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Periodic points and chaotic functions in the unit interval

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It is shown that the set of chaotic self-maps of the unit interval contains an open dense subset of the space of all continuous self-maps of the unit interval. Other aspects of chaotic behaviour are also considered together with some illustrative examples.

1. Introduction

Let I be the unit interval $[0, 1]$ of the real line and let $C(I)$ be the space of all continuous functions from I into itself with the usual norm. For a point $x \in I$ and a function $T \in C(I)$, the orbit of x is the set

$$O_T(x) = \{x, T(x), T^2(x), \ldots\} \text{ where } T^{n+1}(x) = T\left(T^n(x)\right), \quad n = 1, 2, \ldots .$$

For a natural number p, x is called a periodic point of T of order p if $T^p(x) = x$ and $T^i(x) \neq x$, $i = 1, \ldots, p-1$. In this case $O_T(x)$ will be called a periodic orbit of period p. x is called an eventually periodic point of T if for some natural number p, $x_p = T^p(x)$ is a periodic point of T. x is called an asymptotically periodic point of T if there is a periodic point y of T such that $|T^n(x)-T^n(y)| \to 0$ as $n \to \infty$. To avoid unnecessary repetition, we shall suppress reference to the function T in speaking of periodic points, and so on.

There has been much recent interest in functions $T \in C(I)$ which

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possess orbits which are not asymptotically periodic. The study of such functions has been thought possibly to shed light on the complicated behaviour of models exhibiting turbulence of water flow in which there occur so-called strange attractors consisting of highly irregular trajectories [8]. We refer to [1, 2, 3, 7, 10] for some biological motivations for such study.

The term *chaos* has been coined by a number of authors to describe various situations in which irregular orbital behaviour occurs. Motivated by the definition implicit in the paper of Li and Yorke [6] we shall assume the following definition of *chaotic function:*

$T \in C(I)$ is chaotic if there is an uncountable set S consisting of points which are not asymptotically periodic and such that for every pair $x, y \in S$, $x \neq y$, we have

$$0 = \liminf_{n\to\infty} |T^n(x)-T^n(y)| < \limsup_{n\to\infty} |T^n(x)-T^n(y)| .$$

In this paper we shall examine some of the consequences of this definition, both with regard to conditions sufficient for the existence of chaos and with regard to stability properties of chaotic functions.

2. Periodic points and chaos

Li and Yorke show that if there exists a point of period 3 then T is chaotic. In a remarkable paper Sharkovsky [9] showed that there is an ordering $\prec$ of natural numbers such that if $m \prec n$ and T has a point of period m, then T will have a point of period n. The precise ordering is as follows:

$$3 \prec 5 \prec 7 \prec \ldots \prec 2\cdot 3 \prec 2\cdot 5 \prec \ldots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec \ldots \prec 2^n\cdot 3 \prec 2^n\cdot 5 \prec$$
$$\prec 2^{n+1}\cdot 3 \prec 2^{n+1}\cdot 5 \prec \ldots \prec 2^n \prec 2^{n-1} \prec \ldots \prec 4 \prec 2 \prec 1 ;$$

see also [1]. It is easily seen from the definition that T is chaotic if and only if T^n is chaotic. Combining the results of Li and Yorke and Sharkovsky we have the following:

THEOREM 1. *If $T \in C(I)$ has a point of period $m\cdot 2^n$ for some odd integer $m \geq 3$ and for some natural number n then T is chaotic.*

In view of this result it is natural to ask whether there is a natural number n such that the existence of a periodic point of period 2^n implies chaos. We shall show that this question has a negative answer; indeed we shall construct a function $T \in C(I)$ possessing points of all periods of the form 2^n in which every trajectory is eventually periodic.

EXAMPLE 1. A function $T \in C(I)$ which has points of period 2^n, for each natural number n, and such that all points of I are eventually periodic of period 2^i for some i.

First we shall construct, for each natural number n, a continuous function T_n mapping the interval $I_n = [0, 2^n+1]$ on to itself such that T_n fixes the endpoints of I_n, has points of period 2^n, and all points of I_n are periodic of period 2^i for some i with $0 \le i \le n$.

Let A_k, B_k be defined by

$$A_k = \{1, 2, \ldots, 2^k\}, \quad B_k = \{2^k+1, 2^k+2, \ldots, 2^{k+1}\}, \quad k = 1, 2, \ldots .$$

Define $P_1 : A_1 \to A_1$ by $P_1(1) = 2$, $P_1(2) = 1$, and define $P_{k+1} : A_{k+1} \to A_{k+1}$ by

$$\left.\begin{aligned} P_{k+1}(i) &= i + 2^k \quad \text{if } i \in A_k, \\ P_{k+1}(i) &= P_k(i-2^k) \quad \text{if } i \in B_k, \end{aligned}\right\} \quad k = 1, 2, \ldots .$$

Let $T_k(i) = P_k(i)$ if $i \in A_k$, $T_k(0) = 0$, $T_k(2^k+1) = 2^k + 1$, and define $T_k : [0, 2^k+1] \to [0, 2^k+1]$ by piecewise linear extension. For $k = 1$ we have

$$T_1(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 3-x, & 1 < x \le 2, \\ 2x-3, & 2 < x \le 3. \end{cases}$$

It is easily seen that $3/2$ has period 1; each point of $[1, 2]\setminus\{3/2\}$ has period 2 and all other points are eventually periodic of period 1

or 2 .

Suppose that for the function T_n the points $1, 2, \ldots, 2^n$ are periodic of period 2^n and all points of $[0, 2^n]$ are eventually periodic of period 2^i , $0 \le i \le n$. We shall show that this inductive hypothesis extends to the function T_{n+1} . We observe that $P_{n+1} : A_n \to B_n$, $P_{n+1} : B_n \to A_n$, and that $P^2_{n+1}(i) = P_n(i)$ if $i \in A_n$, $P^2_{n+1}(i) = 2^n + P_n(i-2^n)$ if $i \in B_n$. Thus

$$\left.\begin{aligned} P^{2r}_{n+1}(i) &= P^r_n(i) , & i \in A_n , \\ P^{2r}_{n+1}(i) &= 2^n + P^r_n(i-2^n) , & i \in B_n , \end{aligned}\right\} \quad r = 1, 2, \ldots .$$

It follows that $1, 2, \ldots, 2^{n+1}$ are periodic points of period 2^{n+1} for T_{n+1} , and we see that

$$T^2_{n+1}([m, m+1]) = T_n([m, m+1]) , \quad m = 1, 2, \ldots, 2^n-1 ,$$

$$T^2_{n+1}([m, m+1]) = 2^n + T_n([m-2^n, m+1-2^n]) , \quad m = 2^n+1, \ldots, 2^{n+1}-1 .$$

The inductive assumption then yields that any point of $[m, m+1]$, $m \ne 0$, $m \ne 2^n$, is eventually periodic of period 2^i , $0 \le i \le n+1$. Now

$$T_{n+1}(2^n) = 2^{n+1} , \quad T_{n+1}(2^n+1) = 2^{n-1} + 1 .$$

Let $x \in [2^n, 2^n+1]$. If there is a natural number r such that $T^r_{n+1}(x) \notin [2^n, 2^n+1]$ and $T^{r-1}_{n+1}(x) \in [2^n, 2^n+1]$ then we shall have $T^r_{n+1}(x) \in [m, m+1]$ where m is some integer satisfying $1 \le m < 2^{n+1}$, $m \ne 2^n$. Thus x is eventually periodic of period 2^i , $0 \le i \le n+1$. On the other hand, on $[2^n, 2^n+1]$, T_{n+1} has the form

$$T_{n+1}(x) = \alpha + \lambda(x-\alpha)$$

where α is a fixed point of T_{n+1}, $\lambda > 1$; in fact
$\alpha = 2^n\left(1 + \frac{1}{2^{n+1}-2^{n-1}}\right)$, $\lambda = 1 + 2^{n-1} - 2^{n+1}$. Hence if
$T^r_{n+1}(x_0) \in [2^n, 2^n+1]$, $r = 1, 2, \ldots$, we shall have

$$T^r_{n+1}(x_0) = \alpha + \lambda^r(x_0-\alpha) , \quad r = 1, 2, \ldots ,$$

which implies that $x_0 = \alpha$.

Finally we note that $T_{n+1}([0, 1]) = [0, 2^n+1]$ and
$T_{n+1}[2^{n+1}, 2^{n+1}+1] = [1, 2^{n+1}+1]$, and so we conclude that all points of
$[0, 2^{n+1}+1]$ are eventually periodic of period 2^i, $0 \le i \le n+1$, which completes the inductive construction. Now we can scale $T_1(x)$ down on to $[1/2, 1]$, $T_2(x)$ on to $[1/3, 1/2]$, $T_n(x)$ onto $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, $n = 1, 2, \ldots$, and define T on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ to be the "scaled-down" T_n, $T(0)$ to be 0. Since T_n fixes the endpoints of I_n, T is a well-defined continuous function with the required properties.

3. Chaos and stability

In [4], Kloeden has shown that "near" to any function $F \in C(I)$, there are chaotic functions T; in particular it is possible to find a function $T \in C(I)$ with a point of period 3. A natural question to ask is what happens under small perturbations to a function which has a point of period 3. We shall show by an example that the perturbed function need not have points of period 3. However in Theorem 2 we shall show that every continuous function sufficiently close to a given function with a point of period 3 will be chaotic. Thus the property of being chaotic appears to be stable.

EXAMPLE 2. Period 3 can be destroyed by small perturbations. Let $T : [0, 1] \to [0, 1]$ be defined by

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$$T(x) = \begin{cases} x + 1/2 \ , & x \le 1/2 \ , \\ 2 - 2x \ , & x > 1/2 \ . \end{cases}$$

The orbit $\{0, 1/2, 1\}$ is periodic of period 3. Now consider, for fixed $\varepsilon > 0$, the function T_ε defined by

$$T_\varepsilon(x) = \begin{cases} 1/2 + \varepsilon \ , & 0 \le x \le \varepsilon \ , \\ x + 1/2 \ , & \varepsilon < x \le 1/2 \ , \\ 2 - 2x \ , & x > 1/2 \ . \end{cases}$$

We have that $\|T_\varepsilon - T\| < \varepsilon$ and it is easy to see that T_ε has not any point of period 3.

THEOREM 2. *Let T be a continuous real valued function on I, and suppose T has a point of period 3. Then there exists $\varepsilon > 0$ such that if F is continuous and $\|F-T\| < \varepsilon$ then F has at least one point of period 5.*

Proof. Let (x_1, x_2, x_3) be an orbit of period 3 for T. Suppose $x_1 < x_2 < x_3$ (if $x_1 < x_3 < x_2$ the proof is similar). We have $T(x_1) = x_2$, $T(x_2) = x_3$, $T(x_3) = x_1$. Let $y_1 \in [x_1, x_2]$ be such that $T(y_1) = x_2$ and T has no fixed points in $[y_1, x_2]$. Now consider T^2 : $T^2([y_1, x_2]) \supset T([x_2, x_3]) \supset [x_1, x_3]$. There exists $y_2 \in (y_1, x_2)$ such that $T^2(y_2) = x_2$. We remark that $y_2 \ne y_1$, $y_2 \ne x_2$; in fact $T^2(y_1) = x_3$, $T^2(x_2) = x_1$. $T^3([y_2, x_2]) \supset T([x_1, x_2]) \supset [x_2, x_3]$. Thus $T^4([y_2, x_2]) \supset T([x_2, x_3]) \supset [x_1, x_3]$, and therefore $T^5([y_2, x_2]) \supset [x_1, x_3]$. Hence T^5 maps the interval $[y_2, x_2]$ onto a strictly larger interval $[u, v]$ where $u < y_2 < x_2 < v$ so that by continuity arguments there exists an $\varepsilon > 0$ such that if $\|T-F\| < \varepsilon$ then F^5 maps y_2, x_2 onto a larger interval than itself. Thus there exists at least one fixed point x_0; that is, $x_0 = F^5(x_0)$. Since the interval $[y_2, x_2]$ is disjoint from the fixed point set of T, by taking, if necessary, a

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smaller value of ε we can arrange that F has no fixed points in $[y_2, x_2]$. This completes the proof.

From Theorems 1 and 2 and the result in [4] we deduce that the chaotic functions are not only dense in $C(I)$ but also contain an open dense set.

4. The chaotic behaviour of a one-parameter family of continuous functions

Many people have studied the orbit structure of the family of parabolas

$$G_\lambda(x) = 4\lambda x(1-x) \ , \quad 0 \le x \le 1 \ , \quad 0 \le \lambda \le 1 \ ,$$

because of its occurrence in simple models of population growth [1, 2, 3, 7, 10]. It is known that for $\lambda = 0.957 \ldots$ orbits of period 3 begin to appear. The precise value of λ_0 such that chaos occurs for $\lambda > \lambda_0$ but not for $\lambda \le \lambda_0$ is not known; numerical analysis suggests that it is approximately 0.89 . We shall study the onset of chaos for the family

$$T_\lambda(x) = \begin{cases} \lambda x & , \quad 0 \le x \le 1/2 \ , \\ \lambda(1-x) & , \quad 1 \ge x > 1/2 \ , \end{cases}$$

as λ varies from 0 to 2 .

It is obvious that for $\lambda < 1$, there is a single periodic orbit, namely the fixed point 0 , and all orbits converge to 0 . If $\lambda = 1$, all points of $[0, 1/2]$ are fixed points and the points of $[1/2, 1]$ are eventually periodic of period 1 . It is easily seen that for $\lambda = \lambda_1 = (1/2)(1+\sqrt{5})$ there is a bifurcation which produces points of period 3 , and thus chaos for $\lambda \ge \lambda_1$. We shall show that periodic points of period $2^n \cdot 3$ for some n occur as soon as $\lambda > 1$. As a preliminary we need the following lemma.

LEMMA. *Let* T *be a continuous real-valued function on* I . *If there exist two closed intervals* $\bar{I}, \bar{J}$ *such that* $\bar{I} \cap \bar{J}$ *contains at most one point and* $T(\bar{I}) \cap T(\bar{J}) \supset \bar{I} \cup \bar{J}$, *then there exists a point of period* 3 *for* T .

Proof. Since $T(\bar{I}) \supset \bar{J}$, there exists $x_0 \in \bar{I}$ such that

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$T(x_0) = \max\{y : y \in \bar{J}\}$, and let $[x_0, y_0] \subset \bar{I}$ be chosen so that $T([x_0, y_0]) \supset \bar{J}$ and T has no fixed points in $[x_0, y_0]$. $T([x_0, y_0]) \supset \bar{J}$, hence $T^2([x_0, y_0]) \supset T(\bar{J}) \supset \bar{I} \cup \bar{J}$ and $T^3([x_0, y_0]) \supset \bar{I} \cup \bar{J} \supset [x_0, y_0]$. Thus the interval $[x_0, y_0]$ is mapped by T^3 onto a larger interval. Hence there exists in $[x_0, y_0]$ a point fixed under T^3. Since no point of $[x_0, y_0]$ is fixed under T the lemma is proved.

THEOREM 3. *For any* $\lambda > 1$, $T_\lambda(x)$ *has periodic point of period* $3 \cdot 2^n$ *for some* n.

Proof. We shall show that for any fixed $\lambda > 1$ there exist an integer $m = m(\lambda)$ and two closed intervals I_m, J_m such that $T_\lambda^2(I_m) \cap T_\lambda^2(J_m) \supset I_m \cup J_m$ and $I_m \cap J_m$ contains just one point.

It is easily seen that the interval $[T_\lambda(\lambda/2), \lambda/2]$ is mapped onto itself by T. Let $n \geq 1$ be such that

$$\lambda^{2^{n-1}} < 2 , \quad \lambda^{2^n} \geq 2 .$$

Set $T_\lambda(\lambda/2) = a$ and consider T_λ^2. This transformation maps the interval $\left[a, T_\lambda^2(a)\right]$ onto itself and is linear in $[0, 1/2]$, $\left[1/2, T_\lambda^2(a)\right]$. Analogously the transformations $T_\lambda^{2^i}$ $(i \leq n-1)$ map the intervals $\left[a, T_\lambda^{2^i}(a)\right]$ onto themselves and they are linear in (a, x_i) and in $\left(x_i, T_\lambda^{2^i}(a)\right)$, where x_i is the unique point of $\left(a, T_\lambda^{2^i}(a)\right)$ such that $T_\lambda^{2^i}(x_i) = 0$. The slopes of the linear pieces of the curve T^{2^i} are $\pm\lambda^{2^i}$. Since $\lambda^{2^n} \geq 2$ each of the images of intervals $[\dot{a}, x_n]$,

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$[x_n, T^{2^n}(a)]$ by $T_\lambda^{2^n}$ contain $[a, T_\lambda^{2^n}(a)]$, where x_n is the unique point of $[a, T^{2^n}(a)]$ such that $T_\lambda^{2^n}(x) = 0$. This completes the proof.

5. The closure of periodic orbits. A counterexample

In [6] Li and Yorke asked whether or not the closure of the periodic points of $G \in C(I)$ is always a finite union of intervals. We shall now give a negative answer to this question.

Let $I_n = [1-2^{1-n}; 1-2^{-n}]$, $n = 1, 2, \ldots$.

Let $T, U : [0, 1] \to [0, 1]$ be defined by

$$T(x) = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2(1-x), & 1 \ge x \ge 1/2, \end{cases} \qquad U(x) = \begin{cases} 2x, & 0 \le x \le 1/3, \\ \frac{1+x}{2}, & 1/3 \le x \le 1. \end{cases}$$

T has points of period 3 while it is easy to see that U has just two periodic points, namely the two fixed points 0, 1. All other points tend asymptotically to 1. We now define $G : [0, 1] \to [0, 1]$ as follows. On I_{3n-2} define G by

$$G(x) = 1 - 2^{3-3n} + 2^{2-3n} T\left(2^{3n-2}x+2-3^{3n-2}\right).$$

On I_{3n-1} define G by

$$G(x) = 1 - 2^{3-3n} + 3\left(x+2^{2-3n}-1\right).$$

On I_{3n} define G by

$$G(x) = 1 - 2^{1-3n} + 2^{-3n} U\left(2^{3n}x+2-2^{3n}\right),$$

and define $G(1)$ to be 1.

All we have done is to arrange that the graph of G restricted to I_{3n-2} is a suitable "scaled-down" copy of the graph of T and that the graph of G restricted to I_{3n} is a copy of the graph of U. The definition of G on I_{3n-1} assures continuity. G has periodic points in each interval I_{3n-2}, but the interior of each interval I_{3n} is free from

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periodic points. Hence the closure of the periodic point set cannot be a finite union of intervals.

Conclusion

Since the existence of a point of period $3\cdot 2^n$ implies the existence of two closed disjoint sets A, B such that $T^{3\cdot 2^n}(A) \cap T^{3\cdot 2^n}(B) \supset A \cup B$, following [5], period $3\cdot 2^n$ gives us not only chaos in the sense of Li and Yorke, but also the existence of a continuous ergodic measure invariant under T.

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