whose coefficients are functions of λ. Now observe that

$$\begin{aligned} \omega' &= \frac{1}{2}x^{-1/2}(x-1)^{-\frac{1}{2}}(x-\lambda)^{-\frac{3}{2}}\,dx \\ \omega'' &= \frac{3}{4}x^{-\frac{1}{2}}(x-1)^{-\frac{1}{2}}(x-\lambda)^{-\frac{5}{2}}\,dx. \end{aligned}$$

Thus it is reasonable to consider the function

$$f = x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-\lambda)^{-\frac{3}{2}}.$$

Indeed,

$$df = (x-1)\omega' + x\omega' - 2x(x-1)\omega'',$$

which is a relation between ω' and ω''. This is progress, but the coefficients are not functions of λ. Therefore consider the equivalent form

$$\begin{aligned} df &= [(x-\lambda) + (\lambda - 1)]\omega' + [(x-\lambda) + \lambda]\omega' \\ &\quad - 2[(x-\lambda) + \lambda][(x-\lambda) + (\lambda - 1)]\omega'' \end{aligned}$$

and use the relations

$$(x-\lambda)\omega' = \frac{1}{2}\omega, \qquad (x-\lambda)\omega'' = \frac{3}{2}\omega'$$

to obtain

$$-\frac{1}{2}df = \frac{1}{4}\omega + (2\lambda - 1)\omega' + \lambda(\lambda - 1)\omega''.$$

This completes the derivation.

One can find power series solutions of (1.1.17) which converge in a disk Δ_0 around any $\lambda_0 \neq 0, 1$. Analytic continuation of the resulting function produces a multivalued solution defined on $\mathbb{P}^1\backslash\{0, 1, \infty\}$. Now let π_1 and π_2 be two linearly independent solutions defined on Δ_0, and let γ be a loop in $\mathbb{P}^1\backslash\{0, 1, \infty\}$ based at λ_0. Let π_i' be the function on Δ_0 obtained by analytic continuation of π_i along γ. Because the π_i' are also solutions of the differential equation (1.1.17), they can be expressed as linear combinations of π_1 and π_2:

$$\begin{pmatrix} \pi_1' \\ \pi_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

The indicated matrix, which we shall write as $\rho(\gamma)$, depends only on the homotopy class of α, and is called the *monodromy matrix*. We determine this matrix in the next section using a geometric argument. For now we note that the map that sends α to $\rho(\alpha)$ defines a homomorphism

$$\rho : \pi_1(\mathbb{P}^1\backslash\{0, 1, \infty\}, \lambda_0) \longrightarrow \mathrm{GL}(2, \mathbb{C}).$$

It is called the *monodromy representation* and its image is called the *monodromy group*.

The Local Monodromy Representation

To better understand the monodromy representation, consider the family of elliptic curves $\mathcal{E}_s$ defined by

$$y^2 = (x^2 - s)(x - 1).$$

The fiber $\mathcal{E}_0$, given by $y^2 = x^2(x-1)$ has a node at $p = (0, 0)$. As s approaches 0, the fiber undergoes the changes pictured in Fig. 6. A copy of the loop δ is slowly contracted to a point, producing the double point at p. Note that in the limit of $s = 0$, the cycle δ is homologous to 0.

Now restrict this family to the circle $|s| = \epsilon$ and consider the vector field $\partial/\partial\theta$ in the s-plane. It lifts to a vector field ξ on the manifold

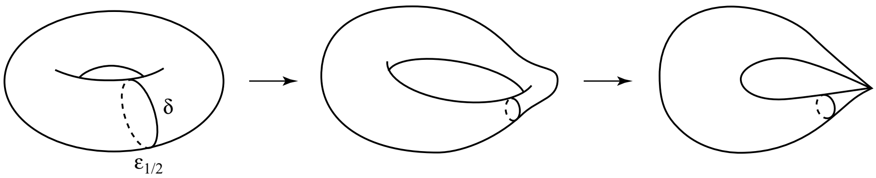

Figure 6. Degeneration of an elliptic curve.

$M = \{(x, y, s) \mid y^2 = (x^2 - s)(x - 1)\}$ which fibers over the circle via $(x, y, s) \mapsto s$. By letting the flow which is tangent to ξ act for time ϕ, one defines a diffeomorphism g_ϕ of the fiber at $\theta = 0$ onto the fiber at $\theta = \phi$. This is illustrated in Fig. 7. (We think of a fluid flow transporting points of $\mathcal{E}_0$ to points of $\mathcal{E}_\phi$, with streamlines tangent to the vector field.)

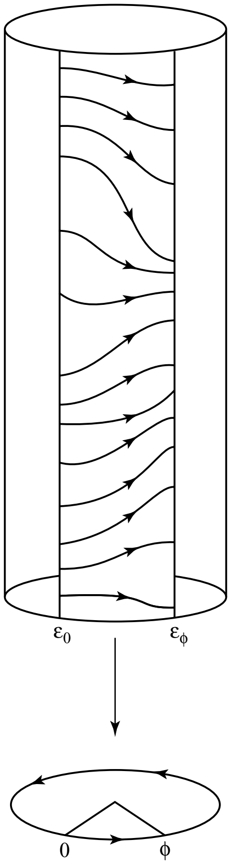

Figure 7. Diffeomorphism $g_\phi : \mathcal{E}_0 \longrightarrow \mathcal{E}_\phi$.