# Periodic Normal Forms for Bifurcations of Limit Cycles in DDEs

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**Abstract**: A recent work [28] by the authors on the existence of a periodic smooth finite-dimensional center manifold near a nonhyperbolic cycle in delay differential equations motivates derivation of periodic normal forms. In this paper, we prove the existence of a special coordinate system on the center manifold that will allow us to describe the local dynamics on the center manifold in terms of these periodic normal forms. Furthermore, we characterize the center eigenspace by proving the existence of time periodic smooth Jordan chains for the original and the adjoint system.

**Keywords**: delay differential equations, dual perturbation theory, sun-star calculus, center manifold, Jordan chains, normal forms, generalized eigenfunctions, nonhyperbolic cycles

**MSC**: 34C20, 34K13, 34K17, 34K18, 34K19

## 1 Introduction

Bifurcation theory allows us to analyze the behavior of complicated high dimensional nonlinear dynamical systems near bifurcations by reducing the system to a low dimensional invariant manifold, called the center manifold. Using normal form theory, the dynamics on the center manifold can be described by a simple canonical equation called the normal form. These bifurcations and normal forms can be categorized, and their properties can be understood in terms of certain coefficients of the normal form, see [27] for more details. Methods to compute these normal form coefficients have been implemented in software like `MatCont` [9] and `DDE-BifTool` [16, 30] to study various classes of dynamical systems.

For bifurcations of limit cycles in continuous-time dynamical systems, there are three generic codimension one bifurcations: fold (or limit point), period-doubling (or flip) and Neimark-Sacker (or torus) bifurcation. These bifurcations are well understood for ordinary differentials equations (ODEs) [22, 23, 26, 33], but for delay differential equations (DDEs) the theory is still lacking. To understand these bifurcations, one should first prove the existence of a center manifold near a nonhyperbolic cycle. The authors proved in [28] that indeed such a center manifold near a nonhyperbolic cycle exists and is sufficiently smooth. The next step is to study the local dynamics near a nonhyperbolic cycle via normal forms.

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The aim of this paper is to show for classical DDEs that the local dynamics near a nonhyperbolic cycle can be studied via periodic normal forms. We generalize the results from Iooss [22, 23] on periodic normal forms for finite-dimensional ODEs towards infinite-dimensional DDEs. This task will be accomplished by using the rigorous perturbation framework of dual semigroups (sun-star calculus). In an upcoming paper, we present explicit computational formulas for the critical normal form coefficients, along the lines of the periodic normalization method [26, 33], for all codimension one bifurcations of limit cycles in classical DDEs, completely avoiding Poincaré maps. Finally, we plan to implement the obtained computational formulas into a software package like `DDE-BifTool`.

## 1.1 Background

Consider a *classical delay differential equation* (DDE)

$$\dot{x}(t) = F(x_t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ for each $t \geq 0$, and

$$x_t(\theta) := x(t+\theta), \quad \theta \in [-h, 0],$$

represents the *history* at time $t$ of the unknown $x$, and $0 < h < \infty$ denotes the upper bound of (finite) delays. The $\mathbb{R}^n$-valued smooth operator $F$ is defined on the Banach space $X := C([-h,0],\mathbb{R}^n)$ consisting of $\mathbb{R}^n$-valued continuous functions defined on the compact interval $[-h,0]$, endowed with the supremum norm. Furthermore, we assume that (1) has a $T$-periodic solution $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that the periodic orbit (cycle) $\Gamma := \{\gamma_t \in X \ : \ t \in \mathbb{R}\}$ is nonhyperbolic.

Using the perturbation framework of dual semigroups, called sun-star calculus, developed in [5, 6, 7, 8, 13], the existence of a periodic smooth finite-dimensional center manifold $\mathcal{W}^c_{\text{loc}}(\Gamma)$ near $\Gamma$ for (1) has been recently rigorously established in [28] by the authors. We also refer to [1, 3, 4, 11, 14, 25] for results on the existence of (parameter-dependent) center manifolds near nonhyperbolic equilibria and cycles in classical, impulsive and abstract DDEs. Furthermore, we mention the work by Hupkes and Verduyn Lunel on the existence and smoothness of center manifolds near equilibria [20] and periodic orbits [21] for so-called functional differential equations of mixed type (MFDEs).

To study the local dynamics of (1) on $\mathcal{W}^c_{\text{loc}}(\Gamma)$, we will characterize the center manifold by proving the existence of a special coordinate system on this manifold. Afterwards, we show that any solution of (1) on this invariant manifold can be locally parametrized in terms of these coordinates. The existence of such a coordinate system for finite-dimensional ODEs was already carried out by Iooss in [22, 23], and so we will generalize his results towards the setting of classical DDEs, using the sun-star calculus framework. It turns out that the linear part of our coordinate system will be closely related to the coordinate system invented by Hale and Weedermann [19], which was used to study perturbations of periodic orbits in classical DDEs. Iooss indicated in [22] that his results would be easily extendable to the infinite-dimensional setting. However, we will show in this paper some results that were truly not expected by the authors. For example, an interplay between history and periodicity for Jordan chains in Theorem 5 was a remarkable observation, since the history concept is not present in ODEs. Furthermore, the proof on the existence of this coordinate system on the center manifold happened to be far more involved, see for example Theorem 10 and especially the role of the sun-star calculus machinery in the proof. As a consequence of the results, the periodic normal forms for bifurcations of limit cycles in classical DDEs are exactly the same as for ODEs.

As already addressed, we will present in an upcoming article explicit computational formulas for the critical normal form coefficients of all codimension one bifurcations of limit cycles in DDEs. To obtain these formulas, it is no surprise that we need to compute (generalized) eigenfunctions and adjoint (generalized) eigenfunctions of a certain operator. The first fundaments to obtain such formulas will be carried out in this paper by proving the existence of time periodic smooth (adjoint) (generalized) eigenfunctions. It turns out that the existence of the periodic smooth (generalized) eigenfunctions are also a necessary for the construction of the coordinate system on $\mathcal{W}^c_{\text{loc}}(\Gamma)$.

## 1.2 Overview

The paper is organized as follows. In Section 2 we review and extend the results from [28] on periodic smooth finite-dimensional center manifolds near nonhyperbolic cycles in the setting of classical DDEs.

In Section 3 we mainly characterize the center eigenspace and its associated adjoint. To do this, we prove that there exists a periodic smooth basis of the center eigenspace by extending the results from [23, Proposition III.1] towards classical DDEs, see Theorem 5. We show an interesting interplay between the history and periodicity of the (generalized) eigenfunctions, a phenomenon that is not present in the setting of ODEs.

In Section 4 we prove the existence of a special coordinate system on $\mathcal{W}^c_{\text{loc}}(\Gamma)$ and generalize the normal form theorems from finite-dimensional ODEs [22, 23] towards infinite-dimensional DDEs, see Theorem 10, Theorem 11 and Theorem 12 for the main results.

# 2 Periodic center manifolds for classical DDEs

In this section, we primarily summarize the results from [28] and secondly recall and extend some results from (time-dependent) dual perturbation theory for which the book [14] together with the article [5] are standard references. All unreferenced claims relating to basic properties of time-dependent perturbations of delays equations can be found in both references.

In the setting of classical DDEs, we work with the real Banach space $X := C([-h,0],\mathbb{R}^n)$ as the state space for some (maximal) finite *delay* $0 < h < \infty$ equipped with the supremum norm $\|\cdot\|_\infty$. Consider for an integer $k \geq 1$ a $C^{k+1}$-smooth operator $F : X \to \mathbb{R}^n$ together with the initial value problem

$$\begin{cases} \dot{x}(t) = F(x_t), & t \geq 0, \\ x_0 = \varphi, & \varphi \in X, \end{cases} \tag{DDE}$$

where the *history of $x$ at time $t \geq 0$*, denoted by $x_t \in X$, is defined as

$$x_t(\theta) := x(t+\theta), \quad \forall \theta \in [-h,0]. \tag{2}$$

By a *solution* of (DDE) we mean a continuous function $x : [-h, t_\varphi) \to \mathbb{R}^n$ for some *final time* $0 < t_\varphi \leq \infty$ that is continuously differentiable on $[0, t_\varphi)$ and satisfies (DDE). When $t_\varphi = \infty$, we call $x$ a *global solution*. We say that a function $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a *periodic solution* of (DDE) if $\gamma$ is a solution of (DDE) and there exists a minimal $T > 0$, called the *period of* $\gamma$, such that $\gamma_T = \gamma_0$. It follows from [18, Corollary 10.3.1] that $\gamma \in C^{k+2}(\mathbb{R},\mathbb{R}^n)$. We call $\Gamma := \{\gamma_t \in X \ : \ t \in \mathbb{R}\}$ a *periodic orbit* or *(limit) cycle* in $X$.

We want to study (DDE) near the periodic solution $\gamma$, and it is therefore convenient to translate $\gamma$ towards the origin. More specifically, if $x$ is a solution of (DDE), then for $y$ defined as $x = \gamma + y$, we have that $y$ satisfies the time-dependent DDE

$$\dot{y}(t) = L(t)y_t + G(t, y_t), \tag{3}$$

where the $C^k$-smooth bounded linear operator $L(t) := DF(\gamma_t) \in \mathcal{L}(X,\mathbb{R}^n)$ denotes the Fréchet derivative of $F$ evaluated at $\gamma_t$ and the $C^k$-smooth operator $G(t,\cdot) := F(\gamma_t + \cdot) - F(\gamma_t) - L(t)$ consists of solely nonlinear terms. Regarding the linear part, it is traditional to apply a vector-valued version of the Riesz representation theorem [14, Theorem 1.1] as

$$L(t)\varphi = \int_0^h d_2\zeta(t,\theta)\varphi(-\theta) =: \langle \zeta(t,\cdot), \varphi \rangle, \quad \forall t \in \mathbb{R},\ \varphi \in X. \tag{4}$$

The *kernel* $\zeta : \mathbb{R} \times [0,h] \to \mathbb{R}^{n\times n}$ is a matrix-valued function, $\zeta(t,\cdot)$ is of bounded variation, right-continuous on the open interval $(0,h)$, $T$-periodic in the first component and normalized by the requirement that $\zeta(t,0) = 0$ for all $t \in \mathbb{R}$. The integral appearing in (4) is of Riemann-Stieltjes type and the subscript in $d$ reflects on the fact that we integrate over the second variable of $\zeta$.

The starting point of applying the sun-star calculus construction towards the setting of classical DDEs is by studying the *trivial DDE*

$$\begin{cases} \dot{x}(t) = 0, & t \geq 0, \\ x_0 = \varphi, & \varphi \in X, \end{cases} \tag{5}$$

which has the unique global solution

$$x(t) = \begin{cases} \varphi(t), & -h \leq t \leq 0, \\ \varphi(0), & t \geq 0. \end{cases} \tag{6}$$

We define the $\mathcal{C}_0$-semigroup $T_0$ on $X$, also called the *shift semigroup*, as

$$(T_0(t)\varphi)(\theta) := \begin{cases} \varphi(t+\theta), & -h \leq t+\theta \leq 0, \\ \varphi(0), & t+\theta \geq 0, \end{cases} \quad \forall t \geq 0,\ \varphi \in X,\ \theta \in [-h, 0].$$

and notice that $T_0$ generates the solution of (6) in the sense of $T_0(t)\varphi = x_t$ for all $t \geq 0$. The shift semigroup has (infinitesimal) generator $A_0 : \mathcal{D}(A_0) \to X$ defined by

$$\mathcal{D}(A_0) = \{\varphi \in C^1([-h,0],\mathbb{R}^n) \ : \ \dot{\varphi}(0) = 0\}, \quad A_0\varphi = \dot{\varphi}.$$

For this specific combination of $X$, $T_0$ and $A_0$, the abstract duality structure can be constructed explicitly, see [14, Section II.5]. We only summarize here the basic results that are needed for the upcoming (sub)sections.

Let us first introduce a convention. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ let $\mathbb{K}^n$ be the linear space of column vectors, while $\mathbb{K}^{n\star}$ denotes the linear space of row vectors, all with components in $\mathbb{K}$. A representation theorem by Riesz [29] enables us to identify $X^\star = C([-h,0],\mathbb{R}^n)^\star$ with the Banach space $\mathrm{NBV}([0,h],\mathbb{R}^{n\star})$ consisting of functions $\zeta : [0,h] \to \mathbb{R}^{n\star}$ that are normalized by $\zeta(0) = 0$, are continuous from the right on $(0,h)$ and have bounded variation. Because $X$ is not reflexive, the *dual semigroup* $T_0^\star$ of $T_0$ is in general only weak$^\star$ continuous on $X^\star$. This is also visible on the generator level, as the adjoint $A_0^\star$ of $A_0$ is only the weak$^\star$ generator of $T_0^\star$ and takes the form

$$\begin{aligned} \mathcal{D}(A_0^\star) = \bigg\{ f \in \mathrm{NBV}([0,h],\mathbb{R}^{n\star}) \ : \ f(\theta) = f(0^+) + \int_0^\theta g(\sigma) d\sigma \text{ for } \theta > 0, \\ g \in \mathrm{NBV}([0,h],\mathbb{R}^{n\star}) \text{ and } g(h) = 0 \bigg\}, \quad A_0^\star f = g, \end{aligned} \tag{7}$$

where $f(0^+) := \lim_{t\downarrow 0} f(t)$ and the function $g$ is called the *derivative* of $f$. The maximal subspace of strong continuity

$$X^\odot := \{x^\star \in X^\star : t \mapsto T_0^\star(t)x^\star \text{ is norm continuous on } [0,\infty)\}$$

is a norm closed $T_0^\star(t)$-invariant weak$^\star$ dense subspace of $X^\star$ and we have the characterization

$$X^\odot = \overline{\mathcal{D}(A_0^\star)}, \tag{8}$$

where the bar denotes the norm closure in $X^\star$. Expression (8) enables us to compute the *sun dual* $X^\odot$ by taking the closure of $\mathcal{D}(A_0^\star)$ with respect to the norm defined on $\mathrm{NBV}([0,h],\mathbb{R}^{n\star})$. As the space consisting of functions that have bounded variation are norm dense in $L^1$, $X^\odot$ has the same description as $\mathcal{D}(A_0^\star)$, but the derivative is allowed to be in $L^1([0,h],\mathbb{R}^{n\star})$.

Let $\mathrm{AC}_0([0,h],\mathbb{R}^{n\star})$ denote the space of $\mathbb{R}^{n\star}$-valued functions that are absolute continuous on $(0,h]$, have zero value at zero and zero derivative at $h$. From the description of the sun dual, it is clear that

$X^{\odot} = \mathrm{AC}_0([0,h],\mathbb{R}^{n\star})$. It turns out that another characterization of the sun dual will be helpful. As an element $f \in X^{\odot}$ is completely specified by $f(0^+) \in \mathbb{R}^{n\star}$ and $g \in L^1([0,h],\mathbb{R}^{n\star})$, we obtain the isometric isomorphism

$$X^{\odot} \cong \mathbb{R}^{n\star} \times L^1([0,h],\mathbb{R}^{n\star}).$$

A representation of the dual space $X^{\odot\star}$ of the sun dual $X^{\odot}$, and its restriction to the maximal subspace of strong continuity $X^{\odot\odot}$ are given by

$$X^{\odot\star} \cong \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n) \text{ and } X^{\odot\odot} \cong \mathbb{R}^n \times C([-h,0],\mathbb{R}^n).$$

The linear canonical embedding $j : X \to X^{\odot\star}$ has action $j\varphi = (\varphi(0),\varphi)$ for all $\varphi \in X$, mapping $X$ onto $X^{\odot\odot}$, meaning that $X$ is $\odot$-*reflexive* with respect to the shift semigroup $T_0$.

Next, we turn our attention to the time-dependent linear DDE

$$\begin{cases} \dot{y}(t) = L(t)y_t, & t \geq s, \\ y_s = \varphi, & \varphi \in X, \end{cases} \tag{T-LDDE}$$

in the setting of time-dependent perturbation theory. Recall that $L(t) = DF(\gamma_t)$ for all $t \in \mathbb{R}$ and $s \in \mathbb{R}$ denotes the *starting time*. For $i = 1,\dots,n$ we denote $r_i^{\odot\star} := (e_i, 0)$, where $e_i$ is the $i$th standard basic vector of $\mathbb{R}^n$. It is conventional and convenient to introduce the shorthand notation

$$wr^{\odot\star} := \sum_{i=1}^{n} w_i r_i^{\odot\star}, \quad \forall w = (w_1,\dots,w_n) \in \mathbb{R}^n,$$

and note that $wr^{\odot\star} = (w,0) \in X^{\odot\star}$. We define the finite rank $T$-periodic *time-dependent bounded linear perturbation* $B$ as

$$B(t)\varphi := [L(t)\varphi]r^{\odot\star}, \quad \forall t \in \mathbb{R},\ \varphi \in X,$$

and since $F \in C^{k+1}(X,\mathbb{R}^n)$, $L(t) = DF(\gamma_t)$, $t \mapsto \gamma_t$ is $T$-periodic and of the class $C^k$, we have that $B \in C^k(\mathbb{R},\mathcal{L}(X,X^{\odot\star}))$ is $T$-periodic and Lipschitz continuous. It is shown in [14, Theorem 3.1] that there is a one-to-one correspondence between solutions of (T-LDDE) and the time-dependent linear abstract integral equation

$$u(t) = T_0(t-s)\varphi + j^{-1}\int_s^t T_0^{\odot\star}(s-\tau)B(\tau)u(\tau)d\tau, \quad \varphi \in X, \tag{T-LAIE}$$

with $t \geq s$ where the integral has to be interpreted as a weak$^\star$ Riemann integral [14, Chapter III] and takes values in $j(X)$ under the running assumption of $\odot$-reflexivity, see [5, Lemma 2.2]. The unique solution of (T-LAIE) on an interval $I_\varphi := [s,t_\varphi) \subset \mathbb{R}$ for some $s < t_\varphi \leq \infty$ is generated by a strongly continuous forward evolutionary system $U := \{U(t,s)\}_{(t,s)\in\Omega_{\mathbb{R}}}$ on $X$ in the sense that $u(t) = U(t,s)\varphi$ for all $t \in I_\varphi$, where $\Omega_J$ is defined by $\{(t,s) \in J \times J \ : \ t \geq s\}$ for some interval $J \subseteq \mathbb{R}$, see [14, Definition XII.2.1 and Theorem XII.2.7].

As we have defined $U(t,s)$ for all $(t,s) \in \Omega_{\mathbb{R}}$, we are interested in the associated (sun) dual(s). It is clear that one can define $U^\star(s,t) := U(t,s)^\star \in \mathcal{L}(X^\star) := \mathcal{L}(X^\star,X^\star)$ and that $U^\star := \{U^\star(s,t)\}_{(s,t)\in\Omega_{\mathbb{R}}^\star}$ forms a backward evolutionary system on $X^\star$, where $\Omega_J^\star$ is defined as $\{(s,t) \in J^2 : t \geq s\}$ for some interval $J \subseteq \mathbb{R}$. Furthermore, the Lipschitz continuity on $B$ ensures that the restriction $U^{\odot}(s,t) := U^\star(s,t)|_{X^{\odot}}$ leaves $X^{\odot}$ invariant and, by construction, $U^{\odot} := \{U^{\odot}(s,t)\}_{(s,t)\in\Omega_{\mathbb{R}}^\star}$ is a strongly continuous backward evolutionary system, see [5, Theorem 5.3]. This allows us to define $U^{\odot\star}(t,s) := (U^{\odot}(s,t))^\star$ and it is clear that $U^{\odot\star} := \{U^{\odot\star}(t,s)\}_{(t,s)\in\Omega_{\mathbb{R}}}$ is a forward evolutionary system on $X^{\odot\star}$ that extends $U$, which was previously defined on $X$.

Let us now characterize the (generalized) generators $A(\tau), A^\star(\tau), A^{\odot}(\tau)$ and $A^{\odot\star}(\tau)$ for all $\tau \in \mathbb{R}$ together with their domains. The weak$^\star$ continuous generator $A^{\odot\star}(\tau)$ takes the form

$$\begin{aligned} \mathcal{D}(A^{\odot\star}(\tau)) &= \{(\alpha,\varphi) \in X^{\odot\star} \ : \ \varphi \in \mathrm{Lip}([-h,0],\mathbb{R}^n) \text{ and } \varphi(0) = \alpha\}, \\ A^{\odot\star}(\tau)j\varphi &= (L(\tau)\varphi,\dot{\varphi}), \quad \forall \tau \in \mathbb{R}. \end{aligned} \tag{9}$$

Because $A(\tau)$ is defined as the preimage under $j$ of the part of $A^{\odot\star}(\tau)$ in $j(X)$, we get

$$\mathcal{D}(A(\tau)) = \{\varphi \in C^1([-h,0],\mathbb{R}^n) \ : \ \dot{\varphi}(0) = L(\tau)\varphi\}, \quad A(\tau)\varphi = \dot{\varphi}, \quad \forall \tau \in \mathbb{R}. \tag{10}$$

The domain and action of the *dual generator* $A^\star(\tau) := [A(\tau)]^\star$ of $A(\tau)$ is treated in the following lemma.

**Lemma 1.** *For the dual generator $A^\star(\tau)$ it holds*

$$\mathcal{D}(A^\star(\tau)) = \mathcal{D}(A_0^\star), \quad A^\star(\tau)f = \dot{f} + f(0^+)\zeta(\tau,\cdot), \quad \forall \tau \in \mathbb{R}.$$

*Proof.* The equality between the domains follows from a sun-variant of [5, Lemma 4.2]. To show the action, we first have to determine $B^\star(\tau) := [B(\tau)]^\star$ and notice that we traditionally restrict this map on $X^\odot$, see [14, page 58]. It follows from the uniqueness of the adjoint that $B^\star(\tau) : X^\odot \to X^\star$ is given by $B^\star(\tau)(c,g) = c\zeta(\tau,\cdot)$ since

$$\langle c\zeta(\tau,\cdot),\varphi\rangle = \langle (\langle \zeta(\tau,\cdot),\varphi\rangle, 0),(c,g)\rangle = \langle B(\tau)\varphi,(c,g)\rangle = \langle B^\star(\tau)(c,g),\varphi\rangle$$

for all $\tau \in \mathbb{R}, \varphi \in X$ and $(c,g) \in X^\odot$. Hence,

$$A^\star(\tau)f = A_0^\star f + B^\star(\tau)f = \dot{f} + f(0^+)\zeta(\tau,\cdot),$$

which completes the proof. ∎

The *sun generator* $A^\odot(\tau)$ is defined as the part of $A^\star(\tau)$ in $X^\odot$ and it is convenient to write this in $X^\odot$-notation as

$$\mathcal{D}(A^\odot(\tau)) = \{(c,g) \in \mathcal{D}(A_0^\star) \ : \ g + c\zeta(\tau,\cdot) \in X^\odot\}, \quad A^\odot(\tau)(c,g) = g + c\zeta(\tau,\cdot).$$

Since $\tau \mapsto \zeta(\tau,\cdot)$ is of the class $C^k$, it is clear from the definitions of the generators that $\tau \mapsto A(\tau), A^\star(\tau), A^\odot(\tau)$ and $A^{\odot\star}(\tau)$ are of the class $C^k$.

Our next aim is to study how the time-dependent nonlinear delay differential equation

$$\begin{cases} \dot{y}(t) = L(t)y_t + G(t,y_t), & t \geq s, \\ y_s = \varphi, & \varphi \in X, \end{cases} \tag{T-DDE}$$

fits naturally in the setting of dual perturbation theory. Recall that $G(t,\cdot) = F(\gamma_t + \cdot) - F(\gamma_t) - L(t)$ and so $G$ is $C^k$-smooth and $T$-periodic in the first component. The $T$-periodic $C^k$-smooth *time-dependent nonlinear perturbation* $R$ is given by

$$R(t,\varphi) := G(t,\varphi)r^{\odot\star}, \quad \forall (t,\varphi) \in \mathbb{R} \times X,$$

and it is shown in [28, Theorem 36] that there is a one-to-one correspondence between solutions of (T-DDE) and the time-dependent nonlinear abstract integral equation

$$u(t) = U(t,s)\varphi + j^{-1}\int_s^t U^{\odot\star}(t,\tau)R(\tau,u(\tau))d\tau, \quad \varphi \in X, \tag{T-AIE}$$

for $t \geq s$ where the integral has to be interpreted as a weak$^\star$ Riemann integral and takes values in $j(X)$ under the running assumption of $\odot$-reflexivity, see [28, Lemma 2]. The $C^k$-smoothness of the nonlinearity $R$ ensures that for any $\varphi \in X$ there exists a unique (maximal) solution $u_\varphi$ of (T-AIE) on some (maximal) interval $I_\varphi = [s,t_\varphi) \subset \mathbb{R}$ with $s < t_\varphi \leq \infty$, see [28, Proposition 2].

Let us now turn our attention towards the construction of the center manifold around the cycle $\Gamma$. The spectrum $\sigma(U(s+T,s))$ of the *monodromy operator* (at time $s$) $U(s+T,s) \in \mathcal{L}(X)$ is a countable set in $\mathbb{C}$, independent of the starting time $s$, consisting of 0 and isolated eigenvalues (called

*Floquet multipliers*) that can possibly accumulate to 0. The number $\sigma \in \mathbb{C}$ satisfying $\lambda = e^{\sigma T}$ is called the *Floquet exponent*, and it is known that 1 is always a Floquet multiplier (called the *trivial Floquet multiplier*) with associated eigenfunction $\dot{\gamma}_s$. The *(generalized) eigenspace* (at time $s$) associated to the Floquet multiplier $\lambda$ is defined as $E_\lambda(s) := \mathcal{N}((\lambda I - U(s+T,s))^{k_\lambda})$, where $k_\lambda$ is the order of a pole of $(\lambda I - U(s+T,s))^{-1}$. Hence, $E_\lambda(s)$ is finite-dimensional, and its dimension, called the *algebraic multiplicity* (of $\lambda$) will be denoted by $m_\lambda$. The *geometric multiplicity* (of $\lambda$) reflects the dimension of the eigenspace. As a consequence, the set of Floquet multipliers on the unit circle $\Lambda_0 := \{\lambda \in \sigma(U(s+T,s)) \ : \ |\lambda| = 1\}$ is finite, say $1 \leq n_0 + 1 < \infty$ counted with algebraic multiplicity. Then, we define the $(n_0+1)$-dimensional *center eigenspace* (at time $s$) as

$$X_0(s) := \bigoplus_{\lambda \in \Lambda_0} E_\lambda(s).$$

In this setting, the periodic local center manifold theorem [28, Corollary 18] for (T-DDE) applies. To be more precise, define the *center fiber bundle* as

$$X_0 := \{(t,\varphi) \in \mathbb{R} \times X \ : \ \varphi \in X_0(t)\}.$$

and denote for any $\delta > 0$ the $\delta$-ball in $X$ centered at the origin by $B_\delta(X)$. Then there exists a $C^k$-smooth map $\mathcal{C} : X_0 \to X$ and a sufficiently small $\delta > 0$ such that the manifold

$$\mathcal{W}^c_{\mathrm{loc}}(\Gamma) := \{\gamma_t + \mathcal{C}(t,\varphi) \in X \ : \ (t,\varphi) \in X_0 \text{ and } \mathcal{C}(t,\varphi) \in B_\delta(X)\} \tag{11}$$

is a $T$-periodic $C^k$-smooth $(n_0+1)$-dimensional locally positively invariant manifold in $X$, called the *(local) center manifold around* $\Gamma$, defined in the vicinity of $\Gamma$ for a sufficiently small $\delta > 0$.

## 3 Periodic spectral computations for classical DDEs

In the upcoming characterization of the center manifold, it turns out that we need a time periodic smooth basis of the center eigenspace $X_0(s)$. Therefore, in Section 3.1, we will prove that such a periodic smooth basis exists, see Theorem 5 for the main result.

To characterize the (dual) (generalized) eigenspaces by periodic (generalized) eigenfunctions, we will study abstract ODEs on the spaces $X^\star$ and $X^{\odot\star}$. As the semigroups and generators on both spaces are only defined in a weak$^\star$ sense, it is no surprise that we have to study the abstract ODEs also in a weak$^\star$ setting. Therefore, we recall for a moment the (partial) weak$^\star$ differential operator.

**Definition 2** ([5, Definition 4.4])**.** Let $E$ be a Banach space, $J \subseteq \mathbb{R}$ an interval and $\Omega \subseteq J \times J$. We say that a function $f : J \to E^\star$ is *weak$^\star$ differentiable* with *weak$^\star$ derivative* $d^\star f : J \to E^\star$ if

$$\frac{d}{dt}\langle f(t), x\rangle = \langle d^\star f(t), x\rangle, \quad \forall x \in E, \ t \in J.$$

If in addition $d^\star f$ is weak$^\star$ continuous, then $f$ is called *weak$^\star$ continuously differentiable*. Furthermore, we say that a function $g : \Omega \to E^\star$ has *partial weak$^\star$ derivatives* $\partial_t^\star g : \Omega \to E^\star$ and $\partial_s^\star g : \Omega \to E^\star$ if

$$\begin{aligned} \frac{\partial}{\partial t}\langle g(t,s), x\rangle &= \langle \partial_t^\star g(t,s), x\rangle, \quad \forall x \in E, \ (t,s) \in \Omega, \\ \frac{\partial}{\partial s}\langle g(t,s), x\rangle &= \langle \partial_s^\star g(t,s), x\rangle, \quad \forall x \in E, \ (t,s) \in \Omega. \end{aligned}$$

If in addition $\partial_t^\star g$ and $\partial_s^\star g$ are weak$^\star$ continuous, then $g$ is called *weak$^\star$ continuously differentiable.*

*Remark* 3. For using spectral theory on the real Banach space $X$, we have to *complexify* $X$ and all discussed operators on $X$. This is not entirely trivial and is discussed in [14, Section III.7 and Section IV.2]. To clarify, by the spectrum of a real (unbounded) operator $L$ defined on (a subspace of) $X$, we mean the spectrum of its complexification $L_\mathbb{C}$ on (a subspace of) the complexified Banach space $X_\mathbb{C}$. For the ease of notation, we omit the additional symbols.

## 3.1 Periodic smooth Jordan chains

Let us focus on a specific Floquet multiplier $\lambda \in \sigma(U(s+T,s))$ for a fixed $s \in \mathbb{R}$. By the construction given in [14, Section IV.4], it is possible to find a basis of $E_\lambda(s)$ that is in *Jordan normal form*. That is, there exists an ordered basis $\{\phi_s^0, \dots, \phi_s^{m_\lambda - 1}\}$ of $E_\lambda(s)$ called a *Jordan chain* such that

$$(U(s+T,s) - \lambda I)\phi_s^i = \begin{cases} 0, & i = 0, \\ \phi_s^{i-1}, & i = 1, \dots, m_\lambda - 1, \end{cases} \tag{12}$$

and $\phi_s^i$ should be interpreted via the translation property (2) for all $i = 0, \dots, m_\lambda - 1$. As the map $U_\lambda(\tau, s) := U(\tau, s)|_{E_\lambda(s)} : E_\lambda(s) \to E_\lambda(\tau)$ is a topological isomorphism [14, Theorem XIII.3.3], we know that $\{\phi_\tau^0, \dots, \phi_\tau^{m_\lambda - 1}\}$ is an ordered basis of $E_\lambda(\tau)$, where $\phi_\tau^i := U_\lambda(\tau, s)\phi_s^i$ for all $\tau \in \mathbb{R}$ and $i = 0, \dots, m_\lambda - 1$. The following lemma shows that this specific basis of $E_\lambda(\tau)$ has additional structure.

**Lemma 4.** *The ordered basis $\{\phi_\tau^0, \dots, \phi_\tau^{m_\lambda - 1}\} \subseteq \mathcal{D}(A(\tau))$ consists of $C^{k+1}$-smooth functions and forms a Jordan chain for $E_\lambda(\tau)$ for all $\tau \in \mathbb{R}$.*

*Proof.* Let $\tau \in \mathbb{R}$ be given. Since $\phi_\tau^i = U_\lambda(\tau, s)\phi_s^i$, it is clear from the computation

$$(U(\tau+T,\tau) - \lambda I)\phi_\tau^i = U(\tau, s)(U(s+T,s) - \lambda I)\phi_s^i = \begin{cases} 0, & i = 0, \\ \phi_\tau^{i-1}, & i = 1, \dots, m_\lambda - 1, \end{cases}$$

that $\{\phi_\tau^0, \dots, \phi_\tau^{m_\lambda - 1}\}$ is a Jordan chain for $E_\lambda(\tau)$. Let us now prove that $\phi_\tau^i \in \mathcal{D}(A(\tau))$, where the domain is defined in (10), by first showing that $\phi_\tau^i \in C^1([-h,0],\mathbb{R}^n)$. Since $\phi_\tau^i = U_\lambda(\tau, s)\phi_s^i$, it follows from [14, Theorem XII.3.1] that $\phi_\tau^i$ satisfies the delay differential equation

$$\dot{\phi}^i(\tau) = L(\tau)\phi_\tau^i. \tag{13}$$

Since $\phi_s^i \in C([-h,0],\mathbb{R}^n) = X$, we know from applying the method of steps onto (13) that its solution for $\tau > s$ is at least continuously differentiable [18, Theorem 1.2.2]. We prove the claim now by induction. If $i = 0$, choose $m \in \mathbb{N}$ large enough to guarantee that $\tau + mT > s + h$ because then the history $\phi_{\tau+mT}^0$ coincides precisely with the solution of (13) on $[\tau + mT - h, \tau + mT] \subset [s, \infty)$, and so $\phi_{\tau+mT} \in C^1([-h,0],\mathbb{R}^n)$. It follows from

$$\lambda^m \phi_\tau^0 = U(\tau+T,\tau)^m \phi_\tau^0 = U(\tau+mT,\tau)\phi_\tau^0 = \phi_{\tau+mT}^0,$$

that $\phi_\tau^0 \in C^1([-h,0],\mathbb{R}^n)$ due to linearity and the fact that $\lambda \neq 0$. The second equality holds due to [14, Corollary XIII.2.2]. Now, assume that $\phi_\tau^0, \dots, \phi_\tau^{i-1}$ are in $C^1([-h,0],\mathbb{R}^n)$ for some $i \in \{1, \dots, m_\lambda - 1\}$. Choose again an $m \in \mathbb{N}$ such that $\tau + mT \geq s + h$ because then by the Jordan chain structure, there exist scalars $c_{m,l}$ for $l \in \{0, \dots, i\}$ such that

$$\lambda^m \phi_\tau^i + \sum_{l=0}^{i-1} c_{m,l}\phi_\tau^l = U(\tau+mT,\tau)\phi_\tau^i = \phi_{\tau+mT}^i \in C^1([-h,0],\mathbb{R}^n).$$

By the induction hypothesis $\phi_\tau^0, \dots, \phi_\tau^{i-1}$ are all in $C^1([-h,0],\mathbb{R}^n)$ and so we conclude that $\lambda^m \phi_\tau^i \in C^1([-h,0],\mathbb{R}^n)$ which proves that $\phi_\tau^i \in C^1([-h,0],\mathbb{R}^n)$ since $\lambda \neq 0$. This completes the induction.

The same arguments can be used now, with choosing $m \in \mathbb{N}$ such that $\tau + mT \geq s + (k+1)h$ and employing the method of steps to increase smoothness of solutions, to conclude that all maps $\tau \mapsto \phi_\tau^i$ are all $C^{k+1}$-smooth.

It remains to show that $\phi_\tau^i$ satisfies the second condition in the domain defined in (10). Since $\phi_\tau^i$ satisfies (13), we get from the translation property that

$$\frac{d}{d\theta}\phi_\tau^i(\theta)\bigg|_{\theta=0} = \frac{d}{d\theta}\phi^i(\tau+\theta)\bigg|_{\theta=0} = \dot{\phi}^i(\tau) = L(\tau)\phi_\tau^i.$$

Hence, $\phi_\tau^i$ satisfies the second condition from (10). $\square$

Let us now take a look at the periodicity of the Jordan chain defined in (12). It is clear from the computation

$$\phi_{s+T}^i - \phi_s^i = \begin{cases} (\lambda - 1)\phi_s^i, & i = 0, \\ (\lambda - 1)\phi_s^i + \phi_s^{i-1}, & i = 1, \dots, m_\lambda - 1, \end{cases}$$

that $\tau \mapsto \phi_\tau^i$ is $T$-periodic if and only if $\lambda = 1$ and $i = 0$. However, in the upcoming characterization of the center manifold, we explicitly need a $T$-periodic $C^{k+1}$-smooth (generalized) eigenbasis. To construct this basis, let us first introduce some notation. For a real or complex Banach space $E$ and integer $l \geq 0$, we define $C_T^l(\mathbb{R}, E)$ as the Banach space consisting of $T$-periodic $C^l$-smooth $E$-valued functions defined on $\mathbb{R}$ equipped with the standard $C^l$-norm. The following result is a generalization from finite-dimensional ODEs [23, Proposition III.1] towards infinite-dimensional DDEs.

**Theorem 5.** *Let $\lambda$ be a Floquet multiplier with $\sigma$ its associated Floquet exponent. Then there exist $\varphi_i \in C_T^{k+1}(\mathbb{R}, X)$ satisfying*

$$\left(\frac{d}{d\tau} - A^{\odot\star}(\tau) + \sigma\right) j\varphi_i(\tau) = \begin{cases} 0, & i = 0, \\ -j\varphi_{i-1}(\tau), & i = 1, \dots, m_\lambda - 1, \end{cases} \tag{14}$$

*or equivalently*

$$\left(\frac{d}{d\tau} - A(\tau) + \sigma\right) \varphi_i(\tau) = \begin{cases} 0, & i = 0, \\ -\varphi_{i-1}(\tau), & i = 1, \dots, m_\lambda - 1, \end{cases} \tag{15}$$

*such that the set of functions $\{\varphi_0(\tau), \dots, \varphi_{m_\lambda - 1}(\tau)\}$ is an ordered basis of $E_\lambda(\tau)$.*

*Proof.* Let $s \in \mathbb{R}$ be an initial starting time and consider the basis $\{\phi_s^0, \dots, \phi_s^{m_\lambda - 1}\} \subseteq \mathcal{D}(A(s))$ of $E_\lambda(s)$ in Jordan normal form, see Lemma 4. We show the claim by induction on $i \in \{0, \dots, m_\lambda - 1\}$. For the base case ($i = 0$), consider the initial value problem

$$\begin{cases} (d^\star - A^{\odot\star}(\tau) + \sigma) j\varphi_0(\tau) = 0, & \tau \geq s, \\ \varphi_0(s) = \phi_s^0, \end{cases} \tag{16}$$

where $d^\star$ denotes the weak$^\star$ differential operator from Definition 2. It follows from the differential equation in (16) that

$$\begin{aligned} d^\star(j \circ e^{\sigma(\cdot - s)}\varphi_0)(\tau) &= \sigma e^{\sigma(\tau - s)} j\varphi_0(\tau) + e^{\sigma(\tau - s)} d^\star j\varphi_0(\tau) \\ &= e^{\sigma(\tau - s)}\left(d^\star + \sigma\right) j\varphi_0(\tau) = A^{\odot\star}(\tau) j(e^{\sigma(\tau - s)}\varphi_0(\tau)). \end{aligned}$$

This differential equation is of the form [5, Equation (4.10)] and hence its unique solution [5, Theorem 4.14] on $[s, \infty)$ is given by

$$e^{\sigma(\tau - s)}\varphi_0(\tau) = U(\tau, s)\varphi_0(s), \tag{17}$$

whenever $\varphi_0(s) \in j^{-1}\mathcal{D}(A_0^{\odot\star})$, because $U$ leaves $j^{-1}\mathcal{D}(A_0^{\odot\star})$ invariant. Since $\varphi_0(s) = \phi_s^0$ the claim follows from Lemma 4 because $\phi_s^0 \in C^1([-h, 0], \mathbb{R}^n) \subset \mathrm{Lip}([-h, 0], \mathbb{R}^n) = j^{-1}\mathcal{D}(A_0^{\odot\star})$. Let us now prove the $T$-periodicity of $\varphi_0$. Choosing $\tau = s + T$ in (17) and using (12) yields

$$e^{\sigma T}\varphi_0(s + T) = U(s + T, s)\varphi_0(s) = \lambda\varphi_0(s).$$

Because $\lambda = e^{\sigma T}$ is non-zero we get $\varphi_0(s + T) = \varphi_0(s)$ and so $\varphi_0$ is $T$-periodic. Hence, $\varphi_0$ extends to $\mathbb{R}$. To prove the smoothness assertion, recall from Lemma 4 that $\tau \mapsto \phi_\tau^0 = U(\tau, s)\phi_s^0$ is $C^{k+1}$-smooth, and because $\tau \mapsto e^{-\sigma(\tau - s)}$ is analytic, it is clear from (17) that the map $\varphi_0$ defined by

$$\varphi_0(\tau) = e^{-\sigma(\tau - s)}\phi_\tau^0, \quad \forall \tau \in \mathbb{R},$$

is $C^{k+1}$-smooth. Hence, the weak$^\star$ differential operator $d^\star$ in (16) can be replaced by $\frac{d}{d\tau}$ for $i=0$. By linearity, we have that $\varphi_0(\tau) \in \mathcal{D}(A(\tau))$ for all $\tau \in \mathbb{R}$ which proves the base case for (14) and (15).

Now to complete the induction, assume that the maps $\varphi_0, \dots, \varphi_{i-1} \in C_T^{k+1}(\mathbb{R}, X)$ are constructed for some $i \in \{1, \dots, m_\lambda - 1\}$ and consider the initial value problem

$$\begin{cases} (d^\star - A^{\odot\star}(\tau) + \sigma) j\varphi_i(\tau) = -j\varphi_{i-1}(\tau), \quad \tau \geq s, \\ \varphi_i(s) = \displaystyle\sum_{k=0}^{i} \alpha_{ik}\phi_s^k, \end{cases} \tag{18}$$

where $\phi_s^0, \dots, \phi_s^i \in \mathcal{D}(A(s))$ are from the Jordan chain. The goal is to find scalars $\alpha_{ik}$ such that $\varphi_i$ becomes $T$-periodic. A similar computation as done for the base case tells us by using (18) that

$$d^\star(j \circ e^{\sigma(\cdot - s)}\varphi_i)(\tau) = e^{\sigma(\tau-s)}\Big(d^\star + \sigma\Big)j\varphi_i(\tau) = A^{\odot\star}(\tau)j(e^{\sigma(\tau-s)}\varphi_i(\tau)) - j(e^{\sigma(\tau-s)}\varphi_{i-1}(\tau)). \tag{19}$$

We will show that the differential equation above admits a unique solution on $[s,\infty)$. Consider the function $w_i : [s, \infty) \to X$ defined by

$$w_i(\tau) := U(\tau, s)\sum_{k=0}^{i} \frac{(s-\tau)^k}{k!}\varphi_{i-k}(s), \quad \forall \tau \in [s, \infty). \tag{20}$$

Because $\phi_s^0, \dots, \phi_s^i \in \mathcal{D}(A(s))$, we have from (18) that $\varphi_0(s), \dots, \varphi_i(s) \in \mathcal{D}(A(s))$ by linearity. It is clear that

$$U(\tau, s)\varphi_j(s) = \sum_{k=0}^{j} \alpha_{jk}\phi_\tau^k \in \mathcal{D}(A(\tau)), \quad \forall \tau \in [s, \infty),\ j \in \{0, \dots, i\},$$

and so it follows that $\tau \mapsto w_i(\tau)$ takes values in $\mathcal{D}(A(\tau)) \subseteq j^{-1}\mathcal{D}(A_0^{\odot\star})$ and is $C^{k+1}$-smooth, which implies the weak$^\star$ differentiability of $w_i$. Clearly $w_i(s) = \varphi_i(s)$ and notice that

$$\begin{aligned} d^\star(j \circ w_i)(\tau) &= A^{\odot\star}(\tau)jU(\tau,s)\sum_{k=0}^{i}\frac{(s-\tau)^k}{k!}\varphi_{i-k}(s) - jU(t,s)\sum_{k=0}^{i-1}\frac{(s-\tau)^k}{k!}\varphi_{i-k}(s) \\ &= A^{\odot\star}(\tau)jw_i(\tau) - jw_{i-1}(\tau), \end{aligned}$$

and so $w_i$ is a solution on $[s,\infty)$ of (19). Since $w_{i-1}$ is at least continuous, it follows from [28, Proposition 32] and by construction of $w_i$ that (19) admits a unique solution $j \circ w_i$ on $[s, \infty)$, where $w_i = e^{\sigma(\cdot - s)}\varphi_i$. As a consequence, $\varphi_i = e^{-\sigma(\cdot - s)}w_i$ is $C^{k+1}$-smooth because $\tau \mapsto e^{-\sigma(\tau - s)}$ is analytic.

Let us now turn our attention towards proving $T$-periodicity. We see from (20) that $\varphi_i(s) = \varphi_i(s+T)$ if and only if

$$(U(s+T,s) - \lambda I)\varphi_i(s) = U(s+T,s)\sum_{k=1}^{i}\frac{(-1)^k T^k}{k!}\varphi_{i-k}(s). \tag{21}$$

Recall from (18) that $\varphi_i(s) = \sum_{k=0}^{i}\alpha_{ik}\phi_s^k$ and retrieving (12) yields

$$\begin{aligned} \sum_{k=1}^{i}\alpha_{ik}\phi_s^{k-1} &= U(s+T,s)\sum_{l=1}^{i}\sum_{k=1}^{i-l}\alpha_{i-l,k}\frac{(-1)^l T^l}{l!}\phi_s^k \\ &= \sum_{l=1}^{i}\sum_{k=0}^{i-l}\alpha_{i-l,k}\frac{(-1)^l T^l}{l!}\begin{cases}\lambda\phi_s^k, & k=0, \\ \lambda\phi_s^k + \phi_s^{k-1}, & k = 1, \dots, i-1. \end{cases} \end{aligned}$$

Because the right-hand side is a known element in the subspace spanned by $\phi_s^0, \dots, \phi_s^{i-1}$, the $\alpha_{ik}$'s are uniquely determined for $k = 0, \dots, i$ and so we have proven that $\varphi_i$ is $T$-periodic. Hence, $\tau \mapsto \varphi_i(\tau)$

extends to a $C^{k+1}$-smooth solution on $\mathbb{R}$, taking values in $\mathcal{D}(A(\tau))$. Similarly, the weak$^\star$ differential operator from (18) can be replaced by $\frac{d}{d\tau}$ and so the formulas (14) and (15) hold.

Furthermore, $\varphi_0(\tau), \dots, \varphi_{m_\lambda}(\tau)$ are all linearly independent because they are all solutions to the abstract ODE

$$\left(\frac{d}{d\tau} - A^{\odot\star}(\tau) + \sigma\right)^{m_\lambda} j\varphi(\tau) = 0, \quad \forall \tau \in \mathbb{R},$$

which completes the proof. □

Let us now take some time to discuss the connection between the $T$-periodicity and the translation property (2) of the (generalized) eigenfunctions. It is clear that $\{\phi_\tau^0, \dots, \phi_\tau^{m_\lambda - 1}\}$ is (in general) a non-$T$-periodic basis of $E_\lambda(\tau)$ that has the translation property because $\phi_\tau^i = U(\tau, s)\phi_s^i$. On the other hand, Theorem 5 shows us that $\{\varphi_0(\tau), \dots, \varphi_{m_\lambda - 1}(\tau)\}$ is a $T$-periodic basis of $E_\lambda(\tau)$, but how is this basis related to the translation property? Notice that the function $\varphi_0(\tau) \in X$ would have the translation property if and only if it satisfies the transport equation

$$\frac{\partial}{\partial \tau}\varphi_0(\tau)(\theta) = \frac{\partial}{\partial \theta}\varphi_0(\tau)(\theta), \tag{22}$$

but a small calculation directly shows that

$$\begin{aligned}
\frac{\partial}{\partial \tau}\varphi_0(\tau)(\theta) &= e^{-\sigma(\tau - s)}(-\sigma + \dot{\phi}_\tau^0(\theta)), \\
\frac{\partial}{\partial \theta}\varphi_0(\tau)(\theta) &= e^{-\sigma(\tau - s)}\dot{\phi}_\tau^0(\theta),
\end{aligned}$$

and so $\varphi_0$ satisfies (22) if and only if $\sigma = 0$ i.e. $\lambda = 1$. A similar analysis for the $T$-periodic generalized eigenfunctions shows that these never have the translation property. Hence, the only $T$-periodic solution of (15) which satisfies the translation property is the map $\tau \mapsto \dot{\gamma}_\tau$. It is however the $T$-periodic basis $\{\varphi_0(\tau), \dots, \varphi_{m_\lambda - 1}(\tau)\}$ of $E_\lambda(\tau)$ that is needed for the characterization of the center manifold $\mathcal{W}_{\mathrm{loc}}^c(\Gamma)$ in Section 4.

In the upcoming construction of the characterization of the center manifold, we also need the *Floquet operator* (at time $\tau$) associated to the Floquet multiplier $\lambda$, defined as the coordinate map $Q_\lambda(\tau) : \mathbb{C}^{m_\lambda} \to E_\lambda(\tau)$ by

$$Q_\lambda(\tau)\xi := \sum_{i=0}^{m_\lambda - 1} \xi_i \varphi_i(\tau), \quad \forall \xi = (\xi_0, \dots, \xi_{m_\lambda - 1}) \in \mathbb{C}^{m_\lambda}. \tag{23}$$

It is clear from Theorem 5 that the map $\tau \mapsto Q_\lambda(\tau)$ is $T$-periodic, $C^{k+1}$-smooth and takes values in $\mathcal{L}(\mathbb{C}^{m_\lambda}, E_\lambda(\tau))$ for all $\tau \in \mathbb{R}$. Furthermore, a direct calculation shows that

$$\left(-\frac{d}{d\tau} + A^{\odot\star}(\tau)\right) j(Q_\lambda(\tau)\xi) = j(Q_\lambda(\tau)M_\lambda\xi), \tag{24}$$

where $M_\lambda$ is the $m_\lambda \times m_\lambda$ Jordan matrix defined by

$$M_\lambda := \begin{pmatrix} \sigma & 1 & \cdots & 0 \\ 0 & \sigma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \sigma \end{pmatrix}. \tag{25}$$

This result is an extension of [23, Proposition III.3] from finite-dimensional ODEs to infinite-dimensional DDEs. Because we are dealing with the real state space $X = C([-h, 0], \mathbb{R}^n)$, the linear operator $M_\lambda$, written in matrix form in (25), should represent a real operator, recall also Remark 3. Depending on the location of the Floquet multiplier $\lambda$ in the complex plane, we have three options [23]:

- If $\lambda$ is real and positive, we choose $\sigma$ and $\varphi_0(\tau), \dots, \varphi_{m_\lambda - 1}(\tau)$ real.
- If $\lambda$ is not real, then its complex conjugate $\overline{\lambda} \neq \lambda$ is also a Floquet multiplier. Hence, we choose $\sigma$ and $\varphi_0(\tau), \dots, \varphi_{m_\lambda - 1}(\tau)$ complex and introduce $\overline{\sigma}$ and $\overline{\varphi_0}(\tau), \dots, \overline{\varphi_{m_\lambda - 1}}(\tau)$ for the complex conjugate Jordan block.
- If $\lambda$ is real and negative, both methods describe above do not succeed. Indeed, the Floquet exponents $\sigma$ are of the form $\frac{\pi}{T} + \frac{2li\pi}{T}$ with $l \in \mathbb{Z}$. The standard way to deal with this situation is to double the period, since if $\lambda \in \sigma(U(s+T, s))$ is a Floquet multiplier then $\lambda^2 \in \sigma(U(s+2T, s))$.

To study this last case, one has to require $T$-antiperiodicity of the maps $\varphi_i$ from Theorem 5. By *$T$-antiperiodicity*, we mean that the maps $\varphi_i$ are $2T$-periodic and satisfy in addition

$$\varphi_i(\tau + T) = -\varphi_i(\tau), \quad \forall \tau \in \mathbb{R},\ i = 0, \dots, m_\lambda - 1.$$

The existence of such $T$-antiperiodic (generalized) eigenfunctions will be proven in the following proposition. In this antiperiodic setting, the associated Floquet exponent $\sigma \in \mathbb{R}$ is defined as $e^{\sigma T} = |\lambda|$.

**Proposition 6.** *Let $\lambda$ be a real and negative Floquet multiplier with $\sigma$ its associated Floquet exponent. Then there exist $T$-antiperiodic maps $\varphi_0, \dots, \varphi_{m_\lambda - 1} \in C^{k+1}_{2T}(\mathbb{R}, X)$ satisfying (14) and (15). Furthermore, there exists a real $T$-periodic projector $P_\lambda : \mathbb{R} \to \mathcal{L}(X)$ onto $E_\lambda(\tau)$. Moreover, the Floquet operator satisfies $Q_\lambda(\tau + T) = -Q_\lambda(\tau)$ and the differential equation (24), where $M_\lambda$ is now a linear operator on $\mathbb{R}^{m_\lambda}$.*

*Proof.* To prove the first assertion, we copy the proof of Theorem 5 but in the $2T$-periodic setting. The proof goes identical up to (17). If we set $\tau = s + T$ in (17) we get

$$|\lambda|\varphi_0(s + T) = e^{\sigma T}\varphi_0(s + T) = U(s + T, s)\varphi_0(s) = \lambda\varphi_0(s)$$

and so $\varphi_0(s + T) = \operatorname{sign}(\lambda)\varphi_0(s) = -\varphi_0(s)$, which proves the $T$-antiperiodicity of $\varphi_0$.

Consider now (18) and suppose that the right-hand side of the differential equation satisfies $\varphi_{i-1}(s+T) = -\varphi_{i-1}(s)$. Our goal now is to find the $\alpha_{ik}$ such that $\varphi_i$ is $T$-antiperiodic. Instead of requiring the $T$-periodicity of $\varphi_i$ we now require $T$-antiperiodicity of $\varphi_i$ and see that this holds if and only if

$$(U(s + T, s) - \lambda I)\varphi_i(s) = U(s + T, s) \sum_{k=1}^{i} \frac{(-1)^k T^k}{k!} \varphi_{i-k}(s)$$

which is precisely (21). Hence, the same procedure in Theorem 5 can be followed to find the associated $\alpha_{ik}$'s uniquely and show that $\varphi_i$ is $T$-antiperiodic.

The real spectral projection $P_\lambda(\tau) \in \mathcal{L}(X)$ onto $E_\lambda(\tau)$ for all $\tau \in \mathbb{R}$ is constructed in the same way as the Dunford integral in [28, Appendix A.2] and so $T$-periodicity follows. For the Floquet operator, it follows from linearity and $\varphi_i(\tau + T) = -\varphi_i(\tau)$ for all $i = 0, \dots, m_\lambda - 1$ that $Q_\lambda(\tau + T) = -Q_\lambda(\tau)$ for all $\tau \in \mathbb{R}$. □

## 3.2 Dual periodic smooth Jordan chains

Our next goal is to repeat the construction above, but now for the adjoint system. Let us first observe that $\sigma(U(s + T, s)) = \sigma(U^\star(s - T, s))$ because $U(s + T, s)$ is a real operator (Remark 3) and $U^\star(s, s + T) = U^\star(s - T, s)$ due to [14, Corollary XIII.2.2]. Hence, it follows from general spectral theory and [15, Proposition IV.2.18] that

$$X = \mathcal{N}((\lambda I - U(s + T, s))^{k_\lambda}) \oplus \mathcal{N}((\lambda I - U^\odot(s - T, s))^{k_\lambda})^\perp,$$

where $\perp$ denotes the annihilator. Notice that the first sum in this direct sum decomposition is precisely the $m_\lambda$-dimensional subspace $E_\lambda(s)$ of $X$ which has been characterized in Section 3.1. The aim of this subsection is to characterize the $m_\lambda$-dimensional subspace $E_\lambda^\odot(s) := \mathcal{N}((\lambda I - U^\odot(s-T,s))^{k_\lambda})$ of $X^\odot$ and therefore we start by constructing a Jordan chain. Again, by the construction given in [14, Section IV.4], there exists an ordered basis $\{\phi_{m_\lambda-1}^\odot(s), \dots, \phi_0^\odot(s)\}$ of $E_\lambda^\odot(s)$ called a *Jordan chain* such that

$$(U^\odot(s-T,s) - \lambda I)\phi_i^\odot(s) = \begin{cases} 0, & i = m_\lambda - 1, \\ \phi_{i+1}^\odot(s), & i = m_\lambda - 2, \dots, 0. \end{cases} \tag{26}$$

Because the $T$-periodic time-dependent perturbation $B$ is Lipschitz continuous, it follows from [5, Remark 5.10] that the backward evolutionary system $U^\odot$ leaves $X^\odot$ invariant, which proves that the Jordan chain in (26) is well-defined. Recall from Section 3.1 that the bounded linear operator $U_\lambda(\tau,s) : E_\lambda(s) \to E_\lambda(\tau)$ is a topological isomorphism. Hence, $U_\lambda^\odot(\tau,s) := (U_\lambda(s,\tau))^\odot : E_\lambda^\odot(s) \to E_\lambda^\odot(\tau)$ is a topological isomorphism and so $\{\phi_{m_\lambda-1}^\odot(\tau), \dots, \phi_0^\odot(\tau)\}$ is an ordered basis of $E_\lambda^\odot(\tau)$, where $\phi_i^\odot(\tau) := U_\lambda^\odot(\tau,s)\phi_i^\odot(s)$ for all $\tau \in \mathbb{R}$ and $i = 0, \dots, m_\lambda - 1$. The following lemma shows that $E_\lambda^\odot(\tau)$ has additional structure.

**Lemma 7.** *The ordered basis $\{\phi_{m_\lambda-1}^\odot(\tau), \dots, \phi_0^\odot(\tau)\} \subseteq \mathcal{D}(A_0^\star)$ consists of $C^{k+1}$-smooth functions and forms a Jordan chain for $E_\lambda^\odot(\tau)$ for all $\tau \in \mathbb{R}$.*

*Proof.* The proof of the Jordan chain structure is analogous to that of Lemma 4. Let $i \in \{m_\lambda - 1, \dots, 0\}$ be given. Because $\phi_i^\odot(s) \in X^\odot$, it follows from [5, Theorem 5.8] that $\tau \mapsto \phi_i^\odot(\tau)$ satisfies

$$\frac{d}{d\tau}\langle \phi_i^\odot(\tau), \varphi\rangle = -\langle A^{\odot\star}(\tau)j\varphi, \phi_i^\odot(\tau)\rangle, \quad \forall \varphi \in j^{-1}\mathcal{D}(A_0^{\odot\star}), \tag{27}$$

where the map $\tau \mapsto \langle \phi_i^\odot(\tau), \varphi\rangle$ is continuously differentiable for all $\tau \le s$. Because $U^\odot$ leaves $X^\odot$ invariant, we know that $\phi_i^\odot(\tau) = (c_i(\tau), g_i(\tau)) \in X^\odot$, where $\phi_i^\odot(\tau)(0^+) =: c_i(\tau) \in \mathbb{C}^{n\star}$ and $g_i(\tau) \in L^1([0,h], \mathbb{C}^{n\star})$ represents the derivative of $\phi_i^\odot$ with respect to the second component. Notice that in the representation of spaces we worked with the complexification (Remark 3). Furthermore, it follows that $g_i(\tau)(h) = 0$ for all $\tau \le s$. As we are working in the setting of classical DDEs, we can evaluate the pairings explicitly [14, Equation (II.5.6)] and so

$$\begin{aligned}
\langle A^{\odot\star}(\tau)j\varphi, \phi_i^\odot(\tau)\rangle &= \langle (L(\tau)\varphi, \dot\varphi), (c_i(\tau), g_i(\tau))\rangle \\
&= \langle c_i(\tau)\zeta(\tau,\cdot), \varphi\rangle + \int_0^h g_i(\tau)(\theta)\dot\varphi(-\theta)d\theta \\
&= \int_0^h d_\theta[c_i(\tau)\zeta(\tau,\theta) + g_i(\tau)(\theta)]\varphi(-\theta),
\end{aligned}$$

where we used (4) and (9) in the second equality, partial integration for Riemann-Stieltjes integrals and the conditions on $g$ coming from (7) in the third equality. The $d_\theta$ refers to Riemann-Stieltjes integration over the $\theta$-variable. We can also express the left-hand side of (27) as

$$\frac{d}{d\tau}\langle \phi_i^\odot(\tau), \varphi\rangle = \dot c_i(\tau)\varphi(0) + \int_0^h \frac{\partial g_i(\tau)(\theta)}{\partial\tau}\varphi(-\theta)d\theta = \int_0^h d_\theta\left[\frac{\partial \phi_i^\odot(\tau)(\theta)}{\partial\tau}\right]\varphi(-\theta).$$

Hence, (27) is equivalent to

$$\int_0^h d_\theta\left[c_i(\tau)\zeta(\tau,\theta) + \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\theta}\right)\phi_i^\odot(\tau)(\theta)\right]\varphi(-\theta) = 0, \quad \forall\varphi \in j^{-1}\mathcal{D}(A_0^{\odot\star}),$$

Clearly, if we can show that $\phi_i^\odot$ satisfies

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\theta}\right)\phi_i^\odot(\tau)(\theta) + c_i(\tau)\zeta(\tau,\theta) = 0, \quad \forall\tau \le s, \ \theta \in (0,h], \tag{28}$$

then (27) is satisfied and the result follows from the uniqueness of (27) when the initial condition $\phi_i^{\odot}(s)$ at starting time $s$ is specified. The inhomogeneous transport equation (28) has the unique solution

$$\phi_i^{\odot}(\tau)(\theta) = c_i(\tau - \theta) - \int_0^\theta c_i(\tau + v - \theta)\zeta(\tau + v - \theta, v)dv, \tag{29}$$

when the initial condition $c_i$ is specified. To determine $c_i$ from $\phi_i^{\odot}(s)$, we will have a look at the map $\theta \mapsto g_i(\tau)(\theta) = \frac{\partial}{\partial\theta}\phi_i^{\odot}(\tau)(\theta)$ that has to satisfy $g_i(\tau)(h) = 0$ for all $\tau \leq s$. Differentiating (29) with respect to $\theta$ by employing the Leibniz integral rule gives

$$\begin{aligned} g_i(\tau)(\theta) = -\dot{c}_i(\tau - \theta) - c_i(\tau)\zeta(\tau, \theta) + \int_0^\theta \dot{c}_i(\tau + v - \theta)\zeta(\tau + v - \theta, v)dv \\ + \int_0^\theta c_i(\tau + v - \theta)D_1\zeta(\tau + v - \theta, v)dv. \end{aligned}$$

Using partial integration for Riemann-Stieltjes integrals on the first integral above leads eventually to

$$g_i(\tau)(\theta) = -\dot{c}_i(\tau - \theta) - \int_0^\theta c_i(\tau + v - \theta)d_2\zeta(\tau + v - \theta, v).$$

To simplify the expression even more, recall that $g_i(\tau - \theta + h)(h) = 0$ for all $\tau \in \mathbb{R}$ and $\theta \in (0, h]$ that satisfy $\tau - \theta + h \leq s$. This is equivalent to

$$0 = -\dot{c}_i(\tau - \theta) - \int_0^h c_i(\tau + v - \theta)d_2\zeta(\tau + v - \theta, v), \tag{30}$$

and so $g_i$ simplifies to

$$g_i(\tau)(\theta) = \int_\theta^h c_i(\tau - \theta + v)d_2\zeta(\tau - \theta + v, v). \tag{31}$$

As $g_i(\tau - h)(h) = 0$ for all $\tau \leq s + h$, it follows from (30) that that $c_i$ satisfies

$$\dot{c}_i(\tau) = -\int_0^h c_i(\tau + v)d_2\zeta(\tau + v, v), \quad \forall\tau \leq s - h, \tag{32}$$

which is an advance differential equation that has to be solved backward in time with an initial condition that still has to be specified. To determine the initial condition, recall that $\phi_i^{\odot}(s)$ is known and so it follows from (29) that $c_i$ has to satisfy

$$c_i(s - \theta) = \phi_i^{\odot}(s)(\theta) + \int_0^\theta c_i(s + v - \theta)\zeta(s + v - \theta, v)dv, \quad \theta \in [0, h].$$

Perform the change of variables: $s - \theta = -\xi, z_i(\xi) = c_i(-\xi), \psi(\xi) = \phi_i^{\odot}(s)(s + \xi)$ and define the map $K$ as $K(\xi - v) := \zeta(v - \xi, v)$. This yields the linear Volterra integral equation of the second kind

$$z_i(\xi) = \psi(\xi) + \int_{-s}^\xi z_i(\eta)K(\xi - \eta)d\eta, \quad \forall\xi \in (-s, -s + h]. \tag{33}$$

Because $\psi \in \mathrm{AC}((-s, -s + h], \mathbb{C}^{n\star})$ and $K \in L^1((0, h], \mathbb{C}^{n\star})$ as the kernel $\zeta$ is $C^k$-smooth in the first component and of bounded variation of the second component, it follows from [17, Theorem 2.3.5] that (33) has a unique solution $\tilde{z}_i = z_i(-s + h + \cdot) \in \mathrm{AC}([-h, 0], \mathbb{C}^{n\star}) \subset X$, where we define $\tilde{z}_i(0) := \lim_{t\downarrow 0} z_i(-s + h + t)$. Perform now the change of variables $\tau = -t, z_i(t) = c_i(-t)$ and $\tilde{\zeta}(t, v) = \zeta(v - t, v)$ in (32) because then it is clear that (32) can be transformed into

$$\begin{cases} \dot{z}_i(t) = \int_0^h z_i(t - v)d_2\tilde{\zeta}(t, v), & t \geq -s + h, \\ z_i(-s + h + \cdot) = \tilde{z}_i, & \tilde{z}_i \in X, \end{cases} \tag{34}$$

which is a delay differential equation with initial condition $\tilde{z}_i$ on starting time $-s+h$. By the Jordan chain structure in (26), we can use similar arguments on (34) as in the proof of Lemma 4 to conclude that $z_i$ is $C^{k+1}$-smooth and extends to $\mathbb{R}$. Hence, $c_i$ is $C^{k+1}$-smooth, and it follows then from (31) that $\tau \mapsto g_i(\tau)$ is $C^{k+1}$-smooth, and the map $\theta \mapsto g_i(\tau)(\theta)$ has bounded variation for all $\theta \in (0,h]$ because of the $C^{k+1}$-smoothness of $c_i$ and the $C^k$-smoothness of $\zeta$ in the first component. As $g_i(\tau)(0) = 0$ for all $\tau \in \mathbb{R}$, we have shown that $g_i(\tau) \in \mathrm{NBV}([0,h], \mathbb{C}^{n\star})$ and so $\phi_i^{\odot}(\tau) \in \mathcal{D}(A_0^\star)$ for all $\tau \in \mathbb{R}$. Hence, we obtain the $C^{k+1}$-smoothness of $\tau \mapsto \phi_i^{\odot}(\tau)$ from (29). $\square$

Let us now take a look at the periodicity of the Jordan chain for the adjoint system. It is clear from the computation

$$\phi_i^{\odot}(s-T) - \phi_i^{\odot}(s) = \begin{cases} (\lambda - 1)\phi_{m_\lambda - 1}^{\odot}(s), & i = m_\lambda - 1, \\ (\lambda - 1)\phi_i^{\odot}(s) + \phi_{i+1}^{\odot}(s), & i = m_\lambda - 2, \dots, 0, \end{cases}$$

that $\tau \mapsto \phi_i^{\odot}(\tau)$ is $T$-periodic if and only if $\lambda = 1$ and $i = m_\lambda - 1$. However, similarly as for the (generalized) eigenfunctions, we explicitly need a $C^{k+1}$-smooth $T$-periodic basis of $E_\lambda^{\odot}(s)$ for the computation of the critical normal coefficients in an upcoming paper.

**Theorem 8.** *Let $\lambda$ be a Floquet multiplier with $\sigma$ its associated Floquet exponent. Then there exist $\varphi_i^{\odot} \in C_T^{k+1}(\mathbb{R}, X^{\odot})$ satisfying*

$$\left(\frac{d}{d\tau} + A^\star(\tau) - \sigma\right)\varphi_i^{\odot}(\tau) = \begin{cases} 0, & i = m_\lambda - 1, \\ \varphi_{i+1}^{\odot}(\tau), & i = m_\lambda - 2, \dots, 0 \end{cases} \tag{35}$$

*such that the set of functions $\{\varphi_{m_\lambda - 1}^{\odot}(\tau), \dots, \varphi_0^{\odot}(\tau)\}$ is an ordered basis of $E_\lambda^{\odot}(\tau)$.*

*Proof.* Let $s \in \mathbb{R}$ be a starting time and consider the order basis $\{\phi_{m_\lambda - 1}^{\odot}(s), \dots, \phi_0^{\odot}(s)\}$ of $E_\lambda^{\odot}(s)$ in Jordan normal form. We show the claim by induction on $i \in \{m_\lambda - 1, \dots, 0\}$. For the base case ($i = m_\lambda - 1$), consider the initial value problem

$$\begin{cases} (d^\star + A^\star(\tau) - \sigma)\varphi_{m_\lambda - 1}^{\odot}(\tau) = 0, & \tau \leq s, \\ \varphi_{m_\lambda - 1}^{\odot}(s) = \phi_{m_\lambda - 1}^{\odot}(s), \end{cases} \tag{36}$$

where $\phi_{m_\lambda - 1}^{\odot}(s)$ is the first basis vector of $E_\lambda^{\odot}(s)$. It follows from the differential equation in (36) that

$$\begin{aligned} d^\star(e^{\sigma(s-\cdot)}\varphi_{m_\lambda - 1}^{\odot})(\tau) &= -\sigma e^{\sigma(s-\tau)}\varphi_{m_\lambda - 1}^{\odot}(\tau) + e^{\sigma(s-\tau)}d^\star\varphi_{m_\lambda - 1}^{\odot}(\tau) \\ &= e^{\sigma(s-\tau)}\left(d^\star - \sigma\right)\varphi_{m_\lambda - 1}^{\odot}(\tau) = -A^\star(\tau)[e^{\sigma(s-\tau)}\varphi_{m_\lambda - 1}^{\odot}(\tau)]. \end{aligned}$$

This differential equation is of the form [5, Equation (5.8)] and so (36) admits a unique solution [5, Theorem 5.8] on $(-\infty, s]$ given by

$$e^{\sigma(s-\tau)}\varphi_{m_\lambda - 1}^{\odot}(\tau) = U^{\odot}(\tau, s)\varphi_{m_\lambda - 1}^{\odot}(s), \quad \forall \tau \leq s, \tag{37}$$

whenever $\varphi_{m_\lambda - 1}^{\odot}(s) \in \mathcal{D}(A_0^\star)$, because $U^{\odot}$ leaves $\mathcal{D}(A_0^\star)$ invariant. Since $\varphi_{m_\lambda - 1}^{\odot}(s) = \phi_{m_\lambda - 1}^{\odot}(s)$, the claim follows from Lemma 7. Let us now prove the $T$-periodicity of $\varphi_{m_\lambda - 1}^{\odot}$. If we set $\tau = s - T$ in (37) and use (26), we get

$$e^{\sigma T}\varphi_{m_\lambda - 1}^{\odot}(s-T) = U^{\odot}(s-T, s)\varphi_{m_\lambda - 1}^{\odot}(s) = \lambda\varphi_{m_\lambda - 1}^{\odot}(s).$$

Because $\lambda = e^{\sigma T}$ is non-zero we get $\varphi_{m_\lambda - 1}^{\odot}(s-T) = \varphi_{m_\lambda - 1}^{\odot}(s)$ and so $\varphi_{m_\lambda - 1}^{\odot}$ is $T$-periodic. Hence, $\varphi_{m_\lambda - 1}^{\odot}$ extends to $\mathbb{R}$. To prove the smoothness assertion, recall from Lemma 7 that $\tau \mapsto \phi_{m_\lambda - 1}^{\odot}(\tau) =$

$U^{\odot}(\tau,s)\phi^{\odot}_{m_\lambda-1}(s)$ is $C^{k+1}$-smooth, and because $\tau \mapsto e^{-\sigma(\tau-s)}$ is analytic, it is clear from (37) that the map $\varphi^{\odot}_{m_\lambda-1}$ defined by

$$\varphi^{\odot}_{m_\lambda-1}(\tau) = e^{-\sigma(s-\tau)}\phi^{\odot}_{m_\lambda-1}(\tau), \quad \forall \tau \in \mathbb{R},$$

is $C^{k+1}$-smooth. Hence, the weak$^\star$ differential operator $d^\star$ in (35) can be replaced by $\frac{d}{d\tau}$ for $i = m_\lambda - 1$. By linearity and Lemma 1, we have that $\varphi^{\odot}_{m_\lambda-1}(\tau) \in \mathcal{D}(A^\star(\tau))$, which proves the base case for (35).

Assume now that the maps $\varphi^{\odot}_{m_\lambda-1},\dots,\varphi^{\odot}_{i+1} \in C^{k+1}_T(\mathbb{R}, X^{\odot})$ are constructed for some $m_\lambda - 2 \geq i \geq 0$ and consider the initial value problem

$$\begin{cases} (d^\star + A^\star(\tau) - \sigma)\varphi^{\odot}_i(\tau) = \varphi^{\odot}_{i+1}(\tau), \quad \tau \leq s, \\ \varphi^{\odot}_i(s) = \displaystyle\sum_{k=i}^{m_\lambda-1} \alpha_{ik}\phi^{\odot}_k(s). \end{cases} \tag{38}$$

The goal is to find scalars $\alpha_{ik}$ such that $\varphi^{\odot}_i$ becomes $T$-periodic. Notice that the differential equation from (38) is equivalent to

$$d^\star(e^{\sigma(s-\cdot)}\varphi^{\odot}_i)(\tau) = e^{\sigma(s-\tau)}\bigg(d^\star - \sigma\bigg)\varphi^{\odot}_i(\tau) = -A^\star(\tau)[e^{\sigma(s-\tau)}\varphi^{\odot}_i(\tau)] + e^{\sigma(s-\tau)}\varphi^{\odot}_{i+1}(\tau). \tag{39}$$

Hence, it suffices to prove that the abstract differential equation above together with the initial condition in (38) admits a unique solution on $(-\infty, s]$. Consider the function $w^{\odot}_i : (-\infty, s] \to X^{\odot}$ defined by

$$w^{\odot}_i(\tau) := U^{\odot}(\tau,s)\sum_{l=i}^{m_\lambda-1}\frac{(\tau-t)^{l-i}}{(l-i)!}\varphi^{\odot}_l(t), \quad \forall \tau \in (-\infty, s]. \tag{40}$$

Since $\phi^{\odot}_{m_\lambda-1}(s),\dots,\phi^{\odot}_i(s) \in \mathcal{D}(A^\star(s))$ we get from (38) that $\varphi^{\odot}_{m_\lambda-1}(s),\dots,\varphi^{\odot}_i(s) \in \mathcal{D}(A^\star(s))$. It is clear that

$$U^{\odot}(\tau,s)\varphi^{\odot}_j(s) = \sum_{k=j}^{m_\lambda-1}\alpha_{ik}\phi^{\odot}_k(\tau) \in \mathcal{D}(A^\star(\tau)), \quad \forall \tau \in (-\infty, s],\ j \in \{m_\lambda - 1, \dots, i\},$$

and so it follows from Lemma 7 that $\tau \mapsto w^{\odot}_i(\tau)$ takes values in $\mathcal{D}(A^\star(\tau))$ and is $C^{k+1}$-smooth, which implies weak$^\star$ differentiability of $w^{\odot}_i$. Clearly $w^{\odot}_i(s) = \varphi^{\odot}_i(s)$ and notice that

$$\begin{aligned} d^\star w^{\odot}_i(\tau) &= -A^\star(\tau)U^{\odot}(\tau,s)\sum_{l=i}^{m_\lambda-1}\frac{(\tau-s)^{l-i}}{(l-i)!}\varphi^{\odot}_l(s) + U^{\odot}(\tau,s)\sum_{l=i+1}^{m_\lambda-1}\frac{(\tau-s)^{l-i-1}}{(l-i-1)!}\varphi^{\odot}_l(s) \\ &= -A^\star(\tau)w_i(\tau) + w^{\odot}_{i+1}(\tau), \end{aligned}$$

and so $w^{\odot}_i$ is a solution on $(-\infty, s]$ of (39). Since $w^{\odot}_{i+1}$ is at least continuous, it follows from Proposition 14 and by construction of $w^{\odot}_i$ that (39) admits a unique solution $w^{\odot}_i$ on $(-\infty, s]$ where $w_i = e^{\sigma(s-\cdot)}\varphi^{\odot}_i$. As a consequence, $\varphi^{\odot}_i = e^{-\sigma(s-\cdot)}w^{\odot}_i$ and because $\tau \mapsto e^{-\sigma(\tau-s)}$ is analytic, we have that $\varphi^{\odot}_i$ is $C^{k+1}$-smooth.

Let us now turn our attention towards proving $T$-periodicity. If we set $\tau = s - T$ in (40), we see that $\varphi^{\odot}_i(s-T) = \varphi^{\odot}_i(s)$ if and only if

$$(U^{\odot}(s-T,s) - \lambda I)\varphi^{\odot}_i(s) = U^{\odot}(s-T,s)\sum_{l=i+1}^{m_\lambda-1}\frac{(-1)^{l-i+1}}{(l-i)!}\varphi^{\odot}_l(s). \tag{41}$$

Recall from (38) that $\varphi_i^\odot(s) = \sum_{k=i}^{m_\lambda-1} \alpha_{ik}\phi_k^\odot(s)$ and retrieving (26) yields

$$\begin{aligned}
\sum_{k=i}^{m_\lambda-2} \alpha_{ik}\phi_{k+1}^\odot(s) &= U^\odot(s-T,s)\sum_{l=i+1}^{m_\lambda-1} \frac{(-1)^{l-i+1}T^{l-i}}{(l-i)!}\varphi_l^\odot(s) \\
&= U^\odot(s-T,s)\sum_{l=i+1}^{m_\lambda-1}\sum_{k=l}^{m_\lambda-1} \alpha_{lk}\frac{(-1)^{l-i+1}T^{l-i}}{(l-i)!}\phi_k^\odot(s) \\
&= \sum_{l=i+1}^{m_\lambda-1}\sum_{k=l}^{m_\lambda-1} \alpha_{lk}\frac{(-1)^{l-i+1}T^{l-i}}{(l-i)!}\begin{cases}\lambda\phi_k^\odot(s), & k = m_\lambda - 1, \\ \lambda\phi_k^\odot(s) + \phi_{k+1}^\odot(s), & k = m_\lambda-2,\dots,i+1.\end{cases}
\end{aligned}$$

Because the right-hand side is a known element in the subspace spanned by $\phi_{m_\lambda-1}^\odot(s),\dots,\phi_{i+1}^\odot(s)$, the $\alpha_{ik}$'s are uniquely determined for $k = m_\lambda-2,\dots,i$ and so we have proven that $\varphi_i^\odot$ is $T$-periodic and so extends to $\mathbb{R}$.

Furthermore, $\varphi_{m_\lambda-1}^\odot(\tau),\dots,\varphi_0^\odot(\tau)$ are all linearly independent because they are all solutions to the abstract ODE

$$\left(\frac{d}{d\tau} + A^\odot(\tau) - \sigma\right)^{m_\lambda}\varphi^\odot(\tau) = 0, \quad \forall \tau \in \mathbb{R},$$

which completes the proof. □

Similar as for the (generalized) eigenfunctions associated to a strictly negative real Floquet multiplier $\lambda$, we need a $T$-antiperiodic version of Theorem 8 for the adjoint (generalized) eigenfunctions.

**Proposition 9.** *Let $\lambda$ be a real strictly negative Floquet multiplier with $\sigma$ its associated Floquet exponent. Then there exist $T$-antiperiodic maps $\varphi_{m_\lambda-1}^\odot,\dots,\varphi_0^\odot \in C_{2T}^{k+1}(\mathbb{R},X^\odot)$ satisfying (35).*

*Proof.* We copy the proof of Theorem 8 but in the $2T$-periodic setting. The proof goes identical up to (37). If we set $\tau = s-T$ in (37) we get

$$|\lambda|\varphi_{m_\lambda-1}^\odot(s-T) = e^{\sigma T}\varphi_{m_\lambda-1}^\odot(s-T) = U^\odot(s-T,s)\psi_{m_\lambda-1}(s) = \lambda\varphi_{m_\lambda-1}^\odot(s)$$

and so $\varphi_{m_\lambda-1}^\odot(s-T) = \operatorname{sign}(\lambda)\varphi_{m_\lambda-1}^\odot(s) = -\varphi_{m_\lambda-1}^\odot(s)$. This proves the $T$-antiperiodicity of $\varphi_{m_\lambda-1}^\odot$.

Consider now (38) and suppose that the right-hand side of the differential equation satisfies $\varphi_{i+1}^\odot(s-T) = -\varphi_{i+1}^\odot(s)$. The goal is to find $\alpha_{ik}$ such that $\varphi_i^\odot$ is $T$-antiperiodic. Instead of requiring the $T$-periodicity of $\varphi_i^\odot$ we require now that $\varphi_i^\odot(s-T) = -\varphi_i^\odot(s)$ and so

$$(U^\odot(t-T,t) - \lambda I)\varphi_i^\odot(t) = U^\odot(t-T,t)\sum_{l=i+1}^{m_\lambda-1}\frac{(-1)^{l-i+1}}{(l-i)!}\varphi_l^\odot(t),$$

which is precisely (41). The same procedure in Theorem 8 can be followed to find the associated $\alpha_{ik}$'s uniquely and obtain $\varphi_i^\odot(s-T) = \varphi_i^\odot(s)$. □

## 4 Characterization of the center manifold and normal forms

In this section, we study the dynamics of (DDE) on the center manifold $\mathcal{W}_{\text{loc}}^c(\Gamma)$ near the *nonhyperbolic* cycle $\Gamma$, meaning that there are, except of the trivial Floquet multiplier, other Floquet multipliers present on the unit circle in the complex plane, or the trivial Floquet multiplier has an algebraic multiplicity higher than one. Recall from Section 1 that there are three generic codimension one bifurcation of limit cycles: the fold bifurcation, where the trivial Floquet multiplier has an algebraic multiplicity 2 and geometric multiplicity 1, the period-doubling bifurcation, where there is a simple

Floquet multiplier at $-1$ and the Neimark-Sacker bifurcation where there is a simple complex conjugate pair of Floquet multipliers present on the unit circle.

To study these bifurcations, we first separate the trivial Floquet multiplier from the rest of the dynamics in Section 4.1. Afterwards, in Section 4.2, we prove the existence of a special coordinate system on the center manifold and provide in addition the periodic (critical) normal forms. These results are an extension of the work by Iooss [22, 23] from finite-dimensional ODEs to infinite-dimensional DDEs. As a consequence, the periodic normal forms obtained in Theorem 10, Theorem 11 and Theorem 12 for DDEs are exactly the same as obtained by Iooss in ODEs, see [23, Theorem III.7, Theorem III.10 and Theorem III.13]. Instead of being interested in only codim 1 bifurcations of limit cycles, the provided framework is also suited to study bifurcations of limit cycles of higher codimension, see for example [33, Table 1] for the periodic normal forms for some codim 2 bifurcations of limit cycles in ODEs, and hence DDEs. It is nevertheless helpful to keep these three codim one bifurcations in mind.

Before we start proving the characterization and periodic normal forms, let us first review some results from [28] on the topological direct sum decompositions of $X$ and $X^{\odot\star}$. It turns out from [28, Hypothesis 1] that we can decompose $X$ as

$$X = X_-(\tau) \oplus X_0(\tau) \oplus X_+(\tau), \quad \forall \tau \in \mathbb{R}, \tag{42}$$

where $X_-(\tau)$ and $X_+(\tau)$ denote the *stable eigenspace* and *unstable eigenspace* (at time $\tau$) respectively, see [28, Section 3.6] for their definitions. Furthermore, it turns out from [28, Appendix A.1] that we can lift the decomposition (42) towards a decomposition in $X^{\odot\star}$ as

$$X^{\odot\star} = X_-^{\odot\star}(\tau) \oplus X_0^{\odot\star}(\tau) \oplus X_+^{\odot\star}(\tau), \quad \forall \tau \in \mathbb{R}, \tag{43}$$

where $X_0^{\odot\star}(\tau) = j(X_0(\tau))$ and $X_+^{\odot\star}(\tau) = j(X_+(\tau))$, see [28, Appendix A.2] for more information.

## 4.1 Separating the dynamics of the periodic orbit

The coordinate system and normal forms we will present consist of two parts and is inspired by [22, 23]. The first part expresses the dynamics along $\Gamma$ by a time-dependent phase and the other part expresses the dynamics transverse to $\Gamma$ in terms of this phase. The normal forms depend on the location and multiplicities of the Floquet multipliers on the unit circle. Let us first separate the dynamics of the periodic orbit via coordinates along this phase and transverse to this phase.

Recall that $X_0(\tau)$ is a $(n_0+1)$-dimensional subspace of $X$ for all $\tau \in \mathbb{R}$. For each $\lambda \in \Lambda_0$, we know that the (generalized) eigenspace $E_\lambda(\tau)$ has a ($T$ or $2T$)-periodic basis that satisfies the conditions from Theorem 5 or Proposition 6, depending on the location of $\lambda$ on the unit circle. Recall that the trivial Floquet multiplier is always present on the unit circle and $\dot{\gamma}_\tau$ is the associated $T$-periodic eigenfunction of $U(\tau+T,\tau)$. We choose $\varphi_0(\tau)$ to be $\dot{\gamma}_\tau$ and denote by $\tilde{X}_0(\tau)$ the space spanned by $\varphi_1(\tau),\dots,\varphi_{n_0}(\tau)$ that forms a ($T$ or $2T$)-periodic $C^{k+1}$-smooth basis as presented in Theorem 5 or Proposition 6. Define for any $\tau \in \mathbb{R}$ the bounded linear operator $\tilde{Q}_0(\tau) : \mathbb{R}^{n_0} \to \tilde{X}_0(\tau)$ as

$$\tilde{Q}_0(\tau)\xi := \sum_{i=1}^{n_0} \xi_i \varphi_i(\tau), \quad \forall \xi = (\xi_1,\dots,\xi_{n_0}) \in \mathbb{R}^{n_0}. \tag{44}$$

With this notation, it is clear that the Floquet operator (at time $\tau$) associated to $\Lambda_0$ is defined as $Q_0(\tau) : \mathbb{R} \times \mathbb{R}^{n_0} \to X_0(\tau)$ has the action

$$Q_0(\tau)(\xi_0,\xi) := \xi_0\dot{\gamma}_\tau + \tilde{Q}_0(\tau)\xi, \quad \forall (\xi_0,\xi) \in \mathbb{R} \times \mathbb{R}^{n_0}.$$

The $(n_0+1)\times(n_0+1)$ matrix $M_0$ from takes the form

$$M_0 = \left(\begin{array}{c|c} 0 & \star \ \cdots \ 0 \\ \hline 0 & \\ \vdots & \tilde{M}_0 \\ 0 & \end{array}\right), \tag{45}$$

where $\star \in \{0,1\}$ depends on the algebraic multiplicity of the trivial Floquet multiplier.

## 4.2 Characterization and normal form theorems

Depending on the algebraic multiplicity of the trivial Floquet multiplier and the location of the other Floquet multipliers on the unit circle, the periodic normal forms will have a different shape and therefore three different normal form theorems will be presented.

The main idea to prove the existence of suitable coordinates on $\mathcal{W}^c_{\text{loc}}(\Gamma)$ is to use the invariance property of $\mathcal{W}^c_{\text{loc}}(\Gamma)$ around the periodic orbit $\Gamma$ to the fullest. We try to parametrize the history $x_t \in \mathcal{W}^c_{\text{loc}}(\Gamma)$ in the vicinity of the periodic orbit $\Gamma$ as

$$x_t = \gamma_\tau + \tilde{Q}_0(\tau)\xi + H(\tau,\xi), \tag{46}$$

where $\tau$ is a function of $t$, expresses the dynamics along $\Gamma$ by a time-dependent phase and $\xi$ is a function of $\tau$ that expresses the dynamics transverse to $\Gamma$ in terms of this phase. Such a coordinate system is visualized for a two-dimensional local center manifold around $\Gamma$ in Figure 1.

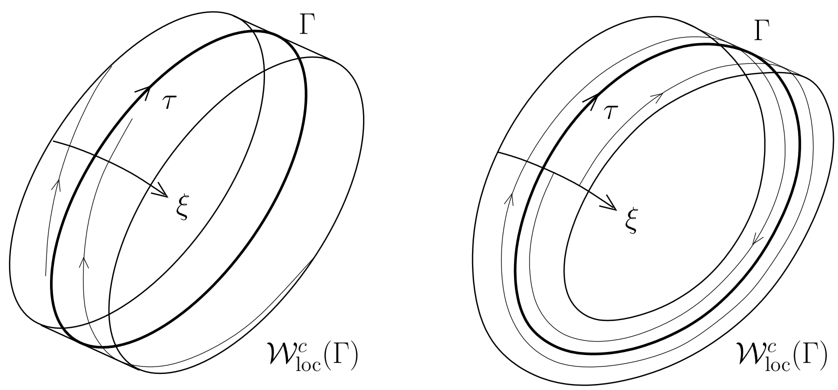


Figure 1: Illustration of two-dimensional center manifolds $\mathcal{W}^c_{\text{loc}}(\Gamma)$ together with the coordinate system $(\tau,\xi)$. The left figure represents the case when $-1 \notin \Lambda_0$ and then $\mathcal{W}^c_{\text{loc}}(\Gamma)$ is locally diffeomorphic to a cylinder in a neighborhood of $\Gamma$, see Theorem 11. The right figure represents the case when $-1 \in \Lambda_0$ and then $\mathcal{W}^c_{\text{loc}}(\Gamma)$ is locally diffeomorphic to a Möbius band in a neighborhood of $\Gamma$, see Theorem 12.

The only unknown in (46) is the nonlinear operator $H : \mathbb{R} \times \mathbb{R}^{n_0} \to X$ and to obtain a Taylor expansion of this operator, we use again the invariance property of $\mathcal{W}^c_{\text{loc}}(\Gamma)$. To be more precise, if we take an initial condition $x_s = \varphi \in \mathcal{W}^c_{\text{loc}}(\Gamma)$, then we must have that $x_t \in \mathcal{W}^c_{\text{loc}}(\Gamma)$ for all $t$ in the time domain of definition, say $I_\varphi \subseteq \mathbb{R}$ with $s \in I_\varphi$. By [7, Theorem 3.6] we know that the history $x_t \in \mathcal{W}^c_{\text{loc}}(\Gamma)$ satisfies the abstract ODE

$$\frac{d}{dt} j(x_t) = A_0^{\odot\star} j(x_t) + G(x_t), \quad t \in I_\varphi, \tag{47}$$

where $G(\psi) = F(\psi) r^{\odot\star}$ for all $\psi \in X$ and $F \in C^{k+1}(X,\mathbb{R}^n)$ for some $k \geq 1$ is the right-hand side of (DDE). The idea is to show the existence of each $q$th order term in the Taylor expansion of $H$ for $q = 2,\dots,k$ by using (47) and the invariance property of $\mathcal{W}^c_{\text{loc}}(\Gamma)$.

First we consider the case where the trivial Floquet multiplier has algebraic multiplicity 1 and there is no Floquet multiplier located at $-1$. This is for example the case in the Neimark-Sacker bifurcation.

**Theorem 10** (Normal Form I)**.** *Assume that the algebraic multiplicity of the trivial Floquet multiplier is one and that $-1$ is not a Floquet multiplier. Then there exist $C^k$-smooth maps $H : \mathbb{R} \times \mathbb{R}^{n_0} \to X, p :$*

$\mathbb{R} \times \mathbb{R}^{n_0} \to \mathbb{R}$ *and* $P : \mathbb{R} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ *such that the history* $x_t \in \mathcal{W}^c_{\text{loc}}(\Gamma)$ *may be represented as*

$$x_t = \gamma_\tau + \tilde{Q}_0(\tau)\xi + H(\tau, \xi), \quad t \in I,$$

*where the time dependence of the coordinates* $(\tau, \xi)$ *describing the dynamics of* (DDE) *on* $\mathcal{W}^c_{\text{loc}}(\Gamma)$ *is defined by the normal form*

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + p(\tau, \xi) + \mathcal{O}(|\xi|^{k+1}), \\ \dfrac{d\xi}{d\tau} = \tilde{M}_0\xi + P(\tau, \xi) + \mathcal{O}(|\xi|^{k+1}). \end{cases}$$

*Here the functions* $H, p,$ *and* $P$ *are* $T$*-periodic in* $\tau$ *and at least quadratic in* $\xi$*. The* $\mathcal{O}$*-terms are also* $T$*-periodic in* $\tau$*. Moreover,* $p$ *and* $P$ *are polynomials in* $\xi$ *of degree less than or equal to* $k$ *such that*

$$\frac{d}{d\tau} p(\tau, e^{-\tau \tilde{M}_0^\star}\xi) = 0 \quad \text{and} \quad \frac{d}{d\tau}\left(e^{\tau \tilde{M}_0^\star} P(\tau, e^{-\tau \tilde{M}_0^\star}\xi)\right) = 0,$$

*for all* $\tau \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^{n_0}$*.*

The proof of this theorem is quite long and technical. Essentially, it is a careful generalization of [23, Theorem III.7]. Therefore, we first sketch the idea of the proof and break it up into several steps. The final goal is to characterize the map $H$ by its Taylor expansion. In **Step 1** of the proof, we assume this Taylor expansion and start in **Step 2** with collecting terms in powers of $\xi^q$ for $q = 0, \dots, k$ from both sides of the resulted equation, obtained from the invariance property of the center manifold. We get for $q = 2, \dots, k$ an equation for the coefficient $H_q$ and we must show that this can be uniquely solved. This will be done via decomposing $H_q$ into the decomposition provided in (42) together with the separation made in Section 4.1, see **Step 3**. Hence, we get for each $q = 2, \dots, k$ the terms $H_q^{00}, \tilde{H}_q^0, H_q^-$ and $H_q^+$ and then we prove the existence of each of these terms in **Step 4** ($H_q^+$), **Step 5** ($H_q^-$) and **Step 6** ($H_q^{00}$ and $\tilde{H}_q^0$). The provided normal forms are partially derived in step 6 of the proof in combination with [23, Theorem III.7].

*Proof of Theorem 10.* We follow the outlined route of the proof as indicated above.

**Step 1: Taylor expansion.** Let us write (DDE) in the form of (47) and notice that

$$G(\gamma_\tau + \varphi) = G(\gamma_\tau) + B(\tau)\varphi + \sum_{q=2}^{k} G_q(\tau, \varphi^{(q)}) + \mathcal{O}(\|\varphi\|_\infty^{k+1}), \quad \forall \varphi \in X, \tag{48}$$

where $B(\tau)\varphi = [DF(\gamma_\tau)\varphi]r^{\odot\star}$ is the time-dependent bounded linear perturbation and the nonlinear terms are given by $G_q(\tau, \varphi^{(q)}) := \frac{1}{q!}D^qF(\gamma_\tau)(\varphi^{(q)})r^{\odot\star}$, where $D^qF(\gamma_\tau) : X^q \to \mathbb{R}^n$ is the $q$th order Fréchet derivative evaluated at $\gamma_\tau$ for $q = 2, \dots, k$ and $\varphi^{(q)} := (\varphi, \dots, \varphi) \in X^q := X \times \dots \times X$. We also expand the maps $H, p$ and $P$ as

$$H(\tau, \xi) = \sum_{q=2}^{k} H_q(\tau, \xi^{(q)}) + \mathcal{O}(|\xi|^{k+1}), \quad p(\tau, \xi) = \sum_{q=2}^{k} p_q(\tau, \xi^{(q)}), \quad P(\tau, \xi) = \sum_{q=2}^{k} P_q(\tau, \xi^{(q)}),$$

with the coefficients $H_q(\tau, \xi^{(q)}) \in X, p_q(\tau, \xi^{(q)}) \in \mathbb{R}$ and $P_q(\tau, \xi^{(q)}) \in \mathbb{R}^{n_0}$, where $\xi^{(q)} := (\xi, \dots, \xi) \in [\mathbb{R}^{n_0}]^q$. As already announced, we will use the invariance property of $\mathcal{W}^c_{\text{loc}}(\Gamma)$ to show existence of the coefficients $H_q(\tau, \xi^{(q)})$ for all $q = 2, \dots, k$. Hence, we compare the expansions of

$$\frac{d}{dt} j(\gamma_\tau + \tilde{Q}_0(\tau)\xi + H(\tau, \xi)) = j\left(\dot{\gamma}_\tau + \frac{\partial \tilde{Q}_0(\tau)}{\partial \tau}\xi + \frac{\partial H(\tau, \xi)}{\partial \tau} + \left(\tilde{Q}_0(\tau) + D_\xi H(\tau, \xi)\right)\frac{d\xi}{d\tau}\right)\frac{d\tau}{dt}$$

and

$$A_0^{\odot\star} j(x_t) + G(x_t) = A_0^{\odot\star} j(\gamma_\tau + \tilde{Q}_0(\tau)\xi + H(\tau,\xi)) + G(\gamma_\tau + \tilde{Q}_0(\tau)\xi + H(\tau,\xi))$$

by substituting

$$\frac{d\tau}{dt} = 1 + p(\tau,\xi) + \mathcal{O}(|\xi|^{k+1}) \quad \text{and} \quad \frac{d\xi}{d\tau} = \tilde{M}_0\xi + P(\tau,\xi) + \mathcal{O}(|\xi|^{k+1}).$$

Using the expansions of $H, p$ and $P$ together with (48), where now $\varphi$ must be substituted by $\tilde{Q}_0(\tau)\xi + H(\tau,\xi)$, we get

$$\begin{aligned}
&j\bigg[\dot{\gamma}_\tau + \frac{\partial \tilde{Q}_0(\tau)}{\partial \tau}\xi + \sum_{q=2}^{k} \frac{\partial H_q(\tau,\xi^{(q)})}{\partial \tau} + \bigg(\tilde{Q}_0(\tau) + \sum_{q=2}^{k} D_\xi H_q(\tau,\xi^{(q)})\bigg)\bigg(\tilde{M}_0 + \sum_{q=2}^{k} P_q(\tau,\xi^{(q)})\bigg)\bigg] \\
&\bigg(1 + \sum_{q=2}^{k} p_q(\tau,\xi^{(q)})\bigg) + \mathcal{O}(|\xi|^{k+1}) \\
&= A_0^{\odot\star} j(\gamma_\tau) + G(\gamma_\tau) + A^{\odot\star}(\tau) j\bigg(\tilde{Q}_0(\tau)\xi + \sum_{q=2}^{k} H_q(\tau,\xi^{(q)})\bigg) \\
&+ \sum_{q=2}^{k} G_q\bigg(\tau, \bigg[\tilde{Q}_0(\tau)\xi + \sum_{p=2}^{k} H_q(\tau,\xi^{(p)})\bigg]^{(q)}\bigg) + \mathcal{O}(|\xi|^{k+1}).
\end{aligned}$$

**Step 2: Collecting terms.** Let us now compare the terms in powers of $\xi$ on both side of this equation. Collecting the $\xi^0$-terms give us

$$\frac{d}{d\tau} j(\gamma_\tau) = A_0^{\odot\star} j(\gamma_\tau) + G(\gamma_\tau),$$

which means that $\gamma$ is a solution (47). This was already known since $\gamma$ is a periodic solution of (DDE). The $\xi^1$-terms give us

$$\bigg(-\frac{\partial}{\partial \tau} + A^{\odot\star}(\tau)\bigg) j(\tilde{Q}_0(\tau)\xi) = j(\tilde{Q}_0(\tau)\tilde{M}_0\xi), \tag{49}$$

which is exactly the result established in (24), but now for all Floquet multipliers on the unit circle and this characterizes the linear part. After collecting the $\xi^{(2)}$-terms, we get the expression

$$\begin{aligned}
&\bigg(-\frac{\partial}{\partial \tau} + A^{\odot\star}(\tau)\bigg) j(H_2(\tau,\xi^{(2)})) \\
&= j(D_\xi H_2(\tau,\xi^{(2)})\tilde{M}_0\xi + p_2(\tau,\xi^{(2)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)P_2(\tau,\xi^{(2)})) - R_2(\tau,\xi^{(2)}),
\end{aligned}$$

where $R_2(\tau,\xi^{(2)}) = G_2(\tau,(\tilde{Q}_0(\tau)\xi)^{(2)})$. Finally, after collecting the $\xi^{(q)}$-terms for $q = 3,\dots,k$ one obtains

$$\begin{aligned}
&\bigg(-\frac{\partial}{\partial \tau} + A^{\odot\star}(\tau)\bigg) j(H_q(\tau,\xi^{(q)})) \\
&= j(D_\xi H_q(\tau,\xi^{(q)})\tilde{M}_0\xi + p_q(\tau,\xi^{(q)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)P_q(\tau,\xi^{(q)})) - R_q(\tau,\xi^{(q)}),
\end{aligned} \tag{50}$$

where $R_q(\tau,\xi^{(q)})$ depends on $G_{q'}(\tau,\xi^{(q')})$ for $2 \leq q' \leq q$ and $j(H_{q'}(\tau,\xi^{(q')}))$, $j(p_{q'}(\tau,\xi^{(q')})\dot{\gamma}_\tau)$ and $j(\tilde{Q}_0(\tau)P_{q'}(\tau,\xi^{(q')}))$ for $q' = 2,\dots,q-1$.

**Step 3: Projecting on subspaces.** We want to project (50) onto the spaces $\mathbb{R}j\dot{\gamma}_\tau, j(\tilde{X}_0(\tau))$ and $X_\pm^{\odot\star}(\tau)$ to show the existence of $H_q$ separately on each individual space. Because $X$ can be decomposed as in (42) where $X_0(\tau) = \mathbb{R}\dot{\gamma}_\tau \oplus \tilde{X}_0(\tau)$ for any $\tau \in \mathbb{R}$, we can decompose for any $q = 2, \ldots, k$ the function $H_q$ as

$$H_q(\tau, \xi^{(q)}) = H_q^{00}(\tau, \xi^{(q)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)\tilde{H}_q^0(\tau, \xi^{(q)}) + H_q^-(\tau, \xi^{(q)}) + H_q^+(\tau, \xi^{(q)}),$$

where $H_q^\pm(\tau, \xi^{(q)}) = P_\pm(\tau)H_q(\tau, \xi^{(q)}) \in X_\pm(\tau)$ together with $H_q^{00}(\tau, \xi^{(q)}) \in \mathbb{R}$ and $\tilde{H}_q^0(\tau, \xi^{(q)}) \in \mathbb{R}^{n_0}$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_0}$. It follows from (43) that for any $\tau \in \mathbb{R}$ there holds

$$R_q(\tau, \xi^{(q)}) = R_q^{00}(\tau, \xi^{(q)})j\dot{\gamma}_\tau + j(\tilde{Q}_0(\tau)\tilde{R}_q^0(\tau, \xi^{(q)})) + R_q^-(\tau, \xi^{(q)}) + R_q^+(\tau, \xi^{(q)}),$$

where $R_q^\pm(\tau, \xi^{(q)}) = P_\pm^{\odot\star}(\tau)R_q(\tau, \xi^{(q)}) \in X_\pm^{\odot\star}(\tau)$ together with $R_q^{00}(\tau, \xi^{(q)}) \in \mathbb{R}$ and $\tilde{R}_q^0(\tau, \xi^{(q)}) \in \mathbb{R}^{n_0}$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_0}$. The definition of the spectral projector $P_\pm^{\odot\star}$ can be found in [28, Appendix A]. Substituting these decompositions into (50) yields for the left-hand side of this equation

$$\begin{aligned}\Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q(\tau, \xi^{(q)})) = &-j\Big(\frac{\partial H_q^{00}(\tau, \xi^{(q)})}{\partial\tau}\dot{\gamma}_\tau + H_q^{00}(\tau, \xi^{(q)})\ddot{\gamma}_\tau\Big)\\ &+ A^{\odot\star}(\tau)j(H_q^{00}(\tau, \xi^{(q)})\dot{\gamma}_\tau)\\ &- j\Big(\frac{\partial\tilde{Q}_0(\tau)}{\partial\tau}\tilde{H}_q^0(\tau, \xi^{(q)}) + \tilde{Q}_0(\tau)\frac{\partial\tilde{H}_q^0(\tau, \xi^{(q)})}{\partial\tau}\Big)\\ &+ A^{\odot\star}(\tau)j(\tilde{Q}_0(\tau)\tilde{H}_q^0(\tau, \xi^{(q)}))\\ &+ \Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q^-(\tau, \xi^{(q)}) + H_q^+(\tau, \xi^{(q)})),\end{aligned}$$

where we twice used the product rule for differentiation. Since $\tau \mapsto \dot{\gamma}_\tau$ is a $T$-periodic eigenfunction we get from Theorem 5 that $(-\frac{d}{d\tau} + A^{\odot\star}(\tau))j\dot{\gamma}_\tau = 0$. Using this in combination with (49), we arrive at

$$\begin{aligned}\Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q(\tau, \xi^{(q)})) = &\, j\Big(-\frac{\partial H_q^{00}(\tau, \xi^{(q)})}{\partial\tau}\dot{\gamma}_\tau\Big)\\ &+ j\Big(\tilde{Q}_0(\tau)\Big(-\frac{\partial}{\partial\tau} + \tilde{M}_0\Big)\tilde{H}_q^0(\tau, \xi^{(q)})\Big)\\ &+ \Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q^-(\tau, \xi^{(q)}) + H_q^+(\tau, \xi^{(q)}))\end{aligned} \tag{51}$$

and this must be equal to the right-hand side of (50). Let us first show existence of $H_q^\pm$ via projecting on the spaces $X_\pm^{\odot\star}(\tau)$. On these subspaces, we get the equation

$$\Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q^\pm(\tau, \xi^{(q)})) = j(D_\xi H_q^\pm(\tau, \xi^{(q)})\tilde{M}_0\xi) - R_q^\pm(\tau, \xi^{(q)}).$$

Substituting $\tau = \theta$ and $\xi = e^{(\theta-\tau)\tilde{M}_0}\xi =: \tilde{\xi}$ leads to

$$-\frac{\partial}{\partial\theta}j(H_q^\pm(\theta, \tilde{\xi}^{(q)})) + A^{\odot\star}(\theta)j(H_q^\pm(\theta, \tilde{\xi}^{(q)})) - j(D_{\tilde{\xi}}H_q^\pm(\theta, \tilde{\xi}^{(q)})\tilde{M}_0\xi) = -R_q^\pm(\theta, \tilde{\xi}^{(q)}).$$

When the operator $-U^{\odot\star}(\tau, \theta)$ acts on both side of the equation, we obtain

$$\begin{aligned}&-U^{\odot\star}(\tau, \theta)\Big[-\frac{\partial}{\partial\theta}j(H_q^\pm(\theta, \tilde{\xi}^{(q)})) + A^{\odot\star}(\theta)j(H_q^\pm(\theta, \tilde{\xi}^{(q)})) - j(D_{\tilde{\xi}}H_q^\pm(\theta, \tilde{\xi}^{(q)})\tilde{M}_0\xi)\Big]\\ &= U^{\odot\star}(\tau, \theta)R_q^\pm(\theta, \tilde{\xi}^{(q)}).\end{aligned} \tag{52}$$

Let us focus on the left-hand-side of this equation. It follows from [5, Theorem 5.5] that

$$-U^{\odot\star}(\tau,\theta)A^{\odot\star}(\theta)j(H_q^{\pm}(\theta,\tilde{\xi}^{(q)})) = -[\partial_\theta^\star U^{\odot\star}(\tau,\theta)]j(H_q^{\pm}(\theta,\tilde{\xi}^{(q)})).$$

Filling this result back into (52) and using the partial weak$^\star$ derivative operator yields

$$U^{\odot\star}(\tau,\theta)[\partial_\theta^\star j(H_q^{\pm}(\theta,\tilde{\xi}^{(q)}))] + [\partial_\theta^\star U^{\odot\star}(\tau,\theta)]j(H_q^{\pm}(\theta,\tilde{\xi}^{(q)})) = U^{\odot\star}(\tau,\theta)R_q^{\pm}(\theta,\tilde{\xi}^{(q)}),$$

where we have used the product rule for differentiation, but essentially in the dual pairings due to the partial weak$^\star$ derivative. Using the product rule again and recalling that $\tilde{\xi} = e^{(\theta-\tau)\tilde{M}_0}\xi$, we get the identity

$$\partial_\theta^\star[U^{\odot\star}(\tau,\theta)j(H_q^{\pm}(\theta,(e^{(\theta-\tau)\tilde{M}_0})^{(q)}))] = U^{\odot\star}(\tau,\theta)R_q^{\pm}(\theta,(e^{(\theta-\tau)\tilde{M}_0})^{(q)}).$$

Using the definition of the weak$^\star$ derivative, we get for all $x^\odot \in X^\odot$ that

$$\frac{\partial}{\partial\theta}\langle j(U(\tau,\theta)H_q^{\pm}(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})),x^\odot\rangle = \langle U^{\odot\star}(\tau,\theta)R_q^{\pm}(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)}),x^\odot\rangle. \tag{53}$$

**Step 4: Existence of $H_q^+$.** Let us first find an expression for $H_q^+(\tau,\xi^{(q)})$. As $X_+(s)$ is finite-dimensional, $U(\tau,s)$ extends to all $\tau,s\in\mathbb{R}$ on the subspace $X_+(s)$. So let $s\geq\tau$ be given and integrate (53) over the interval $[\tau,s]$ to obtain

$$\begin{aligned}\langle j(H_q^+(\tau,\xi^{(q)})),x^\odot\rangle &= \langle j(U(\tau,s)H_q^+(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)})),x^\odot\rangle \\ &\quad - \int_\tau^s \langle U^{\odot\star}(\tau,\theta)R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)}),x^\odot\rangle d\theta.\end{aligned} \tag{54}$$

Let us focus on the first term of the right-hand side. Notice that

$$H_q^+(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)}) = \sum_{|\alpha|=q}\frac{1}{\alpha!}P_+(s)H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)$$

where $H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)\in X$. Then, we get

$$U(\tau,s)H_q^+(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)}) = \sum_{|\alpha|=q}\frac{1}{\alpha!}U(\tau,s)P_+(s)H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)$$

and using the exponential trichotomy property of the forward evolutionary system [28, Hypothesis 1], there is a $b>0$ such that for a given $\varepsilon>0$ there exists a $K_\varepsilon>0$ with the property

$$\|U(\tau,s)H_q^+(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)})\|_\infty \leq K_\varepsilon e^{b(\tau-s)}\sum_{|\alpha|=q}\frac{1}{\alpha!}\|H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)\|_\infty,$$

where the number $N$ from [28, Hypothesis 1] is absorbed in the $K_\varepsilon$ constant. Since the diagonal elements of the matrix $\tilde{M}_0$ have real part zero, $e^{(s-\tau)\tilde{M}_0}\xi$ is a polynomial in $\xi$ and so $\|H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)\|_\infty$ can grow at most polynomially for $s\to\pm\infty$. With this in mind, we get

$$\begin{aligned}|\langle j(U(\tau,s)H_q^+(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)})),x^\odot\rangle| &\leq K_\varepsilon e^{b(\tau-s)}\|x^\odot\|\sum_{|\alpha|=q}\frac{1}{\alpha!}\|H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)\|_\infty \\ &\leq M_\varepsilon e^{b(\tau-s)}\max_{|\alpha|=q}\|H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)\|_\infty \\ &\to 0, \quad \text{as } s\to\infty.\end{aligned}$$

Using this convergence, taking the limit in (54) yields

$$\langle j(H_q^+(\tau,\xi^{(q)})),x^\odot\rangle = \langle \int_\tau^\infty -U^{\odot\star}(\tau,\theta)R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})d\theta, x^\odot\rangle, \tag{55}$$

if we can show that for any $x^\odot \in X^\odot$ and fixed $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_0}$ the map $g_{q,\tau,\xi}^+ : [\tau,\infty) \to \mathbb{R}$ defined by $g_{q,\tau,\xi}^+(\theta) = \langle -U^{\odot\star}(\tau,\theta)R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)}), x^\odot\rangle$ is in $L^1([\tau,\infty),\mathbb{R})$. Let $\tau \in \mathbb{R}, \xi \in \mathbb{R}^{n_0}$ and $x^\odot \in X^\odot$ be given. From [28, Hypothesis 1 and Proposition 18] we get

$$\int_\tau^\infty |g_{q,\tau,\xi}^+(\theta)|d\theta \leq K_\varepsilon N\|x^\odot\|e^{b\tau}\int_\tau^\infty e^{-b\theta}\|R_q(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})\|d\theta.$$

Recall that $e^{(\theta-\tau)\tilde{M}_0}\xi$ is a polynomial in $\xi$ and that $R_q(\tau,\xi^{(q)})$ depends on $G_{q'}(\tau,\xi^{(q')})$ for $2 \leq q' \leq q$ and $j(H_{q'}(\tau,\xi^{(q')}))$, $j(p_{q'}(\tau,\xi^{(q')})\dot{\gamma}_\tau)$ and $j(\tilde{Q}_0(\tau)P_{q'}(\tau,\xi^{(q')}))$ for $q' = 2,\dots,q-1$. Since $G_{q'}$ is periodic in the first variable and evaluated at a polynomial, $G_{q'}$ grows at most polynomially for $2 \leq q' \leq q$. As we can assume that $H_{q'}$ is $T$-periodic in the first variable for $q' = 2,\dots,q-1$ (we will show this later for $q' = q$) and evaluated at a polynomial it follows that $j(H_{q'}(\tau,\xi^{(q')}))$ grows at most polynomially for $q' = 2,\dots,q-1$. As $p_{q'}$ and $P_{q'}$ are $T$-periodic in the first variable for $q' = 2,\dots,q-1$ (we will show this later for $q' = q$), it follows that $j(p_{q'}(\tau,\xi^{(q')})\dot{\gamma}_\tau)$ and $j(\tilde{Q}_0(\tau)P_{q'}(\tau,\xi^{(q')}))$ grow at most polynomially. To conclude, there exists a polynomial $r_{q,\tau,\xi}^+ : \mathbb{R} \to \mathbb{R}$ such that $\|R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})\| \leq r_{q,\tau,\xi}^+(\theta)$ for all $\theta \geq \tau$. Hence,

$$\int_\tau^\infty |g_{q,\tau,\xi}^+(\theta)|d\theta \leq K_\varepsilon N\|x^\odot\|e^{b\tau}\int_\tau^\infty e^{-b\theta}r_{q,\tau,\xi}^+(\theta)d\theta < \infty, \tag{56}$$

because the map $[\tau,\infty) \ni \theta \mapsto e^{-b\theta}g_{q,\tau,\xi}^+(\theta) \in \mathbb{R}$ decays fast enough to zero as $\theta \to \infty$. We have proven that the weak$^\star$ integral in (55) exists. Because $R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)}) \in j(X_+(\theta))$ and (55) holds for any $x^\odot \in X^\odot$, we obtain

$$j(H_q^+(\tau,\xi^{(q)})) = j\int_\tau^\infty -U(\tau,\theta)j^{-1}R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})d\theta.$$

By $\odot$-reflexivity we have that $j$ is an isomorphism on $j(X) = X^{\odot\odot}$ and hence

$$H_q^+(\tau,\xi^{(q)}) = -\int_\tau^\infty U(\tau,\theta)j^{-1}R_q^+(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})d\theta \tag{57}$$

can be evaluated as a standard Riemann integral. It can easily be checked that $H_q^+$ is $T$-periodic in the first variable because $P_+^{\odot\star}$ is $T$-periodic and $R_q$ is $T$-periodic in the first variable. Let us now prove the continuity of the map $H_q^+$. As $U^{\odot\star}(t,\tau)$ restricted to $j(X_+(\tau))$ is invertible, we can adjust the proof from [28, Lemma 1] to prove continuity of the limiting function $v(\tau,\infty,\tau,g)$ (notation from [28, Lemma 1]) for a continuous function $g : [\tau,\infty) \to X^{\odot\star}$ under the assumption that $H_q^+$ is bounded in norm. As it is proved in (56) that $H_q^+$ is bounded in norm and noticing that $P_+^{\odot\star}$ and $R_q$ are continuous for all $q \in \{1,\dots,k\}$, the result follows.

**Step 5: Existence of $H_q^-$.** Now, we can look for an explicit expression of $H_q^-(\tau,\xi^{(q)})$. Integrating (53) over $[s,\tau]$ for a fixed $s \in \mathbb{R}$, yields for any $x^\odot \in X^\odot$, due to the definition of the weak$^\star$ integral

$$\begin{aligned}\langle j(H_q^-(\tau,\xi^{(q)})),x^\odot\rangle &= \langle j(U(\tau,s)H_q^-(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)})),x^\odot\rangle \\ &\quad + \int_\tau^s \langle U^{\odot\star}(\tau,\theta)R_q^-(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)}),x^\odot\rangle d\theta. \end{aligned}\tag{58}$$

Similar to the $H_q^+$-case, we want to show that the first term goes to zero, but now as $s \to -\infty$. Recall that $\|H_s^\alpha((e^{(s-\tau)\tilde{M}_0}\xi)^\alpha)\|_\infty$ can grow at most polynomially for $s \to \pm\infty$ and so due to the exponential trichotomy of the forward evolutionary system [28, Hypothesis 1], there exists an $a < 0$ such that for a given $\varepsilon > 0$ there is a $M_\varepsilon > 0$ with the property

$$
\begin{aligned}
|\langle j(U(\tau,s)H_q^-(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)})),x^\odot\rangle| &\leq M_\varepsilon e^{a(\tau-s)}\max_{|\alpha|=q}\|H_s^\alpha(s,(e^{(s-\tau)\tilde{M}_0}\xi)^{(q)})\|_\infty \\
&\to 0, \quad \text{as } s\to-\infty,
\end{aligned}
$$

where the other constants are already absorbed in $M_\varepsilon$. We conclude that

$$
\langle j(H_q^-(\tau,\xi^{(q)})),x^\odot\rangle = \langle\int_{-\infty}^{\tau} U^{\odot\star}(\tau,\theta)R_q^-(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})d\theta,x^\odot\rangle, \tag{59}
$$

if we are able to show that for any $x^\odot\in X^\odot$ and fixed $\tau\in\mathbb{R}$ and $\xi\in\mathbb{R}^{n_0}$ that the map $g_{q,\tau,\xi}^- : (-\infty,\tau]\to\mathbb{R}$ defined by $g_{q,\tau,\xi}^-(\theta) = \langle U^{\odot\star}(\tau,\theta)R_q^-(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)}),x^\odot\rangle$ is in $L^1((-\infty,\tau],\mathbb{R})$. The exponential trichotomy implies that for a given $\varepsilon>0$ one can find a $K_\varepsilon>0$ such that

$$
\int_\tau^\infty |g_{q,\tau,\xi}^-(\theta)|d\theta \leq K_\varepsilon N\|x^\odot\|e^{a\tau}\int_{-\infty}^{\tau} e^{-a\theta}\|R_q(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})\|d\theta.
$$

From the same reasoning as in the $H_q^+$-case, there exists a polynomial $r_{q,\tau,\xi}^- : \mathbb{R}\to\mathbb{R}$ that satisfies the estimate $\|R_q(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})\|\leq r_{q,\tau,\xi}^-(\theta)$ for all $\theta\leq\tau$. Hence,

$$
\int_\tau^\infty |g_{q,\tau,\xi}^-(\theta)|d\theta \leq K_\varepsilon N\|x^\odot\|e^{a\tau}\int_{-\infty}^{\tau} e^{-a\theta}r_{q,\tau,\xi}^-(\theta)d\theta < \infty, \tag{60}
$$

because the map $\theta\mapsto e^{-a\theta}r_{q,\tau,\xi}^-(\theta)$ decays fast enough to zero as $\theta\to-\infty$. Hence, $g_{q,\tau,\xi}^-\in L^1((-\infty,\tau],\mathbb{R})$ and so the weak$^\star$ integral exists. Since (59) holds for all $x^\odot\in X^\odot$ we get

$$
H_q^-(\tau,\xi^{(q)}) = j^{-1}\int_{-\infty}^{\tau} U^{\odot\star}(\tau,\theta)R_q^-(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})d\theta, \tag{61}
$$

if we can prove that the weak$^\star$ integral takes values in $j(X)$. Notice that we proved in (60) that $H_q^-$ is bounded in norm. With the notation from [28, Lemma 1] we have that $j(H_q^-(\tau,\xi^{(q)})) = v(\tau,\tau,-\infty,g)$ with the continuous map $g$ defined by $g(\theta) = P_-^{\odot\star}(\theta)R_q(\theta,(e^{(\theta-\tau)\tilde{M}_0}\xi)^{(q)})$ for all $\theta\in(-\infty,\tau]$ since $P_-^{\odot\star}$ and $R_q$ are continuous for all $q\in\{1,\dots,k\}$. It follows from [28, Lemma 1] that $H_q$ takes values in $j(X)$ and is continuous. It is not difficult to show that $H_q^-$ is $T$-periodic in the first variable because $P_-^{\odot\star}$ is $T$-periodic and $R_q$ is $T$-periodic in the first variable.

**Step 6: Existence of $H_q^{00}$ and $\tilde{H}_q^0$.** To obtain $H_q^{00}(\tau,\xi^{(q)})$ and $H_q^0(\tau,\xi^{(q)})$, we project (50) onto $\mathbb{R}j\dot{\gamma}_\tau$ and $j(\tilde{X}_0(\tau))$. Since $j$ is an isomorphism on $j(X) = X^{\odot\odot}$ we get from combining (50) and (51) that the coefficients must satisfy

$$
\begin{aligned}
-\frac{\partial H_q^{00}(\tau,\xi^{(q)})}{\partial\tau} - D_\xi H_q^{00}(\tau,\xi^{(q)})\tilde{M}_0\xi &= p_q(\tau,\xi^{(q)}) - R_q^{00}(\tau,\xi^{(q)}),\\
-\frac{\partial\tilde{H}_q^0(\tau,\xi^{(q)})}{\partial\tau} + \tilde{M}_0\tilde{H}_q^0(\tau,\xi^{(q)}) - D_\xi\tilde{H}_q^0(\tau,\xi^{(q)})\tilde{M}_0\xi &= P_q(\tau,\xi^{(q)}) - R_q^0(\tau,\xi^{(q)}).
\end{aligned}
$$

These are precisely the equations obtained in [23, Theorem III.7] and hence from the results of [23, Theorem III.7], the provided normal forms follow. In addition, it is proven in [23, Theorem III.7] that $H_q^{00},\tilde{H}_q^0,p_q,P_q$ are continuous and so we conclude that $H,p$ and $P$ are $C^k$-smooth maps. □

Recall that the map $\tau\mapsto\gamma_\tau$ is $T$-periodic and $C^k$-smooth. Furthermore, from (23) in combination with (44) we have that $\tau\mapsto\tilde{Q}_0(\tau)$ is $T$-periodic and $C^k$-smooth. It also follows from previous theorem that $(\tau,\xi)\mapsto H(\tau,\xi)$ is $T$-periodic in the first component and $C^k$-smooth. Hence, $\mathcal{W}^c_{\text{loc}}(\Gamma)$ can be also written as

$$\mathcal{W}^c_{\text{loc}}(\Gamma)=\{\gamma_\tau+\tilde{Q}_0(\tau)\xi+H(\tau,\xi)\ :\ \tau\in\mathbb{R}\text{ and }\xi\in\mathbb{R}^{n_0}\}\subset X, \tag{62}$$

and has exactly the same properties as the description of $\mathcal{W}^c_{\text{loc}}(\Gamma)$ given in (11). Hence, $\mathcal{W}^c_{\text{loc}}(\Gamma)$ is the center manifold for (DDE) around the periodic orbit $\Gamma$ whenever the Floquet multiplier $\lambda$ fulfills the requirements of Theorem 10. This center manifold is $T$-periodic in the sense that for any $\xi\in\mathbb{R}^{n_0}$ the map $\mathbb{R}\ni\tau\mapsto\gamma_\tau+\tilde{Q}_0(\tau)\xi+H(\tau,\xi)\in X$ is $T$-periodic.

Next we consider the case where the trivial Floquet multiplier has algebraic multiplicity larger than 1 and there is no Floquet multiplier located at $-1$. This is for example the case in the fold bifurcation.

**Theorem 11** (Normal Form II)**.** *Assume that the algebraic multiplicity of the trivial Floquet multiplier is more than one and that $-1$ is not a Floquet multiplier. Then there exist $C^k$-smooth maps $H:\mathbb{R}\times\mathbb{R}^{n_0}\to X, p:\mathbb{R}\times\mathbb{R}^{n_0}\to\mathbb{R}$ and $P:\mathbb{R}\times\mathbb{R}^{n_0}\to\mathbb{R}^{n_0}$ such that the history $x_t\in\mathcal{W}^c_{\text{loc}}(\Gamma)$ may be represented as*

$$x_t=\gamma_\tau+\tilde{Q}_0(\tau)\xi+H(\tau,\xi),\quad t\in I,$$

*where the time dependence of the coordinates $(\tau,\xi)$ describing the dynamics of (DDE) on $\mathcal{W}^c_{\text{loc}}(\Gamma)$ is defined by the normal form*

$$\begin{cases}\dfrac{d\tau}{dt}=1+\xi_1p(\tau,\xi)+\mathcal{O}(|\xi|^{k+1}),\\[2mm] \dfrac{d\xi}{d\tau}=\tilde{M}_0\xi+P(\tau,\xi)+\mathcal{O}(|\xi|^{k+1}).\end{cases}$$

*Here the functions $H,p$ and $P$ are $T$-periodic in $\tau$ and at least quadratic in $\xi$. The $\mathcal{O}$-terms are also $T$-periodic in $\tau$. Moreover, $p$ and $P$ are polynomials in $\xi$ of degree less than or equal to $k$ such that*

$$\frac{d}{d\tau}p(\tau,e^{-\tau\tilde{M}_0^\star}\xi)=0\ \text{and}\ \frac{d}{d\tau}\left(e^{\tau\tilde{M}_0^\star}P(\tau,e^{-\tau\tilde{M}_0^\star}\xi)\right)=0,$$

*for all $\tau\in\mathbb{R}$ and $\xi\in\mathbb{R}^{n_0}$.*

Notice the appearance of the $\xi_1$-term in the normal form description. This is due to the fact that the $\star$ in (45) is now replaced by 1 instead of 0 compared to Theorem 10. The proof of this theorem is very similar to that of Theorem 10.

*Proof of Theorem 11.* We proceed in the same way as the proof of Theorem 10 and start by comparing the expansions of

$$\frac{d}{dt}j(\gamma_\tau+\tilde{Q}_0(\tau)\xi+H(\tau,\xi))=j\left(\dot{\gamma}_\tau+\frac{\partial\tilde{Q}_0(\tau)}{\partial\tau}\xi+\frac{\partial H(\tau,\xi)}{\partial\tau}+\left(\tilde{Q}_0(\tau)+D_\xi H(\tau,\xi)\right)\frac{d\xi}{d\tau}\right)\frac{d\tau}{dt}$$

and

$$A_0^{\odot\star}j(\gamma_\tau+\tilde{Q}_0(\tau)\xi+H(\tau,\xi))+G(\gamma_\tau+\tilde{Q}_0(\tau)\xi+H(\tau,\xi))$$

by substituting

$$\frac{d\tau}{dt}=1+\xi_1+p(\tau,\xi)+\mathcal{O}(|\xi|^{k+1})\ \text{ and }\ \frac{d\xi}{d\tau}=\tilde{M}_0\xi+P(\tau,\xi)+\mathcal{O}(|\xi|^{k+1}).$$

We copy the same notation from the proof of Theorem 10 and use the expansions of $H,p$ and $P$ together with (48) where now $\varphi$ must be substituted by $\tilde{Q}_0(\tau)\xi + H(\tau,\xi)$. Eventually,

$$
\begin{aligned}
& j\Big[\dot{\gamma}_\tau + \frac{\partial \tilde{Q}_0(\tau)}{\partial \tau}\xi + \sum_{q=2}^{k} \frac{\partial H_q(\tau,\xi^{(q)}))}{\partial \tau} + \Big(\tilde{Q}_0(\tau) + \sum_{q=2}^{k} D_\xi H_q(\tau,\xi^{(q)})\Big)\Big(\tilde{M}_0 + \sum_{q=2}^{k} P_q(\tau,\xi^{(q)})\Big)\Big] \\
& \Big(1 + \xi_1 + \sum_{q=2}^{k} p_q(\tau,\xi^{(q)})\Big) + \mathcal{O}(|\xi|^{k+1}) \\
& = A_0^{\odot\star} j(\gamma_\tau) + G(\gamma_\tau) + A^{\odot\star}(\tau) j\Big(\tilde{Q}_0(\tau)\xi + \sum_{q=2}^{k} H_q(\tau,\xi^{(q)})\Big) \\
& + \sum_{q=2}^{k} G_q\Big(\tau, \Big[\tilde{Q}_0(\tau)\xi + \sum_{p=2}^{k} H_q(\tau,\xi^{(p)})\Big]^{(q)}\Big) + \mathcal{O}(|\xi|^{k+1}).
\end{aligned}
$$

Let us now compare the terms in powers of $\xi$ on both side of the equation. The $\xi^0$-terms give us

$$\frac{d}{d\tau} j(\gamma_\tau) = A_0^{\odot\star} j(\gamma_\tau) + G(\gamma_\tau),$$

which means that $\gamma$ is a solution (47). The $\xi^1$-terms tell us

$$\Big(-\frac{\partial}{\partial \tau} + A^{\odot\star}(\tau)\Big) j(\tilde{Q}_0(\tau)\xi) = j((\tilde{Q}_0(\tau)\tilde{M}_0 + \gamma_\tau \Pi_1)\xi), \tag{63}$$

which is exactly the result established in (23), but now for all Floquet multipliers on the unit circle and characterizes the linear part. Here $\Pi_1 : \mathbb{R}^{n_0} \to \mathbb{R}$ is the projection on the first component, defined as $\Pi_1\xi := \xi_1$, where $\xi = (\xi_1, \dots, \xi_{n_0})$. After collecting the $\xi^{(2)}$-terms, we get

$$
\begin{aligned}
& \Big(-\frac{\partial}{\partial \tau} + A^{\odot\star}(\tau)\Big) j(H_2(\tau,\xi^{(2)})) \\
& = j(D_\xi H_2(\tau,\xi^{(2)})\tilde{M}_0\xi + p_2(\tau,\xi^{(2)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)P_2(\tau,\xi^{(2)})) - R_2(\tau,\xi^{(2)}),
\end{aligned}
$$

where $R_2(\tau,\xi^{(2)}) = G_2(\tau,(\tilde{Q}_0(\tau)\xi)^2) - \xi_1(\frac{d\tilde{Q}_0(\tau)}{d\tau}\xi + \tilde{Q}_0(\tau)\tilde{M}_0\xi)$. Finally, after collecting the $\xi^{(q)}$-terms for $q = 3, \dots, k$, we get

$$\Big(-\frac{\partial}{\partial \tau} + A^{\odot\star}(\tau)\Big) j(H_q(\tau,\xi^{(q)})) \tag{64}$$

$$= j(D_\xi H_q(\tau,\xi^{(q)})\tilde{M}_0\xi + p_q(\tau,\xi^{(q)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)P_q(\tau,\xi^{(q)})) - R_q(\tau,\xi^{(q)}), \tag{65}$$

where $R_q(\tau,\xi^{(q)})$ depends on $G_{q'}(\tau,\xi^{(q')})$ for $2 \leq q' \leq q$ and $j(H_{q'}(\tau,\xi^{(q')}))$, $j(p_{q'}(\tau,\xi^{(q')})\dot{\gamma}_\tau)$ and $j(\tilde{Q}_0(\tau)P_{q'}(\tau,\xi^{(q')}))$ for $q' = 2, \dots, q-1$.

We want to project (64) onto the spaces $\mathbb{R}\dot{\gamma}_\tau, \tilde{X}_0(\tau)$ and $X_\pm(\tau)$ to show the existence of $H_q$ separately on each individual space. We decompose for any $q = 2, \dots, k$ the functions $H_q$ and $R_q$ as

$$
\begin{aligned}
H_q(\tau,\xi^{(q)}) &= H_q^{00}(\tau,\xi^{(q)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)\tilde{H}_q^0(\tau,\xi^{(q)}) + H_q^-(\tau,\xi^{(q)}) + H_q^+(\tau,\xi^{(q)}) \\
R_q(\tau,\xi^{(q)}) &= R_q^{00}(\tau,\xi^{(q)})\dot{\gamma}_\tau + \tilde{Q}_0(\tau)\tilde{R}_q^0(\tau,\xi^{(q)}) + R_q^-(\tau,\xi^{(q)}) + R_q^+(\tau,\xi^{(q)}),
\end{aligned}
$$

where $H_q^\pm(\tau,\xi^{(q)}) = P_\pm(\tau)H_q(\tau,\xi^{(q)}) \in X_\pm(\tau)$ and $R_q^\pm(\tau,\xi^{(q)}) = P_\pm^{\odot\star}(\tau)R_q(\tau,\xi^{(q)}) \in X_\pm^{\odot\star}(\tau)$ with coordinates $H_q^{00}(\tau,\xi^{(q)}), R_q^{00}(\tau,\xi^{(q)}) \in \mathbb{R}$ and $\tilde{H}_q^0(\tau,\xi^{(q)}), \tilde{R}_q^0(\tau,\xi^{(q)}) \in \mathbb{R}^{n_0}$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_0}$.

Carrying out the calculations in the same way as the proof of Theorem 11 and recalling that $(-\frac{d}{d\tau} + A^{\odot\star}(\tau))j\dot{\gamma}_\tau = 0$ together with (63), we obtain

$$\begin{aligned}\Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q(\tau,\xi^{(q)})) &= j\Big(-\frac{\partial H_q^{00}(\tau,\xi^{(q)})}{\partial\tau}\dot{\gamma}_\tau + \Pi_1\tilde{H}_q^0(\tau,\xi^{(q)})\dot{\gamma}_\tau\Big)\\ &\quad + j\Big(\tilde{Q}_0(\tau)\Big(-\frac{\partial}{\partial\tau} + \tilde{M}_0\Big)\tilde{H}_q^0(\tau,\xi^{(q)})\Big)\\ &\quad + \Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q^-(\tau,\xi^{(q)}) + H_q^+(\tau,\xi^{(q)}))\end{aligned}$$

and this must be equal to the right-hand side of (64). To obtain the coefficients, we project onto the spaces $\mathbb{R}j\dot{\gamma}_\tau, j(\tilde{X}_0(\tau)), j(X_+(\tau))$ and $X_-^{\odot\star}(\tau)$. This yields the equations

$$\begin{aligned}-\frac{\partial H_q^{00}(\tau,\xi^{(q)})}{\partial\tau}D_\xi H_q^{00}(\tau,\xi^{(q)})\tilde{M}_0\xi &= p_q(\tau,\xi^{(q)}) - \Pi_1\tilde{H}_q^0(\tau,\xi^{(q)}) - R_q^{00}(\tau,\xi^{(q)})\\ -\frac{\partial\tilde{H}_q^0(\tau,\xi^{(q)})}{\partial\tau} + \tilde{M}_0\tilde{H}_q^0(\tau,\xi^{(q)}) - D_\xi\tilde{H}_q^0(\tau,\xi^{(q)})\tilde{M}_0\xi &= P_q(\tau,\xi^{(q)}) - \tilde{R}_q^0(\tau,\xi^{(q)})\\ \Big(-\frac{\partial}{\partial\tau} + A^{\odot\star}(\tau)\Big)j(H_q^\pm(\tau,\xi^{(q)})) &= j(D_\xi H_q^\pm(\tau,\xi^{(q)})\tilde{M}_0\xi) - R_q^\pm(\tau,\xi^{(q)}).\end{aligned}$$

We see that the equations for the $X_\pm^{\odot\star}(\tau)$-component are the same as in the proof of Theorem 10. Hence, we obtain $H_q^\pm$ as in (57) and (61) respectively. To solve the remaining part of this hierarchy of equations, notice these equations are solvable in exactly the same way as the proof of Theorem 10 and the proposed normal forms follow. One should make the observation that $\tilde{H}_q^0$ has to be computed before $H_q^{00}$. □

Under these assumptions on the Floquet multipliers, we have also proven that $\mathcal{W}^c_{\mathrm{loc}}(\Gamma)$ can also be parametrized as (62).

The last normal form theorem is more involved because we have to deal with the Floquet multiplier $-1$ that induces $T$-antiperiodic maps due to Proposition 6. Introduce the decomposition

$$\tilde{X}_0(\tau) = \tilde{X}_0'(\tau) \oplus \tilde{X}_0''(\tau),$$

where $\tilde{X}_0'(\tau)$ is spanned by $T$-periodic maps $\varphi_0(\tau),\dots,\varphi_{n_0'}(\tau)$ and where $\tilde{X}_0''(\tau)$ is spanned by $2T$-periodic maps $\varphi_{n_0'+1}(\tau),\dots,\varphi_{n_0'+n_0''}(\tau)$ and $n_0' + n_0'' = n_0$, corresponding to all (generalized) eigenfunctions of the monodromy operator belonging to the Floquet multiplier $-1$. Define the symmetry $\tilde{S}_0 : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ as

$$(\xi',\xi'') = \xi \mapsto \tilde{S}_0\xi = (\xi',-\xi''),$$

then we have the following theorem, which is for example the case in the period-doubling bifurcation.

**Theorem 12** (Normal form III)**.** *Assume that $-1$ is a Floquet multiplier. Then the results of Theorem 10 or Theorem 11, depending on the location and algebraic multiplicity of other the Floquet multipliers on the unit circle hold with the following modification: the maps $H, p$ and $P$ are $2T$-periodic in the first component and additionally satisfy*

$$H(\tau+T,\xi) = H(\tau,\tilde{S}_0\xi)$$

*and*

$$p(\tau+T,\xi) = p(\tau,\tilde{S}_0\xi), \quad P(\tau+T,\xi) = P(\tau,\tilde{S}_0\xi),$$

*for all $\tau\in\mathbb{R}$ and $\xi\in\mathbb{R}^m$.*

*Proof.* The proof of this theorem is analogous to that of Theorem 10 or Theorem 11 but in a $2T$-periodic setting. Hence, we obtain the results from Theorem 10 or Theorem 11, depending on the location and algebraic multiplicity of the Floquet multipliers on the unit circle where now the maps $H, p$ and $P$ being $2T$-periodic in $\tau$. It remains to show the additional symmetries on the maps $H, p$ and $P$. Because the structure of the parts in the normal form are similar to that of the ODE case, treated in [23, Theorem III.13] this part will be omitted since the proof is identical by making the substitution of $\tau \mapsto \gamma(\tau)$ towards $\tau \mapsto \gamma_\tau$. $\square$

Via this theorem we obtain a $2T$-periodic $(n_0 + 1)$-dimensional $C^k$-smooth manifold $\mathcal{W}^c_{\text{loc}}(\Gamma) \subset X$ that is a center manifold for (DDE) around the periodic orbit $\Gamma$.

# 5 Conclusion and outlook

We have proven that the periodic normal forms to study bifurcations of limit cycles suit naturally in the framework of classical DDEs. This task has been accomplished by proving two principal results: the existence of a periodic smooth basis for the center eigenspace (Theorem 5) and the existence of a special coordinate system on the center manifold (Theorem 10, Theorem 11 and Theorem 12) in which the local dynamics can be described. A paper providing computational formulas to study all codimension one bifurcations of limit cycles in classical DDEs, along the lines of the periodic normalization method [26, 33], is in preparation.

In this paper, we restricted ourselves to the setting of classical DDEs. However, our proof on the existence of a periodic smooth finite-dimensional center manifold near a nonhyperbolic cycle in [28] is given in the general context of dual perturbation theory (sun-star calculus). As a consequence, the results extend to a much broader class of delay equations, such as for example renewal equations [10, 14, 2]. The natural question arises if the results from this paper can also be generalized towards the general context of sun-star calculus. We already see some difficulties in the linear part because the proof on the existence of the periodic smooth (adjoint) (generalized) eigenfunctions rely on Lemma 4 and Lemma 7. Both lemmas are based on the smoothness property of delay differential equations, which is a result that is not present in the general setting of delay equations. Nevertheless, if one requires only weak$^\star$ continuous differentiability of the forward and backward evolutionary systems, we believe that the proven normal form theorems can still be applied (in a weak$^\star$ sense) to the general context of dual perturbation theory, but of course under certain assumptions.

If one is interested in bifurcations of limit cycles for systems consisting of infinite delay [12] or abstract DDEs [24, 25, 32, 31] that describe for example neural fields, it is known that $\odot$-reflexivity is in general lost [32, Theorem 12], and therefore the center manifold theorem for nonhyperbolic cycles from [28] does not directly apply. However, we believe that this technical difficulty can be resolved by employing similar techniques as in [25]. We are convinced that these techniques can also be used to prove the existence of the periodic normal forms in the setting of abstract DDEs and systems consisting of infinite delay because the proof of the periodic normal forms are written in the general setting of sun-star calculus.


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# A Variation-of-constants formula for the adjoint problem

In this section of the appendix, we will prove that solutions of an inhomogeneous perturbed abstract ordinary differential equation are precisely given by an abstract integral equation. This result is important in the proof of Theorem 8.

Let $J \subseteq \mathbb{R}$ be an interval and suppose that $(s,t) \in \Omega_J^\star$. Consider an inhomogeneous perturbation $f : J \to X^\star$ on the generator $A^\star(s)$ to the adjoint problem [5, Equation (5.8)]. This yields the initial value problem

$$\begin{cases} d^\star u(s) = -A^\star(s)u(s) + f(s), & s \leq t, \\ u(t) = \psi, & \psi \in X^\odot, \end{cases} \tag{66}$$

which suggests the variation-of-constants formula

$$u(s) = U^\odot(s,t)\psi + \int_t^s U^\star(s,\tau)f(\tau)d\tau, \quad \psi \in X^\odot, \tag{67}$$

for $t \leq s$, where the integral must be interpreted as a weak$^\star$ integral. This suggestion, with the additional assumptions on $f$, will be verified in this section. Let us first turn our attention towards the weak$^\star$ integral appearing in (67).

**Lemma 13.** *Let $g : J \to X^\star$ be a continuous function and denote the set $\{(s,r,t) \in J^3 \ : \ s \leq r \leq t\}$ by $\Theta_J^\star$. Then the map $v(\cdot,\cdot,\cdot,g) : \Theta_J^\star \to X^\star$ defined as the weak$^\star$ integral*

$$v(s,r,t,g) := \int_t^s U^\star(r,\tau)g(\tau)d\tau, \quad \forall (s,r,t) \in \Theta_J^\star,$$

*is continuous and takes values in $X^\odot$.*

*Proof.* The statement of the theorem is a dual version of the first part of [28, Lemma 1]. The proof is along the same lines as the proof in [28, Lemma 1], where one just has to work with the backward evolutionary system $U^\star$. See also [8, Lemma 3.1] for a semigroup analogue of this lemma. □

As we have proven that the weak$^\star$ integral, appearing in (67) is well-defined, we can turn our attention towards the verification of the variation-of-constants formula. The proof is inspired from [28, Proposition 37] and [25, Proposition 21].

**Proposition 14.** *Let $f : J \to X^\star$ be a continuous function. If $u$ is a solution of (66) on $J$ then $u$ is given by (67).*

*Proof.* Let $(s,t) \in \Omega_J^\star$ with $t > s$ be arbitrary. Define the function $w : [s,t] \to X^\star$ as $w(\tau) := U^\star(s,\tau)u(\tau)$ for all $\tau \in [s,t]$. We claim that $w$ is weak$^\star$ differentiable with weak$^\star$ derivative

$$d^\star w(\tau) = U^\star(s,\tau)d^\star u(\tau) + U^\star(s,\tau)A^\star(\tau)u(\tau). \tag{68}$$

To show this claim, let $\tau \in [s,t]$ and $x \in X$ be given. For any $h \in \mathbb{R}$ such that $\tau + h \in [s,t]$ we have that

$$\begin{aligned} \langle w(\tau+h) - w(\tau), x\rangle &= \langle U^\star(s,\tau+h)u(\tau+h) - U^\star(s,\tau)u(\tau), x\rangle \\ &= \langle U^\star(s,\tau+h)[u(\tau+h) - u(\tau)], x\rangle + \langle [U^\star(s,\tau+h) - U^\star(s,\tau)]u(\tau), x\rangle \\ &= \langle u(\tau+h) - u(\tau), U(\tau+h,s)x\rangle + \langle [U^\star(s,\tau+h) - U^\star(s,\tau)]u(\tau), x\rangle \end{aligned}$$

Because $U$ is a strongly continuous forward evolutionary system, we have that $U(\tau+h,s)x \to U(\tau,s)x$ in norm as $h \to 0$. The definition of the weak$^\star$ derivative implies

$$\frac{1}{h}(u(\tau+h) - u(\tau)) \to d^\star u(\tau) \quad \text{weakly}^\star \text{ as } h \to 0,$$

if we can show that the difference quotients remains bounded in the limit. Since $u$ is a solution to (66), we know that $u$ is weak$^\star$ continuously differentiable and so locally Lipschitz continuous by [25, Remark 16]. Because $[s,t]$ is compact, $u$ is Lipschitz continuous on $[s,t]$ and so the difference quotient remains bounded in the limit. Combining these two facts, we get

$$\frac{1}{h}\langle u(\tau+h)-u(\tau), U(\tau+h,s)x\rangle \to \langle d^\star u(\tau), U(s,\tau)x\rangle \quad \text{as } h \to 0.$$

Furthermore, since $u(\tau) \in \mathcal{D}(A^\star(\tau)) = \mathcal{D}(A_0^\star)$, it follows from [5, Theorem 5.7] that

$$\frac{1}{h}\langle [U^\star(s,\tau+h)-U^\star(s,\tau)]u(\tau), x\rangle \to \langle U^\star(s,\tau)A^\star(\tau)u(\tau), x\rangle \quad \text{as } h \to 0.$$

Hence, we get

$$\frac{1}{h}\langle w(\tau+h)-w(\tau), x\rangle \to \langle U^\star(s,\tau)d^\star u(\tau) + U^\star(s,\tau)A^\star(\tau)u(\tau), x\rangle \quad \text{as } h \to 0,$$

which proves (68). Substituting the differential equation from (66) into (68) yields

$$d^\star w(\tau) = U^\star(s,\tau)f(\tau)$$

and so $d^\star w$ is weak$^\star$ continuous since $f$ was assumed to be norm continuous. For every $x \in X$ we have

$$\langle u(s) - U^\star(s,t)u(t), x\rangle = \langle w(s), x\rangle - \langle w(s), x\rangle = \int_t^s \langle d^\star w(\tau), x\rangle d\tau = \langle \int_t^s U^\star(s,\tau)f(\tau)d\tau, x\rangle.$$

Since $x$ and $(s,t) \in \Omega_J^\star$ were arbitrary, we conclude that

$$u(s) - U^\star(s,t)u(t) = \int_t^s U^\star(s,\tau)f(\tau)d\tau,$$

or equivalently

$$u(s) = U^\star(s,t)\psi + \int_t^s U^\star(s,\tau)f(\tau)d\tau,$$

since $u(t) = \psi$ by assumption. The continuity and range of $f$ ensures from Lemma 13 that the weak$^\star$ integral takes values in $X^\odot$. Since $\psi \in X^\odot$, we have that

$$u(s) = U^\odot(s,t)\psi + \int_t^s U^\star(s,\tau)f(\tau)d\tau,$$

which completes the proof. □

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