

# Periodic Review Inventory Management with Contingent Use of Two Freight Modes with Fixed Costs

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We study a stochastic inventory model of a firm, that sources the product from a make-to-order manufacturer, and can ship orders by a combination of two freight modes. The two freight modes differ in lead-times, and each has a fixed and a quantity proportional cost for each use. The ordering decisions are made periodically; however, the inventory holding and back-order penalty costs are incurred continuously in time. The decision of how to allocate units between the two freight modes utilizes information about demand during the completion of manufacturing. We derive the optimal freight mode allocation policy and show that the optimal ordering policy is not an  $(s, S)$  policy in general. We provide bounds for the optimal policy and perform a stationary analysis of the model assuming an  $(s, S)$  policy. We show that the best  $(s, S)$  policy achieves time average probability of being in-stock equal to the ratio of penalty cost rate and the sum of penalty cost rate and holding cost rate. We carry-out extensive numerical investigations of the properties of the optimal ordering policy and its benefits over the single freight policy, and show that the performance of best  $(s, S)$  policy is comparable to the optimal policy in most cases.

*Key words:* Inventory/Production: Periodic review,  $(s, S)$  Policy, Lead-times. Transportation: Freight mode selection, Transportation economies of scale.

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## 1. Introduction

Over the last few decades, the liberalization of cross-border trade policies has offered firms a unique opportunity to lower procurement costs by sourcing manufactured goods from international locations with low production cost. One of the biggest challenges of such sourcing strategies is managing timely shipments of inventory over far flung supply chains. Thankfully, economic deregulation in various transportation industries has led to entry of several new freight carriers and logistics services providers, resulting in increased number of alternatives for a firm's transportation needs. These alternatives typically offer differing trade-offs between shipping cost and responsiveness: A slower freight mode sacrifices responsiveness to fluctuating demand, but incurs lower shipping costs, whereas a faster freight mode results in higher shipping costs. A firm can enjoy most of

the benefits of both by optimally choosing one or the other, or splitting its shipment across the two. Moreover, with the help of modern information technology a firm can choose between these alternatives dynamically based on the most recent demand information. Since these transportation decisions affect inventory levels, a firm needs to optimize them jointly with inventory decisions to gain the full advantage of low-cost sourcing along with available transportation choices. In this paper we consider this joint optimization for a firm that sources its product from a make-to-order supplier and uses two alternative freight modes for shipping its orders.

In practice, it has not been uncommon for firms to use two freight modes for their transportation needs. For example, firms relying primarily on slower ocean freight, resort to faster and more expensive air freight on an emergency basis. Recently, however, as responsiveness to fluctuating demand has become critical to survive competition, firms have shifted towards combined use of multiple freight modes on a regular basis. For instance, in the last decade globally operating manufacturing firms such as Kodak, Digital Equipment Corporation and Texas Instruments have increased the relative use of air freight along with ocean freight (see *World Trade*, October 1994 and *World Trade*, September 2004). The trend of relying on multiple transportation modes is also evident in the variety of new multi-modal services offered by logistic solution providers and the growing demand for time critical expedited services. Logistic solutions providers, such as UPS, FedEx, DHL and BAX Global now offer single-source multi-modal logistic solutions to firms, which let firms decide the mode of transportation from the solution provider's offerings of air, ocean, rail or road transportation modes (see *Economist*, December 7, 2002, *Wall Street Journal*, November 29, 2004 and *Journal of Commerce*, January 9, 2006). Also gaining ground are information technology enabled "global trade solutions", which among other things help firms manage complex supply chains by facilitating end-to-end visibility and control of shipments. For example, using such a solution provided by TradeBeam, clothing retailer Liz Claiborne Inc. monitors its shipments in pipeline and can expedite their delivery online. Similar solutions are adopted by power tools manufacturer Black & Decker, culinary and kitchen retailer Williams Sonoma, and specialty food importer Liberty Richter to manage their inbound logistics (see *Aberdeen Group Research Study*, January 2006 and *Apparel*, June 2006).

Several papers in the Operations literature consider inventory models with two supply modes with different lead-times and costs. Our work differs from this body of literature in three important ways: (i) First, most of these papers consider a simple quantity proportional cost of sourcing from each supply mode, thus ignoring the economies of scale in transportation cost and resulting in optimal solutions that allow orders to be split in small quantities across different supply modes.

This is contrary to the observation made from the real life data by Thomas and Tyworth (2006), who also question the practicality of always splitting orders. In our model, in addition to the quantity proportional cost, a fixed cost is incurred for each use of any freight mode. Our paper thus incorporates transportation economies of scale and provides a richer analysis of the problem. *(ii)* Furthermore, in most of the papers the allocation of ordered units between different supply modes is static, i.e., it is the same for all orders over time, while in reality recent advances in information technology allow firms to dynamically allocate quantities to different supply modes based on the latest information. Our model, where the optimal transportation decision is made based on latest available information on demand, captures the dynamic nature of the decision. *(iii)* Finally, it is commonly assumed in periodic review models, that the inventory holding and back-order penalty costs are incurred on inventory levels at the end of each review period. Rudi et al. (2004) argue that the inventory costs are predominantly incurred continuously in time, and show that employing end-of-period cost accounting may lead to large errors on the inventory policy and the resulting costs. This end-of-period cost assumption also restricts the analysis to values of lead-times that are integer multiples of the review period length. In order to correctly identify the value of multiple freight modes, we account for inventory costs continuously in time and allow for real values of lead-times.

The organization of the rest of the paper is as follows. Following a brief literature review in the next section, §3 describes the model in detail and introduces notations followed in the rest of the paper. In §4, we derive the optimal policy for allocating ordered quantities between the two freight modes for an order of arbitrary size. In §5 and §6, we analyze ordering decisions given all orders are shipped optimally using the freight modes: In §5 an optimal ordering policy is derived, whereas, the analysis in §6 is performed assuming that a stationary  $(s, S)$  policy is used for placing orders. In §7 we provide numerical illustrations of properties of our model. Finally, §8 concludes the paper. All proofs are provided in Jain et al. (2007), the supplement to this paper available on the on-line companion site.

## 2. Literature Review

In spite of the differences between our paper and previous literature that we discussed in the introduction, this paper is directly related to the stream of literature focusing on inventory models in which managers have some control over lead-times for replenishment from a single source. The model studied by Fukuda (1964) is the earliest such study. In his model at the beginning of each period the manager places orders with a single supplier to be delivered after a time lag of  $k$  and

$k + 1$  review periods, with a higher cost for faster delivery. In the classical multi-echelon inventory model considered by Clark and Scarf (1960), the ability to hold or ship units at each stage implies a certain amount of control over the total lead-time. This model is extended in Lawson and Porteus (2000) by allowing instantaneous transfer of units between two consecutive stages at a higher cost. Muharremoglu and Tsitsiklis (2003) provide further generalization of Lawson and Porteus (2000). In another related work Huggins and Olsen (2003) consider a periodic review production-inventory model, in which the lead-time can be manipulated by either production overtime or expedited delivery, under the requirement of always meeting the complete demand. In addition to two modes of shipment in each period, the model in Sethi et al. (2003) allows demand forecast updating in each period.

All the models considered above are essentially discrete-time models: The values of lead-times are restricted to integer multiples of review period length, and inventory costs are accounted for on the end of period inventory/back-order levels. Unlike these, the models considered in our paper along with those in Groenevelt and Rudi (2002), and Jain et al. (2005) are continuous-time models. The model in Groenevelt and Rudi (2002) is essentially our model without any fixed costs, while Jain et al. (2006) analyze the performance of a  $(r, Q)$  policy for the continuous review version of the model in this paper.

Our work is also related to studies considering inventory models in which orders can be placed with two or more suppliers differing in lead times and costs. One of the earliest works, Barankin (1961), studies a single period model, which is extended to the multi-period case by Daniel (1962) and Neuts (1964). Whittemore and Saunders (1977) is the first paper to consider a case in which lead-times of two supply modes may differ by an arbitrary number of review periods. Recent notable variations of periodic review inventory models with two alternative supply sources are analyzed in Chiang and Gutierrez (1996), Tagaras and Vlachos (2001), and Veeraraghavan and Scheller-Wolf (2003). For a detailed survey of this literature, we refer readers to Minner (2003) and Thomas and Tyworth (2006).

Scarf (1960) establishes optimality of an  $(s, S)$  policy for a periodic inventory model in which the cost of placing an order consists of a fixed and a proportional component. Following this a few papers have provided analytical treatment for models with a different structure to ordering or procurement cost. The foremost such papers are Johnson (1968) and Porteus (1971). Johnson (1968) assumes that when an order is placed to increase inventory position from  $i$  to  $j$ , cost  $M(i) + K(j)$  is incurred. He shows that in the stationary infinite horizon case the optimal policy is an  $(s, S)$  policy under certain conditions. On the other hand in the model analyzed

in Porteus (1971) orders can be placed with multiple suppliers that differ in fixed and quantity proportional components of ordering cost, and hence the cost of placing an order is a piece-wise linear and concave function of the order quantity. Porteus (1971) establishes the optimality of a generalized  $(s, S)$  policy for this model under certain conditions on the demand process. Fox et al. (2004) consider a similar model, with two suppliers of which one does not have a fixed cost. They extend the analysis of Porteus (1971) to less restrictive conditions on the demand process. The model in Fox et al. (2004) (as well as the model in Porteus (1971) when there are only two suppliers) is similar to the special case of our model where manufacturing lead-time is zero and two freight modes have the same lead-time. Of course Porteus (1971) and Fox et al. (2004) use end of period costs and restrict lead-times to integer multiples of a review period. In this paper, we show that with the optimal use of both freight modes the cost associated with placing an order takes a general form, which has not been considered before. In spirit, our analysis of this model is similar to the one offered in Veinott (1966). In a recent notable work, Huh and Janakiraman (2005) extend the analysis in Veinott (1966) to the context of joint inventory-pricing control.

### 3. Model Description

We consider an inventory model, in which the decision of whether to and how much to order is made at the beginning of each review period. Orders are placed with a make-to-order supplier. Irrespective of size, placing an order incurs fixed cost  $K_1$  and its manufacturing takes time  $L_1$ . At the completion of manufacturing, two options are available for shipping: Regular freight mode, which takes time  $L_2$  and has fixed cost  $K_2$ , and express freight mode, which takes time  $l_2$  ( $< L_2$ ), has fixed cost  $k_2$  and additional variable cost  $c_f$  per unit shipped. Hence, the total lead-time is  $l = L_1 + l_2$  for units shipped via express freight and  $L = L_1 + L_2$  for units shipped via regular freight. Each order can be shipped completely by either of the freight modes, or it can be split between the two in any proportion. The demand that arrives to the system during a stock-out, i.e., when it can not be satisfied immediately using on-hand inventory, is back-ordered. For each unit of physical inventory the system incurs holding cost  $h$  per time unit, and for each unit of demand back-ordered it incurs penalty cost  $p$  per time unit. The inventory holding and back-order penalty costs are incurred continuously in time.

Define inventory position as the sum of inventory level (which represents on-hand inventory when positive and back-orders when negative) and outstanding orders. Throughout this paper, the term “review epoch” refers to the instant at the beginning of a review period at which an ordering decision is made, and the term “pre-order inventory position” refers to the inventory position

immediately prior to making the ordering decision at a review epoch. We assume that the length of a review period  $T$  is larger than the lead-time difference between the two freight modes, i.e.,  $T > L_2 - l_2$ , implying that all the units ordered in previous review periods arrive before the units shipped via express freight in the current review period. This in turn implies that the inventory position at a review epoch contains all the relevant information to make the ordering decision at that moment.

The system experiences stochastic demand which has stationary and independent increments. For convenience of analysis, we assume that the demand process has continuous increments. Our results nevertheless are applicable to demand processes with discrete increments. We denote the random demand experienced in time interval  $(t_1, t_2]$  by  $D_{(t_1, t_2]}$ , its cumulative density function by  $F_{(t_1, t_2]}(\cdot)$  and its probability density function by  $f_{(t_1, t_2]}(\cdot)$ . The infinitesimal mean and variance of the demand process are denoted by  $\mu$  and  $\sigma^2$ , respectively. Throughout the paper,  $\mathbb{E}$  denotes the expectation operator, and  $\mathbb{P}(\omega)$  and  $\mathbb{1}_{\{\omega\}}$  denote the probability and the indicator functions, respectively, of an event  $\omega$ .

For initial supply  $x$  and a random cumulative demand  $d$ , define the inventory cost rate function  $G(x, d) = \mathbb{E} \left( h(x - d)^+ + p(d - x)^+ \right)$ . Note that in this definition the expectation is taken over the second argument of  $G$ . For initial supply  $x$ , and assuming that no further replenishments are received until time  $t_2$ , the expected inventory cost incurred in time interval  $(t_1, t_2]$  (with  $0 \leq t_1 \leq t_2$ ) is given by

$$\begin{aligned} \mathcal{G}_{(t_1, t_2]}(x) &= \mathbb{E} \int_{t_1}^{t_2} \left( h(x - D_{(0, t]})^+ + p(D_{(0, t]} - x)^+ \right) dt, \\ &= \int_{t_1}^{t_2} \mathbb{E} \left( h(x - D_{(0, t]})^+ + p(D_{(0, t]} - x)^+ \right) dt = \int_{t_1}^{t_2} G(x, D_{(0, t]}) dt, \end{aligned}$$

where the change in the order of expectation and integration in the second equality is justified by Fubini's theorem. Convexity of  $G(x, d)$  in  $x$  directly follows from the definition and it implies convexity of  $\mathcal{G}_{(t_1, t_2]}(x)$  in  $x$ . To simplify notation, we use the following notational scheme in the remainder of the paper: For a function  $a(\cdot)$  of a single variable,  $\tilde{a}(x) = \mathbb{E}a(x - D_{(0, T]})$ , and for a function  $b(\cdot, \cdot)$  of two variables,  $\tilde{b}(x, y) = \mathbb{E}b(x - D_{(0, T]}, y - D_{(0, T]})$ .

#### 4. The Optimal Freight Mode Decision

Without loss of generality, consider an order placed at time 0 to increase inventory position from  $x$  to  $y$  ( $y > x$ ). In this section, we derive the optimal policy for using the two freight modes for shipping this order. First note that for given  $x$  and  $y$ , the freight mode decision affects inventory levels only in the time interval elapsed between the arrival of express freight and the arrival of regular

freight (i.e., time interval  $(l, L]$ ). Second, since the freight mode decision is made at the completion of manufacturing, the demand incurred in manufacturing lead-time  $D_{(0,L_1]}$  is known at that time. These two facts imply that the optimal freight mode decision minimizes the cost of using freight modes, *plus* the expected inventory cost incurred in time interval  $(l, L]$  conditional on  $D_{(0,L_1]}$ . The optimal value of the relevant cost for the freight mode decision is thus,

$$\Xi(x, y) = \mathbb{E} \left[ \min_{0 \leq q \leq y-x} \left\{ K_2 \mathbb{1}_{\{q < y-x\}} + k_2 \mathbb{1}_{\{q > 0\}} + c_f q + \int_l^L G(x+q - D_{(0,L_1]}, D_{(L_1,t]}) dt \right\} \right]. \quad (1)$$

In the above, the expression inside the curly brackets is the expected relevant cost for the freight mode decision if  $q$  units are shipped via express freight, and it is conditional on  $D_{(0,L_1]}$ . The expression inside the square bracket is the optimal value of this conditional expected cost, where both this cost and the optimal value of  $q$  depend on  $D_{(0,L_1]}$ . Finally, the expectation outside the square brackets is taken over  $D_{(0,L_1]}$  to obtain the expected cost of the optimal freight mode decision.

Define  $g(u) = c_f u + \int_l^L G(u, D_{(L_1,t]}) dt$ . The following properties of the function  $g(u)$  play a key role in the analysis that follows.

LEMMA 1. (a)  $g(u)$  is convex in  $u$  and its minimizer  $z^*$  is given by the unique solution  $z$  to,

$$\frac{1}{L-l} \int_l^L \mathbb{P}(D_{(L_1,t]} < z) dt = \frac{p - \frac{c_f}{(L-l)}}{h+p}, \quad (2)$$

when  $c_f < p(L-l)$ , and is  $z^* = -\infty$ , otherwise.

(b) For  $q \geq 0$ ,  $g(u+q) - g(u)$  is non-decreasing in  $u$ , bounded below by  $q(c_f - p(L-l))$  and bounded above by  $q(c_f + h(L-l))$ .

First consider the special case  $K_2 = k_2 = 0$  of the optimal freight mode decision problem in (1). Groenevelt and Rudi (2002) solves this special case of the problem, and our next lemma restates their result in the setting of our model.

LEMMA 2. *Groenevelt and Rudi (2002)* For  $k_2 = K_2 = 0$  the optimal number of units to be shipped via express freight is given by  $q^* = \min \{y-x, (z^* - x + D_{(0,L_1]})^+\}$  when  $c_f < p(L-l)$ , and  $q^* = 0$  otherwise.

Referring to the quantity  $x+q - D_{(0,L_1]}$  as *the express freight inventory position*, the optimal policy given by the above lemma is to ship-up-to  $z^*$  using express freight, to the extent possible.

REMARK 1. In (1), the sum  $K_2 \mathbb{1}_{\{q < y-x\}} + k_2 \mathbb{1}_{\{q > 0\}}$  is a constant  $K_2 + k_2$  for  $q \in (0, y-x)$ . This implies that a solution to the problem with non-zero  $K_2$  and  $k_2$ , satisfying  $q \in (0, y-x)$  minimizes the sum of the last two terms of the objective function and is given by Lemma 2.

REMARK 2. It follows from Lemma 2 and Remark 1 that when  $c_f \geq p(L-l)$ , the optimal value of  $q \notin (0, y-x)$ . For such cases, the potential solution  $q = y-x$  can also be ruled out when  $K_2 \leq k_2$ .

The next proposition solves (1) for non-zero values of freight mode fixed costs.

PROPOSITION 1. *Define,*

$$Q_e = \frac{k_2 - K_2}{p(L-l) - c_f} \quad \text{and} \quad Q_r = \frac{K_2 - k_2}{h(L-l) + c_f}. \quad (3)$$

*Define  $z_p(Q)$  as the unique solution  $z_0$  to,*

$$g(z_0) - g(z_0 - Q) = K_2 - k_2. \quad (4)$$

*For  $c_f < p(L-l)$  define  $\bar{z}$  as the unique solution  $z_0$  to,*

$$g(z_0) - g(z^*) = K_2, \quad z_0 \geq z^* \quad (5)$$

*and define  $\underline{z}$  as the unique solution  $z_0$  to,*

$$g(z_0) - g(z^*) = k_2, \quad z_0 \leq z^*. \quad (6)$$

*The optimal value of  $q$  solving the optimization problem in (1) is given in Table 1 for different combinations of cost parameters.*

Note that  $c_f$  is the marginal shipping cost incurred on a unit when it is shipped via express freight instead of regular freight, whereas  $p(L-l)$  is the maximum penalty cost that could be saved on the unit. Thus, when  $c_f \geq p(L-l)$ , there is no marginal benefit of shipping a unit via express freight, and only one or the other freight mode is used for shipping the complete order. For the cases with  $c_f < p(L-l)$ , the presence of non-zero fixed costs of using the freight modes restricts the minimum non-zero quantity to be shipped via regular (express) freight to  $\min\{\bar{z} - z^*, y - x\}$  ( $\min\{z^* - \underline{z}, y - x\}$ ). Clearly, when the order size  $y-x$  is smaller than  $\bar{z} - \underline{z}$ , both freight modes are never used simultaneously. In other words, only when the order size is greater than  $\bar{z} - \underline{z}$ , are the savings in inventory and variable shipping costs earned by splitting an order large enough to offset the higher fixed cost incurred in using both freight modes simultaneously. For smaller order sizes, when  $K_2 \leq k_2$  ( $K_2 > k_2$ ),  $Q_e$  ( $Q_r$ ) is the minimum order quantity needed to justify shipping of an order by express freight (regular freight), which has a higher fixed cost. Finally, for  $K_2 > k_2$  and  $c_f \geq p(L-l)$ ,  $Q_e$  is the minimum order quantity needed to justify use of only regular freight, which has a higher fixed but a smaller per unit cost.



**Table 1** The optimal policy for freight mode decision

Cases	$c_f < p(L-l)$	$c_f \geq p(L-l)$
$K_2 \leq k_2$	If $y - x \leq Q_e$ , then $q^* = y - x$ . If $Q_e < y - x \leq \bar{z} - \underline{z}$ , then $\begin{cases} q^* = 0, & \text{if } D_{(0,L_1]} \leq y - z_p(y-x); \\ q^* = y - x, & \text{otherwise.} \end{cases}$ If $\bar{z} - \underline{z} < y - x$ , then $\begin{cases} q^* = 0, & \text{if } D_{(0,L_1]} \leq x - \underline{z}; \\ q^* = z^* - x + D_{(0,L_1]}, & \text{otherwise;} \\ q^* = y - x, & \text{if } D_{(0,L_1]} \geq y - \bar{z}. \end{cases}$	$q^* = 0$
$K_2 > k_2$	If $y - x \leq Q_r$ , then $q^* = 0$ . If $Q_r < y - x \leq \bar{z} - \underline{z}$ , then $\begin{cases} q^* = 0, & \text{if } D_{(0,L_1]} \leq y - z_p(y-x); \\ q^* = y - x, & \text{otherwise.} \end{cases}$ If $\bar{z} - \underline{z} < y - x$ , then $\begin{cases} q^* = 0, & \text{if } D_{(0,L_1]} \leq x - \underline{z}; \\ q^* = z^* - x + D_{(0,L_1]}, & \text{otherwise;} \\ q^* = y - x, & \text{if } D_{(0,L_1]} \geq y - \bar{z}. \end{cases}$	If $y - x \leq Q_r$ , then $q^* = y - x$ . If $Q_r < y - x < Q_e$ , then $\begin{cases} q^* = 0, & \text{if } D_{(0,L_1]} \leq y - z_p(y-x); \\ q^* = y - x, & \text{otherwise.} \end{cases}$ If $Q_e \leq y - x$ , then $q^* = 0$ .

**Table 2** Limiting values of  $z_p(Q)$

Cases	$c_f < p(L-l)$	$c_f \geq p(L-l)$
$K_2 \leq k_2$	$\lim_{Q \downarrow Q_e} z_p(Q) = -\infty$ $\lim_{Q \uparrow \bar{z} - \underline{z}} z_p(Q) = z_p(\bar{z} - \underline{z}) = \bar{z}$	
$K_2 > k_2$	$\lim_{Q \downarrow Q_r} z_p(Q) = \infty$ $\lim_{Q \uparrow \bar{z} - \underline{z}} z_p(Q) = z_p(\bar{z} - \underline{z}) = \bar{z}$	$\lim_{Q \downarrow Q_r} z_p(Q) = \infty$ $\lim_{Q \uparrow Q_e} z_p(Q) = -\infty$

REMARK 3. The values of  $Q$  for which  $z_p(Q)$  is defined in (4) form a convex set. It follows from the implicit function theorem that, inside this set  $z_p(Q)$  is continuous and differentiable at all points. Further,  $z_p(Q)$  has the limiting values given in Table 2.

To facilitate a unified representation of the function  $\Xi(x, y)$  that encompass all possible cases of the optimal freight mode decision in Proposition 1, we define the following.

DEFINITION 1. Define functions  $z_1(\cdot)$  and  $z_2(\cdot)$ , such that for all the cases the optimal freight mode decision is to ship the whole order quantity via regular freight when  $D_{(0,L_1]} < x - z_1(y-x)$ ; to ship the whole order quantity via express freight, when  $D_{(0,L_1]} > y - z_2(y-x)$ ; and to split the order between the two freight modes and ship-up-to  $z^*$  using express freight, otherwise.

In the rest of the paper, we suppress the arguments of the functions  $z_1$  and  $z_2$ , unless it becomes

ambiguous. It directly follows from the above definition and Proposition 1 that:

1. When  $q^*$  is dynamically chosen from 0,  $y - x$  and  $z^* + D_{(0,L_1]} - x$ , then  $z_1 = \underline{z}$  and  $z_2 = \bar{z}$ .
2. When  $q^*$  is dynamically chosen from 0 and  $y - x$  then  $z_1(y - x) = z_p(y - x) - y - x$  and  $z_2(y - x) = z_p(y - x)$ .
3. When  $q^* = 0$ , then  $z_1 = z_2 \rightarrow -\infty$ .
4. When  $q^* = y - x$ , then  $z_1 = z_2 \rightarrow \infty$ .

## 5. Optimal Ordering Decision: Finite Horizon Analysis

In this section, we analyze optimal ordering decisions at review epochs given each order is shipped according to the optimal policy derived in Proposition 1. We consider the inventory model in a finite horizon consisting of  $N$  review epochs indexed  $0, 1, \dots, N - 1$ . Without loss of generality, we assume that the horizon ends  $T + l$  time units after the review epoch  $N - 1$ , and each unit of residual physical inventory and back-ordered demand at that time incurs cost  $h_N$  and  $p_N$ , respectively. The decision of whether to and how much to order at review epoch  $n$  affects the inventory level, hence the inventory costs only after time  $nT + l$ , which is the earliest time at which any of the ordered units can be received. Thus, the relevant cost for the optimal ordering decision at that point is the expected cost incurred from time  $nT + l$  until the end of the horizon at time  $NT + l$ . Let  $V_n(x)$  denote the optimal value of this expected cost. We have  $V_N(x) = h_N \mathbb{E}(x - D_{(0,l]})^+ + p_N \mathbb{E}(D_{(0,l]} - x)^+$ . Let  $U_n(y)$  denote the expected cost from time  $nT + l$  until the end of horizon if no order is placed at review epoch  $n$  and orders are placed optimally at subsequent review epochs. The expression for  $U_n(y)$  is

$$U_n(y) = \mathcal{G}_{(l,l+T]}(y) + \tilde{V}_{n+1}(y), \quad (7)$$

where the first term is the expected inventory cost incurred in time interval  $(l, l + T]$  and the second term is the expected optimal cost of the rest of the time horizon (recall that  $\tilde{V}_{n+1}(y) = \mathbb{E}V_{n+1}(y - D_{(0,T]})$ ). Thus, given pre-order inventory position  $x$  at review epoch  $n$ , if no order is placed the system incurs cost  $U_n(x)$ , and if an order is placed to bring inventory position up to  $y > x$ , then the system incurs cost  $K_1 + \Xi(x, y) + \mathcal{G}_{(L,l+T]}(y) + \tilde{V}_{n+1}(y) = c(x, y) + U_n(y)$ . Where  $c(x, y) = \{K_1 + \Xi(x, y) - \mathcal{G}_{(l,L]}(y)\} \mathbb{1}_{\{y > x\}}$  is the difference between the expected cost if an order has to be placed to reach inventory position  $y$ , and the expected cost of starting at inventory position  $y$  (without having to place an order).  $c(x, y)$  is thus the effective cost of procurement with optimal use of two freight modes. The problem of optimal ordering decision at review epoch  $n$  can now be stated as the following dynamic program,

$$V_n(x) = \min_{y \geq x} \{c(x, y) + U_n(y)\} \quad \forall n = 0 \dots N - 1. \quad (8)$$

REMARK 4. The recursion in (8) also solves models considered in Scarf (1960), where  $c(x, y) = K\mathbb{1}_{\{y>x\}} + c \cdot (y - x)$ ; in Johnson (1968), where  $c(x, y) = M(x) + K(y)$ ; and in Porteus (1971), where  $c(x, y) = \min_i \{K_i\mathbb{1}_{\{y>x\}} + c_i \cdot (y - x)\}$ . The analysis presented here also applies to these models with minor modifications.

The following properties of the function  $c(x, y)$  are instrumental in the further analysis.

LEMMA 3. (a)  $c(x, y)$  is a continuous and differentiable function of  $x$  and  $y$  for  $x < y$ , with a possible discontinuity at  $x = y$  and  $c(x, y) \rightarrow K_1 + \min\{K_2, k_2\}$  as  $y \downarrow x$ .

(b) For given  $x$ ,  $c(x, y)$  is concave in  $y > x$ , and  $\partial c(x, y) / \partial y \rightarrow -h(L - l)$  as  $y \rightarrow \infty$ .

(c) For given  $y$ ,  $c(x, y) + \mathcal{G}_{(l, L]}(y) - \mathcal{G}_{(l, L]}(x)$  is concave in  $x < y$ , and  $\partial (c(x, y) + \mathcal{G}_{(l, L]}(y) - \mathcal{G}_{(l, L]}(x)) / \partial x \rightarrow \max\{-c_f + p(L - l), 0\}$  as  $x \rightarrow -\infty$ .

(d) For given order quantity  $Q$ ,  $c(x, x + Q)$  is non-increasing in  $x$  and  $c(x, x + Q) - \mathcal{G}_{(l, L]}(x) + \mathcal{G}_{(l, L]}(x + Q)$  is non-decreasing in  $x$ .

(e) For  $x_1 \leq x_2 \leq x_3$ ,  $c(x_1, x_3) \leq c(x_1, x_2) + c(x_2, x_3)$ .

Explanations of the above results are now in order. The discontinuity in  $c(x, y)$  at  $x = y$  is due to the fixed costs of ordering and using freight modes. For a very small order size, the complete order is shipped using the freight mode with smaller fixed cost. This explains the limiting value of  $c(x, y)$  as  $y \downarrow x$ . Note that for a given order-up-to level  $y$ ,  $c(x, y) + \mathcal{G}_{(l, L]}(y) - \mathcal{G}_{(l, L]}(x) - K_2$  is the difference in the cost with optimal use of two freights, and with whole order shipped via regular freight. It is thus the marginal cost of express freight at starting inventory position  $x$ . In the optimization problem in (1), a larger order quantity  $y - x$  enables a more efficient allocation of quantities to the two freight modes. The economy of scale inherent in the optimal freight mode decision explains concavity of functions in (b) and (c). For fixed  $x$  and very large value of  $y$ , each incremental unit of an order is shipped via regular freight. Comparing the inventory cost of placing an order with the inventory cost of starting at inventory position  $y$ , the incremental unit saves holding cost  $h(L - l)$ . This explains the limit in part (b). To explain the limit in part (c), note that for fixed order-up-to level  $y$  and very small  $x$ , on each additional unit of inventory  $x$ , the availability of express freight (in addition to regular freight) saves amount  $-c_f + p(L - l)$  if  $c_f < p(L - l)$ , and nothing, otherwise. Part (d) states that the cost of an order of fixed size  $Q$  is larger when it is placed at smaller inventory position  $x$ ; starting at smaller  $x$  increases the likelihood of stock-out, thus resulting in higher cost. This increase in cost at smaller  $x$  is even higher, if all units are to be shipped by regular freight (as the penalty cost of back-orders can not be mitigated by use of express freight), which explains the second result in part (d). Further, the triangle inequality in part (e) is equivalent to

the statement that to increase inventory position from  $x_1$  to  $x_3$ , it is more cost efficient to place a single order than two.

Finally, note that for a nontrivial case of the optimal freight decision, i.e., a case in which each freight mode is employed one time or other, the cost of the optimal freight decision  $\Xi(x, y)$  as well as  $c(x, y)$  are not separable in  $x$  and  $y$ . This makes the analysis of this inventory model and its solution complex in nature. Specifically, the optimal ordering policy is not a simple  $(s, S)$  policy in general. In the rest of this section, we first consider the single period version of our model and illustrate the general nature of the optimal ordering policy. Subsequently, we analyze the multi period case and provide bounds on the optimal policy parameters. Finally, using a numerical example we discuss the structure of the optimal dual freight policy and provide intuition behind it.

### 5.1. Single Period Solution

Consider the optimal ordering decision problem in (8) for the review epoch  $n = N - 1$ . The following lemma directly leads to the optimal ordering policy for this single period problem.

LEMMA 4. *For a given value of  $y$ , there exists at most one solution  $x < y$  to the equation*

$$U_{N-1}(x) - c(x, y) = U_{N-1}(y). \quad (9)$$

The above lemma essentially states that for each order-up-to inventory position  $y$ , there exists a unique threshold value of pre-order inventory position  $x$ , such that placing an order results in reduction of cost if and only if  $x$  is smaller than the threshold value.

For  $n = 0, \dots, N - 1$ , define  $S_n(x) = \arg \min_y \{c(x, y) + U_n(y)\}$ . The following proposition provides the optimal policy for the single period problem.

PROPOSITION 2. *There exists a threshold value  $s_{N-1}$  of pre-order inventory position  $x$ , such that: If  $x < s_{N-1}$  then  $S_{N-1}(x) > x$ , and  $S_{N-1}(x) = x$  otherwise.*

We refer to this optimal policy as a *threshold policy with state dependent order-up-to levels* or simply a *threshold policy* in the rest of the paper. Note that a simple  $(s, S)$  policy as well as a generalized  $(s, S)$  policy studied in Porteus (1971) are special cases of threshold policies. In case of an  $(s, S)$  policy  $S(x)$  is a constant, and in case of a generalized  $(s, S)$  policy  $S(x)$  is a non-increasing step function, for  $x$  smaller than a threshold value. In our case,  $S(x)$  is neither non-increasing nor a step function.

## 5.2. Multi Period Solution

Now consider the problem of optimally placing orders for review epochs  $0 \dots N - 2$ . As illustrated in the single period case, the structure of the optimal ordering policy with two freight modes is not very simple. Moreover, in a large set of (over 50,000) numerically solved instances of our model, we found large variations in the structure of optimal policy from case to case. These observations render the complete characterization of the optimal policy elusive as well as inconsequential. In this section, we first establish optimal ordering policies for special cases of our model. Subsequently, we turn our focus towards providing bounds on the long run optimal policy that are easy to compute and make the computation of the optimal policy considerably faster.

Our analysis requires the following function and its properties illustrated in Lemma 5: Let,

$$\Gamma(x) = \mathbb{E} \left[ \min_{q \geq 0} \left\{ c_f q + \int_l^L G(x + q - D_{(0,L_1]}, D_{(L_1,t]}) dt \right\} \right], \quad (10)$$

and  $S^0 = \arg \min_x \left\{ \tilde{\Gamma}(x) - \Gamma(x) + \mathcal{G}_{(l,l+T]}(x) \right\}$ .

LEMMA 5. (a)  $\Gamma(x)$  and  $\mathcal{G}_{(l,L]}(x) - \Gamma(x)$  are convex in  $x$ .

(b) For a fixed value of  $y$ ,  $d\Gamma(x)/dx \geq \partial c(x, y)/\partial x$ .

(c) For fixed value of  $x$ ,  $d\Gamma(y)/dy \geq -\partial c(x, y)/\partial y$ .

The optimization problem on the right-hand side of (10) is the optimal freight mode decision problem in the absence of fixed costs  $K_2$  and  $k_2$ , and the constraint  $q \leq y - x$ . Using the result of Lemma 2, the optimal solution to this problem is to ship-up-to  $z^*$  with express freight when  $c_f < p(L - l)$ , and ship nothing with express freight, otherwise. It can be shown that when  $K_2 = k_2 = 0$ ,  $c(x, y) = K_1 \mathbb{1}_{\{y > x\}} + \Gamma(x) - \Gamma(y)$ . Furthermore, for the special case of our model with  $K_1 = K_2 = k_2 = 0$ , the optimal policy for placing orders is a base-stock policy. For this case, the long run per period expected cost with base-stock level  $S$  is  $\tilde{\Gamma}(S) - \Gamma(S) + \mathcal{G}_{(l,l+T]}(S)$ , and it is minimized at the optimal base-stock level  $S^0$  (see Groenevelt and Rudi (2002) for detailed analysis of this case).

The following properties of function  $V_n(x)$  are instrumental in further analysis.

LEMMA 6. For  $n = 0, \dots, N - 1$ : (a) for  $x_1 \leq x_2$ ,

$$V_n(x_1) \leq c(x_1, x_2) + V_n(x_2). \quad (11)$$

(b)  $V_n(x)$  is  $K_1 + \max\{K_2, k_2\}$  convex.

Part (a) of the above lemma has the following explanation: For each  $x$ ,  $V_n(x)$  is the result of optimization over all possible opportunities of placing an order at review epoch  $n$ . Thus, this cost can not be further reduced by placing an order (in addition to a potential order that solves the dynamic program) at review epoch  $n$ .

Using Lemma 6, the next proposition provides the optimal policy for the two special cases of our model namely  $K_2 = k_2$  and  $K_2 = k_2 = 0$ .

PROPOSITION 3. *For  $n = 0, \dots, N - 1$ : (a) If  $K_2 = k_2$ , then a threshold policy with state dependent order-up-to levels is optimal at review epoch  $n$ .*

*(b) If  $K_2 = k_2 = 0$ , then the optimal ordering policy is an  $(s, S)$  policy.*

Now we analyze optimal ordering policy for general values of fixed costs  $K_2$  and  $k_2$ . In order to prove Lemma 7, we assume the following property of the terminal cost function  $V_N(x)$ .

ASSUMPTION 1. *For  $x < y \leq S^0$ ,  $V_N(x) \leq \Gamma(x) - \Gamma(y) + V_N(y)$ .*

This assumption implies that if orders were allowed at time  $NT$  without charging any fixed costs, then placing orders would result in cost reduction for all inventory position  $x < S^0$ .

LEMMA 7. *For  $n = 0, \dots, N - 1$ ,  $V_n(x) - \Gamma(x)$  is non-increasing in  $x$  for  $x \leq S^0$*

The above lemma states that at inventory positions smaller than  $S^0$ , placing an additional order would reduce cost if none of the fixed costs were charged. Note that this property holds asymptotically for any fixed  $n$  as  $N$  increases regardless of  $V_N(x)$ . When  $V_N(x)$  does not satisfy Assumption 1, numerical investigation reveals that  $V_n(x)$  rapidly start to satisfy the condition in Lemma 7 as  $n$  decreases.

The next lemma specifies the bounding parameters for the optimal order policy.

LEMMA 8. (a) *Define*

$$\underline{S}(x) = \arg \min_{y \geq x} \left\{ c(x, y) + \tilde{\Gamma}(y) + \mathcal{G}_{(l, l+T]}(y) \right\}, \quad (12)$$

*then, there exists a threshold value  $\underline{s}$ , such that: If  $x < \underline{s}$ , then  $x < \underline{S}(x) \leq S^0$ , and  $\underline{S}(x) = x$  otherwise.*

*(b) There exists a threshold value  $\bar{s}$  satisfying  $\arg \min_u \mathcal{G}_{(l, l+T]}(u) \leq \bar{s} \leq \arg \min_u \mathcal{G}_{(L, L+T]}(u)$ , such that for  $x > \bar{s}$ ,*

$$\mathcal{G}_{(l, l+T]}(x) < c(x, y) - \tilde{c}(x, y) + \mathcal{G}_{(l, l+T]}(y), \forall y > x. \quad (13)$$

*(c) There exists a unique solution  $y = \bar{S} \geq S^0$  to the equation,*

$$\mathcal{G}_{(l, l+T]}(S^0) - \Gamma(S^0) = \mathcal{G}_{(l, l+T]}(y) - \Gamma(y) - \tilde{c}(S^0, y). \quad (14)$$

Using the above lemma, we are now ready to bound the optimal ordering policy for our model.

PROPOSITION 4. *For review epochs  $n = 0, \dots, N - 2$ , the optimal ordering decision satisfies:*

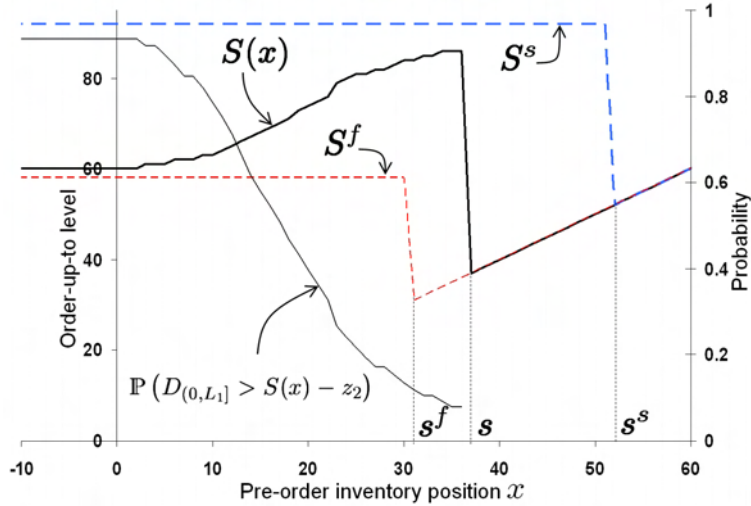
- (a) *If  $x < \underline{s}$ , then  $S_n(x) > x$ , and  $\underline{S}(x) \leq S_n(x) \leq \bar{S}$ .*
- (b) *If  $x > \bar{s}$ , then  $S_n(x) = x$ .*

Note that all the bounding parameters on the optimal policy are solutions to single period problems, hence are easy to compute. Furthermore, when  $K_1 = K_2 = k_2 = 0$ , we have  $\underline{s} = \bar{s} = \bar{S} = S^0$  and  $\underline{S}(x) = S^0, \forall x \leq S^0$ . For this special case of our model, the optimal ordering policy is the base-stock policy with order-up-to level  $S^0$ . Thus, the bounds on the optimal policy presented in Proposition 4 are attained, hence tight.

Although the characterization of optimal ordering policy in Proposition 4 is incomplete (we were unable to prove the existence of  $s_n \in [\underline{s}, \bar{s}]$  such that  $S_n(x) > x$  if and only if  $x \leq s_n$ ). In all the examples we have solved in §7, the optimal policy was a threshold policy.

Given the general nature of the optimal ordering policy, it is now worthwhile to discuss its structure and gain insight into it. Figure 1 illustrates an example of the optimal policy for placing orders in our model. In this figure, we have plotted the values of optimal order-up-to levels  $S(x)$  against pre-order inventory position  $x$ . These optimal order-up-to levels are obtained by solving the dynamic program with a compound Poisson demand process, for a time horizon sufficiently long to achieve stationarity. Thus, the optimal policy depicted in the figure is a long-run stationary policy (hence we drop the subscript  $n$  in the notational scheme). The optimal ordering policies for single freight models, each an  $(s, S)$  policy, are also plotted with dotted lines in the same figure. The optimal reorder levels and order-up-to levels for model with only regular (express) freight are denoted by  $s^s$  and  $S^s$  ( $s^f$  and  $S^f$ ), respectively.

Consider a review epoch at which an order is placed to increase the inventory position from  $x$  to  $y$ . The earliest time from which the order-up-to level  $y$  starts determining the inventory levels in the system is the effective lead-time for the order-up-to level decision. A larger value of the effective lead-time results in a larger difference between inventory level and inventory position, and hence leads to a larger value of the optimal order-up-to level for the ordering decision. The ex-post value of effective lead-time is  $l$  if all the units in the order is shipped by express freight (when  $D_{(0, L_1]} > y - z_2$ ), and is  $L$  otherwise. Thus, the ex-ante optimal order-up-to level depends on the probability  $\mathbb{P}(D_{(0, L_1]} > y - z_2)$ ; a larger value of this probability leads to a higher likelihood of effective lead-time faced by the decision maker being  $l$ , and hence a smaller optimal order-up-to level. In Figure 1, where we also plot values of  $\mathbb{P}(D_{(0, L_1]} > S(x) - z_2)$  along with  $S(x)$ , one can see that a decrease in the probability is accompanied by an increase in  $S(x)$ .



**Figure 1** An example of the optimal policy for placing orders.

For sufficiently small values of  $x$ , orders are sufficiently large in size and  $z_2 = \bar{z}$  and hence, the probability  $\mathbb{P}(D_{(0,L_1]} > y - z_2)$  does not depend on the value of  $x$ . Thus, for such values of  $x$ , the optimal order-up-to level  $S(x)$  is independent of  $x$ , or in other words  $S(x)$  is constant. For larger values of  $x$ ,  $z_2$  becomes  $z_p(y - x)$ , a function of the order quantity  $y - x$ . Consequently, for such values of  $x$ , the probability  $\mathbb{P}(D_{(0,L_1]} > y - z_2)$  depends on  $y$  as well as  $x$ , leading to an optimal order-up-to level  $S(x)$  that changes with  $x$ . Clearly, when  $K_1$  is sufficiently large to ensure that the order size is greater than  $\bar{z} - \underline{z}$ ,  $S(x)$  is a constant whenever an order is placed. Noting a large value of  $K_2$  results in a large value of  $\bar{z} - z^*$  (Equation 5), and a large  $k_2$  results in a large value of  $z^* - \underline{z}$  (Equation 6); it follows that when  $K_1$  is large as compared to  $K_2$  and  $k_2$ , the optimal ordering policy is an  $(s, S)$  policy. This is also substantiated by Proposition 3(c), where the extreme case  $K_2 = k_2 = 0$  of the condition  $K_1 \gg K_2, k_2$  is considered.

## 6. Best $(s, S)$ Policy: Infinite Horizon Analysis

In this section, we characterize the best stationary  $(s, S)$  policy for our model in the infinite horizon setting. Recall that in the finite horizon setting, when the fixed cost of placing an order  $K_1$  is sufficiently larger than the fixed costs of using freight modes  $K_2$  and  $k_2$ , the optimal ordering policy at each review epoch is an  $(s, S)$  policy. As the infinite horizon setting is the limiting case of the finite horizon setting, it can be argued that under the same conditions a stationary  $(s, S)$  policy is optimal for the infinite horizon setting. Moreover, when the optimal policy is not an  $(s, S)$  policy, the best  $(s, S)$  policy may still be very appealing as it is easy to implement, and its cost differs from the optimal cost by a very small margin (in §7, we illustrate this fact using a large set of numerically solved examples).



With a stationary  $(s, S)$  policy, given the *i.i.d.* nature of single period demands  $D_{(0,T]}$ , the inventory position after the ordering decisions at review epochs follows a discrete time regenerative process. The review epochs on which orders are placed, can be set as the regeneration points for this regenerative process. Define  $\mathcal{N}_v = \min\{n : D_{(0,nT]} > v\}$ , then starting a review period at inventory position  $v$  units above the reorder level, the process  $\mathcal{N}_v$  counts the number of review periods elapsed until the next order is placed. Similarly, starting a review period at inventory position  $v$  units above the reorder level  $s$ , let  $\kappa(s, v)$  denote the relevant expected inventory holding and back-order penalty cost incurred until the next order is placed. Conditioning  $\mathbb{E}\mathcal{N}_v$  and  $\kappa(s, v)$  on the demand incurred in the current period leads to the following recursions,

$$\mathbb{E}\mathcal{N}_v = 1 + \int_0^v \mathbb{E}\mathcal{N}_{v-u} dF_{(0,T]}(u), \quad (15)$$

$$\kappa(s, v) = \mathcal{G}_{(l,l+T]}(s+v) + \int_0^v \kappa(s, v-u) dF_{(0,T]}(u). \quad (16)$$

Let  $M(v)$  and  $m(v)$  denote the renewal function and the renewal density function, respectively, associated with single review period demand  $D_{(0,T]}$ . Then, it follows from the renewal theorem (Proposition 3.4 of Ross 1983) that  $\mathbb{E}\mathcal{N}_v = 1 + M(v)$  and

$$\kappa(s, v) = \mathcal{G}_{(l,l+T]}(s+v) + \int_0^v \mathcal{G}_{(l,l+T]}(s+v-u) dM(u), \quad (17)$$

are the unique solutions to recursions (15) and (16), respectively. Define parameter  $\Delta = S - s$ , the minimum order quantity for an  $(s, S)$  policy. Using the renewal reward theorem (Theorem 3.16 of Ross 1983), we obtain the following expression for the long run expected cost per review period of a stationary  $(s, S)$  policy,

$$C(s, \Delta) = \frac{K_1 + \mathbb{E}c(\mathcal{X}_{(s,\Delta)}, s + \Delta) + \kappa(s, \Delta)}{1 + M(\Delta)}, \quad (18)$$

where  $\mathcal{X}_{(s,\Delta)}$  is a random variable representing the pre-order inventory position at the review epochs on which orders are placed and it has a stationary probability distribution that is a function of  $s$  and  $\Delta$ . For notational simplicity we suppress the dependence of  $\mathcal{X}_{(s,\Delta)}$  on  $s$  and  $\Delta$ , unless it is ambiguous. Let  $s^*(\Delta)$  denote the optimal value of the reorder level, for  $\Delta \geq 0$ .

PROPOSITION 5.  $s^*(\Delta)$  satisfies

$$\frac{\left( \begin{array}{c} \mathcal{P}(s, \Delta) \\ - \int_l^L \left( \begin{array}{c} \mathbb{P}(\mathcal{X} < D_{(0,t]} < s + \Delta, D_{(0,L_1]} < \mathcal{X} - z_1) \\ - \mathbb{P}(D_{(L_1,t]} < z^*, \mathcal{X} - z_1 < D_{(0,L_1]} < s + \Delta - z_2) \\ + \mathbb{P}(D_{(0,t]} < s + \Delta, \mathcal{X} - z_1 < D_{(0,L_1]} < s + \Delta - z_2) \end{array} \right) dt \end{array} \right)}{(1 + M(\Delta))T} = \frac{p}{h + p}, \quad (19)$$

where

$$\mathcal{P}(s, \Delta) = \int_l^{l+T} \mathbb{P}(D_{(0,t]} < s + \Delta) dt + \int_0^\Delta \left( \int_l^{l+T} \mathbb{P}(D_{(0,t]} < s + \Delta - u) dt \right) dM(u). \quad (20)$$

Assume an order is placed at time  $t = 0$ , let  $I(t)$  denote inventory level at time  $t$ . Then, the following corollary provides a managerial interpretation to the optimality condition in equation (19).

**COROLLARY 1.** *For a given value of  $\Delta$ ,  $s^*(\Delta)$  satisfies*

$$\frac{1}{(1 + M(\Delta))T} \mathbb{E} \int_l^{l+\mathcal{N}_\Delta T} \mathbb{1}_{\{I(t) > 0\}} dt = \frac{p}{h + p}. \quad (21)$$

The ordering decision at a review epoch affects the inventory levels in the time interval, that starts  $l$  time units following the review epoch and lasts for  $\mathcal{N}_\Delta$  review periods. In the expression on the left-hand side of (21), the denominator  $(1 + M(\Delta))T$  is the expected length of time interval  $(l, l + \mathcal{N}_\Delta T]$ , whereas, the numerator is the total time in the interval  $(l, l + \mathcal{N}_\Delta T]$ , when the system has positive on-hand inventory. Thus, the term on the left-hand side is the time average probability of being in-stock in a replenishment cycle. Applying the renewal reward theorem, this is the same as the long run fraction of time the system has positive on-hand inventory. Thus, Corollary 1 links the optimal reorder level to a measure of service level. Similar observations have been made by Gallego (1998), and Rudi et al. (2004) in single lead-time settings, and by Groenevelt and Rudi (2002), and Jain et al. (2005) in dual lead-time settings such as ours.

For a given value of  $\Delta$ , let  $s^s(\Delta)$  denote the optimal reorder level if only regular freight is used and let  $s^f(\Delta)$  denote the optimal reorder level if only express freight is used.

**LEMMA 9.**  $s^f(\Delta) \leq s(\Delta) \leq s^s(\Delta)$ .

Intuitively, for given values of  $s$  and  $\Delta$ , the average inventory level in the model with two freight modes is greater than that in the model with only regular freight, and is smaller than that in the model with only express freight. However, for all three models the optimal reorder level sets the time average probability of being in-stock to the ratio  $p/(h + p)$ , implying the result stated in Lemma 9.

Let  $\Delta^*$  denote the optimal value of  $\Delta$ , i.e., the minimum order quantity with the best  $(s, S)$  policy for our model.

**PROPOSITION 6.** *Assuming that  $s^*(\Delta)$  is continuous in  $\Delta$ , then  $\Delta^*$  is a solution to*

$$C(s^*(\Delta), \Delta) = \mathcal{G}_{(l, l+T]}(s^*(\Delta)) + \mathbb{E}c(s^*(\Delta) - D_{(0,T]}, s^*(\Delta) + \Delta) - c(s^*(\Delta), s^*(\Delta) + \Delta). \quad (22)$$

The expression  $\mathcal{G}_{(l,l+T]}(s) + \mathbb{E}c(s - D_{(0,T]}, s + \Delta) - c(s, s + \Delta)$  denotes the cost incurred in a review period with pre-order inventory position  $s$ , if placing an order up to  $S$  is delayed to the next period. In other words, it is the marginal cost of delaying placement of the order by one period, at the reorder level. The optimality condition in (22), thus, states that at optimality the marginal cost of delaying the order placement at the reorder point is equal to the average cost.

## 7. Numerical Illustrations

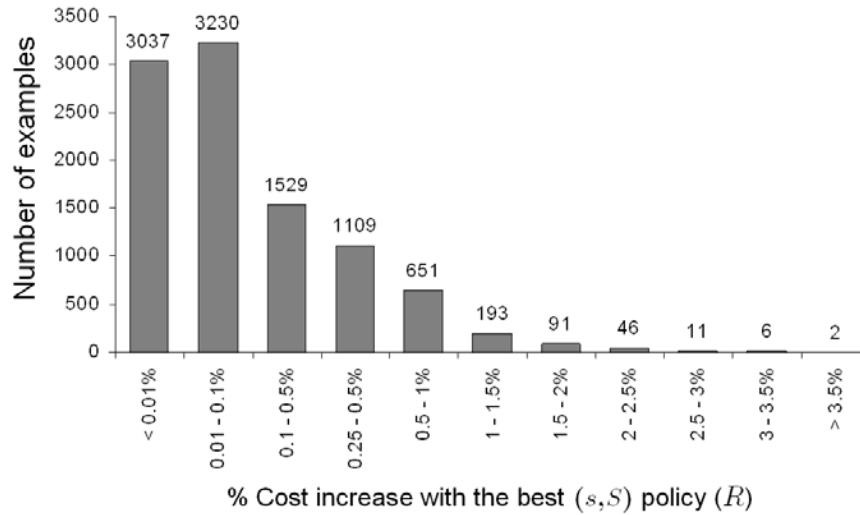
In this section, we illustrate properties of our model using numerical examples. In all the numerical examples discussed in this section, the length of the review period  $T$  is 1, the difference between lead-times  $L_2$  and  $l_2$  is 0.5, and the sum of inventory holding cost rate and back-order penalty cost rate, i.e.,  $h + p$ , is set to 10. The demand process is compound Poisson with individual demand taking values 1, 2, 3, 4 and 5 with probabilities 0.15, 0.25, 0.3, 0.2 and 0.1, respectively.

First, we evaluate the performance of the best  $(s, S)$  policy against the optimal policy for different values of model parameters. For this purpose, we solve 48600 instances of our model for the optimal policy and the best  $(s, S)$  policy. We consider 5 different sets of lead-times,  $(L_1, l_2, L_2) \in \{(0.1, 0.1, 0.6), (0.35, 0.1, 0.6), (0.6, 0.1, 0.6), (0.35, 0.35, 0.85), (0.35, 0.6, 1.1)\}$ . For each set of lead-times, we solve numerical examples with all combinations of the following values of model parameters:  $\mu \in \{5, 25, 50\}$ ,  $h \in \{0.5, 1, 2\}$ ,  $c_f \in \{0.1, 0.3, 0.5\}$ ,  $K_2, k_2 \in \{0, 10, 25, 50, 75, 100\}$  and  $K_1 \in \{0, 10, 25, 50, 75, 100, 250, 500, 750, 1000\}$ . And, for each solved example we calculate the following performance measure for the best  $(s, S)$  policy,

$$R \equiv \frac{C^{(s,S)} - C^*}{C^*} \%,$$

where  $C^{(s,S)}$  is the cost of best  $(s, S)$  policy and  $C^*$  is the cost of the optimal policy.  $R$ , therefore measures the percent increase in cost with the best  $(s, S)$  policy over the optimal cost. Of the 48600 examples that we numerically solve, in 38695 (79.62% of all the examples) the optimal policy is an  $(s, S)$  policy.

For the remaining 9905 examples, the values of performance measure  $R$  are plotted on a frequency-chart in Figure 2. It can be observed from the figure that in only 349 examples (0.72% of all the examples and 3.52% of the examples in which the optimal policy is not an  $(s, S)$  policy), the cost of the best  $(s, S)$  policy is more than 1% higher than the optimal cost. Moreover, in only 65 examples (0.13% of all the examples) this cost increase is more than 2% and in none on the examples is it more than 5%. This suggests that the best  $(s, S)$  policy is a robust heuristic for ordering in our model. Taking a closer look at the 349 instances of our model, in which the best



**Figure 2** Performance of the best  $(s, S)$  policy.

$(s, S)$  policy results in more than a 1% cost increase over the optimal cost, we find that in all these instances the fixed cost of placing an order  $K_1$  is smaller than the fixed costs of using freight modes  $K_2$  and  $k_2$ . Furthermore, in most of these examples the demand rate is 50 and holding cost rate is 2. This indicates that the best  $(s, S)$  policy may lead to larger cost errors when  $K_1 < K_2, k_2$  and the demand rate and the ratio  $h/(h+p)$  are large. Or in other words, when one or more of these conditions are met, the best  $(s, S)$  policy should be carefully compared with the optimal policy before implementation.

Table 3 compares the values of performance measure  $R$  across different sets of lead-times  $(L_1, l_2, L_2)$ . In this table, for each set of lead-times we provide the following: the average value of  $R$ ; and numbers of numerical examples with the value of  $R$  greater than 0.1%, 0.5%, 1% and 2%. Comparing rows 1, 2 and 3, we observe that as  $L_1$  increases (while  $l_2$  and  $L_2$  remain the same), the average value of  $R$  decreases and the number of examples with  $R$  greater than a specific value decreases. This implies that the cost of best  $(s, S)$  policy does not increase proportionately with the optimal cost as  $L_1$  increases. Similar, although somewhat less pronounced, trend is observed in rows 2, 4 and 5, where  $l_2$  and  $L_2$  increase, while  $L_1$  and  $L_2 - l_2$  remain the same. Together these two observations lead us to conclude that the best  $(s, S)$  policy is a better surrogate for the optimal policy at higher values of lead-times.

Figures 3, 4 and 5 illustrate the sensitivity of the optimal policy for our model to freight mode fixed costs  $K_2$  and  $k_2$ , per-unit cost of express freight mode  $c_f$ , and ordering fixed cost  $K_1$ , respectively. The plots in these figures are presented in two columns: In the first column the optimal values of average cost  $C^*$  and fraction of units shipped by express freight  $E_{q^*}/Q^*$  are plotted,

**Table 3** Performance of the best  $(s, S)$  policy for different sets of lead-times.

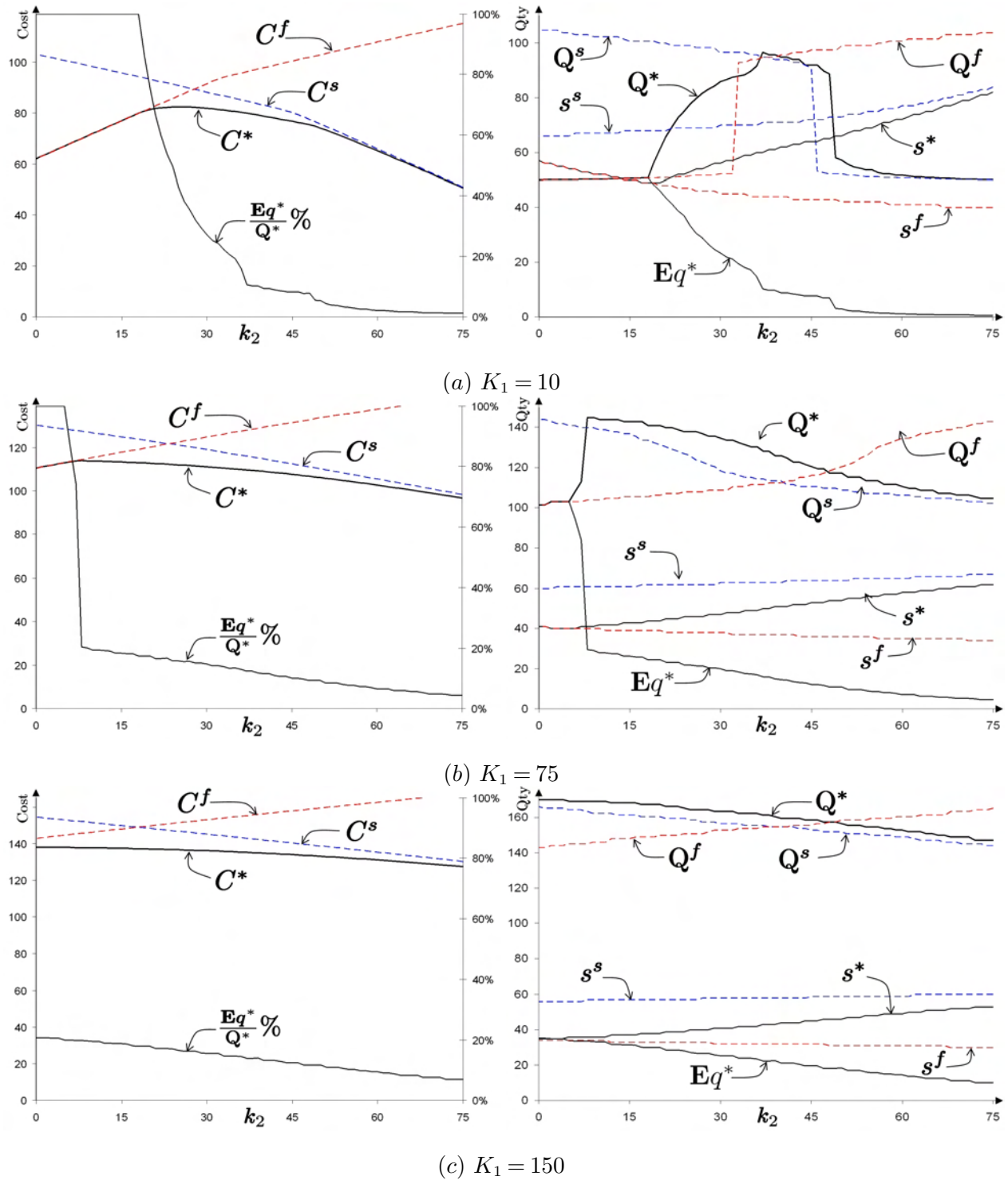
No.	Lead time values ( $L_1, l_2, L_2$ )	Performance of the best $(s, S)$ policy				
		Average $R$	Number of examples with <sup>‡</sup> :			
			$R > 0.1\%$	$R > 0.5\%$	$R > 1\%$	$R > 2\%$
1	(0.1, 0.1, 0.6)	0.048%	843	299	109	26
2	(0.35, 0.1, 0.6)	0.037%	736	195	72	12
3	(0.6, 0.1, 0.6)	0.031%	670	150	50	9
4	(0.35, 0.35, 0.85)	0.035%	707	184	62	9
5	(0.35, 0.6, 1.1)	0.033%	682	172	56	9

<sup>‡</sup>Out of total 9720 examples for each set of lead-times.

while in the second column the optimal values of reorder level  $s^*$ , average order size  $Q^*$  and average number of units shipped via express freight  $\mathbb{E}q^*$  are plotted. To enable a comparison, in each plot, the optimal values of cost, reorder level and average order quantity for the single freight models are plotted using dotted lines. In these plots,  $h = 1$ ,  $p = 9$  and  $\mu = 50$ .

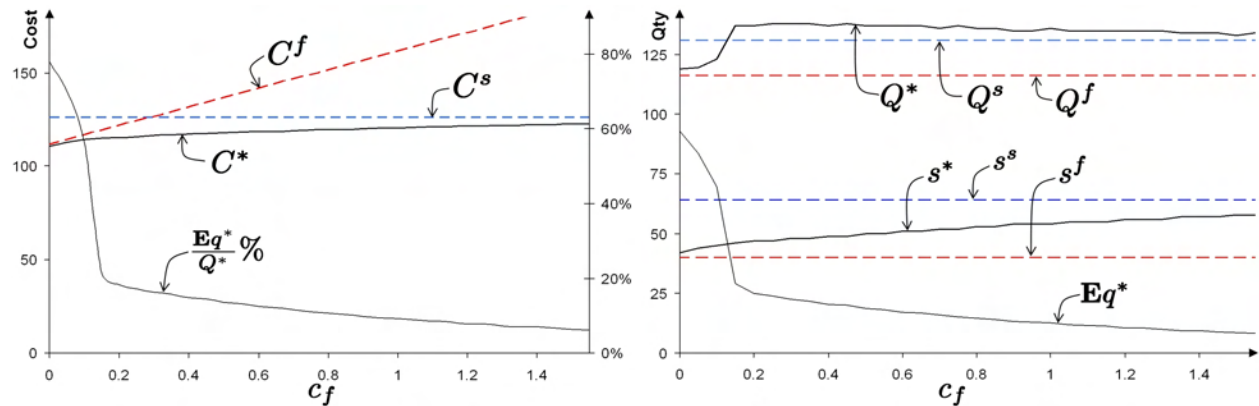
In Figure 3, for three different values of  $K_1$  and the fixed value of the sum  $K_2 + k_2 = 75$ , optimal cost, reorder level and average order quantity are plotted against  $k_2 \in [0, 75]$ . First, consider the two plots for  $K_1 = 10$ : In these plots, at the left and right extremes, (i.e., near  $K_2 = 75, k_2 = 0$  and  $K_2 = 0, k_2 = 75$ ) the optimal policy resembles the optimal policy for the single freight model with smaller fixed cost. At these points, the fraction of an average order shipped by express freight is 1 on the left side, and 0 on the right side, indicating that in these cases the freight mode with smaller fixed cost is almost always used and the other freight mode is used only in extreme cases of the manufacturing lead-time demand. However, for intermediate values of  $K_2$  and  $k_2$ , both freight modes are used for shipping a significant fraction of ordered units, and the optimal cost with dual freight model  $C^*$  is much smaller than optimal costs with both single freight models,  $C^s$  and  $C^f$ . Now, compare the plots for  $K_1 = 10$ , with plots for  $K_1 = 75$  and  $K_1 = 150$ : It can be observed that as  $K_1$  increases, the intermediate region (where the dual freight model outperforms the single models by a significant margin) expands. In other words, for a larger value of  $K_1$ , the dual freight model performs better than the best of the single freight models for wider combinations of  $K_2$  and  $k_2$ , and is less sensitive to the values of  $K_2$  and  $k_2$ .

In Figure 4 for  $K_1 = 75$ ,  $K_2 = 50$  and  $k_2 = 25$ , optimal cost, reorder level and average order quantity are plotted against  $c_f \in [0, 1.5]$ . These plots show that for very small values of  $c_f$ , the dual freight model resembles the model with only express freight; for large values of  $c_f$  it resembles the model with only regular freight; and in the intermediate region it outperforms the single freight models by a significant margin. In Figure 5, where  $K_2 = 50$ ,  $k_2 = 25$  and  $c_f = 0.3$ , optimal cost, reorder level and average order quantity are plotted against  $K_1 \in [0, 500]$ . We note that as  $K_1$

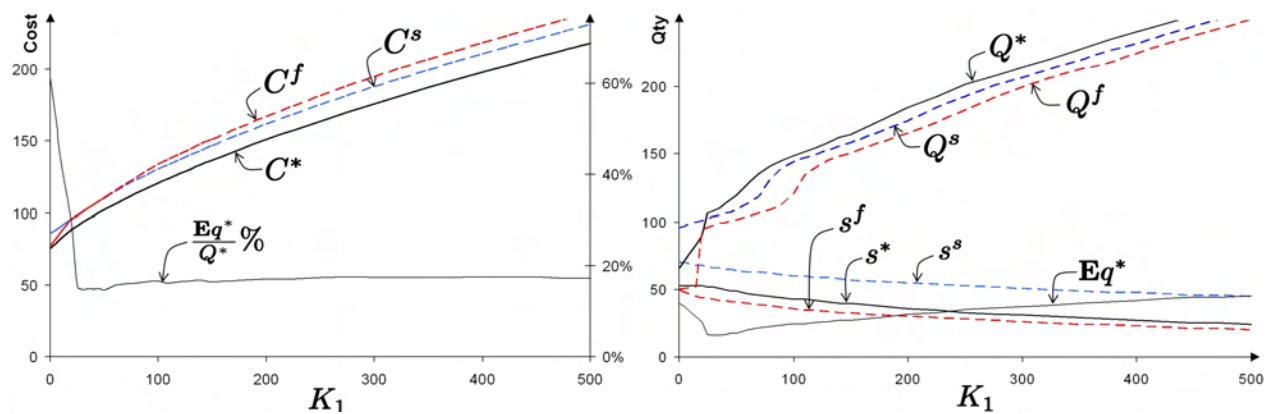


**Figure 3** Optimal cost and policy parameters against  $k_2$ .

increases, the fraction of demand shipped via express freight mode  $Eq^*/Q^*$  remains relatively stable. These plots also show that for sufficiently large values of  $K_1$  the optimal values of cost, reorder level and average order quantity has similar asymptotic behavior as in the single freight



**Figure 4** Optimal cost and policy parameters against  $c_f$ .



**Figure 5** Optimal cost and policy parameters against  $K_1$ .

models.

Summarizing the key observations from Figures 3, 4 and 5: The performance of the model with two freight modes is more sensitive to the fixed cost of freight modes  $K_2$  and  $k_2$ , when  $K_2 + k_2$  is large as compared to the fixed cost of ordering  $K_1$ . On the other hand, the effect of variable cost of express freight  $c_f$  does not change as much with  $K_1$ . Optimal use of two freight modes earns large savings in cost when the costs of single freight models are comparable. However, when one of the single freight models dominates the other by a large margin, the optimal use of the two freight modes performs only marginally better than the single freight model with lower cost.

## 8. Conclusion

This paper studies a periodic review inventory control problem, in which inventory related decisions of when and how much to order and logistics decisions of how to ship ordered units, are optimized simultaneously. Our work contributes to the existing literature on this problem by incorporating the economies of scale in transportation cost and the responsive nature of logistics decisions. In addition, we account for the inventory holding and back-order penalty costs continuously in time

and do not restrict the values of lead-times to integer multiples of the review period length.

Our analysis provides valuable insights into the effects of freight mode fixed costs on the optimal policy for transportation as well as ordering decisions. In the dynamic optimal policy for the freight mode decision, only orders greater than a threshold value, which is determined by the fixed costs, are split across the two freight modes. Orders smaller than the threshold value, are shipped with only one of the freight modes determined by the relative values of their fixed costs and the demand incurred during manufacturing. For the ordering decisions, we show that when each order is shipped optimally using the two freight modes, the optimal policy is not always a simple  $(s, S)$  policy. In our extensive numerical experiments the optimal policy was always of threshold type. Although an analytical proof of this result for the general case of our model remains elusive, we obtain bounds on the optimal policy. Furthermore, we discuss conditions on fixed costs under which the optimal policy becomes an  $(s, S)$  policy. The analytical and numerical findings in this paper indicate that when fixed cost of placing orders is small relative to the fixed costs of freight modes, the transportation costs dominate the savings in inventory costs, and the optimal decisions are similar to the single freight model with smaller cost. However, when the fixed cost of placing orders is large, the variable cost of express freight plays a more dominant role in determining the usage of each freight mode. This suggests that in presence of large economies of scale in transportation costs as compared to ordering cost, it is advisable to rely primarily on the cheaper freight mode and use the other under extreme circumstances.

Our analysis of the periodic review inventory model consider a very general nature of procurement cost  $c(x, y)$ . We provide easily computable bounds on the optimal policy with this procurement cost, which can be extended to other settings. Since the optimal policy with this procurement cost is not very simple hence difficult to implement, an important managerial concern is how well the best  $(s, S)$  policy performs. Using an extensive set of numerically solved problems, we compare the performance of the best  $(s, S)$  policy with the optimal policy, and show that the best  $(s, S)$  policy serves as a robust heuristics in most cases.

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# Supplement to Periodic Review Inventory Management with Contingent Use of Two Freight Modes with Fixed Costs

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This document contains the proofs for Jain et. al. (2006a) and follows the notation introduced there. Any additional notation is presented below.

PROOF OF LEMMA 1. (a) The convexity of  $g(u)$  in  $u$  directly follows from the convexity of function  $G(u, D_{(L_1, t]})$ . Taking the first derivative of  $g(u)$ ,

$$\frac{\partial g(u)}{\partial u} = c_f + (h + p) \int_l^L \mathbb{P}(D_{(L_1, t]} < u) dt - p(L - l) \geq c_f - p(L - l).$$

Thus, when  $c_f \geq p(L - l)$ ,  $g(u)$  is non-decreasing for all values of  $u$ , or  $z^* \rightarrow -\infty$ . When  $c_f < p(L - l)$ , then it follows from the convexity of  $g(u)$  that  $z^*$  is the unique solution to the first order condition in (2).

(b) Taking the first derivative  $g(u + q) - g(u)$  with respect to  $u$  we get

$$(h + p) \int_l^L (\mathbb{P}(D_{(L_1, t]} < u + q) - \mathbb{P}(D_{(L_1, t]} < u)) dt \geq 0,$$

where the inequality follows from the properties of cumulative probability and implies that  $g(u + q) - g(u)$  is non-decreasing in  $u$ . Using this property, it follows that the infimum of  $g(u + q) - g(u)$  is obtained by taking the limit as  $u \downarrow -\infty$  and the supremum is obtained by taking the limit as  $u \uparrow \infty$ . Taking these limits we get the desired bounds.  $\square$

PROOF OF PROPOSITION 1. Remark 1 implies that for all cases the optimal value of  $q$  is one of 0,  $y - x$  and  $z^* - x + D_{(0, L_1]}$ . The last choice needs to be considered only when  $x - z^* \leq D_{(0, L_1]} \leq y - z^*$ , and can be eliminated when  $c_f \geq p(L - l)$  (Remark 2). The values of the objective function with choices  $q = 0$ ,  $y - x$  and  $z^* - x + D_{(0, L_1]}$  are,

$$K_2 + g(x - D_{(0, L_1]}) - c_f(x - D_{(0, L_1]}), \tag{S1}$$

$$k_2 + g(y - D_{(0,L_1]}) - c_f(x - D_{(0,L_1]}), \quad (\text{S2})$$

$$K_2 + k_2 + g(z^*) - c_f(x - D_{(0,L_1]}). \quad (\text{S3})$$

First consider the case  $c_f < p(L-l)$ , for which (S1), (S2) and (S3) have to be compared. The difference between (S1) and (S3), namely  $-k_2 + g(x - D_{(0,L_1]}) - g(z^*)$ , is non-decreasing in  $D_{(0,L_1]}$  for  $x - D_{(0,L_1]} \leq z^*$  (Lemma 1(a)), and vanishes at  $x - D_{(0,L_1]} = \underline{z}$  (the definition of  $\underline{z}$  in (6)). Similarly, the difference between the expressions (S2) and (S3), namely  $-K_2 + g(y - D_{(0,L_1]}) - g(z^*)$ , is non-increasing in  $D_{(0,L_1]}$  for  $y - D_{(0,L_1]} \geq z^*$  (Lemma 1(a)) and vanishes at  $y - D_{(0,L_1]} = \bar{z}$  (the definition of  $\bar{z}$  in (5)). It follows from these two comparisons that, when  $x - D_{(0,L_1]} \geq \underline{z}$  then  $q^* = 0$ , when  $y - D_{(0,L_1]} \leq \bar{z}$  then  $q^* = y - x$  and  $q^* = z^* - x + D_{(0,L_1]}$  otherwise. However, when  $y - x \leq \bar{z} - \underline{z}$  the last case does not occur, and for such values of  $y - x$  choices  $q = 0$  and  $q = y - x$  have to be compared. The difference between (S2) and (S1) is,

$$k_2 - K_2 + g(y - D_{(0,L_1]}) - g(x - D_{(0,L_1]}). \quad (\text{S4})$$

As  $y > x$  the above expression is non-increasing in  $D_{(0,L_1]}$  (Lemma 1 (b)) and is equal to 0 when  $y - D_{(0,L_1]} = z_p(y - x)$  (the definition of  $z_p(\cdot)$  in (4)). Clearly, when  $y - D_{(0,L_1]} > z_p(y - x)$  then  $q^* = 0$  and  $q^* = y - x$  otherwise. However, the expression in (S4) is bounded above by  $k_2 - K_2 + (y - x)(c_f + h(L-l))$  and is bounded below by  $k_2 - K_2 + (y - x)(c_f - p(L-l))$  (Lemma 1(b)). This implies that for  $k_2 > K_2$  and  $y - x \leq Q_e$ ,  $q^* = 0$ . And for  $k_2 < K_2$  and  $y - x \leq Q_r$ ,  $q^* = y - x$ . This completes proof for the first two cases.

For  $c_f \geq p(L-l)$ , only expressions (S1) and (S2) have to be compared. For  $K_2 \leq k_2$ , the expression in (S4) is always non-negative and hence  $q^* = 0$ . For  $K_2 > k_2$ : The expression (S4) is non-positive for all values of  $D_{(0,L_1]}$ , if  $y - x \leq Q_r$ , implying  $q^* = y - x$ . Similarly the expression is non-negative for all values of  $D_{(0,L_1]}$  if  $y - x \geq Q_e$  implying  $q^* = 0$ . For the intermediate values of  $y - x$  the optimal policy follows from the definition of function  $z_p(\cdot)$ . This completes the proof for all the cases.  $\square$

PROOF OF LEMMA 3. (a) First note that the expression for  $\Xi(x, y)$  in (1) can be alternatively written as  $\mathbb{E}[\xi(x - D_{(0,L_1]}, y - D_{(0,L_1]}) - c_f(x - D_{(0,L_1]})]$ , where

$$\xi(u, v) = \min_{u \leq z \leq v} \{K_2 \mathbf{1}_{\{z < v\}} + k_2 \mathbf{1}_{\{z > v\}} + g(z)\}. \quad (\text{S5})$$

It follows directly from the its above definition that for  $u < v$ ,  $\xi(u, v)$  is continuous in  $u$  and  $v$ .

This leads to continuity of  $\Xi(x, y)$  in  $x$  and  $y$  for  $x < y$ . Furthermore, for  $u < v$ ,  $\partial\xi(u, v)/\partial u$  and  $\partial\xi(u, v)/\partial v$  have finitely many discontinuities. Since  $D_{(0, L_1]}$  is a continuous random variable,  $\partial\Xi(x, y)/\partial x$  and  $\partial\Xi(x, y)/\partial y$  exist. Continuity and differentiability of  $c(x, y)$  in  $x$  and  $y$ , for  $x < y$ , then follows immediately from its definition. It follows from Proposition 1, that as  $y \downarrow x$ ,  $\Xi(x, y) = \min\{K_2, k_2\} + \mathcal{G}_{(l, L]}(x)$ . This combined with the definition of  $c(x, y)$  then gives the desired limit, and it implies a discontinuity at  $x = y$  when  $K_1 + \min\{K_2, k_2\} > 0$ .

To prove parts (b), (c) and (d), we use the following expression for  $c(x, y)$ , which results from the definitions of  $z_1$  and  $z_2$

$$c(x, y) = K_1 + c_f(y - x) + \int_{-\infty}^{x-z_1} \{K_2 + g(x - u) - g(y - u)\} dF_{(0, L_1]}(u) \\ + \int_{x-z_1}^{y-z_2} \{K_2 + k_2 + g(z^*) - g(y - u)\} dF_{(0, L_1]}(u) + k_2 \int_{y-z_2}^{\infty} dF_{(0, L_1]}(u). \quad (\text{S6})$$

(b) Consider the following four cases: (i) When  $z_1 \rightarrow -\infty$  and  $z_2 \rightarrow -\infty$ , then  $c(x, y) = K_1 + K_2 + \mathcal{G}_{(l, L]}(x) - \mathcal{G}_{(l, L]}(y)$ , a concave function of  $y$ . (ii) When  $z_1 \rightarrow \infty$  and  $z_2 \rightarrow \infty$ , then  $c(x, y) = K_1 + k_2 + c_f(y - x)$ , a linear and hence concave function of  $y$ . For the cases, when  $z_1$  and  $z_2$  take finite values, the first and the second derivatives of  $c(x, y)$  with respect to  $y$  are

$$\frac{\partial c(x, y)}{\partial y} = c_f - \int_{-\infty}^{y-z_2(y-x)} g'(y - u) dF_{(0, L_1]}(u), \\ \frac{\partial^2 c(x, y)}{\partial y^2} = - \left( 1 - \frac{\partial z_2(y-x)}{\partial y} \right) g'(z_2(y-x)) f_{(0, L_1]}(y - z_2(y-x)) \\ - \int_{-\infty}^{y-z_2(y-x)} g''(y - u) dF_{(0, L_1]}(u).$$

In the expression for the second derivative the last term is non-positive as  $g(x)$  is convex in  $x$ . We evaluate the multiplier of  $f_{(0, L_1]}(y - z_2(y-x))$  in the first term for the last two cases: (iii) When  $z_2(y-x) = z_p(y-x)$ , using the definition of  $z_p(y-x)$  in (4),

$$- \left( 1 - \frac{\partial z_2(y-x)}{\partial y} \right) g'(z_2(y-x)) = - \frac{(g'(z_p(y-x)))^2}{g'(z_p(y-x)) - g'(z_p(y-x) - y + x)} \leq 0,$$

where the last inequality follows from the convexity of  $g$ . (iv) When  $z_2(y-x) = \bar{z}$ ,

$$- \left( 1 - \frac{\partial z_2(y-x)}{\partial y} \right) g'(z_2(y-x)) = -g'(\bar{z}) \leq 0,$$

where the inequality follows as  $\bar{z} \geq z^*$ . Thus, for all four cases  $\partial^2 c(x, y)/\partial y^2 \leq 0$ . This combined with differentiability of  $c(x, y)$  in  $y$  implies its concavity in  $y$  for  $y > x$ . Finally, letting  $y \rightarrow \infty$  in the expression for  $\partial c(x, y)/\partial y$ , we obtain  $-h(L-l)$ .

(c) The proof of this part is analogous to that of part (b), and follows from showing non-positivity of the second derivative of  $c(x, y) + \mathcal{G}_{(l, L]}(y) - \mathcal{G}_{(l, L]}(x)$  with respect to  $x$  for four different cases. The first derivative of the function with respect to  $x$  is,

$$\frac{\partial (c(x, y) + \mathcal{G}_{(l, L]}(y) - \mathcal{G}_{(l, L]}(x))}{\partial x} = - \int_{x-z_1(y-x)}^{\infty} g'(x-u) dF_{(0, L_1]}(u).$$

When  $x \rightarrow -\infty$ , then  $x - z_1(y-x) \rightarrow -\infty$  if  $c_f < p(L-l)$ , and  $x - z_1(y-x) \rightarrow \infty$ , otherwise. Using this, while letting  $x \rightarrow -\infty$  in the expression for  $\partial c(x, y) / \partial x$  leads to the desired limit.

(d) For a given value of  $Q$ ,

$$\frac{dc(x, x+Q)}{dx} = \int_{-\infty}^{x-z_1(Q)} (g'(x-u) - g'(x+Q-u)) dF_{(0, L_1]}(u) - \int_{x-z_1(Q)}^{x+Q-z_2(Q)} g'(x+Q-u) dF_{(0, L_1]}(u).$$

The first term in the above expression is always non-positive as  $g'(x-u) \leq g'(x+Q-u)$ . The second term vanishes for all the cases, except when  $z_1 = \underline{z}$  and  $z_2 = \bar{z}$ , in which case the integrand is non-negative over the range of integration. Thus,  $\partial c(x, x+Q) / \partial x \leq 0$ . Similarly,

$$\begin{aligned} \frac{\partial (c(x, x+Q) - \mathcal{G}_{(l, L]}(x) + \mathcal{G}_{(l, L]}(x+Q))}{\partial x} &= - \int_{x-\infty}^{x+Q-z_2(Q)} g'(x-u) dF_{(0, L_1]}(u) \\ &+ \int_{x+Q-z_1(Q)}^{\infty} (g'(x+Q-u) - g'(x-u)) dF_{(0, L_1]}(u). \end{aligned}$$

The first term in this expression vanishes for all the cases except when  $z_1(Q) = \underline{z}$  and  $z_2(Q) = \bar{z}$ , in which case the integrand is non-positive over the range of integration, and the second term is always non-negative. This implies non-negativity of the derivative on the left hand side.

(e) The inequality follows immediately from the definition of  $c(x, y)$ , when  $x_1 = x_2$  or  $x_2 = x_3$ . Thus, we consider values of  $x_1, x_2$  and  $x_3$  satisfying  $x_1 < x_2 < x_3$ . For such cases, the inequality is equivalent to  $\Xi(x_1, x_3) - K_1 + \mathcal{G}_{(l, L]}(x_2) \leq \Xi(x_1, x_2) + \Xi(x_2, x_3)$ . Noting the definition of  $\xi(u, v)$  in (S5), it follows that proving the inequality

$$\xi(u_1, u_3) - \xi(u_2, u_3) \leq \xi(u_1, u_2) - g(u_2), \quad (\text{S7})$$

for  $u_1 < u_2 < u_3$ , is equivalent to proving an stronger form of the desired inequality for each realization of  $D_{(0, L_1]}$ . We prove (S7) for the case  $K_2 \geq k_2$  and  $c_f < p(L-l)$ . The proofs for other cases are analogous. When  $K_2 \geq k_2$  and  $c_f < p(L-l)$ ,

$$\xi(u, v) = \begin{cases} k_2 + g(v), & \text{if } u \leq \underline{z} \text{ and } v \leq \bar{z}, \\ K_2 + k_2 + g(z^*), & \text{if } u \leq \underline{z} \text{ and } v > \bar{z}, \\ k_2 + g(v), & \text{if } u > \underline{z} \text{ and } v \leq \hat{z}(u), \\ K_2 + g(u), & \text{if } u > \underline{z} \text{ and } v > \hat{z}(u), \end{cases}$$

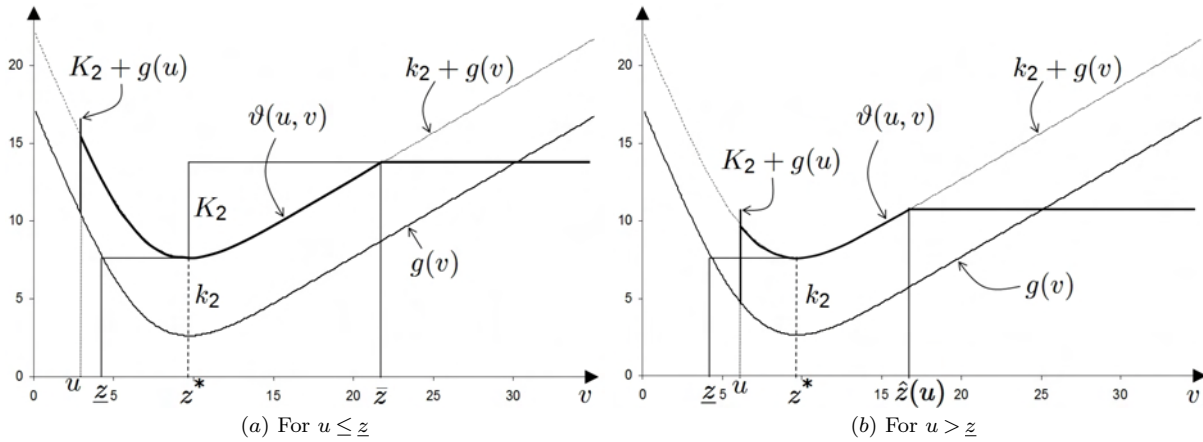


Figure S1  $\xi(u, v)$  plotted against  $v$  for the case  $K_2 > k_2$  and  $c_f < p(L-l)$ .

where  $\hat{z}(u)$  is the unique solution to  $g(z) - g(u) = K_2 - k_2$ ,  $z \geq u$ . In Figure S1 we provide plots of function  $\xi(u, v)$  against  $v$  for  $u \leq \underline{z}$  and for  $u > \underline{z}$ . Using the expression for  $\xi(u, v)$ , we now verify (S7) on case-by-case basis.

**Case 1**,  $u_1 < u_2 \leq \underline{z}$ : In this case  $\xi(u_1, u_2) = k_2 + g(u_2)$  and  $\xi(u_1, u_3) = \xi(u_2, u_3) \forall u_3 > u_2$ , implying that  $\xi(u_1, u_3) - \xi(u_2, u_3) = 0 \leq k_2 = \xi(u_1, u_2) - g(u_2)$ .

**Case 2**,  $u_1 \leq \underline{z} \leq u_2 \leq \bar{z}$ : In this case,  $\xi(u_1, u_2) = k_2 + g(u_2)$ , so we need to verify that  $\xi(u_1, u_3) - \xi(u_1, u_2) \leq k_2$ . If  $u_3 \leq \hat{z}(u_2) \leq \bar{z}$ , then  $\xi(u_1, u_3) = \xi(u_2, u_3) = k_2 + g(u_3)$ ; otherwise  $\xi(u_1, u_3) \leq K_2 + k_2 + g(z^*)$  and  $\xi(u_2, u_3) = K_2 + g(u_2) \geq K_2 + g(z^*)$ . Hence,  $\xi(u_1, u_3) - \xi(u_1, u_2) \leq k_2$ .

**Case 3**,  $u_1 \leq \underline{z} \leq \bar{z} \leq u_2$ : In this case  $\xi(u_1, u_3) = \xi(u_1, u_2) = K_2 + k_2 + g(z^*)$  and  $\xi(u_2, u_3) \geq k_2 + g(u_2) \geq g(u_2)$ . This implies (S7).

**Case 4**,  $\underline{z} \leq u_1 \leq u_2 \leq \bar{z}(u_1)$ : In this case  $\xi(u_1, u_2) = k_2 + g(u_2)$  and we need to verify that  $\xi(u_1, u_3) - \xi(u_2, u_3) \leq k_2$ . If  $u_3 \leq \min\{\hat{z}(u_1), \hat{z}(u_2)\}$ , then  $\xi(u_1, u_3) = \xi(u_2, u_3) = k_2 + g(u_3)$ ; otherwise  $\xi(u_1, u_3) - \xi(u_2, u_3) = g(u_1) - g(u_2) \leq k_2$  as  $u_2 \leq \hat{z}(u_1)$ .

**Case 5**,  $\underline{z} \leq u_1 \leq \bar{z}(u_1) \leq u_2$ : In this case  $\xi(u_1, u_3) = \xi(u_1, u_2) = K_2 + g(u_1)$  and  $\xi(u_2, u_3) \geq k_2 + g(u_2)$ , implying (S7).

□

PROOF OF LEMMA 4. The right-hand side in (9) can be also expressed as,

$$-c(x, y) + \mathcal{G}_{(l, L]}(x) + \mathcal{G}_{(L, l+T]}(x) + h_N \mathbf{E}(x - D_{(0, T+l]})^+ + p_N \mathbf{E}(D_{(0, T+l]} - x)^+$$

In the above expression the sum of the first two terms is convex for  $x < y$  (Lemma 3(c)), and each of the last three terms is convex in  $x$ . Thus, the whole expression is convex in  $x$  for  $x < y$ . Further,

it follows from the limit in Lemma 3(c) that,

$$\lim_{x \rightarrow -\infty} \frac{\partial(-c(x, y) + U_{N-1}(x))}{\partial x} = -\max\{c_f - p(L-l), 0\} - p(T - (L-l)) - p_N < 0,$$

which implies that for sufficiently small values of  $x$ ,  $U_{N-1}(x) - c(x, y) > U_{N-1}(y)$ . Also,  $U_{N-1}(x) - c(x, y) \rightarrow U_{N-1}(y) - K_1 - \min\{K_2, k_2\} \leq U_{N-1}(y)$  as  $x \uparrow y$ . These two facts along with convexity of function  $U_{N-1}(x) - c(x, y)$  in  $x$  lead to the desired result.  $\square$

PROOF OF PROPOSITION 2. First,  $\partial(U_{N-1}(y) + c(x, y))/\partial y = h(T - (L-l)) + h_0$  as  $y \rightarrow \infty$ , implying that  $S_{N-1}(x)$  is bounded above for all values of  $x$  and hence is finite. This limit also implies that for very large values of  $x$ ,  $U_{N-1}(y) + c(x, y)$  is increasing in  $y$  for all  $y > x$ . Further,  $U_{N-1}(y) + c(x, y) \rightarrow U_{N-1}(x) + K_1 + \min\{K_2, k_2\} \geq U_{N-1}(x)$ , as  $y \downarrow x$ . Combining these facts leads to the conclusion that for large values of  $x$ ,  $S_{N-1}(x) = x$ . Let  $s_{N-1}$  be the supremum of all values of  $x$  satisfying  $S_{N-1}(x) > x$ , i.e., for all  $x > s_{N-1}$ ,  $S_{N-1}(x) = x$ .

Now consider  $x_1 < s_{N-1}$ , such that  $S_{N-1}(x_1) > x_1$ , or  $U_{N-1}(x_1) - c(x_1, S_{N-1}(x_1)) > U_{N-1}(S_{N-1}(x_1))$ . Then using Lemma 4, for all  $x < x_1$

$$U_{N-1}(x) - c(x, S_{N-1}(x_1)) \geq U_{N-1}(x_1) - c(x_1, S_{N-1}(x_1)) > U_{N-1}(S_{N-1}(x_1)),$$

or,  $U_{N-1}(x) > c(x, S_{N-1}(x_1)) + U_{N-1}(S_{N-1}(x_1)) \geq c(x, S_{N-1}(x)) + U_{N-1}(S_{N-1}(x))$ . Further  $S_{N-1}(x) > x$ ; if not then  $U_{N-1}(x) > c(x, x) + U_{N-1}(x) = U_{N-1}(x)$ , which is contradictory. Thus, for all values  $x < s_{N-1}$ ,  $S_{N-1}(x) > x$ .  $\square$

PROOF OF LEMMA 5. Using Lemma 2,

$$\Gamma(x) = -c_f x + \int_{-\infty}^{x-z^*} g(x-u) dF_{(0, L_1]}(u) + \int_{x-z^*}^{\infty} g(z^*) dF_{(0, L_1]}(u).$$

Taking the second derivative with respect to  $x$ ,

$$\frac{\partial^2 \Gamma(x)}{\partial x^2} = \int_{-\infty}^{x-z^*} g''(x-u) dF_{(0, L_1]}(u) \geq 0,$$

where the inequality follows from the convexity of function  $g(x)$ . Similarly,

$$\frac{\partial^2 (\mathcal{G}_{(l, L]}(x) - \Gamma(x))}{\partial x^2} = \int_{x-z^*}^{-\infty} g''(x-u) dF_{(0, L_1]}(u) \geq 0.$$

The above two inequalities prove part (a). Next, for fixed  $y$ ,

$$\frac{\partial c(x, y)}{\partial x} = -c_f + \int_{-\infty}^{x-z_1} g'(x-u) dF_{(0, L_1]}(u) \leq -c_f + \int_{-\infty}^{x-z^*} g'(x-u) dF_{(0, L_1]}(u) = \frac{\partial \Gamma(x)}{\partial x},$$



and for fixed  $x$ ,

$$\frac{\partial c(x, y)}{\partial y} = c_f - \int_{-\infty}^{y-z_2} g'(y-u) dF_{(0, L_1]}(u) \geq c_f - \int_{-\infty}^{y-z^*} g'(y-u) dF_{(0, L_1]}(u) = -\frac{\partial \Gamma(y)}{\partial y}.$$

The above two inequalities follow as  $g'(x)$  is an increasing function of  $x$  and is equal to 0 at  $x = z^*$ .

This completes the proof for parts (b) and (c).  $\square$

PROOF OF LEMMA 6. Part (a) is the direct result of the following string of inequalities,

$$\begin{aligned} V_n(x_1) &= \min_{y \geq x_1} \{c(x_1, y) + U_n(y)\}, \\ &\leq \min_{y \geq x_2} \{c(x_1, y) + U_n(y)\}, \\ &\leq \min_{y \geq x_2} \{c(x_1, x_2) + c(x_2, y) + U_n(y)\}, \\ &= c(x_1, x_2) + \min_{y \geq x_2} \{c(x_2, y) + U_n(y)\} = c(x_1, x_2) + V_n(x_2). \end{aligned}$$

In the above, the inequality in second line follows as the function is being minimized over a subset  $[x_2, \infty)$  of the original set  $[x_1, \infty)$ , and the inequality in third line follows from Lemma 3(e).

We prove part (b) by the principle of mathematical induction. We use the properties of  $K$ -convexity listed in Lemma 7.1 and Lemma 7.2 of Porteus (2002). For  $n = N$ ,  $V_N(x) = h_N \mathbb{E}(x - D_{(0, l]})^+ + p_N \mathbb{E}(D_{(0, l]} - x)^+$  is convex, hence  $K_1 + \max\{K_2, k_2\}$  convex. Next we show that given  $V_{k+1}(x)$  is  $K_1 + \max\{K_2, k_2\}$  convex, then so is  $V_k(x)$ . Properties of  $K_1 + \max\{K_2, k_2\}$  convexity imply that  $U_k(x) = \mathcal{G}_{(l, l+T]}(x) + \tilde{V}_{k+1}(x)$  is  $K_1 + \max\{K_2, k_2\}$  convex. Let  $S_k(x)$  denote the optimal order-up-to level for inventory position  $x$  at review epoch  $k$ , then an expression for  $V_k(x)$  is,

$$V_k(x) = \begin{cases} U_k(x), & \text{if } S_k(x) = x, \\ c(x, S_k(x)) + U_k(S_k(x)), & \text{if } S_k(x) > x. \end{cases} \quad (\text{S8})$$

For values of  $x$  with  $S_k(x) = x$ ,  $K_1 + \max\{K_2, k_2\}$  convexity of  $U_k(x)$  immediately implies the same for  $V_k(x)$ . Consider the case when  $S_k(x) > x$ : Since  $V_{k+1}(y)$  is a continuous function with finite number of discontinuities in its first derivatives,  $\tilde{V}_{k+1}(y) = \mathbb{E}V_{k+1}(y - D_{(0, T]})$  is continuous and differentiable in  $y$  for non-deterministic and continuous demand  $D_{(0, T]}$ . This implies that for all  $y > x$ ,  $c(x, y) + U_k(y)$  is continuous and differentiable in  $y$ . Thus, the optimal order-up-to level  $S_k(x) > x$  must satisfy the first order condition of the decision variable  $y$ . Using this to fact to evaluate the first derivative of  $V_k(x)$  at points where  $S_k(x) > x$ , we obtain

$$\frac{dV_k(x)}{dx} = \left( \frac{\partial c(x, y)}{\partial x} \Big|_{y=S_k(x)} + \frac{\partial \{c(x, y) + \tilde{U}_k(y)\}}{\partial y} \Big|_{y=S_k(x)} \frac{dS_k(x)}{dx} \right) = \frac{\partial c(x, y)}{\partial x} \Big|_{y=S_k(x)}. \quad (\text{S9})$$

The relation in equation (S9) can be extended to,

$$\begin{aligned}
(y-x)\frac{dV_k(x)}{dx} &= (y-x)\left.\frac{\partial c(x,y)}{\partial x}\right|_{y=S_k(x)}, \\
&= (y-x)\int_{-\infty}^{x-z_1(S_k(x)-x)} g'(x-u)dF_{(0,L_1]}(u) - c_f(y-x), \\
&\leq \int_{-\infty}^{x-z_1(S_k(x)-x)} (g(y-u) - g(x-u))dF_{(0,L_1]}(u) - c_f(y-x), \\
&= -\left[K_1 + c_f(\mu L_1 - x) + \int_{-\infty}^{x-z_1} (K_2 + g(x-u))dF_{(0,L_1]}(u)\right. \\
&\quad \left.+ \int_{x-z_1}^{-\infty} (k_2 + g(y-u))dF_{(0,L_1]}(u)\right] + \mathcal{G}_{(l,L]}(y) \\
&\quad + K_1 + K_2 F_{(0,L_1]}(x-z_1) + k_2(1 - F_{(0,L_1]}(x-z_1)), \\
&\leq -c(x,y) + K_1 + K_2 F_{(0,L_1]}(x-z_1) + k_2(1 - F_{(0,L_1]}(x-z_1)), \\
&\leq -V_k(x) + V_k(y) + K_1 + K_2 F_{(0,L_1]}(x-z_1) + k_2(1 - F_{(0,L_1]}(x-z_1)), \\
&= -V_k(x) + V_k(y) + K_1 + \max\{K_2, k_2\}.
\end{aligned}$$

In the above, the second equality follows from the expression for  $c(x,y)$  in (S6); the inequality then follows from convexity of  $g(x)$ ; the next equality follows from rearranging terms (here we have suppressed the argument  $S_k(x) - x$  of  $z_1$ ); the next inequality follows as the term in square bracket is the relevant cost for freight mode decision with sub-optimal decision and is greater than  $K_1 + \Xi(x,y)$ ; finally the last inequality follows from part (a), and implies  $K_1 + \max\{K_2, k_2\}$  convexity of  $V_n(x)$ , thus completing the proof of part (b).  $\square$

PROOF OF PROPOSITION 3. When  $K_2 = k_2$ ,  $\max\{K_2, k_2\} = K_2$ . Lemma 3(c) and Lemma 6(b) imply that for a given  $y$ ,  $-c(x,y) + U_n(x) = -c(x,y) + \mathcal{G}_{(l,l+T]}(x) + \tilde{V}_{n+1}(x)$  is  $K_1 + K_2$  convex in  $x$ . Assume the pre-order inventory position  $x$  and order-up-to level  $y (> x)$  are such that  $U_n(x) > c(x,y) + U_n(y)$ , or in other words at inventory position  $x$ , placing an order-up-to  $y$  leads to reduced cost. For  $x_0 < x$ , let  $\lambda \in (0,1)$  be such that  $x_0 \leq x = (1-\lambda)x_0 + \lambda y \leq y$ . Then from the definition of  $K$ -convexity, we have,

$$\begin{aligned}
-c(x,y) + U_n(x) &\leq (1-\lambda)(-c(x_0,y) + U_n(x_0)) + \lambda \left( K_1 + K_2 - \lim_{x \uparrow y} c(x,y) + U_n(y) \right), \\
&= (1-\lambda)(-c(x_0,y) + U_n(x_0)) + \lambda U_n(y),
\end{aligned}$$

where the last equality follows from Lemma 3(a). Combining the above with  $U_n(x) > c(x,y) + U_n(y)$ , we obtain  $U_n(x_0) > c(x_0,y) + U_n(y)$ , i.e., for  $x_0 < x$  order-up-to  $y$  results in cost reduction. This property, analogous to the proof of Proposition 2, immediately implies the optimality of a threshold

policy for review epoch  $n$ . This proves part (a).

Using part (a), a threshold policy is optimal for  $K_2 = k_2 = 0$ . Additionally, for this case  $c(x, y) = K_1 \mathbb{1}_{\{y > x\}} + \Gamma(y) - \Gamma(x)$ , implying that for all values of  $x \leq s_n$ ,  $S_n(x)$  takes the value that minimizes  $-\Gamma(y) + U(y)$ . In other words the threshold policy becomes an  $(s, S)$  policy.  $\square$

PROOF OF LEMMA 7. We prove this lemma by the principle of mathematical induction: For  $k = N$ , it follows from the Assumption 1 that  $\tilde{V}_k(x) - \tilde{\Gamma}(x)$  is non-increasing in  $x$  for  $x \leq S^0$ . For  $k < N$ , suppose  $V_k(x) - \Gamma(x)$  is non-increasing in  $x$  for  $x \leq S^0$ , which in turn implies that  $\tilde{V}_k(x) - \tilde{\Gamma}(x)$  also satisfies the property. Given this, we need to show that  $V_{k-1}(x) - \Gamma(x)$  is non-increasing in  $x$  for  $x \leq S^0$ .

The function  $U_{k-1}(x) - \Gamma(x)$  can be decomposed into  $\left\{ \tilde{\Gamma}(x) - \Gamma(x) + \mathcal{G}_{(l, l+T]}(x) \right\} + \left\{ \tilde{V}_k(x) - \tilde{\Gamma}(x) \right\}$ . As both the functions inside the curly brackets are non-increasing in  $x$  for  $x \leq S^0$ , so is  $U_{k-1}(x) - \Gamma(x)$ . Next, note that  $V_{k-1}(x) - \Gamma(x)$  is either  $U_{k-1}(x) - \Gamma(x)$  or  $c(x, S_{k-1}(x)) + U_{k-1}(S_{k-1}(x)) - \Gamma(x)$ . In the former case, the assertion is directly satisfied. In the latter case the first derivative of  $V_{k-1}(x)$  from (S9) is,

$$\frac{dV_{k-1}(x)}{dx} = \left. \frac{\partial c(x, y)}{\partial x} \right|_{y=S_{k-1}(x)} \leq \frac{d\Gamma(x)}{dx},$$

where the inequality follows from Lemma 5(a), and implies that  $V_{k-1}(x) - \Gamma(x)$  is non-increasing in  $x$ . This completes the induction argument hence the proof.  $\square$

PROOF OF LEMMA 8.(a) Consider the single period inventory problem,

$$\min_{y \geq x} \left\{ c(x, y) + \tilde{\Gamma}(y) + \mathcal{G}_{(l, l+T]}(y) \right\}. \quad (\text{S10})$$

Analogous to the proof of Lemma 4 and Proposition 2, convexity of the function  $-c(x, y) + \mathcal{G}_{(l, l+T]}(x) + \tilde{\Gamma}(x)$  in  $x$  implies that a threshold policy is optimal for the problem in (S10). In other words, there exists  $\underline{s}$  such that  $S(x) > x$  if and only if  $x < \underline{s}$ . Further, as the function  $c(x, y) + \tilde{\Gamma}(y) + \mathcal{G}_{(l, l+T]}(y)$  is continuous and differentiable in  $y$  for  $y > x$ , its minimizer  $\underline{S}(x) > x$  must satisfy the first order condition. Using Lemma 5(b) and the definition of  $S^0$ , for  $y > S^0$ ,

$$\frac{\partial \left( c(x, y) + \tilde{\Gamma}(y) + \mathcal{G}_{(l, l+T]}(y) \right)}{\partial y} \geq \frac{\partial \left( -\Gamma(y) + \tilde{\Gamma}(y) + \mathcal{G}_{(l, l+T]}(y) \right)}{\partial y} \geq 0,$$

implying that all points satisfying the first order condition of problem (S10) are bounded above by  $S^0$ , or for  $x < \underline{s}$ ,  $\underline{S}(x) \leq S_0$ .

(b) For  $x_1 < \arg \min_u \mathcal{G}_{(l,l+T]}(u)$ , there exists some  $y_1 > x_1$  satisfying  $\mathcal{G}_{(l,l+T]}(x_1) \geq \mathcal{G}_{(l,l+T]}(y_1)$ . Lemma 3(d) implies  $c(x_1 - D_{(0,T]}, y_1 - D_{(0,T]}) \geq c(x_1, y_1)$ , which on taking expectation over  $D_{(0,T]}$  gives  $\tilde{c}(x_1, y_1) \geq c(x_1, y_1)$ . Combining these two inequality, we get  $\mathcal{G}_{(l,l+T]}(x_1) \geq c(x_1, y_1) - \tilde{c}(x_1, y_1) + \mathcal{G}_{(l,l+T]}(y_1)$ . Thus,  $\bar{s} \geq \arg \min_u \mathcal{G}_{(l,l+T]}(u)$ .

Now consider  $x_2$ , for which there exists some  $y_2 > x_2$ , satisfying,

$$\mathcal{G}_{(l,l+T]}(x_2) \geq c(x_2, y_2) - \tilde{c}(x_2, y_2) + \mathcal{G}_{(l,l+T]}(y_2). \quad (\text{S11})$$

Lemma 3(d) implies,  $c(x_2, y_2) - \tilde{c}(x_2, y_2) \geq (\mathcal{G}_{(l,L]}(x_2) - \mathcal{G}_{(l,L]}(y_2)) - (\mathcal{G}_{(l+T,L+T]}(x_2) - \mathcal{G}_{(l+T,L+T]}(y_2))$ . Combining this with (S11) and rearranging terms we obtain that  $\mathcal{G}_{(L,L+T]}(x_2) \geq \mathcal{G}_{(L,L+T]}(y_2)$ . Clearly, this can not be satisfied for  $x_2 > \arg \min_u \mathcal{G}_{(L,L+T]}(u)$ . Thus,  $\bar{s} \leq \arg \min_u \mathcal{G}_{(L,L+T]}(u)$ .

(c) The expression on the right-hand side of (14) is convex in  $y$ , since  $\mathcal{G}_{(l,l+T]}(y) - \Gamma(y)$  is convex in  $y$  and  $\tilde{c}(S^0, y)$  is concave in  $y$ . Further, as  $y \downarrow S^0$ , the expression becomes  $\mathcal{G}_{(l,l+T]}(S^0) - \Gamma(S^0) - K_1 - \min\{K_2, k_2\} \leq \mathcal{G}_{(l,l+T]}(S^0) - \Gamma(S^0)$ , and its first derivative with respect to  $y$  approaches  $hT$  as  $y \rightarrow \infty$ . These facts and convexity of the expression in  $y$  imply the desired result.  $\square$

PROOF OF PROPOSITION 4. Using Lemma 8(a), for  $x < \underline{s}$  we have  $\underline{S}(x) > x$  and

$$\tilde{\Gamma}(x) + \mathcal{G}_{(l,l+L]}(x) > c(x, \underline{S}(x)) + \tilde{\Gamma}(\underline{S}(x)) + \mathcal{G}_{(l,l+L]}(\underline{S}(x)).$$

Since  $\underline{S}(x) \leq S^0$ , using Lemma 7(b),  $\tilde{V}_{n+1}(x) - \tilde{\Gamma}(x) \geq \tilde{V}_{n+1}(\underline{S}(x)) - \tilde{\Gamma}(\underline{S}(x))$ . Combining these two inequalities we obtain

$$U_n(x) > c(x, \underline{S}(x)) + U_n(\underline{S}(x)) \geq c(x, S_n(x)) + U_n(S_n(x)),$$

implying  $S_n(x) > x$  for  $x < \underline{s}$ . Next, it follows from the definition of  $\underline{S}(x)$  that for  $y < \underline{S}(x)$ ,

$$c(x, y) + \tilde{\Gamma}(y) + \mathcal{G}_{(l,l+T]}(y) \geq c(x, \underline{S}(x)) + \tilde{\Gamma}(\underline{S}(x)) + \mathcal{G}_{(l,l+T]}(\underline{S}(x)).$$

And since  $y < \underline{S}(x) \leq S^0$ ,  $\tilde{V}_{n+1}(y) - \tilde{\Gamma}(y) \geq \tilde{V}_{n+1}(\underline{S}(x)) - \tilde{\Gamma}(\underline{S}(x))$ . Combining these two inequalities,

$$c(x, y) + U_n(y) \geq c(x, \underline{S}(x)) + U_n(\underline{S}(x)), \text{ for } y < \underline{S}(x),$$

implying  $\underline{S}(x) \leq S_n(x)$ . Using Lemma 8(c), for all  $y > \bar{S}$ ,

$$\mathcal{G}_{(l,l+T]}(S^0) - \Gamma(S^0) \leq \mathcal{G}_{(l,l+T]}(y) - \Gamma(y) - \tilde{c}(S^0, y).$$

For  $y \geq S^0$ , we have  $\tilde{V}_{n+1}(S^0) \leq \tilde{c}(S^0, y) + \tilde{V}_{n+1}(y)$  (obtained by subtracting  $D_{(0,T]}$  from  $x_1$  and  $x_2$  in Lemma 7(a), and then taking expectation over  $D_{(0,T]}$ ), and  $\Gamma(y) - \Gamma(S^0) \geq c(x, S^0) - c(x, y)$  (Lemma 5(b)). Combining these three inequalities, we get for  $y \geq \bar{S}$ ,

$$c(x, y) + U_n(y) \geq c(x, S^0) + U_n(S^0),$$

implying sub-optimality of ordering-up-to  $y \geq \bar{S}$ . Thus,  $S_n(x) \leq \bar{S}$ .

(b) From the definition of  $\bar{s}$  in Lemma 8(a), for all  $x > \bar{s}$

$$\mathcal{G}_{(l,l+T]}(x) \leq c(x, y) - \tilde{c}(x, y) + \mathcal{G}_{(l,l+T]}(y), \forall y > x.$$

Combining this with  $\tilde{V}_{n+1}(x) \leq \tilde{V}_{n+1}(y) + \tilde{c}(x, y)$ , we get the following inequality for all  $x > \bar{s}$ ,

$$\mathcal{G}_{(l,l+T]}(x) + \tilde{V}_{n+1}(x) \leq c(x, y) + \mathcal{G}_{(l,l+T]}(y) + \tilde{V}_{n+1}(y), \forall y > x,$$

implying  $S_n(x) = x$  for  $x > \bar{s}$ .  $\square$

PROOF OF PROPOSITION 5. In (18), only the numerator namely  $\mathbb{E}c(\mathcal{X}, s + \Delta) + \kappa(s, \Delta)$  depends on  $s$ . As both the terms in this expression are continuous and differentiable functions of  $s$ , the first order condition of  $s$  is a necessary condition for optimal  $s$ . Evaluating the first derivative of  $\kappa(s, \Delta)$  with respect to  $s$ ,

$$\begin{aligned} \frac{\partial \kappa(s, \Delta)}{\partial s} &= (h+p) \int_l^{l+T} \mathbb{P}(D_{(0,t]} < s + \Delta) dt - pT \\ &\quad + \int_0^\Delta \left( (h+p) \int_l^{l+T} \mathbb{P}(D_{(0,t]} < s + \Delta - u) dt - pT \right) dM(u), \\ &= (h+p)\mathcal{P}(s, \Delta) - p(1 + M(\Delta))T. \end{aligned}$$

Note that  $\mathcal{X}_{(s,\Delta)} = s - \mathcal{U}_\Delta$ , where  $\mathcal{U}_\Delta$  is the random undershoot below the reorder-level that depends only on  $\Delta$ . Thus,  $\mathcal{X}_{(s,\Delta)}$  is linear in  $s$ . Denoting the partial derivative of  $c(x, y)$  with respect to its first and second arguments by  $c_1(x, y)$  and  $c_2(x, y)$ , respectively, the first derivative of  $\mathbb{E}c(\mathcal{X}, \Delta)$  with respect to  $s$  can be expressed as  $\mathbb{E}(c_1(\mathcal{X}, s + \Delta) + c_2(\mathcal{X}, s + \Delta))$ . Further,

$$\begin{aligned} c_1(x, y) + c_2(x, y) &= \int_{-\infty}^{x-z_1} g'(x-u) dF_{(0,L_1]}(u) - \int_{-\infty}^{y-z_2} g'(y-u) dF_{(0,L_1]}(u), \\ &= - \int_{-\infty}^{x-z_1} (g'(y-u) - g'(x-u)) dF_{(0,L_1]}(u) - \int_{x-z_1}^{y-z_2} (g'(y-u) - g'(z^*)) dF_{(0,L_1]}(u), \\ &= -(h+p) \int_l^L \left( \begin{array}{l} \mathbb{P}(x < D_{(0,t]} < y, D_{(0,L_1]} < x - z_1) \\ -\mathbb{P}(D_{(L_1,t]} < z^*, x - z_1 < D_{(0,L_1]} < y - z_2) \\ +\mathbb{P}(D_{(0,t]} < y, x - z_1 < D_{(0,L_1]} < y - z_2) \end{array} \right) dt, \end{aligned}$$

where the first equality follows from taking partial derivatives of the expression given in (S6), the second equality follows from rearranging terms and noting that  $x - z_1 \leq y - z_2$  and  $g'(z^*) = 0$  and the third equality follows from writing  $g'(\cdot)$  in terms of probabilities and combining all the time integrals. Substituting  $x = \mathcal{X}$  and  $y = s + \Delta$  in the last expression, combining it with  $\partial\kappa(s, \Delta)/\partial s$  and equating the sum to 0 leads in the desired optimality condition.  $\square$

PROOF OF COROLLARY 1. First note that  $\mathcal{P}(s, \Delta)$  in (20) can also be expressed as,

$$\mathcal{P}(s, \Delta) = \mathbb{E} \int_l^{l+\mathcal{N}_\Delta T} \mathbb{1}_{\{D_{(0,t]} < s+\Delta\}} dt.$$

Using this, the numerator of the expression on the right hand-side of (19) can be written as,

$$\mathbb{E} \int_l^L \left( \begin{array}{c} \mathbb{1}_{\{D_{(0,t]} < \mathcal{X}, D_{(0,L_1]} < \mathcal{X}-z_1\}} \\ + \mathbb{1}_{\{D_{(L_1,t]} < z^*, \mathcal{X}-z_1 < D_{(0,L_1]} < s+\Delta-z_2\}} \\ + \mathbb{1}_{\{D_{(0,t]} < s+\Delta, D_{(0,L_1]} > s+\Delta-z_2\}} \end{array} \right) dt + \mathbb{E} \int_{L+T}^{l+\mathcal{N}_\Delta T} \mathbb{1}_{\{D_{(0,t]} < s+\Delta\}} dt. \quad (\text{S12})$$

In the above the change in the order of expectation and time integration is justified by Fubini's theorem. Now consider the inventory process  $I(t)$  in the interval  $(l, l + \mathcal{N}_\Delta T]$ . For  $t \in (l, L]$ :  $I(t) = \mathcal{X} - D_{(0,t]}$ , if the order is shipped completely by regular freight (when  $D_{(0,L_1]} < \mathcal{X} - z_1$ );  $I(t) = s + \Delta - D_{(0,t]}$ , if the order is shipped completely by express freight (when  $D_{(0,L_1]} > s + \Delta - z_2$ ); and  $I(t) = z^* - D_{(L_1,t]}$ , otherwise. For  $t \in (L, l + \mathcal{N}_\Delta T]$ ,  $I(t) = s + \Delta - D_{(0,t]}$ . A careful observation of the expression in (S12) in conjunction with process  $I(t)$  then leads to (21).  $\square$

PROOF OF PROPOSITION 6. When  $s^*(\Delta)$  is continuous in  $\Delta$ ,  $C(s^*(\Delta), \Delta)$  is a continuous and differentiable function of  $\Delta$ . Further,  $C(s^*(\Delta), \Delta) \rightarrow \infty$  as  $\Delta \rightarrow \infty$  and hence the first order condition is a necessary condition for an optimal  $\Delta$ . Taking the first derivative of  $C(s^*(\Delta), \Delta)$  with respect to  $\Delta$ ,

$$\frac{dC(s^*(\Delta), \Delta)}{d\Delta} = \frac{\partial C(s, \Delta)}{\partial s} \Big|_{s=s^*(\Delta)} \frac{\partial s^*(\Delta)}{\partial \Delta} + \frac{\partial C(s, \Delta)}{\partial \Delta} \Big|_{s=s^*(\Delta)} = \frac{\partial C(s, \Delta)}{\partial \Delta} \Big|_{s=s^*(\Delta)},$$

where the second equality follows as the first term of the derivative vanishes at  $s = s^*(\Delta)$ . The partial derivative with respect to  $\Delta$  is,

$$\frac{\partial C(s, \Delta)}{\partial \Delta} = \frac{1}{1 + M(\Delta)} \left( \frac{\partial (\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta) + \kappa(s, \Delta))}{\partial \Delta} - m(\Delta)C(s, \Delta) \right). \quad (\text{S13})$$

Thus, we need to evaluate the first partial derivative inside the parentheses. Using (17),

$$\frac{\partial \kappa(s, \Delta)}{\partial \Delta} = \mathcal{G}'_{(l, l+T]}(s + \Delta) + \int_0^\Delta \mathcal{G}'_{(l, l+T]}(s + y - u) dM(u) + \mathcal{G}_{(l, l+T]}(s) m(\Delta),$$

$$= \frac{\partial \kappa(s, \Delta)}{\partial s} + \mathcal{G}_{(l, l+T]}(s) m(\Delta). \quad (\text{S14})$$

Define  $H(u, \Delta) = \mathbb{P}(\mathcal{U}_\Delta < u)$ , the cdf of undershoot  $\mathcal{U}_\Delta$  below reorder level  $s$ , and  $h(u, \Delta)$ , the corresponding pdf,

$$h(u, \Delta) = f_{(0, T]}(u + \Delta) + \int_0^\Delta f_{(0, T]}(u + \Delta - v) dM(v).$$

Noting that  $\mathcal{X}_{(s, \Delta)} = s - \mathcal{U}_\Delta$ , we obtain

$$\begin{aligned} \mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta) &= \int_0^\infty c(s - u, s + \Delta) h(u, \Delta) du, \\ \frac{d\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta)}{d\Delta} &= \mathbb{E}c_2(\mathcal{X}(s, \Delta), s + \Delta) + \int_0^\infty c(s - u, s + \Delta) \frac{\partial h(u, \Delta)}{\partial \Delta} du. \end{aligned}$$

It follows from the expression for  $h(u, \Delta)$  that,

$$\frac{\partial h(u, \Delta)}{\partial \Delta} = \frac{\partial h(u, \Delta)}{\partial u} + f_{(0, T]}(u) m(\Delta),$$

which on substituting in the second term of the expression for  $d\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta)/d\Delta$  gives,

$$\mathbb{E}c_2(\mathcal{X}(s, \Delta), s + \Delta) + \int_0^\infty c(s - u, s + \Delta) \frac{\partial h(u, \Delta)}{\partial u} du + m(\Delta) \int_0^\infty c(s - u, s + \Delta) f_{(0, T]}(u) du.$$

The second term in the above expression on integration becomes, upon integration

$$\int_0^\infty c(s - u, s + \Delta) \frac{\partial h(u, \Delta)}{\partial u} du = -c(s, s + \Delta) m(\Delta) + \mathbb{E}c_1(\mathcal{X}(s, \Delta), s + \Delta).$$

Finally, combining all the terms we get,

$$\frac{d\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta)}{d\Delta} = \frac{d\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta)}{ds} + (\mathbb{E}c(s - D_{[0, T]}, s + \Delta) - c(s, s + \Delta)) m(\Delta).$$

The above combined with (S14) gives,

$$\begin{aligned} \frac{\partial (\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta) + \kappa(s, \Delta))}{\partial \Delta} &= \frac{\partial (\mathbb{E}c(\mathcal{X}(s, \Delta), s + \Delta) + \kappa(s, \Delta))}{\partial s} \\ &+ (\mathcal{G}_{(l, l+T]}(s) + \mathbb{E}c(s - D_{(0, T]}, s + \Delta) - c(s, s + \Delta)) m(\Delta). \end{aligned}$$

Substituting this to (S13), evaluating it at  $s = s^*(\Delta)$  and equating it to 0 gives the desired optimality condition.  $\square$

PROOF OF LEMMA 9. When only regular freight is available for shipping,  $c(x, y)$  becomes  $(K_1 + K_2)\mathbf{1}_{\{y > x\}} + \mathcal{G}_{(l, L]}(x) - \mathcal{G}_{(l, L]}(y)$ , and when only express freight is available it becomes  $(K_1 +$

$k_2)\mathbb{1}_{\{y>x\}} + c_f(y-x)$ . Using these, we obtain the following expressions for cost functions  $C^s(s, \Delta)$  and  $C^f(s, \Delta)$  with only regular and express freight modes,

$$C^s(s, \Delta) = \frac{K_1 + K_2 + \mathbb{E}\mathcal{G}_{(l,L]}(\mathcal{X}) - \mathcal{G}_{(l,L]}(s + \Delta) + \kappa(s, \Delta)}{1 + M(\Delta)}, \quad (\text{S15})$$

$$C^f(s, \Delta) = \frac{K_1 + k_2 + c_f\mathbb{E}(s + \Delta - \mathcal{X}) + \kappa(s, \Delta)}{1 + M(\Delta)}. \quad (\text{S16})$$

Thus,  $s^s(\Delta) = \min_s C^s(s, \Delta)$  and  $s^f(\Delta) = \min_s C^f(s, \Delta)$ . As  $C^s(s, \Delta)$  and  $C^f(s, \Delta)$  are convex functions of  $s$ ,  $s^s(\Delta)$  and  $s^f(\Delta)$  are the unique solutions to the respective first order conditions.

Thus,  $s^s(\Delta)$  solves,

$$\frac{\mathcal{P}(s, \Delta) - \int_0^L \mathbb{P}(\mathcal{X} < D_{(0,t]} < s + \Delta) dt}{(1 + M(\Delta))T} = \frac{p}{h + p},$$

and  $s^f(\Delta)$  solves,

$$\frac{\mathcal{P}(s, \Delta)}{(1 + M(\Delta))T} = \frac{p}{h + p}.$$

The optimality condition in equation (19), the above two optimality conditions and the following inequality imply the desired result,

$$\mathbb{P}(\mathcal{X} < D_{(0,t]} < s + \Delta) dt \geq \left( \begin{array}{l} \mathbb{P}(\mathcal{X} < D_{(0,t]} < s + \Delta, D_{(0,L_1]} < \mathcal{X} - z_1) \\ -\mathbb{P}(D_{(L_1,t]} < z^*, \mathcal{X} - z_1 < D_{(0,L_1]} < s + \Delta - z_2) \\ +\mathbb{P}(D_{(0,t]} < s + \Delta, \mathcal{X} - z_1 < D_{(0,L_1]} < s + \Delta - z_2) \end{array} \right) \geq 0.$$

□

## References

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