Proyecciones Journal of Mathematics
Vol. 41, N^o 6, pp. 1251-1271, December 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5017

Periodic parabolic problem with discontinuous coefficients: Mathematical analysis and numerical simulation

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Received : July 2021. Accepted : November 2021

Abstract

This work presents a new approach for the mathematical analysis and numerical simulation of a class of periodic parabolic equations with discontinuous coefficients. Our technique is based on the minimization of a least squares cost function. By the means of variational calculus, we prove that the considered optimization problem admits an optimal solution. Using the Lagrangian method, we compute the gradient of the cost function associated with our problem. Finally, we give several numerical simulations that show the efficiency and robustness of our method.

Subjclass: *35B10, 35K55, 35K59.*

Keywords: *Weak periodic solutions, discontinuous coefficients, optimization, Lagrangian.*

1. Introduction

During the last forty years, a large number of researchers have been interested in the study of partial differential equations, their numerical simulations and their applications. A particular interest has been shown for equations with nonlinear terms. Several methods have been developed to answer different questions about the considered solutions. In particular, questions of existence, uniqueness, regularity, stability, asymptotic behavior and numerical simulation. For more details, we refer the reader to see the works [1, 2, 3, 4, 5, 16, 17, 20, 28].

In this work, we are concerned with a periodic parabolic equation with discontinuous coefficients modeled as

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x)\nabla u) = f(t,x) & \text{in } \ Q_T \\ u(0,\cdot) \ = \ u(T,\cdot) & \text{in } \ \Omega \\ u(t,x) = 0 & \text{on } \ \Sigma_T, \end{cases} \tag{1.1}$$

where Ω is an open regular bounded subset of $\mathbf{R}^N$, with smooth boundary $\partial\Omega$, $T>0$ is the period, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, f is a measurable function, periodic in time with period T and belonging to certain Lebesgue space and $A(t,x) = (a_{ij}(t,x))_{1\le i,j\le N}$ is a periodic bounded matrix.

A large literature exists on periodic equations and several researchers have been interested in the subject, we quote in particular [8, 6, 11, 12, 13, 15, 14, 18, 19, 23, 24]. We start by referring the readers to see the book [24] for a major and comprehensive introduction to periodic parabolic equations with regular data. Amann has also been interested in the same subject with regular data. In his work [7], the author proved the existence of classical periodic solution via the method of sub and super solution. In [25], Lions studied the well-posedness of weak solutions of a class of periodic parabolic equations involving Leray-Lions type operators. He used the theory of maximal monotone operators to prove the existence, uniqueness and regularity properties of the solutions. Deuel and Hess in D-H were interested in the quasilinear case with a critical growth nonlinearity with respect to the gradient. They established the existence and regularity property of a weak solution by using the techniques of sub- and super-solutions. The work [18] was generalized by Alaa et al in their paper [11], the authors examined the existence of a weak solution to a nonlinear parabolic equation with L^1 data. They combined the truncation method with the sub-and super-solution techniques to obatin SOLA solution (Solution Obtained as the Limit of Approximation). However, there are also quite a few papers that are devoted to simulate numerically the periodic solutions for parabolic boundary problems, we refer the readers to see [10, 26, 27]. To detail the discussion, let us start with the work of Carasso [10], where the author used the least squares method to numerically simulate periodic solutions. When the time period is unknown, Lust et al presented an iterative construction scheme for the periodic solutions of an ordinary differential system. Another approach is given in [27] where the authors formulated the problem (1.1) as an evolution equation in a suitable Banach space and showed the existence of a periodic solution via semigroup theory and fixed point theorems. They studied a nonlinear heat conduction problem and used Newton's method for the numerical simulation of the periodic solutions. Note that all of the above works are concerned with numerical simulations for periodic parabolic equations with continuous coefficient. In this work, we develop a new approach able to numerically construct the periodic solution of (1.1). Our method is based on the formulation of the periodic problem (1.1) into a minimization problem associated with a least squares cost function. We prove that the optimization problem is well posed in an appropriate space of admissible functions. Then, we use Lagrange's method to explicitly compute the derivative of the considered cost function through an intermediate state called adjoint equation. Thus, the derivative of the cost function allows us to develop an iterative algorithm to numerically simulate the considered optimization problem.

The rest of our paper is structured as follows: In Section 2, we introduce the necessary assumptions and state the definition of the weak periodic solution of the problem (1.1). In Section 3, we first formulate the existence problem (1.1) into an equivalent optimization problem by means of a least square cost function. Then, we prove the existence of an optimal solution to the optimization problem in an appropriate admissible space. Subsequently, we use the Lagrange method to compute the derivative of the cost function with respect to the state variable. Section 4 is devoted to the discretization of our finite element problem and the presentation of the proposed numerical algorithm to solve the optimization problem. In Section 5, we give some numerical examples to illustrate the efficiency of the proposed approach.

2. Mathematical Preliminaries and definitions

We start initially this section by introducing necessary assumptions to solve (1.1).

2.1. Assumptions

Throughout this paper, we assume that $A = (a_{ij})_{1\le i,j\le N}$ is a periodic bounded elliptic matrix, namely

(H_1) $a_{ij} \in L^\infty(Q_T)$, for all $1 \le i, j \le N$ and periodic with period T.

(H_2) there exists $\alpha > 0$ such that

$$A(t,x)\xi \cdot \xi \ge \alpha |\xi|^2,$$

for all $\xi \in \mathbf{R}^N$, for almost every $(t,x) \in Q_T$.

(H_3) f is a measurable function periodic with period T and belonging to $L^2(Q_T)$.

2.2. Functional framework and definition

Let us introduce the functional framework involving our work, we set

$$\mathcal{V}_T := L^2(0,T; H_0^1(\Omega)),$$

we equipped with the following norm

$$\|u\|_{\mathcal{V}_T} := \left(\int_{Q_T} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Furthermore, we set

$$\mathcal{V}_T^* := L^2(0,T; H^{-1}(\Omega)),$$

the dual space of $\mathcal{V}_T$. The above spaces lead to define the following functional space

$$\mathcal{W}_T := \{u \in \mathcal{V}_T, \quad \partial_t u \in \mathcal{V}_T^*\},$$

we equipped with the following norm

$$\|u\|_{\mathcal{W}_T} := \|u\|_{\mathcal{V}_T} + \|\partial_t u\|_{\mathcal{V}_T^*}.$$

In what follows, we will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Let us define the notion of a weak periodic solution which we will be used in the resolution of problem (1.1).

Definition 1. *A measurable function $u : Q_T \to \mathbf{R}$ is said to be a weak periodic solution to (1.1) if it satisfies*

$$
\begin{aligned}
&u \in \mathcal{W}_T, \quad u(0,x) = u(T,x) \text{ in } L^2(\Omega), \\
&\textstyle\int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} A(t,x)\nabla u.\nabla\varphi = \int_{Q_T} f\varphi,
\end{aligned} \tag{2.1}
$$

for every test function $\varphi \in \mathcal{V}_T$.

Remark 2.1. *In accordance with assumptions (H_1), (H_2) and (H_3), we can easy to verify that all the terms in (2.1) are well defined. Moreover, by employing the following continuous embedding*

$$\mathcal{W}_T \hookrightarrow \mathcal{C}([0,T]; L^2(\Omega)),$$

we deduce that the periodic condition makes a sense in Definition 1.

From a theoretical point of view, the existence and uniqueness of a weak periodic solution to problem (1.1) can be obtained by using monotone operators theory see Theorem 1.1 p.316 of [25]. Here, we propose a method based on the minimization of a cost function because this will help us to build an algorithm to simulate numerically our periodic solution.

3. Statement of the minimization problem

The purpose of this section is to formulate the theoretical question about the existence of a weak periodic solution to (1.1) into a research of a minimum of a well-posed optimization problem. To deal with this, we consider a cost function of the least-squares type defined as follows

$$\mathcal{J}(v) = \frac{1}{2}\int_\Omega (u(T,x) - v(x))^2\, dx, \tag{3.1}$$

where u is the weak solution to the following initial problem

$$
\begin{cases}
\partial_t u - \operatorname{div}(A(t,x)\nabla u) = f(t,x) & \text{in } Q_T \\
u(0,x) = v(x) & \text{in } \Omega \\
u(t,x) = 0 & \text{on } \Sigma_T.
\end{cases} \tag{3.2}
$$

We recall that for any $v \in L^2(\Omega)$, problem (3.2) has a unique weak solution u which satisfying the following variational formulation

$$
\begin{aligned}
&u \in \mathcal{W}_T, \quad u(0,x) = v(x) \text{ in } L^2(\Omega), \\
&\textstyle\int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} A(t,x)\nabla u.\nabla\varphi = \int_{Q_T} f\varphi,
\end{aligned} \tag{3.3}
$$

for all $\varphi \in \mathcal{V}_T$. Note that the existence and uniqueness of the weak solution to (3.2) can be directly obtained by applying the result of Theorem 1.2 p. 162 in [25]. Therefore, we can deduce that the cost function $\mathcal{J}$ is well-defined. In accordance with the above discussion, we introduce the minimization problem as follows

$$\begin{cases} \text{Find } v^* \in \mathcal{U}_{ad} \\ \mathcal{J}(v^*) = \min\limits_{v \in \mathcal{U}_{ad}} \mathcal{J}(v), \end{cases} \tag{3.4}$$

where $\mathcal{U}_{ad}$ is the set of admissible functions which will be detailed later. It is clear that when the cost function $\mathcal{J}$ converges to zero we obtain that u is the weak periodic solution of (1.1). Hence, we can easy to verify that the minimum of $\mathcal{J}$ on $\mathcal{U}_{ad}$ it is only the weak periodic to (1.1), which proves the equivalence between the existence problem (1.1) and the minimization problem (3.4).

3.1. Existence of an optimal solution

We are concerned with the existence of an optimal solution to the minimization problem (3.4). As we can see, the choice of the set $\mathcal{U}_{ad}$ plays a crucial role in the well-posedness of the problem (3.4). Moreover, in view to (3.1) and (3.2), it is imposed to choose $L^2(\Omega)$ as a space of admissible functions, but to get a good compactness result, it is recommended to consider

$$\mathcal{U}_{ad} := \{v \in H^1(\Omega),\ \|v\|_{H^1(\Omega)} \leq C\}, \tag{3.5}$$

where C is a strictly positive constant. We use on $\mathcal{U}_{ad}$ the topology defined by the strong convergence in $L^2(\Omega)$.

Theorem 1. *Assume that* (H_1), (H_2) *and* (H_3) *hold true. Then, the optimization problem (3.4) has at least one solution in* $\mathcal{U}_{ad}$.

Proof. From Rellich-Kondrachov injection [9], we have

$$\mathcal{U}_{ad} \overset{\text{compact}}{\hookrightarrow} L^2(\Omega).$$

Hence, the existence of an optimal solution to (3.4) requires to check the continuity of the cost function $\mathcal{J}$ in $L^2(\Omega)$. To do this, let (v_n) a sequence in $L^2(\Omega)$ such that (v_n) converges to v strongly in $L^2(\Omega)$. We shall prove that $\mathcal{J}(v_n)$ converges to $\mathcal{J}(v)$. Let us recall that

$$\mathcal{J}(v_n) = \frac{1}{2}\int_\Omega (u_n(T,x) - v_n(x))^2\,dx, \tag{3.6}$$

where u_n is the unique weak solution to the following problem

$$\begin{cases} \partial_t u_n - \operatorname{div}(A(t,x)\nabla u_n) = f & \text{in } \ Q_T \\ u_n(0,.) \ = v_n & \text{in } \ \Omega \\ u_n = 0 & \text{on } \ \Sigma_T. \end{cases} \tag{3.7}$$

Multiplying the first equation of (3.7) by u_n and integrating over Q_T, one obtains

$$\tfrac{1}{2}\textstyle\int_\Omega |u_n(T)|^2 + \int_{Q_T} A(t,x)\nabla u_n.\nabla u_n = \int_{Q_T} f u_n + \frac{1}{2}\int_\Omega |v_n|^2. \tag{3.8}$$

Thanks to the coercivity condition (H_2) and by applying Hölder's inequality, the relation (3.8) becomes

$$\alpha\|u_n\|^2_{\mathcal{V}_T} \le \|f\|_{L^2(Q_T)}\|u_n\|_{L^2(Q_T)} + \|v_n\|^2_{L^2(\Omega)}. \tag{3.9}$$

Since (v_n) convergences strongly in $L^2(Q_T)$, one may deduce that is it bounded in $L^2(Q_T)$. Furthermore, by using Young's inequality in the right-hand side of (3.9), we conclude that (u_n) is bounded in $\mathcal{V}_T$. On the other hand, using the equation satisfied by (u_n) and the growth conditions (H_1), we can obtain that $(\partial_t u_n)$ is bounded $\mathcal{V}_T^*$. Then, a direct application of Aubin compactness Theorem (see e.g [25]) permit us to deduce the existence of $u \in \mathcal{V}_T$ and a subsequence of (u_n) still denoted by (u_n) for simplicity such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.$$

Hence, by applying the last convergences, it comes that

$$\begin{array}{ll} u_n & \rightharpoonup u \text{ weakly in } \mathcal{V}_T, \\ \partial_t u_n & \rightharpoonup \partial_t u \text{ weakly in } \mathcal{V}_T^*. \end{array}$$

By passing to the limit in the weak formulation of (3.7), one gets

$$\textstyle\int_0^T \langle \partial_t u, \varphi\rangle + \int_{Q_T} A(t,x)\nabla u.\nabla\varphi = \int_{Q_T} f\varphi. \tag{3.10}$$

Which proves that u is a weak solution to the problem (3.3). On the other hand, using the uniqueness of the weak solution to (3.3), one may deduce that

$$\lim_{n\to\infty} \mathcal{J}(v_n) = \mathcal{J}(v).$$

Which is equivalent to say that $\mathcal{J}$ is continuous on $L^2(\Omega)$. Furthermore, a simple application of the calculus of variations theory [21] permits us to deduce the existence of an optimal solution to (3.4). □

3.2. Calculus of the derivative of the cost function

Our numerical approach requires the utilization of the gradient of the cost function $\mathcal{J}$. So, we are concerned in this paragraph by the calculation of the derivative of $\mathcal{J}$. To deals with this, we will use the Lagrangian method which gives a rapid derivative of $\mathcal{J}$. The principle of this method is based on the construction of a functional $\mathcal{L}$ called the Lagrangian. The role of the latter is to separate the dependence of the direct state variables (u) with the variable to optimize (v) this through the introduction of a secondary equation called the adjoint state.

Theorem 2. *Under the assumptions* (H_1), (H_2) *and* (H_3) *the cost function* $\mathcal{J}$ *is differentiable on* $L^2(\Omega)$. *Furthermore, for all* $\eta \in L^2(\Omega)$ *we have*

$$\mathcal{J}'(v).\eta = \int_\Omega \Big(v - u(T) - p(0) \Big) \eta, \tag{3.11}$$

with u *is the solution of the state equation (3.2) and* p *is the solution of the following adjoint equation*

$$\begin{cases} \partial_t p + div(A^*(t,x)\nabla p) = 0 & in\ Q_T \\ p(T) = v - u(T) & in\ \Omega \\ p = 0 & in\ \Sigma_T, \end{cases} \tag{3.12}$$

where A^* *is the transpose matrix of* A.

Proof. In order to establish the result of Theorem 2, we introduce the Lagrangian $\mathcal{L}$ for all $(u,p,v,\sigma) \in \mathcal{W}_T \times \mathcal{W}_T \times L^2(\Omega) \times L^2(\Omega)$ as follows

$$\begin{aligned} \mathcal{L}(u,p,v,\sigma) := \ & \tfrac{1}{2} \textstyle\int_\Omega (u(T)-v)^2 + \int_0^T \langle \partial_t u, p \rangle + \int_{Q_T} A(t,x)\nabla u.\nabla p \\ & - \textstyle\int_{Q_T} fp + \int_\Omega \sigma(u(0) - v). \end{aligned}$$

Note that the expression of the function σ will be fixed later to get the initial boundary condition for the adjoint equation. To obtain the adjoint equation, we derive the Lagrangian $\mathcal{L}$ with respect to u, for all direction $\varphi \in \mathcal{W}_T$, we have

$$\left\langle \frac{\partial \mathcal{L}}{\partial u}, \varphi \right\rangle = \int_\Omega \varphi(T)(u(T)-v) + \int_0^T \langle \partial_t \varphi, p \rangle + \int_{Q_T} A(t,x)\nabla\varphi.\nabla p + \int_\Omega \sigma\varphi(0).$$

After integration by part, one obtains

$$\begin{aligned} \left\langle \frac{\partial \mathcal{L}}{\partial u}, \varphi \right\rangle = & \int_\Omega \varphi(T)(u(T)-v) - \int_0^T \langle \partial_t p, \varphi \rangle + \int_\Omega \Big(p(T)\varphi(T) - p(0)\varphi(0) \Big) \\ & - \int_0^T \langle div(A^*(t,x)\nabla p, \varphi \rangle + \int_\Omega \sigma\varphi(0), \end{aligned} \tag{3.13}$$

where A^* is the transpose matrix of A. By taking φ with compact support in (3.13), we get the following equation

$$\partial_t p + \operatorname{div}(A^*(t,x)\nabla p) = 0 \text{ in } Q_T. \tag{3.14}$$

It remains to obtain the initial condition for the adjoint state. To deal with this we take $\sigma = p(0)$ in (3.13), one may deduce that

$$p(T) = v - u(T) \text{ in } \Omega. \tag{3.15}$$

In accordance with (3.14)-(3.15), we conclude that the adjoint equation is given by the following problem

$$\begin{cases} \partial_t p + \operatorname{div}(A^*(t,x)\nabla p) = 0 & \text{in } Q_T \\ p(T) = v - u(T) & \text{in } \Omega \\ p = 0 & \text{in } \Sigma_T. \end{cases}$$

Let us derive the Lagrangian $\mathcal{L}$ with respect to v, for a direction $\eta \in L^2(\Omega)$ one gets

$$\left\langle \frac{\partial \mathcal{L}}{\partial v}, \eta \right\rangle = -\int_\Omega (u(T) - v)\eta - \int_\Omega p(0)\eta = \int_\Omega \left(v - u(T) - p(0) \right) \eta.$$

In addition, to obtain the derivative of the cost function $\mathcal{J}$, we take u as the solution of the state equation (3.3), we obtain

$$\mathcal{L}(u, p, v, \sigma) = \mathcal{J}(v).$$

We therefore have

$$\mathcal{J}'(v).\eta = \int_\Omega \left(v - u(T) - p(0) \right) \eta,$$

where $p(0)$ is the solution of the adjoint equation (3.12) evaluated at the instant $t = 0$ and $u(T)$ is the solution of the state equation (3.2) at the final time T. $\square$

4. The finite element approximation

Throughout this section we assume that Ω is a bounded convex N-polyhedron, that is a bounded interval if $N = 1$, a convex polygon if $N = 2$ and a convex polyhedron if $N = 3$. For $h > 0$, we consider $\mathcal{T}_h$ a regular triangulation of Ω which covers Ω exactly. The $P1$ finite element space is

$$V_h = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega}), v_h \text{ is affine on every N-simplex of } \mathcal{T}_h \right\}.$$

The space V_h is a finite dimensional subspace of $V = H^1(\Omega)$. The finite element approximation of problem (3.4) reads:

$$\left\{ \begin{array}{l} \text{Find } v_h^* \in \mathcal{U}_{ad}^h \\ \mathcal{J}_h(v_h^*) = \min\limits_{v_h \in \mathcal{U}_{ad}^h} \mathcal{J}_h(v_h), \end{array} \right. \tag{4.1}$$

where $\mathcal{U}_{ad}^h := \{v_h \in V_h,\ \|v_h\|_{H^1(\Omega)} \leq C\}$ is the set of admissible functions and

$$\mathcal{J}_h(v_h) = \frac{1}{2} \int_\Omega (u_h(T,x) - v_h(x))^2 dx, \tag{4.2}$$

with u_h is the solution to the following initial problem

$$\left\{ \begin{array}{l} u_h(0,x) = v_h(x) \quad a.e.\ x \in \Omega \\ \forall t \in]0,T[, \forall \phi_h \in V_h : \\ \dfrac{d}{dt} \displaystyle\int_{\mathcal{T}_h} u_h(t,x)\phi_h(x) + \int_{\mathcal{T}_h} A(t,x)\nabla u_h(t,x).\nabla\phi_h(x)dx = \int_{\mathcal{T}_h} f(t,x)\phi_h(x)dx. \end{array} \right. \tag{4.3}$$

According to the previous paragraph, the expression of the differential of $\mathcal{J}_h$ is given by:

$$D\mathcal{J}_h(v_h)(x) = v_h(x) - p_h(0,x) - u_h(T,x), \tag{4.4}$$

where p_h is a solution of the adjoint model:

$$\left\{ \begin{array}{l} p_h(T,x) = v_h(x) - u_h(T,x) \quad a.e.\ x \in \Omega \\ \forall t \in]0,T[, \forall \phi_h \in V_h : \\ \dfrac{d}{dt} \displaystyle\int_{\mathcal{T}_h} p_h(t,x)\phi_h(x) - \int_{\mathcal{T}_h} A^*(t,x)\nabla p_h(t,x).\nabla\phi_h(x)dx = 0, \end{array} \right. \tag{4.5}$$

where u_h is the solution of (4.3).

5. Numerical simulations

We performed numerical simulations with the software FreeFem ++ ([22]) in two spatial dimensions. Our algorithm reads as follows, for a bounded domain Ω of $\mathbf{R}^2$ with smooth boundary and fix $\mu > 0$ a step of descent (see Algorithm **??**).

Algorithm 1

Input: a mesh $\mathcal{T}_h$ which gives a triangulation of Ω_h (a polygonal approximation of Ω) and an initial estimate $u_0^0 \in V_h$ (for example a constant C_0). Compute $\mathcal{J}_h^0 = \mathcal{J}_h(u_0^0)$ For $k = 0, ..., k_{max} - 1$; Solve the state equation

$$\begin{cases} u_h^k(0,x) = u_0^k(x) \quad a.e.\, x \in \Omega, \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h} u_h^k(t,x)\phi_h(x) + \int_{\mathcal{T}_h} A(t,x)\nabla u_h^k(t,x).\nabla\phi_h(x)dx = \int_{\mathcal{T}_h} f(t,x)\phi_h(x)dx, \\ \forall t \in]0,T[, \forall \phi_h \in V_h. \end{cases} \tag{5.1}$$

Compute the value of $u_h^k(T,x)$; Solve the adjoint equation

$$\begin{cases} p_h^k(T,x) = u_0^k(x) - u_h^k(T,x) \quad a.e.\, x \in \Omega, \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h} p_h^k(t,x)\phi_h(x) - \int_{\mathcal{T}_h} A^*(t,x)\nabla p_h^k(t,x).\nabla\phi_h(x)dx = 0, \\ \forall t \in]0,T[, \forall \phi_h \in V_h. \end{cases} \tag{5.2}$$

Update the new initial function u_0^{k+1} and a new value of J_h by computing

$$u_0^{k+1}(x) = (1-\mu)u_0^n(x) + \mu(p_h^n(0,x) + p_h^k(T,x)).$$
$$\mathcal{J}_h^{k+1} = \mathcal{J}_h(u_0^{k+1}).$$

Output: $u_h^{k_{max}}, \mathcal{J}_h^{k_{max}}$.

We use an implicit method in time to solve the equation (4.3). In the same way, we use an implicit method in time for the resolution of the linear retrograde adjoint equation (4.5).

5.1. A numerical simulation

In order to illustrate our method, we computed the numerical solution obtained on the following two examples:

5.1.1. Example: A radial test case with regular coefficients

We consider now problem (1.1) on the unit disc Ω in $\mathbf{R}^2$

$$\Omega = \left\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 1\right\},$$

with

$$A(x,y) = \frac{1}{\sqrt{1+x^2+y^2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $r = \sqrt{x^2+y^2}$ and

$$u(x,y) = 1 - r^2, \quad f(t,x,y) = 2\frac{2+r^2}{(1+r^2)^{\frac{3}{2}}}.$$

Then u is the exact solution of (1.1) with $T = 1$.

Table 5.1: L^2 error and mesh characteristics for Example 5.1.1

Nb vertices	3633	14003	31564
$h_{\min}$	0.025	0.012	0.008
$h_{\max}$	0.053	0.032	0.019
L^2 error	0.09032	0.09035	0.09037
J_h	$8.719.e-05$	$8.719.e-05$	$8.718.e-05$

In Table 1, we present the L^2 error $= \left\| u - u_h^{kmax} \right\|_{L^2(\Omega_h)}$ and $\mathcal{J}_h = \mathcal{J}_h^{kmax}$ obtained for different value of the mesh size h and for $k_{max} = 100$. The initial guess is taken $u_h^0 = 1$. The Table 1 gives also the number of vertices in the mesh $\mathcal{T}_h$ as well as the minimum and maximum length of the edges of the used triangulation. The solutions corresponding to the initial $u_h^{kmax}(0,\cdot)$ and the final time $u_h^{kmax}(T,\cdot)$ are shown respectively in Figure 1 and Figure 2.

Figure 3 shows objective function $\mathcal{J}_h$ value decreases along with the increase of the iteration number.

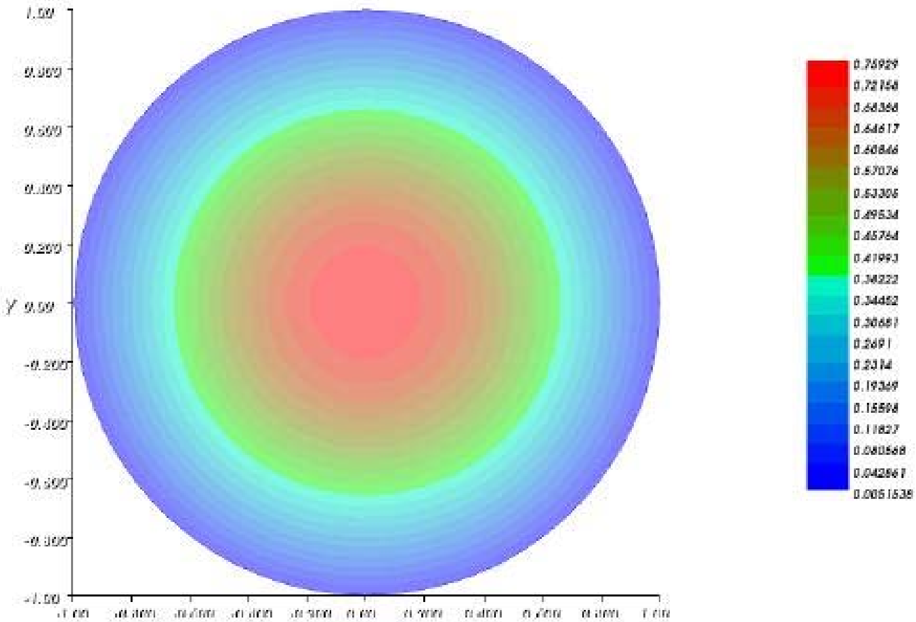

Figure 1: Output initial $u_h^{kmax}(0,\cdot)$.

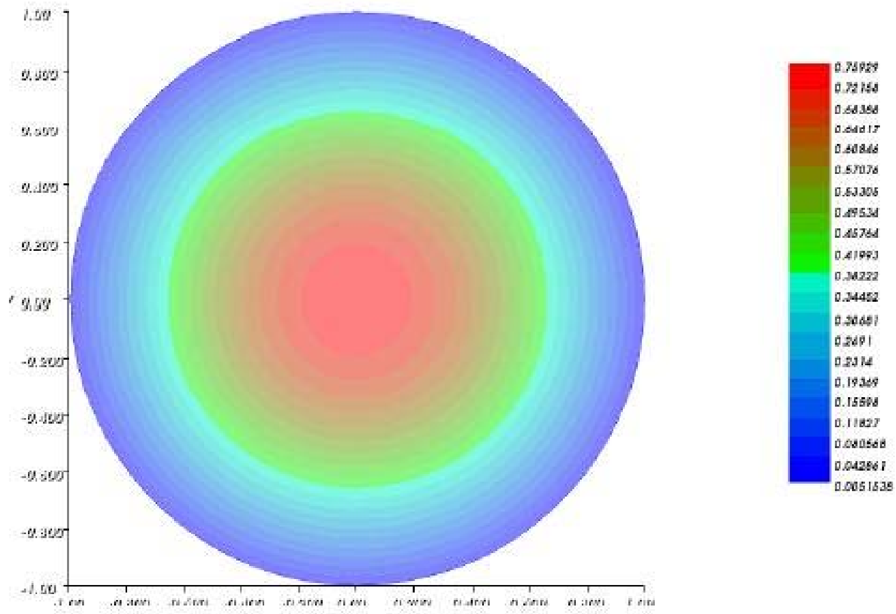

Figure 2: Output initial $u_h^{kmax}(T,\cdot)$.

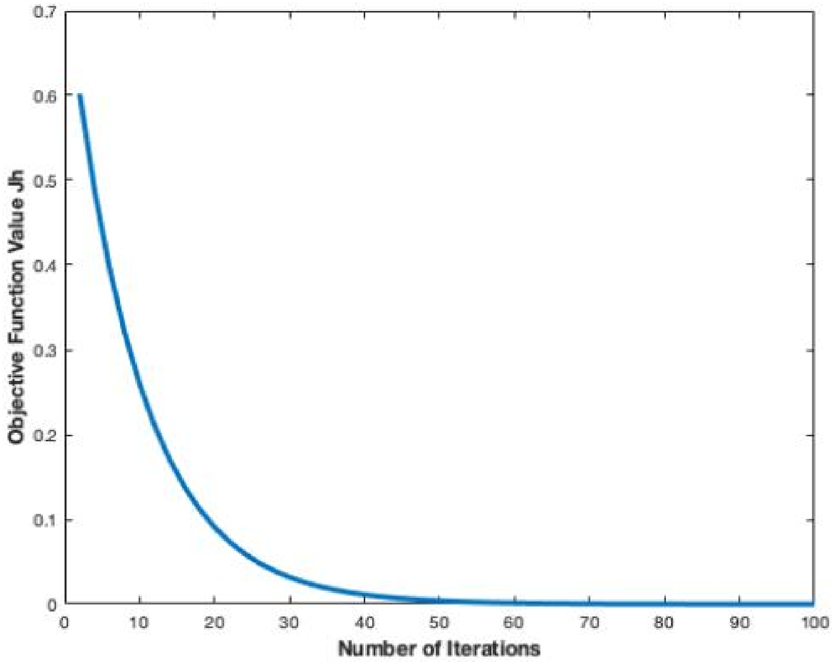

Figure 3: The decrease of the objective function $\mathcal{J}_h$ according to the number of iterations.

5.1.2. Example: Numerical simulation with a discontinuous matrix

In order to illustrate our method in the case of a discontinuous elliptic matrix, we computed the numerical solution obtained on the unit disc Ω in $\mathbf{R}^2$ for the values

$$A(x,y) = a(x,y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with

$$a(x,y) = \begin{cases} 0.2 & \text{if } x^2+y^2 < 0.2^2 \\ a_1(x,y) & \text{if } x^2+y^2 \geq 0.2^2, \end{cases}$$

where

$$a_1(x,y) = \begin{cases} 1.05 - y & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 2.10 - 2x & \text{if } x \geq 0 \text{ and } y < 0 \\ 1.05 + x & \text{if } x < 0 \text{ and } y \geq 0 \\ 2.10 + 2y & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

We note that a is discontinuous accross the circle of radius 0.2 and along the x and axis when $0.2 < |x| < 1$ or $0.2 < |y| < 1$. By taking $u(t,x,y) =$

$cos(\pi t)(1-x^2-y^2)$ and the period $T=2$, one has

$$f(t,x,y)=\begin{cases} \pi * sin(\pi * t) * (1-x^2-y^2) + 0.8 * cos(\pi * t) & \text{if } x^2+y^2 < 0.2^2 \\ f_1(t,x,y) & \text{if } x^2+y^2 \geq 0.2^2, \end{cases}$$

where

$$f_1(t,x,y)=\begin{cases} -\pi * sin(\pi * t) * (1-x^2-y^2) + 2 * cos(\pi * t) * (2.1 - 3 * y) & \text{if } x \geq 0 \text{ and } y \geq 0 \\ -\pi * sin(\pi * t) * (1-x^2-y^2) + 2 * cos(\pi * t) * (4.2 - 6 * x) & \text{if } x \geq 0 \text{ and } y < 0 \\ -\pi * sin(\pi * t) * (1-x^2-y^2) + 2 * cos(\pi * t) * (2.1 + 3 * x) & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi * sin(\pi * t) * (1-x^2-y^2) + 2 * cos(\pi * t) * (4.2 + 6 * y) & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

Table 5.2: L^2 error and mesh characteristics for Example 5.1.2

Nb vertices	1766	4518	7064
$h_{\min}$	0.055	0.033	0.025
$h_{\max}$	0.010	0.063	0.053
L^2 error	0.090	0.0.095	0.0.0937
J_h	$4.32.e-10$	$4.31.e-10$	$4.3.e-10$

In Table 2, we present the L^2 error $= \left\| u - u_h^{k_{max}} \right\|_{L^2(\Omega_h)}$ and $\mathcal{J}_h = \mathcal{J}_h^{k_{max}}$ obtained for different value of the mesh size h and for $k_{max} = 150$. The initial guess is taken $u_h^0 = 1$. The Table 2 shows also the number of vertices in the mesh T_h as well as the minimum and maximum length of the edges of the triangulation. The corresponding output solution $u_h^{kmax}(0,\cdot)$ and the final time $u_h^{kmax}(T,\cdot)$ are presented respectively in Figure 4 and 5. Figure 6 shows objective function $\mathcal{J}_h$ value decreases along with the increase of the iteration number.

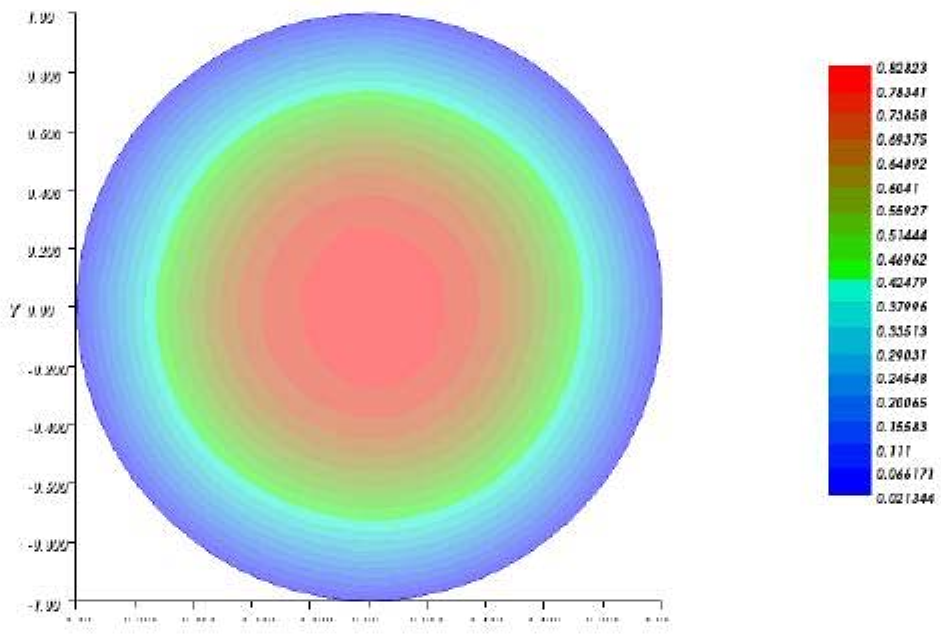

Figure 4: Output initial $u_h^{kmax}(0,\cdot)$.

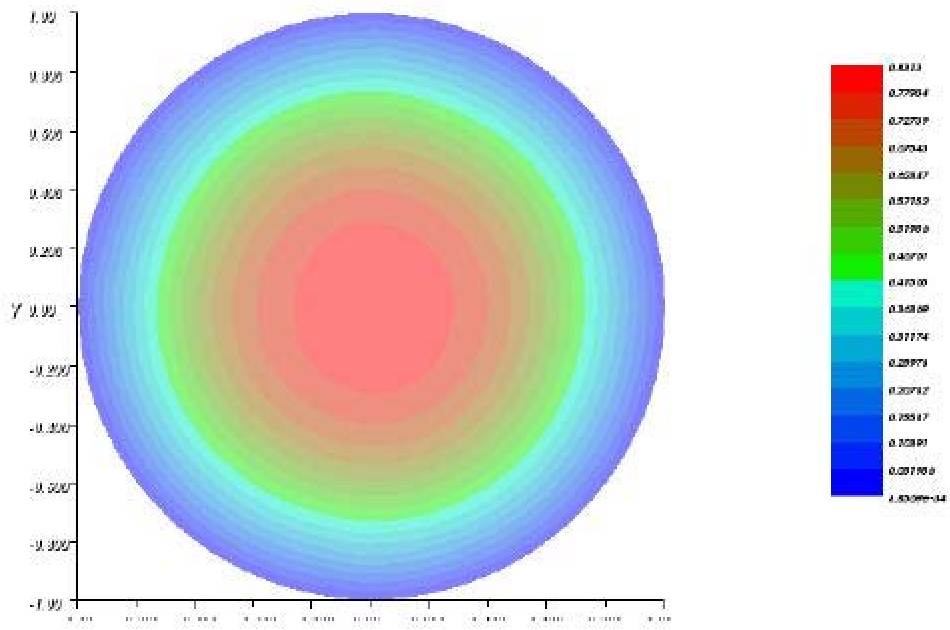

Figure 5: Output initial $u_h^{kmax}(T,\cdot)$.

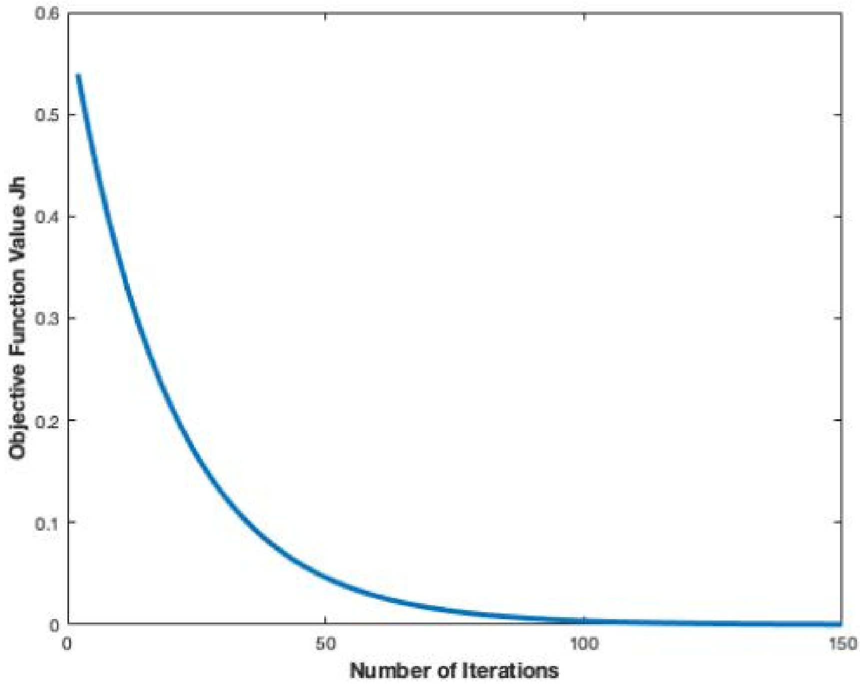

Figure 6: The decrease of the objective function $\mathcal{J}_h$ according to the number of iterations.

Conclusions and Future Works

In this work, we are interested in periodic solutions generated by a differential operator with discontinuous coefficients. Under certain reasonable assumptions, we prove the existence and uniqueness of the positive periodic solution via the minimization of a cost function associated to our problem in an appropriate space. Using a Lagrangian formulation, we computed the gradient of this cost function. This helped us to develop a gradient descent algorithm to compute our periodic solution. After discretizing our equations using a finite element method, we wrote the approximate varitional formulations as well as the numerical algorithm to determine an approximate periodic solution. Finally, we performed several numerical simulations to confirm the results of our approach. In the near future, we will apply this original approach to the search and simulation of periodic solutions for problems with nonlinear boundary conditions.

Acknowledgment

The authors would like to express their sincere gratitude to the anonymous referees and the handling editor for their careful reading of the manuscript.

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