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Periodic points of maps of degree one of a circle

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Abstract. There is a theorem of Sarkovskii which characterizes the sets of periods of periodic points for continuous maps of an interval into itself. We give a similar characterization for continuous maps of degree one of a circle into itself.

Introduction

The aim of this paper is to give a full characterization of the set of periods for a continuous map of degree one of a circle into itself.

By the period of a periodic point we always mean its smallest period. By $P(f)$ we denote the set of periods of all periodic points of a given map f.

Consider the Sarkovskii ordering of the set $\mathbb{Z}^+$ of all positive integers: 3, 5, 7, $9, \ldots, 2\cdot 3,\ 2\cdot 5,\ 2\cdot 7,\ 2\cdot 9, \ldots, 2^2\cdot 3,\ 2^2\cdot 5,\ 2^2\cdot 7,\ 2^2\cdot 9, \ldots, 2^4, 2^3, 2^2, 2, 1$. For $n\in\mathbb{Z}^+$ we shall denote by $Ш(n)$ the set consisting of n and all numbers standing to the right of n in this ordering. By $Ш(2^\infty)$ we denote the set $\{\ldots, 2^4, 2^3, 2^2, 2, 1\}$ of all powers of 2. The famous Sarkovskii theorem says that for a continuous map f of an interval into itself, $P(f)=Ш(n)$ for some $n\in\mathbb{Z}^+\cup\{2^\infty\}$, and for all such n there exists f with $P(f)=Ш(n)$ ([**11**], see also [**12**], [**4**]).

In the case of a circle instead of an interval, i.e. for a continuous map $f: S^1\to S^1$, it is known that:

(i) if $1\in P(f)$ then $P(f)=B\cup S$, where

$$B=\{n\in\mathbb{Z}^+ : n\ge b\}\quad\text{and}\quad S=Ш(s)$$

for some $b\in\mathbb{Z}^+$ and $s\in\mathbb{Z}^+\cup\{2^\infty\}$ ([**3**]);

(ii) if $|\deg f|\ge 2$ then $P(f)=\mathbb{Z}^+$ with one exception: if $\deg f=-2$ it is possible also that $P(f)=\mathbb{Z}^+\setminus\{2\}$ ([**4**], [**7**]);

(iii) if $\deg f=0$ then $P(f)=Ш(n)$ for some $n\in\mathbb{Z}^+\cup\{2^\infty\}$ ([**4**], [**7**]);

(iv) if $\deg f=-1$ then $P(f)=Ш(n)$ for some $n\in\mathbb{Z}^+\cup\{2^\infty\}$ ([**7**], [**3**]).

The remaining, but most complicated, case is of $\deg f=1$ (notice that if f has no fixed points then $\deg f=1$). For this case, some partial results were obtained by Bernhardt [**1**], [**2**], Jefremova [**7**] and Falbo [private communication from D. Fried].

In this paper, we prove the following theorem:

THEOREM. (*a*) *For a continuous map $f: S^1\to S^1$ of degree one, there exist $a, b\in\mathbb{R}$ and $l, r\in\mathbb{Z}^+\cup\{2^\infty\}$ with $a\le b$ and such that $P(f)=M(a,b)\cup S(a,l)\cup S(b,r)$,* *where:*

$$M(a,b)=\{n\in\mathbb{Z}^+ : \text{there exists } k\in\mathbb{Z} \text{ such that } a<k/n<b\};$$

$$S(a,l)=\begin{cases}\varnothing & \text{if } a \text{ is irrational,}\\ \{ns : s\in Ш(l)\} & \text{if } a=k/n \text{ for } k \text{ and } n \text{ coprime,}\end{cases}$$

and $S(b,r)$ *analogously.*

(*b*) *For every set* $A\subset\mathbb{Z}^+$ *of the above form, there exists a map of class* C^∞, $f: S^1\to S^1$ *of degree one, with* $P(f)=A$.

Remark. For $l, r\in\mathbb{Z}^+\cup\{2^\infty\}$ either $Ш(l)\subset Ш(r)$ or $Ш(r)\subset Ш(l)$. Hence, if $a=b$, then we can take r and l equal.

In the following, f will be a continuous map $S^1\to S^1$, F its lifting to $\mathbb{R}$, e the natural projection $\mathbb{R}\to S^1(e(X)=\exp(2\pi iX))$.

We shall use the techniques of [4]. For maps of degree one, one of the main tools is the theory of rotation numbers ([**8**], [**4**]). Since we are interested only in periodic points, we present this theory along the lines of [**4**] rather than [**8**].

For a periodic point $x\in S^1$ of period n and $X\in\mathbb{R}$ with $e(X)=x$, we have $F^n(X)=X+k$ for some $k\in\mathbb{Z}$. We call the number $\rho(F,x)=k/n$ the *rotation number* of x. It does not depend on the choice of X from $e^{-1}(\{x\})$. Rotation numbers have the following properties ([**8**], [**4**]):

(i) for a lifting $G=F^p+m$ of f^p,

$$\rho(G,x)=p\cdot\rho(F,x)+m;$$

(ii) if k and n are coprime and $\rho(F,x)\le k/n\le\rho(F,y)$ for some $x,y\in S^1$ then there exists a periodic point $z\in S^1$ with $\rho(F,z)=k/n$.

The following fact follows from [**6**] and [**8**]. We shall give its simple proof in the spirit of [**4**].

LEMMA 1. *Let* $x_m\in S^1$ $(m=1,2,3,\dots)$ *be periodic points and let* $\lim_m\rho(F,x_m)=k/n$. *Then there exists a periodic point* x_0 *with* $\rho(F,x_0)=k/n$.

Proof. Set $G=F^n-k$. Suppose that G has no fixed points. Then the number $c=\inf_{X\in\mathbb{R}}|G(X)-X|$ is positive. Hence, if p is a period of x_m and $e(X_m)=x_m$, we have

$$|G^p(X_m)-X_m|\ge pc.$$

Consequently,

$$\left|\rho(F,x_m)-\frac{k}{n}\right|=\frac{1}{n}|\rho(G,x_m)|\ge\frac{c}{n}\quad\text{for all } m$$

which is a contradiction.

Thus, G has a fixed point, say X_0. The point $x_0=e(X_0)$ has rotation number

$$\rho(F,x_0)=\frac{1}{n}(\rho(G,x_0)+k)=\frac{k}{n}. \qquad \square$$

From (ii) and lemma 1 it follows that if f has a periodic point then the set $\{\rho(F,x): X \text{ is periodic}\}$ is an intersection of some closed interval (perhaps degenerated to a single point) with the rational numbers. We shall call this interval the *rotation* *interval* and denote its endpoints by $a(F)$ (the lefthand one) and $b(F)$ (the righthand one). In the degenerate case, $a(F)=b(F)$.

In the case when f has no periodic points, it is easy to see (for any k/n look at the graph of F^n-k) that the limit $a=\lim_n (1/n)(F^n(X)-X)$ exists, does not depend on $X\in\mathbb{R}$, and is irrational. Then we set $a(F)=b(F)=a$.

Consider a continuous map $t\mapsto f_t$ from an interval I to the space $C_1^0(S^1,S^1)$ of continuous maps of S^1 into itself of degree one. Then we can find a continuous map $t\mapsto F_t$ from I to $C^0(\mathbb{R},\mathbb{R})$ (with the topology of uniform convergence) such that for every t, F_t is a lifting of f_t.

Simple arguments, based on the fact that $a(F)\le k/n\le b(F)$ if and only if $G=F^n-k$ has a fixed point, show that the following lemma holds ([**8**]):

LEMMA 2. *In the above situation, the maps $t\mapsto a(F_t)$ and $t\mapsto b(F_t)$ are continuous.* □

For homeomorphisms, the notion of the rotation number was known long ago [**10**]. In this case the rotation interval always reduces to one point.

LEMMA 3. *Let $f,g:S^1\to S^1$ be continuous maps with liftings F,G respectively. If $G\ge F$ and f or g is a homeomorphism, then $a(G)\ge b(F)$.*

Proof. Suppose that f is a homeomorphism (the other case is analogous). Take $X\in\mathbb{R}$. Then, since F is increasing,

$$G^n(X)\ge F(G^{n-1}(X))\ge F^2(G^{n-2}(X))\ge\cdots\ge F^n(X).$$

Since f is a homeomorphism, we have (see e.g. [**9**])

$$\lim_n \frac{1}{n}(F^n(X)-X)=b(F).$$

Hence,

$$\liminf_n \frac{1}{n}(G^n(X)-X)=\liminf_n \frac{1}{n}G^n(X)\ge\lim_n \frac{1}{n}F^n(X)$$
$$=\lim_n \frac{1}{n}(F^n(X)-X)=b(F).$$

But if $e(X)$ is periodic for g then

$$\rho(G,e(X))=\lim_n \frac{1}{n}(G^n(X)-X),$$

and hence $a(G)\ge b(F)$. □

Remark. (A) If F is non-decreasing then it is easy to see that for every X,

$$\lim_n \frac{1}{n}(F^n(X)-X)=a(F)=b(F).$$

(B) In a general case, the maps α_F and β_F, defined as

$$\alpha_F(X)=\inf_{Y\in[X,\infty)}F(Y),\qquad \beta_F(X)=\sup_{Y\in(-\infty,X]}F(Y),$$

are also liftings of continuous maps of a circle and are non-decreasing. Moreover,

$$a(\alpha_F)=a(F)\quad\text{and}\quad b(\beta_F)=b(F).$$

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(C) In view of (A), (B) and lemma 3, we obtain that in the general case, $G \geq F$ implies $a(G) \geq a(F)$ and $b(G) \geq b(F)$. However, we shall only need to use lemma 3, and not this more general result.

Before proving the theorem, we have to prove several auxiliary facts. We shall be interested also in obtaining some estimates of the topological entropy $h(f)$.

PROPOSITION 4. *If $a(F) < 0 < b(F)$ then f has periodic points of all periods with rotation number* 0, *and* $h(f) \geq \log 3$.

Proof. Take x, y periodic with $\rho(F, y) < 0 < \rho(F, x)$, and $X, Y \in \mathbb{R}$ with $e(X) = x$, $e(Y) = y$, $X < Y$. Set:

$$m = \max\{k \geq 0: F^k(X) < Y\}; \quad Z = \min\{F^k(X): k > m\};$$

$$n = \max\{k \geq 0: F^k(Y) > F^m(X)\}; \quad T = \max\{F^k(Y): k > n\}.$$

Then $F(F^m(X)) \geq Z$, $F(Z) > Z$, $F(F^n(Y)) \leq T$ and $F(T) < T$. Therefore for the intervals $I = [T, F^m(X)]$, $J = [F^m(X), F^n(Y)]$ and $K = [F^n(Y), Z]$, we get (cf. figure 1):

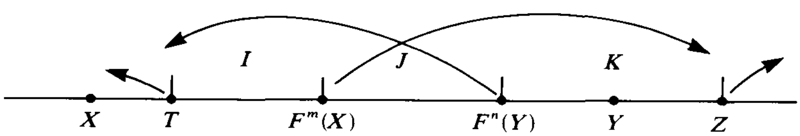

FIGURE 1

$F(I) \supset I \cup J \cup K$, $F(J) \supset I \cup J \cup K$ and $F(K) \supset I \cup J \cup K$. Hence, F has periodic points of all periods (cf. [**4**]). Their images under e are periodic for f and have rotation number zero. Consequently, they have the same periods for F and f (see [**4**, theorem 3.2]).

The set $C = \{W: F^k(W) \in [T, Z] \text{ for all } k \geq 0\}$ is F-invariant and closed. We have $h(F|_C) \geq \log 3$. Since $e|_C$ is at most p–to–1 for some integer p, we get

$$h(f) \geq h(f|_{e(C)}) = h(F|_C) \geq \log 3. \qquad \square$$

LEMMA 5. *If x is periodic of period s for f^n and $\rho(F^n, x) = k \in \mathbb{Z}$, and k and n are coprime, then $\rho(F, x) = k/n$ and x has period sn for f.*

Proof. By property (i) of rotation numbers, $\rho(F, x) = k/n$. Therefore the f-period of x is pn for some integer p. But then the f^n-period of x is p, and hence $p = s$. $\square$

PROPOSITION 6. *If $a(F) < k/n < b(F)$ and k and n are coprime then f has periodic points of all periods sn, $s \in \mathbb{Z}^+$, with rotation number k/n. The topological entropy of f is at least* $1/n \log 3$.

Proof. For the lifting $G = F^n - k$ of f^n, we have $a(G) < 0 < b(G)$. By proposition 4 for all $s \in \mathbb{Z}^+$ there exist periodic points of period s with G-rotation number 0. Their F^n-rotation number is k, and by lemma 5, their F-rotation number is k/n and their f-periods are sn. By proposition 4, $h(f^n) \geq \log 3$, and hence $h(f) \geq 1/n \log 3$. $\square$

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Since $k/n = ks/ns$, we get the following corollary:

COROLLARY 7. *If $a(F) < k/n < b(F)$ then f has a periodic point of period n.* □

Proof of theorem

Denote by P_0 the periods of periodic points with rotation number in the open interval $(a(F), b(F))$, by P_1 the periods of periodic points with rotation number $a(F)$, and by P_2 the periods of periodic points with rotation number $b(F)$. Then obviously $P(f) = P_0 \cup P_1 \cup P_2$.

Set $a = a(F)$, $b = b(F)$. By corollary 7, $M(a, b) \subset P_0$, and by the definition of a and b, $P_0 \subset M(a, b)$. Therefore $P_0 = M(a, b)$. If a is irrational then, clearly, $P_1 = \varnothing$. If $a = k/n$ where k and n are coprime, then we look at the lifting $G = F^n - k$ of f^n. By the Sarkovskii theorem, $P(G) = Ш(l)$ for some $l \in \mathbb{Z}^+ \cup \{2^\infty\}$. As in the proof of proposition 4, there is one-to-one correspondence (preserving periods) between periodic points of f^n with G-rotation number zero and periodic points of G. Hence, by lemma 5, $P_1 = n \cdot Ш(l)$. In both cases, $P_1 = S(a, l)$. Analogously, $P_2 = S(b, r)$ for some $r \in \mathbb{Z}^+ \cup \{2^\infty\}$. This completes the proof of (a).

To prove (b), we have to describe for given a, b, l and r how to construct f with $P(f) = M(a, b) \cup S(a, l) \cup S(b, r)$. We start with a homeomorphism f_0 (of class C^∞) with $a(F_0) = a$. If a is irrational, we require also that the set of non-wandering points $\Omega(f_0)$ is a Cantor set (an example of such a homeomorphism for irrational a was given recently by Hall [**5**]). If a is rational, we require that $\Omega(f_0)$ consists of only one periodic orbit, attracting from the left and repelling from the right (this is very easy to construct). Then we deform f_0 (in a C^∞ way) outside $\Omega(f_0)$ to get a fixed point whose rotation number is an integer $k > a$ (we 'push a part of the graph up'), all the time keeping the lifting larger than or equal to F_0. By lemma 2, for some f_t, we get $b(F_t) = b$. In view of lemma 3, $a(F_t) = a(F_0) = a$ (we did not change the map on $\Omega(f_0)$). Clearly, we may start from $b(F)$ and then 'push a part of the graph down' to obtain the required $a(F)$.

We make the above construction if at least one of the numbers a or b is irrational. If only one of them is irrational, then we start from the other side. If both a and b are rational, we take instead a map f (of class C^∞) such that the graph of F lies between the lines $Y = X + a$ and $Y = X + b$ and touches each of them along a lifting of one periodic orbit of f.

In such a way in all cases when a (or b) is rational, there is only one orbit with that rotation number. If l (or r) is different from 1, then we replace this orbit by a periodic interval and then put on it a corresponding Sarkovskii example (see [**11**], [**12**]; it is easy to do it in a C^∞ manner). □

Our theorem describes all possible sets $P(f)$. However, one can ask further questions about the structure of the sets $M(a, b)$, $S(a, l)$ and $S(b, r)$. The structure of the sets $S(a, l)$ and $S(b, r)$ is very simple and needs no further explanation. Hence, we are left with the question about the structure of $M(a, b)$. This is a number-theoretical problem: given $a < b$, what are the (positive) denominators of fractions from (a, b)?

If $a<0<b$, we have all denominators $(0/n=0)$. If $a=0$ then we have all denominators larger than $E(1/b)$, where E denotes the integer part. The situation for $b=0$ is analogous – we have all denominators larger than $E(-1/a)$. The case $b<0$ may be reduced to the case $a>0$ by replacing (a, b) by $(-b, -a)$.

Thus, we are left with the case $0<a<b$. In this case, the answer is not obvious. Moreover, it is even not quite obvious what should be considered as a satisfactory answer. One of the possibilities is to find an easy way of determining, for given a, b and n, whether there exists an integer k such that $a<k/n<b$.

PROPOSITION 8. *Let $0<a<b$ and let n be a positive integer. Then there exists an integer k with $a<k/n<b$ if and only if $E(na)+1<nb$.*

Proof. If: We set $k=E(na)+1$ and assume that $k<nb$. Then $a<k/n<b$.
Only if: We assume that $a<k/n<b$ for some integer k, then $na<k$, and hence $E(na)+1\le k<nb$. □

Another possibility is to find an easy way of writing the denominators down (in some order). This can be done, when we notice that $a<k/n<b$ if and only if $k/b<n<k/a$. Hence, to obtain all the denominators, we take all integers from $(k/b, k/a)$, $k=1, 2, 3, \ldots$. We can end this procedure when we reach $1/(b-a)$, since if $n>1/(b-a)$ then $b-a>1/n$ and consequently there is some k with $a<k/n<b$. Hence, past $1/(b-a)$ we already obtain all integers.

If a and b are rational, we can provide different kinds of answers. Unfortunately, they are only partial.

LEMMA 9. *Let l, m, p, q, r and s be positive integers such that $l/m<p/q$. Then*

$$\frac{l}{m}<\frac{rl+sp}{rm+sq}<\frac{p}{q}.$$

Proof. The number

$$\frac{rl+sp}{rm+sq}=\frac{rm(l/m)+sq(p/q)}{rm+sq}$$

is a weighted average of l/m and p/q. □

The above lemma allows us to deduce the result of [**1**] from proposition 6. If the smallest elements m, q of $P(f)$ are coprime and larger than 1 then we can distinguish two cases. Either the corresponding periodic points have the same rotation numbers – then this rotation number is an integer and we use the properties of the Sarkovskii ordering – or these rotation numbers are of the form l/m and p/q and we use lemma 9 and proposition 6. In both cases we deduce that f has periodic points of periods $rm+sq$ for all $r, s>1$.

Under an additional assumption, we can reach a stronger conclusion than that in lemma 9.

PROPOSITION 10. *Let $l/m<p/q$ and let m and q be coprime. Then for a positive integer n there exists an integer k such that $l/m<k/n<p/q$ if and only if there exist positive integers r and s such that $n(mp-lq)=rm+sq$.*

Proof. If: We assume that such r and s exist. Then, by lemma 9, for

$$k=\frac{rl+sp}{mp-lq}$$

we get

$$\frac{l}{m}<\frac{k}{n}<\frac{p}{q}.$$

Since $mp-lq$ divides $rm+sq$, it also divides the following integers:

$$p(rm+sq)-r(mp-lq)=q(rl+sp)$$

and

$$l(rm+sq)+s(mp-lq)=m(rl+sp).$$

Therefore, since q and m are coprime, $mp-lq$ divides $rl+sp$. Consequently, k is an integer.

Only if: We let $l/m<k/n<p/q$. Then for $r=np-kq$ and $s=mk-ln$, we have $n(mp-lq)=rm+sq$. □

REFERENCES

[1] Ch. Bernhardt. Periodic orbits of continuous mappings of the circle. Preprint. Carbondale, 1980.

[2] Ch. Bernhardt. Periodic points and topological entropy of maps of the circle. Preprint. Carbondale, 1980.

[3] L. Block. Periods of periodic points of maps of the circle which have a fixed point. *Proc. Amer. Math. Soc.* **82** (1981), 481–486.

[4] L. Block, J. Guckenheimer, M. Misiurewicz & L.-S. Young. Periodic points and topological entropy of one-dimensional maps. *Global Theory of Dynamical Systems. Lecture Notes in Mathematics* No. 819, Springer, Berlin, 1980, pp. 18–34.

[5] G. R. Hall. A C^∞ Denjoy counterexample. *Ergod. Th. & Dynam. Sys.* **1** (1981), 261–272.

[6] R. Ito. Rotation sets are closed. *Math. Proc. Camb. Phil. Soc.* **89** (1981), 107–111.

[7] L. S. Jefremova. Periodic points of a continuous map of a circle. *Proc. IX int. conf. on Non-linear Oscillations, Kiev 1981,* p. 121. (In Russian.)

[8] S. Newhouse, J. Palis & F. Takens. Stable families of dynamical systems, I: Diffeomorphisms. *Publ. I.H.E.S.* (To appear.)

[9] Z. Nitecki. *Differentiable Dynamics.* MIT Press: Cambridge, Mass., 1971.

[10] H. Poincaré. Oeuvres completes, t. 1. Gauthier-Villars: Paris 1952, pp. 137–158.

[11] A. N. Sarkovskii. Coexistence of cycles of a continuous map of a line into itself. *Ukr. Math. Zurn.* **16** (1964), 61–71. (In Russian.)

[12] P. Štefan. A theorem of Sarkovskii on the existence of continuous endomorphisms of the real line. *Commun. Math. Phys.* **54** (1977), 237–248.