

# Periodic Event-Triggered Control for Linear Systems

W. P. M. H. (Maurice) Heemels, *Senior Member, IEEE*, M. C. F. (Tijs) Donkers, and Andrew R. Teel, *Fellow, IEEE*

**Abstract**—Event-triggered control (ETC) is a control strategy that is especially suited for applications where communication resources are scarce. By updating and communicating sensor and actuator data only when needed for stability or performance purposes, ETC is capable of reducing the amount of communications, while still retaining a satisfactory closed-loop performance. In this paper, an ETC strategy is proposed by striking a balance between conventional periodic sampled-data control and ETC, leading to so-called periodic event-triggered control (PETC). In PETC, the event-triggering condition is verified periodically and at every sampling time it is decided whether or not to compute and to transmit new measurements and new control signals. The periodic character of the triggering conditions leads to various implementation benefits, including a minimum inter-event time of (at least) the sampling interval of the event-triggering condition. The PETC strategies developed in this paper apply to both static state-feedback and dynamical output-based controllers, as well as to both centralized and decentralized (periodic) event-triggering conditions. To analyze the stability and the  $\mathcal{L}_2$ -gain properties of the resulting PETC systems, three different approaches will be presented based on 1) impulsive systems, 2) piecewise linear systems, and 3) perturbed linear systems. Moreover, the advantages and disadvantages of each of the three approaches will be discussed and the developed theory will be illustrated using a numerical example.

**Index Terms**—Control systems, digital control, event-triggered control, linear feedback, networked control systems.

## I. INTRODUCTION

IN many digital control applications nowadays, the control task consists of sampling the outputs of the plant and computing and implementing new actuator signals. Typically, the control task is executed periodically, since this allows the closed-loop system to be analyzed and the controller to be designed using the well-developed theory on sampled-data systems. Although periodic sampling is preferred from an analysis and design point of view, it is sometimes less preferable from a resource utilization point of view. Namely, executing

the control task at times when no disturbances are acting on the system and the system is operating desirably is clearly a waste of communication resources. This is especially disadvantageous in applications where the measured outputs and/or the actuator signals have to be transmitted over a shared (and possibly wireless) communication network, where the bandwidth of the network and the power sources of the wireless devices are constrained. To mitigate the unnecessary waste of communication resources, it is of interest to consider an alternative control paradigm, namely event-triggered control (ETC), which has been proposed in the late nineties [1]–[5]. Various ETC strategies have been proposed since then, see, e.g., [6]–[17]. In ETC, the control task is executed after the occurrence of an event, generated by some well-designed event-triggering condition, rather than the elapse of a certain fixed period of time, as in conventional periodic sampled-data control. In this way, ETC is capable of significantly reducing the number of control task executions, while retaining a satisfactory closed-loop performance.

The main difference between the afore-cited papers [1]–[17] and the ETC strategy that will be proposed in this paper is that in the former the event-triggering condition has to be monitored continuously, while in the latter the event-triggering condition is verified only periodically, and at every sampling time it is decided whether or not to transmit new measurements and control signals. Only when necessary to guarantee stability or performance requirements, the communication resources are used. The resulting control strategy aims at striking a balance between periodic sampled-data and event-triggered control and, therefore, we will use the term *periodic event-triggered control* (PETC) for this class of ETC, while we will use the term continuous event-triggered control (CETC) to indicate the existing approaches that require monitoring of the event-triggering conditions continuously. By mixing ideas from ETC and periodic sampled-data control, the benefits of reduced resource utilisation are preserved in PETC as transmissions and controller computations are *not* performed periodically, while the event-triggering conditions are evaluated only periodically. The latter aspect leads to several benefits, including a guaranteed minimum inter-event time of (at least) the sampling interval of the event-triggering condition. Furthermore, as already mentioned, the event-triggering condition has to be verified only at periodic sampling times, making PETC better suited for practical implementations as it can be implemented in more standard time-sliced embedded software architectures. In fact, often CETC will eventually be implemented using a discretized version based on a sufficiently high sampling period resulting in a PETC strategy after all. This fact provides a further motivation for direct analysis and design of PETC instead of obtaining them in a final implementation stage as a discretized approximation of a CETC strategy. Another advantage of PETC is that

Manuscript received September 02, 2011; accepted April 27, 2012. Date of publication September 21, 2012; date of current version March 20, 2013. This work was supported by the Innovational Research Incentives Scheme under the VICI grant "Wireless control systems: A new frontier in automation" (No. 11382) awarded by NWO (The Netherlands Organization for Scientific Research) and STW (Dutch Science Foundation), and the European 7th Framework Network of Excellence "Highly-complex and networked control systems" (HYCON2) (grant agreement no. 257462), the Air Force Office of Scientific Research (AFOSR) under FA9550-09-1-0203, and the National Science Foundation (NSF) under Grants ECCS-0925637 and CNS-0720842. A preliminary version of this paper was presented at the Joint IEEE Conference on Decision and Control and the European Control Conference 2011. Recommended by Associate Editor P. Tabuada.

W. P. M. H. Heemels and M. C. F. Donkers are with the Hybrid and Networked Systems Group, Department of Mechanical Engineering, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands. (e-mail: m.heemels@tue.nl; m.c.f.donkers@tue.nl).

A. R. Teel is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560 USA (e-mail: teel@ece.ucsb.edu).

Color versions of one or more of the figures in this paper are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TAC.2012.2220443

it can be transformed more easily into a self-triggered control variant [18]–[20] (at least in the case that the controller is in a state-feedback form). Initial work in the direction of PETC was taken in [2], [6], [7], [21], which focused on restricted classes of systems, controllers, and/or (different) event-triggering conditions without providing a general analysis framework. Recently, the interest in what we call here PETC is growing, see, e.g., [22]–[26] and [27, Sec. 4.5], although these approaches start from a discrete-time plant model instead of a continuous-time plant, as we will do here.

In this paper, we will provide a general framework for a broad class of PETC in the context of linear systems that allows to carry out stability and performance analysis. In fact, we will provide three different modeling and analysis approaches, namely: 1) an impulsive system approach; 2) a discrete-time piecewise linear (PWL) system approach; and 3) a discrete-time perturbed linear (PL) system approach. The first approach adopts impulsive systems [28], [29] that explicitly include the intersample behavior, which is not the case for the previously mentioned PETC approaches [21]–[24] and [27, Sec. 4.5]. Based on the impulsive system paradigm, we are able to provide guarantees on performance in terms of  $\mathcal{L}_2$ -gains, besides guaranteeing stability. In the second method, we exploit PWL models, which can be obtained as time-discretizations of the corresponding impulsive systems, and use piecewise quadratic (PWQ) Lyapunov functions that lead to LMI-based stability conditions for the PETC system. The third method, which is based on PL systems, can be seen as a discrete-time counterpart of the work in [11], in which CETC was studied. The essence of this approach is that the difference between the control signal obtained by a standard periodic controller and its event-triggered counterpart can be modeled as a disturbance, resulting in a PL system, see also [22]. This insight will be used to derive a sufficient condition for stability of the PETC system based on the  $\mathcal{H}_\infty$ -norm of the PL system. This provides a simple stability test, which is, however, more conservative than the stability conditions based on the second approach. Interestingly, the PL system approach provides insights that justify an emulation-based controller synthesis method, as we will discuss in detail.

In the first part of the paper, we will present the three mentioned approaches for the basic setup of state-feedback controllers. However, as in many practical situations not all the states are available for feedback, it is of interest to study output-based dynamic controllers as well, which we will do in the second part of the paper. Another important issue is handling the situation in which sensors, actuators and controllers are physically distributed over a wide area. In fact, a centralized event-triggering mechanism can be prohibitive in this case, as the conditions that generate events would need access to all the plant and controller outputs at every sampling time, which can be an unrealistic assumption in large-scale systems. To resolve this issue, in the second part of the paper we will also propose decentralized periodic event-triggered conditions for output-based dynamic controllers (which may be decentralized themselves).

The remainder of this paper is organized as follows. After introducing the necessary notational conventions, we introduce PETC and give the problem formulation in Section II. In Sec-

tion III, the impulsive system approach, the PWL system approach, and the PL system approach are presented, together with a discussion on their advantages and disadvantages. In Section IV, we provide emulation-based design considerations for PETC. In Section V, we will extend the ideas presented in the first part of this paper towards output-based dynamic controllers and decentralized periodic event-triggered conditions. Before providing the conclusions in Section VII, we will provide a numerical example in Section VI illustrating the main developments in this paper. The Appendix contains the more technical proofs of the lemmas and theorems.

### A. Nomenclature

For a vector  $x \in \mathbb{R}^n$ , we denote by  $\|x\| := \sqrt{x^\top x}$  its 2-norm, and by  $x_{\mathcal{J}}$  the subvector formed by all components of  $x$  in the index set  $\mathcal{J} \subseteq \{1, \dots, n\}$ . For a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ ,  $\lambda_{\max}(A)$  and  $\lambda_{\min}(A)$  denote the maximum and minimum eigenvalue of  $A$ , respectively. For a matrix  $A \in \mathbb{R}^{n \times m}$ , we denote by  $A^\top \in \mathbb{R}^{m \times n}$  the transposed of  $A$ , and by  $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$  its induced 2-norm. Furthermore, by  $A_{\mathcal{J}, \bullet}$  and  $A_{\bullet, \mathcal{J}}$ , we denote the submatrices formed by taking all the rows of  $A$  in the index set  $\mathcal{J} \subseteq \{1, \dots, n\}$ , and by all the columns of  $A$  in the index set  $\mathcal{J} \subseteq \{1, \dots, m\}$ , respectively. By  $\text{diag}(A_1, \dots, A_N)$ , we denote a block-diagonal matrix with the entries  $A_1, \dots, A_N$  on the diagonal, and for the sake of brevity we sometimes write symmetric matrices of the form  $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$  as  $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$  or  $\begin{bmatrix} A & \star \\ B^\top & C \end{bmatrix}$ . We call a matrix  $P \in \mathbb{R}^{n \times n}$  positive definite and write  $P \succ 0$ , if  $P$  is symmetric and  $x^\top P x > 0$  for all  $x \neq 0$ . Similarly, we use  $P \succeq 0$ ,  $P \prec 0$  and  $P \preceq 0$  to denote that  $P$  is positive semidefinite, negative definite and negative semidefinite, respectively. For a locally integrable signal  $w : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , where  $\mathbb{R}_+$  denotes the set of nonnegative real numbers, we denote by  $\|w\|_{\mathcal{L}_2} = (\int_0^\infty \|w(t)\|^2 dt)^{1/2}$  its  $\mathcal{L}_2$ -norm, provided the integral is finite. Furthermore, we define the set of all locally integrable signals with a finite  $\mathcal{L}_2$ -norm as  $\mathcal{L}_2$ . For a signal  $w : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , we denote the limit from above at time  $t \in \mathbb{R}_+$  by  $w^+(t) = \lim_{s \downarrow t} w(s)$ .

## II. PERIODIC EVENT-TRIGGERED CONTROL

In this section, we introduce periodic event-triggered control (PETC) and give a precise formulation of the stability and performance analysis problems we aim to solve in this paper.

### A. Periodic Event-Triggered Control System

To introduce PETC, let us consider a linear time-invariant (LTI) plant, given by

$$\frac{d}{dt}x = A^p x + B^p \hat{u} + B^w w \quad (1)$$

where  $x \in \mathbb{R}^{n_x}$  denotes the state of the plant,  $\hat{u} \in \mathbb{R}^{n_u}$  is the input applied to the plant, and  $w \in \mathbb{R}^{n_w}$  is an unknown disturbance. In a conventional sampled-data state-feedback setting, the plant is controlled using a controller

$$\hat{u}(t) = Kx(t_k), \quad \text{for } t \in (t_k, t_{k+1}] \quad (2)$$

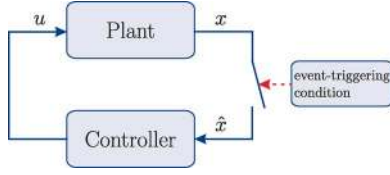


Fig. 1. Event-triggered control schematic.

where  $t_k, k \in \mathbb{N}$ , are the sampling times, which are periodic in the sense that  $t_k = kh, k \in \mathbb{N}$ , for some properly chosen sampling interval  $h > 0$ .

Instead of using conventional periodic sampled-data control, we propose here to use PETC meaning that at each sampling time  $t_k = kh, k \in \mathbb{N}$ , state measurements are transmitted over a communication network and the control values are updated only when certain event-triggering conditions are satisfied. This modifies the controller from (2) to

$$\hat{u}(t) = K\hat{x}(t), \text{ for } t \in \mathbb{R}_+ \quad (3)$$

where  $\hat{x}$  is a left-continuous signal<sup>1</sup>, given for  $t \in (t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ , by

$$\hat{x}(t) = \begin{cases} x(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) > 0 \\ \hat{x}(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) \leq 0 \end{cases} \quad (4)$$

and some initial value for  $\hat{x}(0)$ . Hence, considering the configuration in Fig. 1, the value  $\hat{x}(t)$  can be interpreted as the most recently transmitted measurement of the state  $x$  to the controller at time  $t$ . Whether or not new state measurements are transmitted to the controller is based on the event-triggering condition  $\mathcal{C} : \mathbb{R}^{n_\xi} \rightarrow \mathbb{R}$  with  $n_\xi := 2n_x$ . In particular, if at time  $t_k$  it holds that  $\mathcal{C}(x(t_k), \hat{x}(t_k)) > 0$ , the state  $x(t_k)$  is transmitted over the network to the controller and  $\hat{x}$  and the control value  $\hat{u}$  are updated accordingly. In case  $\mathcal{C}(x(t_k), \hat{x}(t_k)) \leq 0$ , no new state information is sent to the controller, in which case the input  $\hat{u}$  is not updated and kept the same for (at least) another sampling interval implying that no control computations are needed and no new state measurements and control values have to be transmitted. In the next sections, we focus on centralized event-triggering conditions, which requires that a central coordinator has access to the full state  $x(t_k)$ , while later in Section V we consider decentralized event-triggering conditions.

### B. Quadratic Event-Triggering Conditions

In this paper, we focus on *quadratic* event-triggering conditions, i.e.,  $\mathcal{C}$ , as in (4), is given by

$$\mathcal{C}(\xi(t_k)) = \xi^\top(t_k) Q \xi(t_k) > 0 \quad (5)$$

where  $\xi := [x^\top \hat{x}^\top]^\top \in \mathbb{R}^{n_\xi}$ , for some symmetric matrix  $Q \in \mathbb{R}^{n_\xi \times n_\xi}$ . To show that these event-triggering conditions form a relevant class, we will review some existing event-triggering conditions that have been applied in the context of *continuous* event-triggered control (CETC), and show how they can

<sup>1</sup>A signal  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is called left-continuous, if for all  $t > 0$ ,  $\lim_{s \uparrow t} x(s) = x(t)$ .

be written as quadratic event-triggering conditions for PETC as in (5).

1) *Event-Triggering Conditions Based on the State Error:* An important class of event-triggering conditions, which has been applied to CETC in [10], [11], are given by

$$\|\hat{x}(t_k) - x(t_k)\| > \sigma \|x(t_k)\| \quad (6)$$

for  $k \in \mathbb{N}$ , where  $\sigma > 0$ . Clearly, (6) is of the form (5) with

$$Q = \begin{bmatrix} (1 - \sigma^2)I & -I \\ -I & I \end{bmatrix}. \quad (7)$$

2) *Event-Triggering Conditions Based on the Input Error:* In [15], where the objective was to develop *output-based* CETC, an event-triggering condition was proposed that would translate for state-feedback-based PETC systems to

$$\|K\hat{x}(t_k) - Kx(t_k)\| > \sigma \|Kx(t_k)\| \quad (8)$$

where  $\sigma > 0$ . Condition (8) is equivalent to  $\|\hat{u}(t_k) - u(t_k)\| > \sigma \|u(t_k)\|$  in which  $u(t_k) = Kx(t_k)$  is the control value determined on the basis of  $x(t_k)$  as in standard periodic state-feedback (see (2)). The event-triggering condition (8) is equivalent to (5), in which

$$Q = \begin{bmatrix} (1 - \sigma^2)K^\top K & -K^\top K \\ -K^\top K & K^\top K \end{bmatrix}. \quad (9)$$

3) *Event-Triggering Conditions as in [20]:* A PETC version of the condition used in [20] is

$$\|\hat{u}(t_k) - u(t_k)\|^2 > (1 - \beta^2)\|x(t_k)\|^2 + \|\hat{u}(t_k)\|^2 \quad (10)$$

where  $0 < \beta \leq 1$  and, again,  $u(t_k) = Kx(t_k)$ , which results in an event-triggering condition (5) with  $Q = \begin{bmatrix} (\beta^2 - 1)I + K^\top K & -K^\top K \\ -K^\top K & 0 \end{bmatrix}$ , as  $\hat{u}(t_k) = K\hat{x}(t_k)$ ,  $k \in \mathbb{N}$ .

4) *Event-Triggering Conditions Based on Lyapunov Functions:* In [30] and [31] in the context of CETC and in [19] in the context of self-triggered control [18], Lyapunov-based event-triggering conditions have been proposed. For PETC, a Lyapunov-based event-triggering condition can be derived using the discretization of (1), with  $w = 0$ , given by

$$x(t_{k+1}) = Ax(t_k) + B\hat{u}_k \quad (11)$$

in which

$$A := e^{A^p h} \text{ and } B := \int_0^h e^{A^p s} B^p ds \quad (12)$$

and  $\hat{u}_k := \lim_{t \downarrow t_k} \hat{u}(t)$  taken as  $Kx(t_k)$ ,  $k \in \mathbb{N}$ , as in (2). In case  $K$  is designed such that  $A + BK$  has all its eigenvalues inside the open unit circle, there exists a quadratic Lyapunov function of the form  $V(x) = x^\top Px$ ,  $x \in \mathbb{R}^{n_x}$ , with

$$P \succ 0 \text{ and } (A + BK)^\top P (A + BK) \preceq \lambda P \quad (13)$$

for some  $0 \leq \lambda < 1$ . This implies the decrease of the Lyapunov function in the sense that  $V(x(t_{k+1})) \leq \lambda V(x(t_k))$  for all  $k \in \mathbb{N}$  along the solutions of (11) and (2). In [19] and [30], [31] an event-triggering condition has been proposed (in the context of CETC) based on the existence of  $V$  by selecting  $0 \leq \beta < 1$  and only updating  $\hat{x}$  at time  $t_k$  to  $x(t_k)$  when

$$(Ax(t_k) + BK\hat{x}(t_k))^T P(Ax(t_k) + BK\hat{x}(t_k)) > \beta x^T(t_k) P x(t_k). \quad (14)$$

Hence, only when the current input  $\hat{u}(t_k) = K\hat{x}(t_k)$  no longer guarantees a decrease of the Lyapunov function  $V$  with a factor  $\beta$ , the signals  $\hat{x}$  and  $\hat{u}$  are updated. Obviously, (14) can be written as in (5) by taking  $Q = \begin{bmatrix} A^T P A - \beta P & A^T P B K \\ (BK)^T P A & (BK)^T P B K \end{bmatrix}$ . The interest in [19], [30], [31] for this event-triggering condition is motivated by the fact that for any choice of  $0 \leq \beta < 1$ ,  $V$  is a Lyapunov function for the PETC system (1), with  $w = 0$ , (3) and (4) with event-triggering condition (14), and thus stability of the resulting PETC system is inherently guaranteed. In fact, it is easily seen that for this scheme it holds that  $V(x(t_{k+1})) \leq \max(\lambda, \beta)V(x(t_k))$  for all  $k \in \mathbb{N}$ .

The four mentioned examples show the relevance of the class of *quadratic* event-triggering conditions (5), as their CETC counterparts have been considered in the literature extensively.

### C. Closed-Loop Model and Objective of the Paper

To obtain a complete model of the PETC system, we combine (1), (3), (4) and (5), we use, as before,  $\xi = [x^T \hat{x}^T]^T$  and define

$$\bar{A} := \begin{bmatrix} A^p & B^p K \\ 0 & 0 \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} B^w \\ 0 \end{bmatrix}, \quad J_1 := \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix}, \quad J_2 := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (15)$$

to arrive at an impulsive system [28], [29] given by

$$\frac{d}{dt} \begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} \bar{A}\xi + \bar{B}w \\ 1 \end{bmatrix}, \quad \text{when } \tau \in [0, h] \quad (16a)$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{cases} \begin{bmatrix} J_1 \xi \\ 0 \end{bmatrix}, & \text{when } \xi^T Q \xi > 0, \tau = h \\ \begin{bmatrix} J_2 \xi \\ 0 \end{bmatrix}, & \text{when } \xi^T Q \xi \leq 0, \tau = h \end{cases}. \quad (16b)$$

$$z = \bar{C}\xi + \bar{D}w \quad (16c)$$

where  $z \in \mathbb{R}^{n_z}$  is a performance output with the matrices  $\bar{C}$  and  $\bar{D}$  chosen appropriately, and the state  $\tau$  keeps track of the time elapsed since the last sampling time.

Besides the introduction of new classes of PETC schemes, the main objective of this paper is to provide analysis and design techniques for event-triggering conditions of the form (5) such that the corresponding closed-loop system (1), (3), (4) and (5) is stable and has a certain closed-loop performance, both defined in an appropriate sense, while the number of transmissions between the plant and the controller is kept small. To make precise what we mean by stability and performance, let us define the notion of global exponential stability and  $\mathcal{L}_2$ -performance.

**Definition II.1:** The PETC system (16) is said to be globally exponentially stable (GES), if there exist  $c > 0$  and  $\rho > 0$  such that for any initial condition  $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$  all corresponding solutions to (16) with  $\tau(0) \in [0, h]$  and  $w = 0$  satisfy  $\|\xi(t)\| \leq ce^{-\rho t} \|\xi_0\|$  for all  $t \in \mathbb{R}_+$ . In this case, we call  $\rho$  a (lower bound on the) decay rate.

**Definition II.2:** The PETC system (16) is said to have an  $\mathcal{L}_2$ -gain from  $w$  to  $z$  smaller than or equal to  $\gamma$ , if there is a function  $\delta : \mathbb{R}^{n_\xi} \rightarrow \mathbb{R}_+$  such that for any  $w \in \mathcal{L}_2$ , any initial state  $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$  and  $\tau(0) \in [0, h]$ , the corresponding solution to (16) satisfies

$$\|z\|_{\mathcal{L}_2} \leq \delta(\xi_0) + \gamma \|w\|_{\mathcal{L}_2}. \quad (17)$$

## III. STABILITY AND $\mathcal{L}_2$ -GAIN ANALYSIS OF THE PETC SYSTEM

In this section, we analyze stability and performance of the PETC system (16) using three different approaches, namely: 1) an impulsive system approach; 2) a discrete-time piecewise linear (PWL) system approach; and 3) a discrete-time perturbed linear (PL) system approach. In particular, the first approach allows to analyze both GES and  $\mathcal{L}_2$ -gain properties, while the latter two approaches will focus on GES only.

### A. Impulsive System Approach

In this section, we will analyze the stability and the  $\mathcal{L}_2$ -gain of the impulsive system model (16), directly. To do so, let us consider a Lyapunov/storage function of the form

$$V(\xi, \tau) = \xi^T P(\tau) \xi \quad (18)$$

for  $\xi \in \mathbb{R}^{n_\xi}$  and  $\tau \in [0, h]$ , where  $P : [0, h] \rightarrow \mathbb{R}^{n_\xi \times n_\xi}$  with  $P(\tau) \succ 0$ , for  $\tau \in [0, h]$ . The choice of Lyapunov function is inspired by [28], [32]. The function  $P : [0, h] \rightarrow \mathbb{R}^{n_\xi \times n_\xi}$  will be chosen such that it becomes a candidate storage function for (16) with the supply rate  $\gamma^{-2} z^T z - w^T w$ . In particular, we will select  $P$  to satisfy the Riccati differential equation

$$\begin{aligned} \frac{d}{d\tau} P = & -\bar{A}^T P - P \bar{A} - 2\rho P - \gamma^{-2} \bar{C}^T \bar{C} \\ & - (P \bar{B} + \gamma^{-2} \bar{C}^T \bar{D}) M (\bar{B}^T P + \gamma^{-2} \bar{D}^T \bar{C}) \end{aligned} \quad (19)$$

provided the solution exists on  $[0, h]$  for a desired convergence rate  $\rho > 0$ , in which  $M := (I - \gamma^{-2} \bar{D}^T \bar{D})^{-1}$  is assumed to exist and to be positive definite, which means that  $\gamma^2 > \lambda_{\max}(\bar{D}^T \bar{D})$ . As we will show in the proof of Theorem III.2, this choice for the matrix function  $P$  yields

$$\frac{d}{dt} V \leq -2\rho V - \gamma^{-2} z^T z + w^T w \quad (20)$$

during the flow (16a). Combining inequality (20) with the conditions

$$V(J_1 \xi, 0) \leq V(\xi, h), \quad \text{for all } \xi \text{ with } \xi^T Q \xi > 0, \quad (21a)$$

$$V(J_2 \xi, 0) \leq V(\xi, h), \quad \text{for all } \xi \text{ with } \xi^T Q \xi \leq 0, \quad (21b)$$

which imply that the storage function does not increase during the jumps (16b) of the impulsive system (16), we can guarantee that the  $\mathcal{L}_2$ -gain from  $w$  to  $z$  is smaller than or equal to  $\gamma$ , see, e.g., [33]. The result that we present below, is based on verifying the satisfaction of (21) by relating  $P_0 := P(0)$  to  $P_h := P(h)$ . To do so, we introduce the Hamiltonian matrix

$$H := \begin{bmatrix} \bar{A} + \rho I + \gamma^{-2} \bar{B} M \bar{D}^\top \bar{C} & \bar{B} M \bar{B}^\top \\ -\bar{C}^\top L \bar{C} & -(\bar{A} + \rho I + \gamma^{-2} \bar{B} M \bar{D}^\top \bar{C})^\top \end{bmatrix} \quad (22)$$

with  $L := (\gamma^2 I - \bar{D} \bar{D}^\top)^{-1}$ , which is positive definite if again  $\gamma^2 > \lambda_{\max}(\bar{D}^\top \bar{D}) = \lambda_{\max}(\bar{D} \bar{D}^\top)$ . In addition, we introduce the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix} \quad (23)$$

allowing us to provide the explicit solution to the Riccati differential (19), yielding

$$P_0 = (F_{21}(h) + F_{22}(h)P_h)(F_{11}(h) + F_{12}(h)P_h)^{-1} \quad (24)$$

provided that the solution (24) is well defined on  $[0, h]$ , see, e.g., [34, Lem. 8.2]. To guarantee this, we will use the following assumption.

*Assumption III.1:*  $F_{11}(\tau)$  is invertible for all  $\tau \in [0, h]$ .

Before presenting the main result, observe that Assumption III.1 is always satisfied for sufficiently small  $h$ . Namely,  $F(\tau) = e^{-H\tau}$  is a continuous function and we have that  $F_{11}(0) = I$ . Let us also introduce the notation  $\bar{F}_{11} := F_{11}(h)$ ,  $\bar{F}_{12} := F_{12}(h)$ ,  $\bar{F}_{21} := F_{21}(h)$  and  $\bar{F}_{22} := F_{22}(h)$ , and a matrix  $\bar{S}$  that satisfies  $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$ . A matrix  $\bar{S}$  exists under Assumption III.1, because this assumption will guarantee that the matrix  $-\bar{F}_{11}^{-1}\bar{F}_{12}$  is positive semidefinite, as we will show in the proof of the theorem presented below.

*Theorem III.2:* Consider the impulsive system (16) and let  $\rho > 0$ ,  $\gamma > \sqrt{\lambda_{\max}(\bar{D}^\top \bar{D})}$ , and Assumption III.1 hold. Suppose that there exist a matrix  $P_h \succ 0$ , and scalars  $\mu_i \geq 0$ ,  $i \in \{1, 2\}$ , such that for  $i \in \{1, 2\}$

$$\begin{bmatrix} P_h + (-1)^i \mu_i Q & J_i^\top \bar{F}_{11}^{-\top} P_h \bar{S} & J_i^\top (\bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21} \bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P_h \bar{S} & 0 \\ \star & \star & \bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21} \bar{F}_{11}^{-1} \end{bmatrix} \succ 0. \quad (25)$$

Then, the PETC system (16) is GES with convergence rate  $\rho$  (when  $w = 0$ ) and has an  $\mathcal{L}_2$ -gain from  $w$  to  $z$  smaller than or equal to  $\gamma$ .

The results of Theorem III.2 guarantee both GES (for  $w = 0$ ) and an upper bound on the  $\mathcal{L}_2$ -gain. In case disturbances are absent (i.e.,  $w = 0$ ), the conditions of Theorem III.2 simplify and GES can be guaranteed using the following corollary.

*Corollary III.3:* Consider the impulsive system and let  $\rho > 0$  be given. Assume there exist a matrix  $P_h \succ 0$  and scalars  $\mu_i \geq 0$ ,  $i \in \{1, 2\}$ , such that

$$\begin{bmatrix} e^{-2\rho h} P_h + (-1)^i \mu_i Q & J_i^\top e^{\bar{A}^\top h} P_h \\ \star & P_h \end{bmatrix} \succ 0, \quad i \in \{1, 2\}. \quad (26)$$

Then, the PETC system (16) is GES (for  $w = 0$ ) with decay rate  $\rho$ .

## B. Piecewise Linear System Approach

In this section, we will obtain less conservative conditions for GES (when  $w = 0$ ), if compared to the impulsive system approach. These conditions will be obtained based on a discrete-time PWL model, which is obtained by discretizing the impulsive system (16) at the sampling times  $t_k = kh$ ,  $k \in \mathbb{N}$ , where we take  $\tau(0) = h$  and  $w = 0$ . The fact that we use a discretized model, thereby losing exact information on the inter-sample behavior, might make an  $\mathcal{L}_2$ -gain analysis complicated. Therefore, we focus on GES and  $w = 0$ . By defining the state variable  $\xi_k := \xi(t_k)$  (and assuming  $\xi$  to be left-continuous), the discretization leads to the bimodal PWL model

$$\xi_{k+1} = \begin{cases} A_1 \xi_k, & \text{when } \xi_k^\top Q \xi_k > 0, \\ A_2 \xi_k, & \text{when } \xi_k^\top Q \xi_k \leq 0, \end{cases} \quad (27)$$

where

$$A_1 := e^{\bar{A}h} J_1 = \begin{bmatrix} A + BK & 0 \\ I & 0 \end{bmatrix}, \quad A_2 := e^{\bar{A}h} J_2 = \begin{bmatrix} A & BK \\ 0 & I \end{bmatrix} \quad (28)$$

with  $A$  and  $B$  as in (12).

Using the PWL model (27) and a piecewise quadratic (PWQ) Lyapunov function of the form

$$V(\xi) = \begin{cases} \xi^\top P_1 \xi, & \text{when } \xi^\top Q \xi > 0, \\ \xi^\top P_2 \xi, & \text{when } \xi^\top Q \xi \leq 0, \end{cases} \quad (29)$$

we can guarantee GES of the PETC system given by (1), (3) (4) and (5) under the conditions given next.

*Theorem III.4:* The PETC system (16) is GES with decay rate  $\rho$ , if there exist matrices  $P_1, P_2$  and scalars  $\alpha_{ij} \geq 0$ ,  $\beta_{ij} \geq 0$  and  $\kappa_i \geq 0$ ,  $i, j \in \{1, 2\}$ , satisfying

$$e^{-2\rho h} P_i - A_i^\top P_j A_i + (-1)^i \alpha_{ij} Q + (-1)^j \beta_{ij} A_i^\top Q A_i \succeq 0, \quad \text{for all } i, j \in \{1, 2\}, \quad \text{and} \quad (30a)$$

$$P_i + (-1)^i \kappa_i Q \succ 0, \quad \text{for all } i \in \{1, 2\}. \quad (30b)$$

As stated in the beginning of this section, the impulsive system approach can never outperform the PWL system approach in terms of the stability analysis. To formally prove this statement, we substitute (28) into (26), and apply a Schur complement to (26), yielding that  $e^{-2\rho h} P_h + (-1)^i \mu_i Q - A_i^\top P_h A_i \succ 0$ ,  $i \in \{1, 2\}$  and  $P_h \succ 0$  and  $\mu_i \geq 0$ ,  $i \in \{1, 2\}$ . As these conditions are equivalent to the LMIs (30a), with  $P_1 = P_2 = P_h$ ,  $\alpha_{ij} = \mu_i$  and  $\beta_{ij} = 0$ ,  $i, j \in \{1, 2\}$ , this shows that if the LMIs (26) are feasible, then the LMIs (30a) are feasible. In addition, since  $P_h \succ 0$  the LMIs (30b) hold with  $\kappa_1 = \kappa_2 = 0$ . Hence, we have proven the following result.

*Theorem III.5:* Let  $\rho > 0$  be fixed. Suppose that (26) is satisfied for some  $P_h \succ 0$ ,  $\mu_1 \geq 0$  and  $\mu_2 \geq 0$ . Then (30a) and (30b) are satisfied for some  $P_1, P_2$  and constants  $\alpha_{ij} \geq 0$ ,  $\beta_{ij} \geq 0$ , and  $\kappa_i$ ,  $i, j \in \{1, 2\}$ .

<sup>2</sup>Note that for (16) and in the stability definition (Def. II.1) the initial conditions are allowed to have any value  $\tau(0) \in [0, h]$ , while in the discretization we take  $\tau(0) = h$ . Due to the linearity of the flow dynamics (16a) and the fact that  $\tau(0)$  lies in a bounded set, it is straightforward to see that GES for initial conditions with  $\tau(0) = h$  implies GES for all initial conditions with  $\tau(0) \in [0, h]$ .

Hence, in case the impulsive system approach guarantees GES with convergence rate  $\rho$  of the PETC system (16) using Corollary III.3, then the PWL system approach using Theorem III.4, proves GES with convergence rate  $\rho$  of the PETC system as well.

*Remark III.6:* In case Corollary III.3 is applied for stability analysis, the resulting conditions are equivalent to the existence of a quadratic Lyapunov function for the corresponding PWL system. This explains that exploiting the impulsive system approach for stability analysis does not improve upon the results obtained by the PWL system approach directly. However, theoretically one can show that any Lyapunov function  $V$  that proves GES based on the discrete-time PWL system (27) can be converted into a Lyapunov function  $W$  for the impulsive system (16) (with  $w = 0$ ) given by  $W(\xi, \tau) = e^{-\varepsilon\tau} V(e^{\tilde{A}(h-\tau)}\xi)$ ,  $\tau \in [0, h]$  and  $\xi \in \mathbb{R}^{n_\xi}$ , for a sufficiently small positive value of  $\varepsilon$ . However, to construct a Lyapunov function of the form  $W(\xi, \tau) = e^{-\varepsilon\tau} V(e^{\tilde{A}(h-\tau)}\xi)$  by a tractable computational method directly on the basis of (16) is complicated.

### C. A Perturbed Linear System Approach

For the particular case where the event-triggering conditions are in the form of (6) or (8), more easily verifiable conditions for GES can be obtained at the cost of being more conservative than the PWL system approach, see Theorem III.8 below. These conditions will be obtained through a PL system approach and can be based on standard  $\ell_2$ -gain techniques allowing for a simple maximization of  $\sigma$  in (6) or (8) subject to the sufficient GES-conditions.

The GES analysis will be based on the discrete-time PL system

$$x_{k+1} = (A + BK)x_k + BK e_k \quad (31)$$

where  $x_k = x(t_k)$ ,  $\hat{x}_k = \lim_{t \downarrow t_k} \hat{x}(t)$  [recall that the signal  $\hat{x}$  is piecewise constant and left-continuous, cf. (4)],  $e_k := \hat{x}_k - x_k$ ,  $k \in \mathbb{N}$ , and  $A, B$  as in (12). The system (31) is obtained by discretizing (1), with  $w = 0$ , and combining it with (3). The system expresses how the plant (1) with the event-triggered controller (3) is perturbed when compared to the original periodic sampled-data control system given by (1) and (2).

The following stability result relies on the concepts of dissipativity, storage functions and supply rates, see, e.g., [35], [36]. Note that the result we present below uses the event-triggering condition (6). A similar result can be obtained for event-triggering condition (8) by modifying (31) into  $x_{k+1} = (A + BK)x_k + B e_k$  and then  $e_k := \hat{u}_k - u_k$ ,  $k \in \mathbb{N}$ .

*Theorem III.7:* Suppose that the PL system (31) admits a storage function  $\bar{V}(x) = x^\top \bar{P} x$  with  $\bar{P}$  a symmetric positive definite matrix for supply rate<sup>3</sup>  $-\theta^2 \|x\|^2 + \|e\|^2$  with  $\theta > 0$ , i.e., the dissipation inequality

$$\bar{V}(x_{k+1}) - \bar{V}(x_k) \leq -\theta^2 \|x_k\|^2 + \|e_k\|^2, \quad k \in \mathbb{N} \quad (32)$$

is satisfied for any disturbance sequence  $\{e_k\}_{k \in \mathbb{N}}$  and all corresponding solutions  $\{x_k\}_{k \in \mathbb{N}}$ . Then the PETC system (16) with  $Q$  as in (6) is GES for any  $0 < \sigma < \theta$ .

<sup>3</sup>We scaled the constant in front of  $\|e\|^2$  to 1. Note that this is without loss of generality as  $P$  and  $\theta$  can be scaled as well.

*Proof:* It is possible to give a direct proof on the basis of (32) along the lines of [11], [22]. For reasons of brevity, we will not give a direct proof, but point out that the proof follows from Theorem III.8 together with Theorem III.4. ■

Observe that the existence of a storage function  $\bar{V}$  satisfying the dissipation inequality (32) is equivalent to feasibility of the LMIs

$$\begin{bmatrix} \bar{P} - (A+BK)^\top \bar{P} (A+BK) - \theta^2 I & \star \\ -(BK)^\top \bar{P} (A+BK) & I - (BK)^\top \bar{P} BK \end{bmatrix} \succ 0 \quad \text{and } \bar{P} \succ 0. \quad (33)$$

In fact, feasibility of (33) is equivalent to the system (31) having an  $\ell_2$ -gain smaller than or equal to  $\theta^{-1}$  from  $e$  to  $x$ . To obtain the largest minimal inter-event times, it follows from (6) that  $\sigma$  should be as large as possible and thus that  $\theta$  should be maximized, while satisfying (32) (in order to have a GES guarantee). Hence, this results in the convex optimization problem of maximizing  $\theta^2$  subject to (33). Interestingly, provided that  $A+BK$  has all its eigenvalues within the unit circle, the maximal value  $\theta^*$  obtained in this way is such that the true  $\ell_2$ -gain of (31) from  $e$  to  $x$  is equal to  $(\theta^*)^{-1}$ , which is equal to the  $\mathcal{H}_\infty$ -norm given by  $\sup_{\omega \in [0, \pi]} \|(e^{j\omega} I - A - BK)^{-1} BK\|$ , see, e.g., [35]. Hence, the supremal  $\theta^*$  satisfying (33) gives rise to the  $\ell_2$ -gain of  $(\theta^*)^{-1}$  and guarantees stability of the PETC system (1), (3), (4), with (6) for any  $0 < \sigma < \theta^* = 1/(\sup_{\omega \in [0, \pi]} \|(e^{j\omega} I - A - BK)^{-1} BK\|)$ . Hence, a standard  $\mathcal{H}_\infty$ -norm calculation for a linear system provides stability bounds in terms of  $\sigma$  for the event-triggering condition (6) [or (8)].

We will now formally show that for the particular event-triggering condition (6) (or (8)), the PL system approach can never do better than the PWL system approach in terms of the range of  $\sigma$  for which GES of the PETC system can be proven.

*Theorem III.8:* Let  $\sigma > 0$  be given. Suppose that (33) is satisfied for some  $\bar{P} \succ 0$  and  $\theta > \sigma$ . Then (30a) and (30b) are satisfied for some  $P_1, P_2$ , constants  $\alpha_{ij} \geq 0, \beta_{ij} \geq 0, \rho > 0$ , and  $\kappa_i \geq 0, i, j \in \{1, 2\}$  and with  $Q$  as in (7) with the same  $\sigma$ .

Hence, in case the PL system approach guarantees GES of the PETC system (16) with  $Q$  as in (7) [or (9)] for some  $\sigma > 0$  using Theorem III.7, then the PWL system approach also guarantees GES of the PETC system for the same  $\sigma > 0$  based on Theorem III.4.

### D. Discussion on the Different Approaches

When comparing the different analysis approaches, several observations can be made. The first observation is that the PL system approach provides, in case of event-triggering conditions (6) and (8), stability guarantees of the PETC system via a simple  $\mathcal{H}_\infty$ -norm computation, or, alternatively, via direct maximization of  $\sigma$  subject to the sufficient GES conditions in terms of the LMIs (33). Both these computations are of a lower complexity than the computational tests required for the impulsive and PWL system approaches. In particular, considering the PWL system approach, the maximization of  $\sigma$  subject to the sufficient GES-conditions (30) are not directly LMIs as  $\sigma$  is included through  $Q$ , see (7) and (9), which is multiplied by  $\alpha_{ij}$  and  $\beta_{ij}$ ,  $i, j \in \{1, 2\}$ . However, when fixing  $\sigma$  the conditions



(30) become LMIs and hence, a line search in  $\sigma$  is needed to maximize  $\sigma$  subject to (30). Similar comments apply for the impulsive system approach.

The second observation is that the impulsive system approach is the only one of the three approaches that allows the  $\mathcal{L}_2$ -gain from  $w$  to  $z$  to be studied at this point, which makes this approach important for PETC as well. Although one might attempt to use the PWL and PL system approaches to obtain upper bounds on the  $\mathcal{L}_2$ -gain by including the intersample behavior, doing so might be difficult.

Finally, the PWL system approach is relevant since, when comparing it to the PL system and the impulsive system approach, we can show that for stability analysis (when  $w = 0$ ), the PWL system approach never yields more conservative results than the other two approaches (Theorem III.5 and Theorem III.8). However, the PWL system approach is computationally (somewhat) more involved than the PL system approach, as already mentioned.

#### IV. DESIGN CONSIDERATIONS

In this section, we will provide guidelines for the design of PETC strategies, which consists of the proper selection of the sampling period  $h$ , the feedback gain  $K$  in (3), and the matrix  $Q$  in the event-triggering condition (5). In general, the design of the PETC loop will be a tradeoff between control properties (stability and performance, e.g.,  $\mathcal{L}_2$ -gains) and resource utilization.

The *joint* design of the controller and the event-triggering condition is a hard problem, both in the context of CETC and PETC. In fact, most of the existing CETC design methods follow a so-called *emulation-based* approach. Based on the general analysis framework provided in Section III, one can show that a similar emulation-based approach can be taken in the context of PETC as well. To explain this in more detail, we focus on the PETC conditions given by (6) or (8), so that the design process consists of selecting  $h$ ,  $K$  and  $\sigma$ .

In the emulation-based approach, two phases can be distinguished. In the first phase, the controller is assumed to be implemented in a standard periodic sampled-data fashion and, therefore, standard sampled-data controller design tools, see, e.g., [37], [38] can be used to select  $h$  and  $K$  such that the resulting closed-loop system, given by (1), (2), and  $t_k = kh$ ,  $k \in \mathbb{N}$ , is GES and has a satisfactory  $\mathcal{L}_2$ -gain. In this first phase, the selection of  $h$  is directly incorporated and balanced with stability and performance requirements. The importance of the selection of  $h$  for the eventual PETC law is that it provides directly a lower bound on the time difference between two consecutive updates of the control signal in the PETC system (1), (3), (4) and (5). In CETC it is rather difficult to tune the parameters in the controller and the event-triggering condition in order to guarantee an *a priori* specified lower bound on the inter-event times. Only indirect tuning knobs are available for CETC, and several iterations selecting different parameter settings might be needed to obtain a desirable lower bound (if possible at all). In PETC, the lower bound  $h$  can be selected directly in the design, which is a benefit of PETC over CETC.

*Remark IV.1:* The *minimum inter-event time*, being the largest lower bound on the time differences between two consecutive control updates in the PETC system (1), (3), (4) and

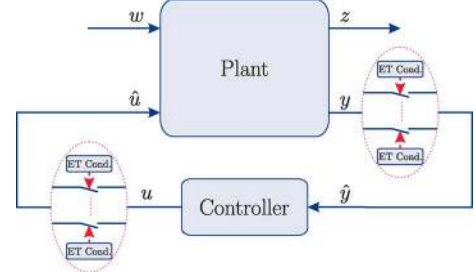


Fig. 2. Decentralized event-triggered control schematic.

(5), might actually be larger than  $h$ . If we restrict ourselves to the disturbance-free case ( $w = 0$ ), the minimum inter-event time can be computed exactly [39, Ch. 5], and is given by  $ht_{\min,h}^*$  with

$$t_{\min,h}^* := \inf \{l \in \mathbb{N}_{\geq 1} \mid \lambda_{\max}((A_2^{l-1}A_1)^\top Q A_2^{l-1}A_1) > 0\} \quad (34)$$

where  $A_1$  and  $A_2$  were defined in (28).

In the second phase of the emulation-based design process (when  $h$  and  $K$  are already given) an appropriate value for  $\sigma$  has to be chosen in order to appropriately balance control performance and resource utilization. Based on the analysis framework in Section III, it can be investigated for which values of  $\sigma$  GES and certain upper bounds on the  $\mathcal{L}_2$ -gain can still be guaranteed. Of course, a first requirement when increasing  $\sigma$  is that GES is preserved. Interestingly, based on the results in Section III-C using the PL system approach (see Theorem III.7 and the succeeding discussion), we immediately have that if the controller implemented in a conventional periodic sampled-data fashion stabilises the system (i.e., if  $A + BK$  has all its eigenvalues inside the unit circle), then the PETC system (16) with  $w = 0$  and the event-triggering condition (6) or (8) remains to be GES for sufficiently small values of  $\sigma > 0$ . Indeed, if  $A + BK$  has all its eigenvalues inside the unit circle, the  $\mathcal{H}_\infty$ -norms given by  $(\theta^*)^{-1} = \sup_{\omega \in [0, \pi]} \|(e^{j\omega}I - A - BK)^{-1}BK\|$  in case of (6), and  $(\theta^*)^{-1} = \sup_{\omega \in [0, \pi]} \|K(e^{j\omega}I - A - BK)^{-1}B\|$  in case of (8) are finite and hence, for any  $0 < \sigma < \theta^*$ , GES of the corresponding PETC systems is guaranteed. Using the PWL system approach the values of  $\sigma$  for which GES can be guaranteed can even be enlarged.

Similarly, starting from appropriate  $\mathcal{L}_2$ -gain properties for the conventional periodic sampled-data loop, one can investigate how the guaranteed upper bound on the  $\mathcal{L}_2$ -gain based on Theorem III.2 varies as a function  $\sigma$ . An illustration will be given for a numerical example later leading to Fig. 3(a) below. Based on such curves one can decide how much of the  $\mathcal{L}_2$ -gain one would like to trade for less resource utilization. The larger  $\sigma$  is taken, the larger the  $\mathcal{L}_2$ -gain typically becomes [Fig. 3(a)], and the larger the reduction in resource utilization will be [Fig. 3(d)].

#### V. OUTPUT-BASED DECENTRALIZED PETC

In this section, we will extend the previous results in two directions, namely towards dynamical output-based controllers and towards *decentralized* event-triggering conditions. As already indicated in the introduction, the motivation for the study of output-based controllers is that often not all the states are

available for feedback in practice. The focus on decentralized event-triggering conditions is motivated by the fact that sensors, actuators and controllers can be physically distributed over a wide area. In this case, a centralized event-triggering mechanism can be prohibitive, as the coordinator that verifies the event-triggering conditions would need access to all the plant and controller outputs at every sampling time, which can be an unrealistic assumption in large-scale systems. To resolve this issue, decentralized periodic event-triggered conditions based on only local information are of interest.

#### A. Description of Output-Based PETC With Decentralized Event-Triggering

Let us consider the linear time-invariant (LTI) plant given by

$$\begin{cases} \dot{x}^p = A^p x^p + B^p \hat{u} + B^w w \\ y = C^p x^p \end{cases} \quad (35)$$

where  $x^p \in \mathbb{R}^{n_p}$  denotes the state<sup>4</sup> of the plant,  $\hat{u} \in \mathbb{R}^{n_u}$  the input applied to the plant,  $w \in \mathbb{R}^{n_w}$  an unknown disturbance, and  $y \in \mathbb{R}^{n_y}$  the output of the plant. The plant is controlled using a discrete-time LTI controller

$$\begin{cases} x_{k+1}^c = A^c x_k^c + B^c \hat{y}_k \\ u_k = C^c x_k^c + D^c \hat{y}_{k-1} \end{cases} \quad (36)$$

where  $x^c \in \mathbb{R}^{n_c}$  denotes the state of the controller,  $\hat{y} \in \mathbb{R}^{n_y}$  the input of the controller, and  $u \in \mathbb{R}^{n_u}$  the output of the controller. As before, at the sampling times  $t_k = kh$ ,  $k \in \mathbb{N}$ , where  $h > 0$  is again the sampling interval, the outputs of the plant  $y(t_k)$  and controller  $u(t_k) = u_k$  are sampled. At a sampling time  $t_k$ , a *decentralized* event-triggering condition will determine which values in  $y(t_k)$  and  $u(t_k)$  will be transmitted and which are not. This will determine the updates of  $\hat{u}$  and  $\hat{y}$ . Just as  $\hat{x}$  was the most recently received version of  $x$  in the state-feedback case in Section II,  $\hat{u}$  and  $\hat{y}$  are now the most recently received versions of  $u$  and  $y$ , see Fig. 2. To formalize this, we need a few conventions.

The states of the controller  $x_{k+1}^c$  are updated based on  $\hat{y}_k$ . To implement the discrete-time controller (36) in practice, the update of the state  $x_k^c$  to  $x_{k+1}^c$  should occur somewhere in the time interval  $(t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ , although in the mathematical model we adopt the convention that for  $t \in (t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ , it holds that

$$x^c(t) = x_{k+1}^c = A^c x_k^c + B^c \hat{y}_k \quad (37)$$

indicating that the updates of  $x^c$  take place right after  $t_k$ ,  $k \in \mathbb{N}$ . Observe that  $x^c$  is a left-continuous signal. In addition, the control value  $u(t_k) = u_k$  at time  $t_k$  is computed on the basis of  $\hat{y}_{k-1}$ , which will be equal to  $\hat{y}(t_k)$ , being the most recently received output at the plant at  $t_k$ , as we define for  $t \in (t_k, t_{k+1}]$

$$\hat{y}(t) = \hat{y}_k \quad \text{and} \quad \hat{u}(t) = \hat{u}_k. \quad (38)$$

Hence,  $\hat{u}$  and  $\hat{y}$  are also left-continuous,  $\hat{y}_k := \lim_{t \downarrow t_k} \hat{y}(t)$ , and  $\hat{u}_k := \lim_{t \downarrow t_k} \hat{u}(t)$ . In this way,  $u(t_k) = C^c x^c(t_k) + D^c \hat{y}(t_k)$ ,  $k \in \mathbb{N}$ .

<sup>4</sup>We added superscript  $p$  here to denote the state of the plant (cf. (1)), as now we have to distinguish between the plant state  $x^p$  and the controller state  $x^c$ .

Finally, to introduce the *decentralized* event-triggering conditions to determine which signals will be transmitted at  $t_k$ , we define  $v = [y^\top u^\top]^\top \in \mathbb{R}^{n_v}$  and  $\hat{v} = [\hat{y}^\top \hat{u}^\top]^\top \in \mathbb{R}^{n_v}$  with  $n_v := n_y + n_u$ , and assume that the outputs of the plant and controller, i.e., the entries in  $v$  and  $\hat{v}$ , are grouped into  $N$  nodes, see also Fig. 2. The entries in  $v$  and  $\hat{v}$  corresponding to node  $j \in \{1, \dots, N\}$  are denoted by  $v^j$  and  $\hat{v}^j$ , respectively. To introduce the adopted decentralized event-triggering conditions, we focus on (6), although alternative event-triggering conditions can be used as well, see Remark V.1. By focussing on (6), the decentralized event-triggering condition and update of the signals  $\hat{v}$  can be described as

$$\hat{v}^j(t) = \begin{cases} v^j(t_k), & \text{if } \|v^j(t_k) - \hat{v}^j(t_k)\| > \sigma_j \|v^j(t_k)\| \\ \hat{v}^j(t_k), & \text{if } \|v^j(t_k) - \hat{v}^j(t_k)\| \leq \sigma_j \|v^j(t_k)\|, \end{cases} \quad (39)$$

for  $t \in (t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ , in which  $\sigma_j \geq 0$ ,  $j \in \{1, \dots, N\}$ , are given constants. Hence, (39) expresses that at a sampling time  $t_k$ ,  $k \in \mathbb{N}$ , each node  $j$  samples the respective outputs of plant and controller and verifies if the difference  $v^j(t_k) - \hat{v}^j(t_k)$  is too large with respect to  $v^j(t_k)$  (determined by  $\sigma_j$ ). In case the difference is too large, node  $j$  will transmit its corresponding signals  $v^j(t_k)$ , and  $\hat{v}^j$  is updated accordingly just after  $t_k$ . In this setup, each node has its own *local* event-triggering condition, which invokes transmission of  $v^j$  if

$$\|v^j(t_k) - \hat{v}^j(t_k)\| > \sigma_j \|v^j(t_k)\|. \quad (40)$$

Note that in this setup it is possible that several nodes may transmit at the same time. If communication constraints prohibit that multiple nodes transmit simultaneously, extensions of the presented framework are possible. The interested reader is referred to [39, Sec. 5.6.3] for a short discussion on this issue.

Evidently, each of the local event-triggering conditions in (40) can be reformulated as the quadratic event-triggering condition

$$\xi^\top(t_k) Q_j \xi(t_k) > 0 \quad (41)$$

in terms of  $\xi = [x^p{}^\top x^c{}^\top \hat{v}^\top]^\top = [x^p{}^\top x^c{}^\top \hat{y}^\top \hat{u}^\top]^\top$  by proper choice of  $Q_j$ ,  $j \in \{1, \dots, N\}$ . To show how this can be accomplished, we introduce some notational conventions. For an index set  $\mathcal{J} \subseteq \{1, \dots, N\}$  we define the diagonal matrices  $\Gamma_{\mathcal{J}} \in \mathbb{R}^{n_v \times n_v}$

$$\Gamma_{\mathcal{J}} = \text{diag}(\gamma_{\mathcal{J}}^1, \dots, \gamma_{\mathcal{J}}^{n_y+n_u}) \quad (42)$$

where the elements  $\gamma_{\mathcal{J}}^l$ , with  $l \in \{1, \dots, n_y\}$ , are equal to 1 if  $l \in \mathcal{J}$ , elements  $\gamma_{\mathcal{J}}^{l+n_y}$ , with  $l \in \{1, \dots, n_u\}$ , are equal to 1 if  $l + n_y \in \mathcal{J}$ . The element  $\gamma_{\mathcal{J}}^l$  is 0 otherwise. We will also sometimes use the diagonal submatrices  $\Gamma_{\mathcal{J}}^y \in \mathbb{R}^{n_y \times n_y}$  and  $\Gamma_{\mathcal{J}}^u \in \mathbb{R}^{n_u \times n_u}$  of  $\Gamma_{\mathcal{J}}$  that satisfy  $\Gamma_{\mathcal{J}} = \text{diag}(\Gamma_{\mathcal{J}}^y, \Gamma_{\mathcal{J}}^u)$ . Furthermore, we use the notation  $\Gamma_j = \Gamma_{\{j\}}$ ,  $\Gamma_j^y = \Gamma_{\{j\}}^y$  and  $\Gamma_j^u = \Gamma_{\{j\}}^u$  for  $j \in \{1, \dots, N\}$ ,  $C := \text{diag}(C^p, C^c)$ , and  $D := \begin{bmatrix} 0 & 0 \\ D^c & 0 \end{bmatrix}$ , to obtain for  $k \in \mathbb{N}$  that

$$\begin{aligned} \|v^j(t_k)\| &= \|\Gamma_j [C \ D] \xi(t_k)\| \quad \text{and} \\ \|\hat{v}^j(t_k) - \hat{v}^j(t_k)\| &= \|\Gamma_j [C \ D - I] \xi(t_k)\| \end{aligned} \quad (43)$$



which allow us to rewrite (40) as (41) with

$$Q_j := \begin{bmatrix} (1-\sigma_j)C^\top \Gamma_j C & (1-\sigma_j)C^\top \Gamma_j D - C^\top \Gamma_j \\ (1-\sigma_j)D^\top \Gamma_j C - \Gamma_j C & (D-I)^\top \Gamma_j (D-I) - \sigma_j D^\top \Gamma_j D \end{bmatrix}. \quad (44)$$

Moreover, now we can compactly write the updates of  $\hat{v}$  just after time  $t_k$  as

$$\begin{aligned} \hat{v}^+(t_k) &= \Gamma_{\mathcal{J}(\xi(t_k))} v(t_k) + (I - \Gamma_{\mathcal{J}(\xi(t_k))}) \hat{v}(t_k) \\ &= [\Gamma_{\mathcal{J}(\xi(t_k))} C \quad \Gamma_{\mathcal{J}(\xi(t_k))} D + I - \Gamma_{\mathcal{J}(\xi(t_k))}] \xi(t_k) \end{aligned} \quad (45)$$

where for  $\xi \in \mathbb{R}^{n_\xi}$

$$\mathcal{J}(\xi) := \{j \in \{1, \dots, N\} \mid \xi^\top Q_j \xi > 0\}. \quad (46)$$

*Remark V.1:* Note that any decentralized event-triggering conditions that can be written in the form (41), e.g., the decentralized equivalents of (10) or (14), can also be analyzed with the tools presented below without any modification.

To obtain an impulsive system model of the decentralized PETC system, given by (35), (36), (38), (37), and (40), we observe that due to the definition of  $\mathcal{J}(\xi)$  in (46) we have for  $k \in \mathbb{N}$  that  $\mathcal{J}(\xi(t_k)) = \mathcal{J}$  if and only if

$$\xi(t_k)^\top Q_j \xi(t_k) > 0, \quad j \in \mathcal{J} \quad \text{and} \quad \xi(t_k)^\top Q_j \xi(t_k) \leq 0, \quad j \in \mathcal{J}^c \quad (47)$$

where we denote for any arbitrary set  $J \subseteq \{1, \dots, N\}$  its complement by  $\mathcal{J}^c := \{1, \dots, N\} \setminus J$ . Based on the above, we can obtain the impulsive model

$$\begin{bmatrix} \dot{\xi} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} \bar{A}\xi + \bar{B}w \\ 1 \end{bmatrix}, \quad \text{when } \tau \in [0, h] \quad (48a)$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} J_{\mathcal{J}} \xi \\ 0 \end{bmatrix}, \quad \text{when } \tau = h, \xi^\top Q_j \xi > 0, j \in \mathcal{J} \quad (48b)$$

$$\text{and } \xi^\top Q_j \xi \leq 0, j \in \mathcal{J}^c \quad (48b)$$

$$z = \bar{C}\xi + \bar{D}w \quad (48c)$$

where  $z \in \mathbb{R}^{n_z}$  is a performance output, similar to (16c). The matrices  $Q_j, j \in \{1, \dots, N\}$ , are given as in (44), and

$$\bar{A} = \begin{bmatrix} A^p & 0 & 0 & B^p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B^w \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (49a)$$

$$J_{\mathcal{J}} = \begin{bmatrix} I & 0 & 0 & 0 \\ B^c \Gamma_{\mathcal{J}}^y C^p & A^c & B^c (I - \Gamma_{\mathcal{J}}^y) & 0 \\ \Gamma_{\mathcal{J}}^y C^p & 0 & (I - \Gamma_{\mathcal{J}}^y) & 0 \\ 0 & \Gamma_{\mathcal{J}}^u C^c & \Gamma_{\mathcal{J}}^u D^c & I - \Gamma_{\mathcal{J}}^u \end{bmatrix}. \quad (49b)$$

### B. Impulsive System Approach

In a similar fashion as the developments in Section III-A, we can obtain the following result, whose proof is omitted due to space limitations.

*Theorem V.2:* Let  $\rho > 0$ ,  $\gamma > \sqrt{\lambda_{\max}(D^\top D)}$  and Assumption III.1 hold, and suppose that there are a ma-

trix  $P_h \succ 0$  and scalars  $\mu_{\mathcal{J}j} \geq 0$ ,  $\mathcal{J} \subseteq \{1, \dots, N\}$ ,  $j \in \{1, \dots, N\}$ , such that

$$\begin{bmatrix} P_h - \sum_{j \in \mathcal{J}} \mu_{\mathcal{J}j} Q_j + \sum_{j \in \mathcal{J}^c} \mu_{\mathcal{J}j} Q_j & J_{\mathcal{J}}^\top \bar{F}_{11}^{-\top} P_h \bar{S} & \Xi(\mathcal{J}) \\ \star & I - \bar{S}^\top P_h \bar{S} & 0 \\ \star & \star & \Theta \end{bmatrix} \succ 0 \quad (50)$$

for all  $\mathcal{J} \subseteq \{1, \dots, N\}$ , with  $\Xi(\mathcal{J}) := J_{\mathcal{J}}^\top (\bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21} \bar{F}_{11}^{-1})$  and  $\Theta := \bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21} \bar{F}_{11}^{-1}$ , where  $\bar{F}_{ij} = F_{ij}(h)$ ,  $i, j \in \{1, 2\}$  as in (23) with  $H$  in (22) for  $\bar{A}$ ,  $\bar{B}$  as in (49), and a matrix  $\bar{S}$  satisfying  $\bar{S} \bar{S}^\top = -\bar{F}_{11}^{-1} \bar{F}_{12}$ . Then, the PETC system (48) is GES with decay rate  $\rho$  (when  $w = 0$ ) and has an  $\mathcal{L}_2$ -gain from  $w$  to  $z$  smaller than or equal to  $\gamma$ .

### C. Piecewise Linear System Approach

To arrive at a discrete-time PWL model (for the case  $w = 0$ ), we discretize the impulsive system (48), with  $\tau(0) = h$  and  $w = 0$ , at the sampling times  $t_k = kh$ ,  $k \in \mathbb{N}$ , as before (see footnote 2). Following now the same rationale used to derive the PWL system (27), we again define the state  $\xi_k := \xi(t_k)$ , and obtain the model

$$\begin{aligned} \xi_{k+1} &= A_{\mathcal{J}} \xi_k, \quad \text{when } \xi_k^\top Q_j \xi_k > 0, j \in \mathcal{J} \\ \text{and } \xi_k^\top Q_j \xi_k &\leq 0, j \in \mathcal{J}^c \end{aligned} \quad (51)$$

where

$$\begin{aligned} A_{\mathcal{J}} &:= e^{\bar{A}h} J_{\mathcal{J}} \\ &= \begin{bmatrix} A & B \Gamma_{\mathcal{J}}^u C^c & B \Gamma_{\mathcal{J}}^u D^c & B(I - \Gamma_{\mathcal{J}}^u) \\ B^c \Gamma_{\mathcal{J}}^y C^p & A^c & B(I - \Gamma_{\mathcal{J}}^y) & 0 \\ \Gamma_{\mathcal{J}}^y C^p & 0 & I - \Gamma_{\mathcal{J}}^y & 0 \\ 0 & \Gamma_{\mathcal{J}}^u C^c & \Gamma_{\mathcal{J}}^u D^c & I - \Gamma_{\mathcal{J}}^u \end{bmatrix} \end{aligned} \quad (52)$$

with  $A$  and  $B$  as in (12).

In a similar fashion as we derived Theorem III.4 for the state-feedback case, we can obtain the following result using the piecewise quadratic Lyapunov function  $V : \mathbb{R}^{n_\xi} \rightarrow \mathbb{R}_+$  given by

$$\begin{aligned} V(\xi) &= \xi^\top P_{\mathcal{J}} \xi, \quad \text{when } \xi^\top Q_j \xi > 0, j \in \mathcal{J} \\ \text{and } \xi^\top Q_j \xi &\leq 0, j \in \mathcal{J}^c \end{aligned} \quad (53)$$

with  $2^N$  regions. The proof is omitted due to space limitations.

*Theorem V.3:* The PETC system (48) is GES with decay rate  $\rho > 0$ , if there exist symmetric matrices  $P_{\mathcal{J}}$ ,  $\mathcal{J} \subseteq \{1, \dots, N\}$ , and scalars  $\alpha_{\mathcal{J}\tilde{\mathcal{J}}j} \geq 0$ ,  $\beta_{\mathcal{J}\tilde{\mathcal{J}}j} \geq 0$  and  $\kappa_{\mathcal{J}j} \geq 0$ ,  $\mathcal{J}, \tilde{\mathcal{J}} \subseteq \{1, \dots, N\}$ ,  $j \in \{1, \dots, N\}$ , such that for all  $\mathcal{J}, \tilde{\mathcal{J}} \subseteq \{1, \dots, N\}$

$$\begin{aligned} A_{\mathcal{J}}^\top P_{\tilde{\mathcal{J}}} A_{\mathcal{J}} - e^{-2\rho h} P_{\mathcal{J}} + \sum_{j \in \mathcal{J}} \alpha_{\mathcal{J}\tilde{\mathcal{J}}j} Q_j - \sum_{j \in \mathcal{J}^c} \alpha_{\mathcal{J}\tilde{\mathcal{J}}j} Q_j \\ + A_{\mathcal{J}}^\top \left( \sum_{j \in \tilde{\mathcal{J}}} \beta_{\mathcal{J}\tilde{\mathcal{J}}j} Q_j - \sum_{j \in \tilde{\mathcal{J}}^c} \beta_{\mathcal{J}\tilde{\mathcal{J}}j} Q_j \right) A_{\mathcal{J}} \preceq 0 \end{aligned} \quad (54)$$

and for all  $\mathcal{J} \subseteq \{1, \dots, N\}$

$$P_{\mathcal{J}} - \sum_{j \in \mathcal{J}} \kappa_{\mathcal{J}j} Q_j + \sum_{j \in \mathcal{J}^c} \kappa_{\mathcal{J}j} Q_j \succ 0.$$

## VI. NUMERICAL EXAMPLE

In this section, we illustrate the presented theory using a numerical example based on a state-feedback controller. For an example using a dynamic output-based controller and decentralized event-triggering conditions, we refer the interested reader to [39, Ch. 5].

Let us consider the example taken from [11] with plant (1) given by

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 1 \\ 0 \end{bmatrix}w \quad (55)$$

and state-feedback controller (3), where we take  $K = [1 \ -4]$  and  $t_k = kh$ ,  $k \in \mathbb{N}$ , with sampling interval  $h = 0.05$ . We consider the event-triggering condition given by (8). For this PETC system, we will apply all the three developed approaches for stability analysis (for  $w = 0$ ), and the impulsive system approach for performance analysis. For all three approaches, we aim at constructing the largest value of  $\sigma$  in (6) and (8) such that GES or a certain  $\mathcal{L}_2$ -gain can be guaranteed. The reason for striving for large values of  $\sigma$  is that then large (minimum) inter-event times are obtained, due to the forms of (6) and (8).

The PWL system approach (using Theorem III.4) yields a maximum value for  $\sigma$  of  $\sigma_{\text{PWL}} = 0.2550$ , while still guaranteeing stability of the PETC system. The PL system approach gives a maximum value of  $\sigma_{\text{lin}} = 0.2506$ , while the impulsive system approach results in the maximum  $\sigma_{\text{IS}} = 0.2532$  in this case. Hence, as expected based on Theorem III.5 and Theorem III.8, we see that  $\sigma_{\text{lin}} \leq \sigma_{\text{PWL}}$  and  $\sigma_{\text{IS}} \leq \sigma_{\text{PWL}}$ , although the values are rather close.<sup>5</sup> In fact, the minimum inter-event time according to (34) is equal to  $h = 0.05$  for all values  $\sigma_{\text{PWL}}$ ,  $\sigma_{\text{IS}}$  and  $\sigma_{\text{lin}}$  in the event-triggering condition (8). For the equivalent CETC scheme also using (8), there does not exist a strictly positive minimum inter-event time (see also [15]), even in absence of disturbances. In fact, accumulations of update times (Zeno behavior) occur, indicating that the corresponding CETC strategy is not useful.

When analyzing the  $\mathcal{L}_2$ -gain from the disturbance  $w$  to the output variable  $z$  as in (16c) where  $z = [0 \ 1 \ 0 \ 0]\xi$ , we obtain Fig. 3(a), in which the smallest upper bound on the  $\mathcal{L}_2$ -gain that can be guaranteed on the basis of Theorem III.2 is given as a function of  $\sigma$ . This figure clearly demonstrates that better guarantees on the control performance (i.e., smaller  $\gamma$ ), necessitates more updates (i.e., smaller  $\sigma$ ), allowing us to make tradeoffs between these two competing objectives (see also the discussion in Section IV on emulation-based design perspectives). Note that for  $\gamma \rightarrow \infty$  (meaning no performance requirements), the value of  $\sigma$  approaches the stability boundary obtained by using Corollary III.3, which is equal to  $\sigma_{\text{IS}} = 0.2532$ . On the other hand, for

<sup>5</sup>In case the event-triggering condition (6) would have been used, the differences would be larger, namely  $\sigma_{\text{lin}} := 0.1728$  and  $\sigma_{\text{PWL}} = \sigma_{\text{IS}} = 0.2425$ . In absence of disturbances the expression (34) leads for  $\sigma = \sigma_{\text{PWL}} = \sigma_{\text{IS}} = 0.2425$  to the exact minimum inter-event time  $3h = 0.15$  and for  $\sigma = \sigma_{\text{lin}}$  to  $2h = 0.10$  [39, Ch. 5].

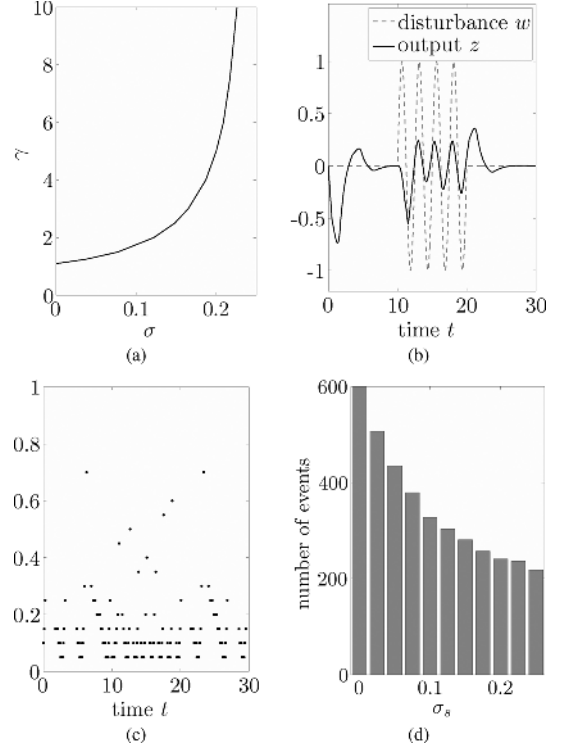


Fig. 3. Figures corresponding to numerical example. (a) Upperbound  $\mathcal{L}_2$ -gain as a function of  $\sigma$ . (b) The evolution of the disturbances  $w$  and the output  $z$  as a function of time for  $\sigma = 0.2$ . (c) The inter-event times as a function of time for  $\sigma = 0.2$ . (d) The number of events as a function of  $\sigma$ .

$\sigma \rightarrow 0$ , we recover the  $\mathcal{L}_2$ -gain for the periodic sampled-data system, given by (1) of the controller (2) with sampling interval  $h = 0.05$  and  $t_k = kh$ ,  $k \in \mathbb{N}$ . Hence, this figure can be used to get information on closed-loop performance for various values of  $\sigma$ .

Fig. 3(b) shows the response of the performance output  $z$  of the PETC system with  $\sigma = 0.2$ , initial condition  $\xi(0) = [1 \ 0 \ 0 \ 0]^T$  and a disturbance  $w$  as also depicted in Fig. 3(b). For the same situation, Fig. 3(c) shows the evolution of the inter-event times. We see inter-event times ranging from  $h = 0.05$  up to  $0.85$  (17 times the sampling interval  $h$ ) indicating a significant reduction in the number of transmissions. To more clearly illustrate this reduction, Fig. 3(d) depicts the number of transmissions for this given initial condition and disturbance, as a function of  $\sigma$ . Using this figure and Fig. 3(a), it can be shown that the increase of the guaranteed  $\mathcal{L}_2$ -gain, through an increased  $\sigma$ , leads to fewer transmissions, which demonstrates the tradeoff between the closed-loop performance and the number of transmissions that has to be made. Conclusively, using the PETC instead of the periodic sampled-data controller for this example yields a significant reduction in the number of transmissions/controller computations, while still preserving closed-loop stability and performance to some degree.

## VII. CONCLUSIONS

In this paper, we proposed a novel class of event-triggered control (ETC) strategies, which aim at combining the benefits that both periodic sampled-data control and ETC offer. In particular, the ETC strategy is based on the idea of having an event-triggering condition that is verified only periodically, instead

of continuously as in most existing ETC schemes. This control strategy, for which we used the term *periodic event-triggered control* (PETC), preserves the benefits of reduced resource utilisation as transmissions and controller computations are not performed periodically, while the event-triggering condition still has a periodic character. The latter aspect leads to several benefits as the event-triggering condition has to be verified only at the periodic sampling times, instead of continuously, which makes it suitable for implementation in standard time-sliced embedded system architectures. Moreover, the strategy has an inherently guaranteed minimum inter-event time of (at least) one sampling interval of the event-triggering condition, which is easy to tune directly.

We developed PETC for both static state-feedback controllers, and dynamical output-based controllers, and both centralized and decentralized event-triggering conditions. To analyze the stability and  $\mathcal{L}_2$ -gain properties of the PETC systems, we used three approaches: 1) an impulsive system approach; 2) a discrete-time piecewise linear (PWL) system approach; and 3) a discrete-time perturbed linear (PL) system approach. We discussed the advantages and disadvantages of all the three approaches, showing that each of the three presented modeling approaches is of independent interest. Namely, the PWL system approach provides the least conservative stability guarantees, the PL system approach has the lowest computational complexity and provides useful insights for emulation-based controller synthesis, while the impulsive system approach provides  $\mathcal{L}_2$ -gain analyses. We illustrated the theory using a numerical example and showed that PETC is indeed capable of reducing the utilization of communication and computation resources significantly, while still realizing satisfactory closed-loop behavior.

## APPENDIX

*Proof of Theorem III.2:* The proof is based on showing that (19) guarantees that (20) holds, and that the hypotheses of the theorem guarantee that the conditions in (21) hold and that (18) is a well-defined storage function candidate. This would complete the proof as, provided that (18) is a well-defined storage function candidate, (20) and (21) prove GES and an upper bound on the  $\mathcal{L}_2$ -gain of  $\gamma$ , see, e.g., [33]. Proving that (19) guarantees that (20) holds for the function (18) will be the first step in the proof. To prove that the hypotheses of the theorem guarantee that the conditions in (21) hold, we need to relate  $P_0 := P(0)$  to  $P_h := P(h)$  and doing so will be the second step of the proof. The final step in the proof is to show how this relation can be used to show that the conditions in the theorem guarantee that (21) hold and that (18) is indeed a well-defined storage function candidate.

To show that (19) yields that the derivative of (18) along the flow part (16a) of the impulsive system (16) satisfies (20), we observe that

$$\begin{aligned} \frac{d}{dt}V &= w^\top \bar{B}^\top P \xi + \xi^\top P \bar{B} w - \xi^\top (2\rho P + \gamma^{-2} \bar{C}^\top \bar{C}) \xi \\ &\quad - \xi^\top (P \bar{B} + \gamma^{-2} \bar{C}^\top \bar{D}) M (\bar{B}^\top P + \gamma^{-2} \bar{D}^\top \bar{C}) \xi. \end{aligned} \quad (56)$$

Now using the fact that  $(G\xi - M^{-1}w)^\top M(G\xi - M^{-1}w) \geq 0$ , with  $G := \bar{B}^\top P + \gamma^{-2} \bar{D}^\top \bar{C}$ , we have that  $-\xi^\top G^\top M G \xi \leq -\xi G^\top w - w^\top G \xi + w^\top M^{-1}w$ , and, therefore, it holds that

$$\begin{aligned} \frac{d}{dt}V &\leq -\xi^\top (2\rho P + \gamma^{-2} \bar{C}^\top \bar{C}) \xi \\ &\quad - \gamma^{-2} \xi^\top \bar{C}^\top \bar{D} w - \gamma^{-2} w^\top \bar{D}^\top \bar{C} \xi + w^\top M^{-1}w \end{aligned} \quad (57)$$

or, equivalently, due to (16c), this gives (20). This completes the first step in the proof.

We will now relate  $P_0 := P(0)$  to  $P_h := P(h)$ . To do so, we first reverse the time in the Riccati differential (19) by introducing  $\tilde{P}(\tau) := P(h - \tau)$ ,  $\tau \in [0, h]$ . This results in

$$\begin{aligned} \frac{d}{d\tau} \tilde{P} &= (\bar{A} + \rho I)^\top \tilde{P} + \tilde{P}(\bar{A} + \rho I) + \gamma^{-2} \bar{C}^\top \bar{C} \\ &\quad + (\tilde{P} \bar{B} + \gamma^{-2} \bar{C}^\top \bar{D}) M (\bar{B}^\top \tilde{P} + \gamma^{-2} \bar{D}^\top \bar{C}) \end{aligned} \quad (58)$$

or equivalently

$$\begin{aligned} \frac{d}{d\tau} \tilde{P} &= (\bar{A} + \rho I + \gamma^{-2} \bar{B} M \bar{D}^\top \bar{C})^\top \tilde{P} \\ &\quad + \tilde{P}(\bar{A} + \rho I + \gamma^{-2} \bar{B} M \bar{D}^\top \bar{C}) \\ &\quad + \gamma^{-2} \bar{C}^\top (I + \gamma^{-2} \bar{D} M \bar{D}^\top) \bar{C} + \tilde{P} \bar{B} M \bar{B}^\top \tilde{P} \end{aligned} \quad (59)$$

in which we have exploited the fact that  $M$  is symmetric. Note that

$$\begin{aligned} I + \gamma^{-2} \bar{D} M \bar{D}^\top &= I + \gamma^{-2} \bar{D} (I - \gamma^{-2} \bar{D}^\top \bar{D})^{-1} \bar{D}^\top \\ &= I + \gamma^{-2} \bar{D} \bar{D}^\top (I - \gamma^{-2} \bar{D} \bar{D}^\top)^{-1} \end{aligned}$$

because for any matrix  $Z$  it holds that  $(I + Z^\top Z)^{-1} Z^\top = Z^\top (I + Z Z^\top)^{-1}$ . Furthermore, because for any matrix  $Z$  for which  $I - Z$  is invertible, it holds that  $I + Z(I - Z)^{-1} = (I - Z)^{-1}$ , we have that  $I + \gamma^{-2} \bar{D} \bar{D}^\top (I - \gamma^{-2} \bar{D} \bar{D}^\top)^{-1} = L$ , where  $L := (I - \gamma^{-2} \bar{D} \bar{D}^\top)^{-1}$  as was also used in the definition of the Hamiltonian matrix (22). Furthermore, observe that  $\tilde{P}(0) := \tilde{P}_0 = P_h$  and  $\tilde{P}(h) := \tilde{P}_h = P_0$ . To link  $\tilde{P}_0$  to  $\tilde{P}_h$ , and thereby  $P_h$  to  $P_0$ , we use the Hamiltonian matrix (22), which allows us to find explicitly the solution to the Riccati differential (59). Indeed by using (23), we can express the solution to (59) as

$$\tilde{P}(\tau) = (F_{21}(\tau) + F_{22}(\tau) \tilde{P}_0)(F_{11}(\tau) + F_{12}(\tau) \tilde{P}_0)^{-1} \quad (60)$$

which requires that  $F_{11}(\tau) + F_{12}(\tau) \tilde{P}_0$  is invertible, see, e.g., [34, Lemma 8.2]. Since (60) relates  $\tilde{P}_0$  to  $\tilde{P}_h$  (by taking  $\tau = h$ ), provided that (60) is well defined for all  $\tau \in [0, h]$ , and thereby relates  $P_h$  to  $P_0$ , this completes the second step of the proof.

It now only remains to show how the expression (60) and the hypotheses can be used to show that the candidate storage function is well defined on  $[0, h]$  and satisfies (21). To do so, we will use the fact that  $F(\tau)$  is symplectic, i.e.,  $F^\top(\tau) \Omega F(\tau) = \Omega$  for all  $\tau \in [0, h]$ , where  $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ , and thus  $\Omega^{-1} = -\Omega$ .

This fact follows from observing that  $F^\top(0) \Omega F(0) = \Omega$  holds and  $(d/d\tau) F^\top(\tau) \Omega F(\tau) = 0$  for all  $\tau \in \mathbb{R}$ , by exploiting the structure in the Hamiltonian (22) giving  $H^\top \Omega + \Omega H = 0$ . From

$F^\top(\tau)\Omega F(\tau) = \Omega$ , we obtain that (omitting the  $\tau$ -dependence for shortness)

$$\begin{aligned} (i) \quad & 0 = F_{11}^\top F_{21} - F_{21}^\top F_{11}, \\ (ii) \quad & 0 = F_{22}^\top F_{12} - F_{12}^\top F_{22} \\ (iii) \quad & I = F_{11}^\top F_{22} - F_{21}^\top F_{12}. \end{aligned} \quad (61)$$

We will use these relations to rewrite (60). In particular, under Assumption III.1, we have for all  $\tau$  for which  $F_{11}(\tau) + F_{12}(\tau)\tilde{P}_0 = F_{11}(\tau) + F_{12}(\tau)P_h$  is invertible

$$\begin{aligned} P(h - \tau) &= \tilde{P}(\tau) \stackrel{(60)}{=} (F_{21} + F_{22}P_h)(F_{11} + F_{12}P_h)^{-1} \\ &= (F_{21} + F_{22}P_h)(I + F_{11}^{-1}F_{12}P_h)^{-1}F_{11}^{-1} \\ &\stackrel{(61)(iii)}{=} (F_{21} + F_{11}^{-\top}(I + F_{21}^\top F_{12})P_h)(I + F_{11}^{-1}F_{12}P_h)^{-1}F_{11}^{-1} \\ &\stackrel{(61)(i)}{=} (F_{21} + F_{11}^{-\top}P_h + F_{21}F_{11}^{-1}F_{12}P_h)(I \\ &\quad + F_{11}^{-1}F_{12}P_h)^{-1}F_{11}^{-1} \\ &= (F_{21}(I + F_{11}^{-1}F_{12}P_h) + F_{11}^{-\top}P_h)(I \\ &\quad + F_{11}^{-1}F_{12}P_h)^{-1}F_{11}^{-1} \\ &= F_{21}F_{11}^{-1} + F_{11}^{-\top}P_h(I + F_{11}^{-1}F_{12}P_h)^{-1}F_{11}^{-1} \\ &= F_{21}F_{11}^{-1} + F_{11}^{-\top}(P_h - P_h(I \\ &\quad + F_{11}^{-1}F_{12}P_h)^{-1}F_{11}^{-1}F_{12}P_h)F_{11}^{-1} \end{aligned} \quad (62)$$

where the fact that  $X^{-1} = X^{-1}Y^{-1}Y = (XY)^{-1}X$  for  $X = F_{11}$  and  $Y = F_{11} + F_{12}P_h$  was used in the third equality and the fact that  $(I - Z)^{-1} = I + (I - Z)^{-1}Z$  for  $Z = -F_{11}^{-1}F_{12}P_h$  was used in the last equality. Now since Lemma A.1 states that  $-F_{11}^{-1}F_{12}$  is positive semidefinite, there exists a ( $\tau$ -dependent) matrix  $S$  that satisfies  $-F_{11}^{-1}F_{12} = SS^\top$ . This leads to

$$P(h - \tau) = F_{21}F_{11}^{-1} + F_{11}^{-\top}(P_h + P_hS(I - S^\top P_hS)^{-1}S^\top P_h)F_{11}^{-1} \quad (63)$$

for all  $\tau$  for which  $P(h - \tau)$  is defined, as  $(I - SS^\top P_h)^{-1}SS^\top = S(I - S^\top P_hS)^{-1}S^\top$ .

Now (63) will be used to show that the candidate Lyapunov/storage function is well defined for all  $\tau \in [0, h]$  and that the conditions in (21) hold, if the hypotheses in the theorem are satisfied. Namely, (63) shows that having  $P(h - \tau)$  defined for all  $\tau \in [0, h]$  is equivalent to the existence of  $(I - S^\top(\tau)P_hS(\tau))^{-1}$  for all  $\tau \in [0, h]$ . Now observe that the existence of the inverse of  $I - S^\top(\tau)P_hS(\tau)$  for all  $\tau \in [0, h]$  is guaranteed if  $I - S^\top(\tau)P_hS(\tau) \succ 0$  for all  $\tau \in [0, h]$ , or, equivalently (by applying a Schur complement twice and using that  $P_h \succ 0$ ), if  $P_h^{-1} - S(\tau)S^\top(\tau) \succ 0$  for all  $\tau \in [0, h]$ . The fact that Lemma A.1 guarantees that  $U(\tau) = -F_{11}^{-1}(\tau)F_{12}(\tau)$  is nondecreasing, i.e.,  $S(\tau)S^\top(\tau) \preceq S(h)S^\top(h)$  for all  $\tau \leq h$ , gives that  $P_h^{-1} - S(h)S^\top(h) \succ 0$  implies that  $P_h^{-1} - S(\tau)S^\top(\tau) \succ 0$ , for all  $\tau \in [0, h]$ . Furthermore, using the reasoning in the proof of Proposition 8.1 of [34], we can show that  $P(\tau) \succ 0$  for  $\tau \in [0, h]$  under Assumption III.1. Now since by the hypotheses of the theorem (in particular, (25)),  $P_h \succ 0$  and  $I - S^\top(h)P_hS(h) \succ 0$ , the function (18) satisfies  $c_1\|\xi\|^2 \leq V(\tau, \xi) \leq c_2\|\xi\|^2$  for some  $0 < c_1 \leq c_2$ , for all

$\xi \in \mathbb{R}^{n_\xi}$  and for all  $\tau \in [0, h]$ , and is therefore a well-defined storage function candidate.

It now only remains to show that the conditions in the theorem guarantee that the conditions (21) hold. To do so, we choose  $\tau = h$  in (63) to obtain

$$P_0 = \bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top}(P_h + P_h\bar{S}(I - \bar{S}^\top P_h\bar{S})^{-1}\bar{S}^\top P_h)\bar{F}_{11}^{-1} \quad (64)$$

where  $\bar{F}_{ij} := F_{ij}(h)$  for  $i, j \in \{1, 2\}$ , and  $\bar{S} := S(h)$ . Substituting (64) into (21a), and using an S-procedure to encode that  $\xi^\top Q\xi > 0$ , yield that (21a) with  $i = 1$  holds if

$$P_h - \mu_1 Q - J_1^\top [\bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top}(P_h + P_h\bar{S}(I - \bar{S}^\top P_h\bar{S})^{-1}\bar{S}^\top P_h)\bar{F}_{11}^{-1}]J_1 \quad (65)$$

is positive semidefinite for some  $\mu_1 \geq 0$ , which is implied by (25) for  $i = 1$ . Using a similar reasoning, satisfaction of (21b) is implied by (25) with  $i = 2$  as  $\mu_2 \geq 0$ .  $\square$

**Lemma A.1:** Consider  $F(\tau)$  as in (23). Under Assumption III.1, it holds that  $U(\tau) := -F_{11}^{-1}(\tau)F_{12}(\tau)$  and  $R(\tau) := F_{21}(\tau)F_{11}^{-1}(\tau)$  are positive semidefinite and nondecreasing for all  $\tau \in [0, h]$ , i.e.,  $U(\tau_1) \preceq U(\tau_2)$  and  $R(\tau_1) \preceq R(\tau_2)$ , when  $0 \leq \tau_1 \leq \tau_2 \leq h$ .

**Proof of Lemma A.1:** Note that  $R(\tau)$  is the solution to (59) for  $\tilde{P}(0) = R(0) = 0$  according to (60). In particular

$$\frac{d}{d\tau}R = \tilde{A}^\top R + R\tilde{A} + \tilde{C}^\top L\tilde{C} \quad (66)$$

where  $\tilde{A} = \bar{A} + \rho I + \gamma^{-2}\bar{B}M\bar{D}^\top C + (1/2)\bar{B}M\bar{B}^\top R$ , which depends on  $\tau$ . Applying now Proposition 8.1 of [34] yields that  $R(\tau) \succeq 0$ ,  $\tau \in [0, h]$ . Since  $R(\tau)$  satisfies (66),  $(d/d\tau)R(\tau)$  satisfies

$$\begin{aligned} \frac{d^2}{d\tau^2}R &= \tilde{A}^\top \frac{dR}{d\tau} + \frac{d\tilde{A}^\top}{d\tau}R + \frac{dR}{d\tau}\tilde{A} + R\frac{d\tilde{A}}{d\tau} \\ &= \left( \bar{A} + \rho I + \gamma^{-2}\bar{B}M\bar{D}^\top C + \frac{1}{2}\bar{B}M\bar{B}^\top R \right)^\top \frac{dR}{d\tau} \\ &\quad + \left( \frac{1}{2}\bar{B}M\bar{B}^\top \frac{dR}{d\tau} \right)^\top R \\ &\quad + \frac{dR}{d\tau} \left( \bar{A} + \rho I + \gamma^{-2}\bar{B}M\bar{D}^\top C + \frac{1}{2}\bar{B}M\bar{B}^\top R \right) \\ &\quad + R \left( \frac{1}{2}\bar{B}M\bar{B}^\top \frac{dR}{d\tau} \right) \\ &= (\bar{A} + \rho I + \bar{B}M\bar{D}^\top \bar{C} + \bar{B}M\bar{B}^\top R)^\top \frac{dR}{d\tau} \\ &\quad + \frac{dR}{d\tau}(\bar{A} + \rho I + \bar{B}M\bar{D}^\top \bar{C} + \bar{B}M\bar{B}^\top R). \end{aligned} \quad (67)$$

Since  $(d/d\tau)R(0) \stackrel{(66)}{=} \bar{C}^\top L\bar{C} \succeq 0$ , applying Proposition 8.1 of [34] once more shows that  $(d/d\tau)R(\tau) \succeq 0$  for  $\tau \in [0, h]$ . This shows that  $R$  is nondecreasing.

A similar reasoning applies to  $U(\tau)$  for a somewhat modified Riccati differential equation corresponding to the Hamiltonian  $\tilde{H} = V^{-1}H^\top V$  with  $V = \text{diag}(I, -I)$ . Defining

$$\tilde{F}(\tau) := e^{-\tilde{H}\tau} = \begin{bmatrix} \tilde{F}_{11}(\tau) & \tilde{F}_{12}(\tau) \\ \tilde{F}_{21}(\tau) & \tilde{F}_{22}(\tau) \end{bmatrix} \quad (68)$$

and using that  $\tilde{F}(\tau) = V^{-1}F(\tau)^\top V$ , we can show that  $\tilde{R}(\tau) := \tilde{F}_{21}(\tau)\tilde{F}_{11}^{-1}(\tau) = -F_{11}^{-1}(\tau)F_{12}(\tau) = U(\tau)$  (exploiting symmetry of solutions to Riccati differential equations of the type (59) for symmetric initial conditions). Applying the same reasoning to  $\tilde{R}(\tau)$  as for  $R(\tau)$  using the Riccati differential equation corresponding to the Hamiltonian  $\tilde{H}$ , the facts that  $U(\tau) \succeq 0$  and  $U(\tau)$  is nondecreasing follow.  $\square$

*Proof of Corollary III.3:* The proof follows from a slight modification of the reasoning in Section III-A and the proof of Theorem III.2. Namely, instead of the dissipation inequality (20), we require  $(d/dt)V = -2\rho V$  along the solutions of (16a) with  $w = 0$ . Using the same Lyapunov function candidate as in (18), this is satisfied if the matrix differential equation  $(d/dt)P = -(\bar{A}^\top + \rho I)P - P(\bar{A} + \rho I)$  holds, which has the solution  $P(\tau) = e^{2\rho(h-\tau)}\bar{A}^\top(h-\tau)P_h e^{\bar{A}(h-\tau)}$  and thus  $P(0) = e^{2\rho h}\bar{A}^\top h P_h e^{\bar{A}h}$ . Substituting this in the jump conditions (21) yields

$$e^{2\rho h}\xi^\top J_1^\top e^{\bar{A}^\top h}P_h e^{\bar{A}h}J_1\xi \leq \xi^\top P_h\xi \quad (69)$$

when  $\xi^\top Q\xi > 0$ , and

$$e^{2\rho h}\xi^\top J_2^\top e^{\bar{A}^\top h}P_h e^{\bar{A}h}J_2\xi \leq \xi^\top P_h\xi \quad (70)$$

when  $\xi^\top Q\xi \leq 0$ . These conditions are guaranteed by the hypotheses of the theorem.  $\square$

*Proof of Theorem III.4:* From (30b), it follows that  $P_i \succ -(-1)^i \kappa_i Q$  for  $i \in \{1, 2\}$ . Since this implies that  $\xi^\top (P_1 - \kappa_1 Q)\xi \geq \lambda_{\min}(P_1 - \kappa_1 Q)\|\xi\|^2$  and that  $\xi^\top (P_2 + \kappa_2 Q)\xi \geq \lambda_{\min}(P_2 + \kappa_2 Q)\|\xi\|^2$ , with  $\lambda_{\min}(P_1 - \kappa_1 Q), \lambda_{\min}(P_2 + \kappa_2 Q) > 0$ , we have that

$$\begin{aligned} V(\xi) &= \xi^\top P_1\xi \geq \lambda_{\min}(P_1 - \kappa_1 Q)\|\xi\|^2 + \kappa_1 \xi^\top Q\xi \\ &\geq \lambda_{\min}(P_1 - \kappa_1 Q)\|\xi\|^2 \end{aligned}$$

for all  $\xi$  satisfying  $\xi^\top Q\xi > 0$  and that

$$\begin{aligned} V(\xi) &= \xi^\top P_2\xi \geq \lambda_{\min}(P_2 + \kappa_2 Q)\|\xi\|^2 - \kappa_2 \xi^\top Q\xi \\ &\geq \lambda_{\min}(P_2 + \kappa_2 Q)\|\xi\|^2 \end{aligned}$$

for all  $\xi$  satisfying  $\xi^\top Q\xi \leq 0$ . This proves that for the candidate Lyapunov function (29), there exists a  $c_1 = \min\{\lambda_{\min}(P_1 - \kappa_1 Q), \lambda_{\min}(P_2 + \kappa_2 Q)\} > 0$  and some  $c_2 \geq c_1$  such that  $c_1\|\xi\|^2 \leq V(\xi) \leq c_2\|\xi\|^2$  for all  $\xi \in \mathbb{R}^{n_\xi}$ .

Furthermore, note that if  $V(\xi) = \xi^\top P_i\xi$ , it holds that  $(-1)^i \xi^\top Q\xi \leq 0$ , and if  $V(A_i\xi) = (A_i\xi)^\top P_j(A_i\xi)$ , then  $(-1)^j \xi^\top A_i^\top Q A_i\xi \leq 0$ , for  $i, j \in \{1, 2\}$ . Hence, using this and (30a)

$$\begin{aligned} V(A_i\xi) &= \xi^\top A_i^\top P_j A_i\xi \leq e^{-2\rho h}\xi^\top P_i\xi \\ &\quad + (-1)^i \alpha_{ij}\xi^\top Q\xi + (-1)^j \beta_{ij}\xi^\top A_i^\top Q A_i\xi \\ &\leq e^{-2\rho h}\xi^\top P_i\xi = e^{-2\rho h}V(\xi) \end{aligned} \quad (71)$$

where in the latter inequality we used that  $\alpha_{ij}, \beta_{ij} \geq 0$ . By standard Lyapunov arguments this proves GES of the discrete-time PWL system (27) with decay factor  $e^{-\rho h}$ . Now, by including the intersample behavior in a straightforward fashion, as was done in, e.g., [40] following [41], this also implies GES with decay rate  $\rho$  of the (continuous-time) PETC system (16).  $\square$

*Proof of Theorem III.8:* We will only give the proof for the triggering condition (6), as the proof is similar for (8). The proof will be based on showing that if the LMIs (33) are feasible for  $\bar{P} \succ 0$  for some  $\sigma < \theta$ , then

$$P_1 = P_2 = P := \begin{bmatrix} \bar{P} & 0 \\ 0 & \delta I \end{bmatrix} \quad (72)$$

is a solution to the LMIs (30a) and (30b), with the matrix  $Q$  as in (7), for some constants  $\alpha_{ij} \geq 0, \beta_{ij} \geq 0$ , and  $\kappa_i \geq 0, i, j \in \{1, 2\}$ , for some (sufficiently small)  $\delta > 0$ . To do so, we first observe that (30b) is satisfied for all  $\delta > 0$  with  $\kappa_1 = \kappa_2 = 0$ . Focussing on (30a) with  $i = 1$ , we observe that for  $\alpha_{11} = \alpha_{12} = \beta_{11} = \beta_{12} = 0$ , (30a) with  $i = 1$  and  $P_1$  as in (72) is equivalent to

$$\begin{aligned} e^{-2\rho h}P - A_1^\top P A_1 \\ = \begin{bmatrix} e^{-2\rho h}\bar{P} - (A + BK)^\top \bar{P}(A + BK) - \delta I & 0 \\ 0 & e^{-2\rho h}\delta I \end{bmatrix} \\ \succeq 0 \end{aligned} \quad (73)$$

where  $A_1$  is given as in (28). Clearly, due to (33), for sufficiently small  $\delta > 0$  and  $\rho > 0$ , we have that  $e^{-2\rho h}\bar{P} - (A + BK)^\top \bar{P}(A + BK) - \delta I$  and  $e^{-2\rho h}\delta I$  are positive semidefinite matrices. Hence, the matrix inequality in (73) is satisfied and thus (30a) with  $i = 1$  is satisfied for  $P$  as in (72) for  $\alpha_{11} = \alpha_{12} = \beta_{11} = \beta_{12} = 0$  and a sufficiently small value of  $\delta > 0$ . Focussing now on (30a) with  $i = 2$ , we observe that by taking  $\alpha_{21} = \alpha_{22} = 1$  and  $\beta_{21} = \beta_{22} = 0$ , we obtain

$$\begin{aligned} e^{-2\rho h}P + Q - A_2^\top P A_2 \\ = \begin{bmatrix} e^{-2\rho h}\bar{P} + (1 - \sigma^2)I - A^\top \bar{P} A & -I - A^\top \bar{P} B K \\ -I - (BK)^\top \bar{P} A & I + (e^{-2\rho h} - 1)\delta I - (BK)^\top \bar{P} B K \end{bmatrix} \\ \succeq 0 \end{aligned} \quad (74)$$

where  $A_2$  is given as in (28) and  $Q$  is given as in (6). This inequality is equivalent to

$$\begin{aligned} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} R(\sigma) \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \\ \succeq \begin{bmatrix} (1 - e^{-2\rho h})\bar{P} & 0 \\ 0 & (1 - e^{-2\rho h})\delta I \end{bmatrix} \succ 0 \end{aligned} \quad (75)$$

with

$$R(\sigma) := \begin{bmatrix} \bar{P} - \sigma^2 I - (A + BK)^\top \bar{P}(A + BK) & * \\ -(BK)^\top \bar{P}(A + BK) & I - (BK)^\top \bar{P} B K \end{bmatrix}. \quad (76)$$

To guarantee now that (75) is satisfied for some (arbitrary small)  $\rho > 0$ , we have to show that  $R(\sigma) \succ 0$  for the given  $\sigma$ . Since  $\sigma < \theta$ , (33) implies that  $R(\sigma) \succ 0$  and, hence, that the matrix inequality in (75) is satisfied and thus (30a) with  $i = 2$  is satisfied for  $P$  as in (72) for  $\alpha_{21} = \alpha_{22} = 1$  and  $\beta_{21} = \beta_{22} = 0$ . This completes the proof.  $\square$

## REFERENCES

- [1] K. Åström and B. Bernhardsson, "Comparison of periodic and event based sampling for first order stochastic systems," in *Proc. IFAC World Conf.*, 1999, pp. 301–306.
- [2] K.-E. Arzén, "A simple event-based PID controller," in *Preprints IFAC World Conf.*, 1999, vol. 18, pp. 423–428.







**M. C. F. (Tijs) Donkers** received the M.Sc. and the Ph.D. degrees (both *summa cum laude*) in systems and controls from Eindhoven University of Technology, Eindhoven, the Netherlands, in 2008 and 2011, respectively.

In 2010, he was a Visiting Researcher at the Cyber-Physical Systems Laboratory of the University of California at Los Angeles. His current research interests include networked and event-driven control, distributed control, and switched systems.



**Andrew R. Teel** (S'91–M'92'SM'99–F'02) received the A.B. degree in engineering sciences from Dartmouth College, Hanover, NH, in 1987, and the M.S. and Ph.D. degrees in electrical engineering from the University of California, Berkeley, in 1989 and 1992, respectively.

After the Ph.D. degree, he was a Postdoctoral Fellow at the Ecole des Mines de Paris, Fontainebleau, France. In 1992, he joined the faculty of the Electrical Engineering Department, University of Minnesota, where he was an Assistant Professor until 1997. Subsequently, he joined the faculty of the Electrical and Computer Engineering Department, University of California, Santa Barbara, where he is currently a Professor. His research interests are in nonlinear and hybrid dynamical systems, with a focus on stability analysis and control design.

Dr. Teel received NSF Research Initiation and CAREER Awards, the 1998 IEEE Leon K. Kirchmayer Prize Paper Award, the 1998 George S. Axelby Outstanding Paper Award, and was the recipient of the first SIAM Control and Systems Theory Prize in 1998. He was the recipient of the 1999 Donald P. Eckman Award and the 2001 O. Hugo Schuck Best Paper Award, both given by the American Automatic Control Council, and also received the 2010 IEEE Control Systems Magazine Outstanding Paper Award. He is an area editor for *Automatica*, and a Fellow of the IFAC.