Periodic Solidification in a Rectangular Duct Due to Velocity Modulation; One-dimensional Analysis*

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Key Words: *Phase change; dynamic solidification; perturbation solution; convection heat transfer; duct flows, e-learning.*

Abstract*. In the present study, we report on the solid phase dynamic response due to time-varying duct flows when a portion of a duct wall is cooled to below the liquidus temperature, along which unidirectional solidification from the cooling duct wall, perpendicular to the flow direction, is assumed. A one-dimensional numerical model for the average solid phase thickness has been formulated employing the boundary tracking method. It is shown that a quasi-steady state temperature in the solid layer allows us to develop an analytical solution, making use of perturbation technique. The afore-mentioned perturbation analysis identifies important three nondimensional parameters, i.e. the Biot number based on the solid phase thickness at steady state, the Stefan number based on the temperature difference between the cooling wall and the liquidus temperatures, and the Stefan number based on the liquidus and the flowing liquid temperatures. Results obtained by both approaches agree well in general, and the time-variation trends of solid phase thickness and its phase delay have been obtained as a function of the non-dimensional angular frequency of the modulating duct flow velocity, with the above three non-dimensional parameters. Various applications in practical engineering and in engineering education have been identified and are being addressed by the developed Graphical Interface Framework for Educational and Engineering Support (GIFEES).*

Introduction

Solidification is widely seen both in our daily life and in large scale processes pertinent to geophysics and planetary physics. The formation of Earth's crust at mid-ocean ridges and crystallization of magma in magma chambers are of great significance for understanding the structures of Earth's interior and plate kinematics [1,2]. The ice formation and melting in the Earth Polar Regions is very important for analyzing the global climate changes, due to its great thermal impact on the Earth's heat balance [3]. The amount of snow fall and its melting in the high mountain regions has a significant influence over the hydrological circulation in the limnology, of which our environmental, agricultural and industrial water resources are greatly dependent [4]. The solidification also plays an important role in different industrial applications, in particular in such fields as metal castings and production of single crystal silicon. Through multi-component molten phase solidification [5] various metal alloys can be formed and produced on commercial basis.

As mentioned above, a wide spectrum of solidification phenomena can be observed and studied within the scope of regular human activities. Most of the solidification researches, however, have been conducted for fixed thermal boundary conditions, i.e. the cooling and the liquid phase temperatures are assumed constant throughout the solidification processes. In reality, either of them is seldom satisfied in a strict sense. Periodic temperature change, for example, is one of the typical cases of temperature variations. Solidification in a duct with a cooled wall has been investigated for the past few decades in order to understand freeze-shut phenomena in water running pipes [6,7]. Freezing in a duct is often an undesirable phenomenon in many occasions. However, in certain cases, a formation of finite solid layer covering the vessel and duct walls that carry high temperature molten materials is necessary to prevent erosions and corrosions of the wall material from the highly reactive liquid substances at extreme temperatures.

In the present study, we investigate the stability of a thin solid layer formed on the wall, when the flowing velocity in the duct has a modulation. We first describe the basic governing equations of a one-dimensional formulation, focusing on the average solid layer thickness. A numerical solution is carried out by employing the boundary tracking method, which enables us to handle the moving boundary problem. A perturbation analysis is then performed under the condition that the velocity modulation is small, compared with the average flow velocity, and that the modulation frequency is also small, in comparison with the transversal diffusion time in the solid layer. An analytical form describing the response of the solid layer to velocity modulations in terms of the velocity modulation frequency and the Stefan number based on the cooling temperature is presented. A comparison made between the numerical and the perturbation solutions shows a very good agreement, demonstrating the usefulness of the analytical form due to perturbation solution.

Mathematical Formulation and Numerical Solution

Problem Statement and Formulation. A schematic diagram of the present problem is shown in *figure 1*, in which a relatively thin solid layer is formed on the side wall of the duct by cooling the wall to a temperature below the liquidus of the flowing liquid. A vertically descending fully-developed laminar flow is assumed, and the thickness of the

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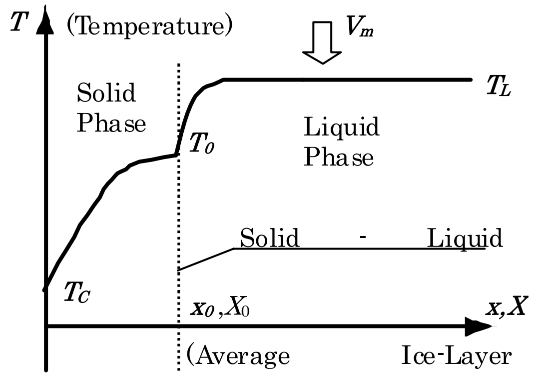

Figure 1. Schematic diagram of a one-dimensional model for formation of a solid layer on a cold duct wall with liquid flowing next to it

solidified layer remains small in respect to the duct width *W*. As demonstrated by Kimura and Kanev [12], a one-dimensional model can be a very good simplification for estimating the average solid layer thickness.

The energy balance on the solid-liquid boundary and the transversal conduction equation in the solidified layer are given by the following equations

$$(1)\quad \rho_S L\frac{dx_0}{dt}+h(T_L-T_0)=k_S\left.\frac{\partial T}{\partial x}\right|_{x=x_0};$$

$$(2)\quad \frac{\partial T_S}{\partial t}=\alpha_S\frac{\partial^2 T_S}{\partial x^2}$$

where ρ_s, L, x, t, h, k_s, T_c, T_o, T_L, T_s denote the solid-layer density, the latent heat, the horizontal coordinate, the time, the convective heat transfer coefficient, the thermal conductivity of the solid layer, the cooling wall temperature, the liquidus temperature, the upstream liquid temperature, and the solid layer temperature, respectively.

If one introduces non-dimensional quantities and applies a coordinate transformation, the above two equations can take the following form

$$(3)\quad \left.\begin{array}{c}\theta_S=\dfrac{T_S-T_0}{T_0-T_C},X_0=\dfrac{x_0}{W},X=\dfrac{x}{W},\tau=\dfrac{V_m}{W}t\\ \mathrm{Re}=\dfrac{V_mW}{\nu},S=\dfrac{c_S(T_0-T_C)}{L},\eta=\dfrac{X}{X_0}\end{array}\right\}$$

and

$$(4)\quad \frac{\partial X}{\partial\tau}=\frac{S}{\mathrm{Re}\Pr}\frac{\alpha_S}{\alpha_L}\left\{\frac{\partial\eta}{\partial X}\frac{\partial\theta_S}{\partial\eta}-\frac{h(T_L-T_0)W}{k_S(T_0-T_C)}\right\};$$

$$(5)\quad \frac{\partial\theta_S}{\partial\tau}+\frac{\partial\theta_S}{\partial\eta}\frac{\partial\eta}{\partial\tau}=\frac{1}{\mathrm{Re}\Pr}\frac{\alpha_S}{\alpha_L}\left(\frac{\partial\eta}{\partial X}\right)^2\frac{\partial\theta_S{}^2}{\partial\eta^2}.$$

The corresponding initial condition for *X* in equation (4) is provided as a very thin solid layer, 0.1% of the duct width *W*, and the thermal boundary conditions appropriate for θ_S in equation (5) are

$\theta_S=-1$ at $\eta=0$, and $\theta_S=0$ at $\eta=1$

Equations (4) and (5) are discretized by finite differences, and solved simultaneously. An important parameter that must be specified in the above formulation is the average convective heat transfer coefficient *h* between the flowing liquid and the solidified layer over the length *H* of the solidified layer. We obtain it through the Graetz solution for the duct flow

$$(6)\quad h=1.6\left\{\frac{3}{2}\frac{k_L}{de}1.077\left(\frac{H}{de}\frac{1}{\mathrm{Re}_d\Pr}\right)^{-1/3}\right\}$$

where *de* and *H* represent the hydraulic diameter of the duct and the vertical solid layer length respectively. The coefficient 1.6 in the front ensures that the heat transfer coefficients computed for various Reynolds numbers accord with the experimental results.

Perturbation Solution

Quasi-Steady State One-Dimensional Model. Assuming a quasi-steady temperature state in the solid layer greatly simplifies the mathematical formulation, since it eliminates the need of a second-order partial differential equation for the heat diffusion. This assumption is valid as long as the solid-liquid boundary moves more slowly than the heat diffusion time within the solid layer. The condition can be expressed in a strict sense by

$$(7)\quad \frac{l^2}{\alpha_S}\ll\omega^{-1}$$

where α_s, l, ω are the thermal diffusivity of the solid, the average solid layer thickness at steady state, and the modulation angular frequency of the flowing velocity, respectively. If the above condition is satisfied, the right hand side of equation (1) is replaced by the heat flux due to the linear temperature distribution in the solid layer.

$$(8)\quad \rho_S L\frac{dx_0}{dt}+h(T_L-T_0)=k_S\frac{T_0-T_C}{x_0}$$

The flowing velocity modulation affects the convective heat transfer coefficient *h* in eq. (6), and it can be taken into account in the following form, by recognizing that *h* is a function of the Reynolds number

$$(9)\quad h_0=1.6\left\{\frac{3}{2}\frac{k_L}{de}1.077\left(\frac{H}{de}\frac{1}{\Pr}\right)^{-1/3}\right\}\mathrm{Re}_{d0}^{1/3}$$

(10) $$h(t)=h_0\left(1-\varepsilon\, f(t)\right)^{1/3}\cong h_0\left(1-\frac{1}{3}\varepsilon\, f(t)\right)$$

where $f(t)$ is a function of time that determines how the velocity is modulated. In the present work we assume a rectangular wave shape for the velocity modulation in the velocity versus time graph. Therefore, $f(t)$ can be expressed by a Fourier series expansion as follows

(11) $$f(t)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi t}{(P/2)}$$

where P is the period of the velocity modulation. Before seeking a solution for eq. (8), the equation should be nondimensionalized in an appropriate way. Recognizing the existence of a solid layer thickness l corresponding to the (non-modulating) base duct flow we adjust the previously employed (eq.(3)) nondimensionalization procedure. This yields the following nondimensional form of eq. (8)

(12) $$\frac{\partial X_{PV}}{\partial \tau_{PV}}+Ste*Bi=\frac{S}{X_{PV}}$$

in which the following nondimensionalization has been performed,

(13) $$\left.\begin{array}{l}\ell=\dfrac{k_S(T_0-T_C)}{h(T_L-T_0)},\tau_{PV}=\dfrac{t}{(\ell^2/\alpha_S)},X_{PV}=\dfrac{x}{\ell}\\ Bi=\dfrac{h\ell}{k_S},Ste=\dfrac{c_S(T_L-T_0)}{L},S=\dfrac{c_S(T_0-T_C)}{L}\end{array}\right\}.$$

We assume that the solution for eq. (12) can be expanded as $X_{PV}=X_{PV0}+\varepsilon X_{PV1}+O(\varepsilon^2)$, where ε is the relative amplitude of the velocity modulation in respect to the magnitude of the base flow. Substituting this series into eq. (12), we have

$$\frac{\partial X_{PV0}}{\partial\tau_{PV}}+\varepsilon\frac{\partial X_{PV1}}{\partial\tau_{PV}}+\cdots\cdots-S\left(\frac{1}{X_{PV0}}-\varepsilon\frac{X_{PV1}}{X_{PV0}^{\ 2}}+O(\varepsilon^2)\right)$$

(14) $$=-C_0\left(1-\frac{1}{3}\varepsilon\, f(\tau_{PV})\right)$$

where

$$C_0=Bi_0Ste=\frac{h_0\ell}{k_S}\frac{c_S(T_L-T_0)}{L}=\frac{h_0(T_L-T_0)}{k_S(T_0-T_C)}\ell\frac{c_S(T_0-T_C)}{L}$$

(15) $= S$.

0th order Solution. By collecting the 0th order terms in eq.(14), the following equation can be obtained

(16) $$\frac{\partial X_{PV0}}{\partial\tau_{PV0}}-\frac{S}{X_{PV0}}=-C_0.$$

At steady state $X_{PV0}=1$ by the definition of eq. (13), and the condition specified by eq. (15) is sufficient to determine the average solid layer thickness ℓ due to a base flow.

(17) $$\frac{S}{Bi_0Ste}=1.$$

1st order Correction Term. By collecting the 1st order terms, and taking $X_{PV0}=1$ into consideration, the 1st order correction term can be obtained as a solution of eq. (18).

(18) $$\frac{\partial X_{PV1}}{\partial\tau_{PV}}+SX_{PV1}=\frac{1}{3}S\, f(\tau_{PV}).$$

Substituting the square wave velocity modulation of eq. (11) in the right hand side of the above equation, the 1st order correction term takes the following form:

(19) $$X_{PV1}=\frac{1}{3}S\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\left\{\frac{\omega_{2n-1}\,\varepsilon^{-S\tau_{PV}}}{S^2+\omega_{2n-1}^2}+\frac{S\sin\omega_{2n-1}\tau_{PV}-\omega_{2n-1}\cos\omega_{2n-1}\tau_{PV}}{S^2+\omega_{2n-1}^2}\right\}.$$

The first term in the right hand side of eq. (19) will decay exponentially, and the second term will thus become dominant, which is a steady periodic solution.

(20) $$X_{PV1}=\frac{1}{3}S\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\frac{1}{\sqrt{S^2+\omega_{2n-1}^{\ 2}}}\sin(\omega_{2n-1}\tau_{PV}-\beta_{2n-1});$$

(21) $$\tan\beta_{2n-1}=\frac{\omega_{2n-1}}{S};\ \omega_{2n-1}=\frac{(2n-1)\pi}{(P_{PV}/2)},\ \omega_{PV}=\frac{1}{(P_{PV}/2)}.$$

The above solution indicates that the Stefan number based on the cooling temperature and the frequency of the modulation velocity are the parameters that determine the amplitude of the oscillation of the solid layer thickness. It is worth noting that eq.(18) admits various solutions due to different forcing functions of $f(\tau_{PV})$, such as sinusoidal or triangular velocity modulations, all of which are expressed by Fourier series expansions.

Comparison Between Perturbation and Numerical Solutions

Oscillating Solid Layer as a Function of Time. Solid layer oscillations due to duct flow velocity modulations are shown in *figure 2*.

The oscillation amplitude of the solid layer is given in the ordinate axis, while the nondimensional time in terms of period of velocity modulation P is given in the abscissa. The perturbation solutions agree well with the numerical solutions, even in case of a relatively large modulation amplitude in the heat transfer coefficient.

Oscillation Amplitudes of Solid-Layer Thickness. Oscillation amplitudes of solid-layer thickness are shown as a function of nondimensional angular frequency for different

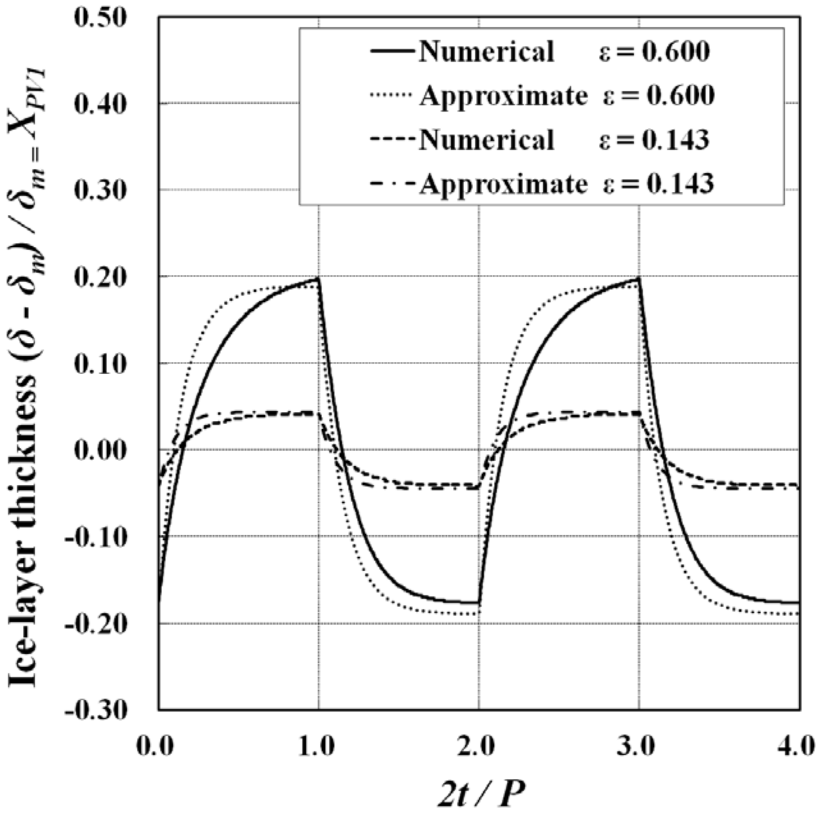

Figure 2. Perturbation analysis of an oscillating solid-layer amplitude as a function of nondimentional time compared with numerical results for different values of ε

values of *S* in *figure 3*. Perturbation predictions are indicated with solid lines for several values of *S*, while the numerical solutions are shown with dotted lines. Quite good agreement between the analytical and the numerical results can be seen in the *figure*.

Phase-Delay of Ice-Layer Oscillation. Another impor-

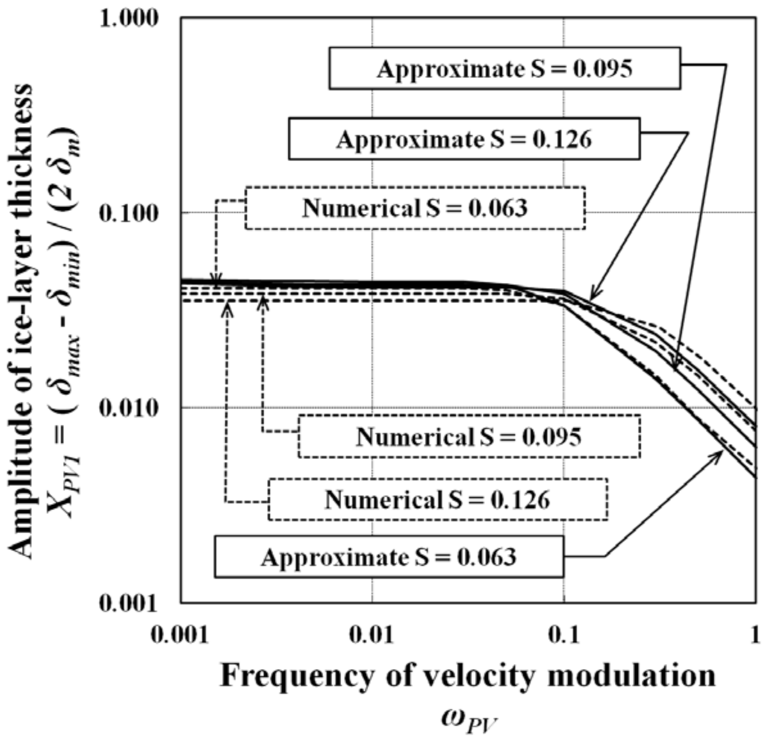

Figure 3. Perturbation analysis of oscillating solid-layer amplitude as a function of frequency w_{PV} compared with numerical results for different values of S–(e = 0.143)

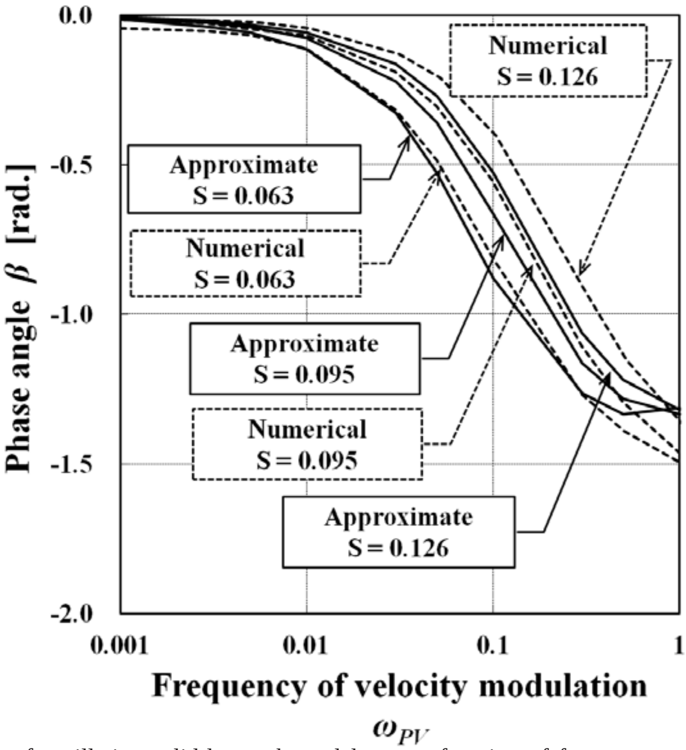

Figure 4. Perturbation analysis of oscillating solid-layer phase delay as a function of frequency ω_{PV} compared with numerical results for different values of S (ε = 0.143)

tant observation is the phase-delay of the oscillating solid-layer thickness relative to the velocity modulation in the duct. Again, the phase-delays predicted by perturbation analysis are shown with solid lines, while the numerical results are displayed with dotted lines for three different values of S in *figure 4*. Here the numerical results also agree well with the perturbation results.

Region where the Perturbation Solutions are Accurate. The region in terms of the Stefan number S and the velocity modulation frequency ω_{PV} is displayed in *figure 5*. The lower area in the graph is the region where the perturbation solution is accurate within 10% in comparison with the numerical solutions.

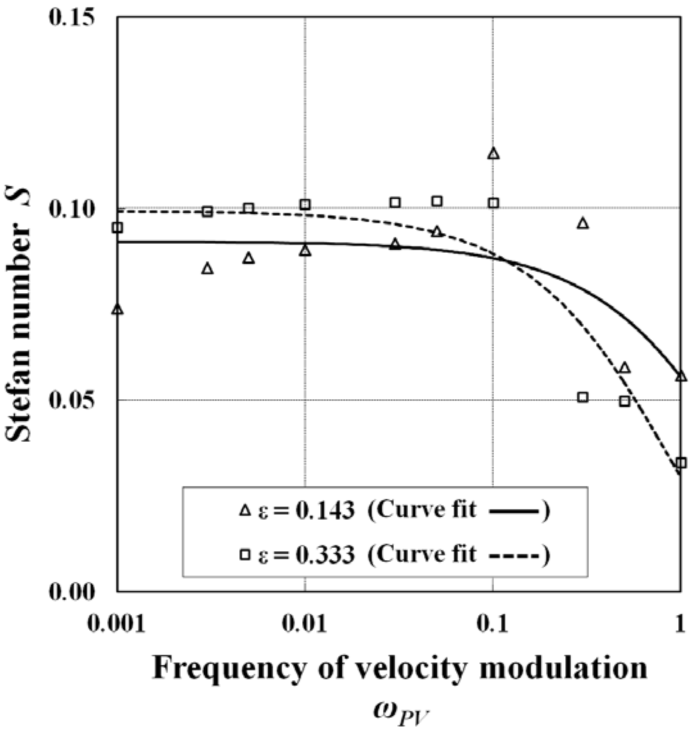

Figure 5. Applicability limitation of Stefan number in perturbation analysis as a function of ω_{PV} for two different values of ε

Heat Transfer e-learning Environment

As already demonstrated in [8], visualizing the dynamic behavior of a solid layer due to the adjacent flow velocity modulation is vital in order to grasp the complex phase change phenomena and to develop appropriate intuition regarding the heat transfer. The one-dimensional model discussed in this paper extends our previous work while the analytical solution delineates the important nondimensional parameters as well as their impacts. This opens the way to building new powerful tools for augmented heat transfer simulations with consecutive visualization and their integration into the Graphical Interface Framework for Educational Support (GIFES).

When adopted in Fluid Dynamics and Heat and Mass Transfer classes, GIFES can effectively support engineering students on their path of learning [8]. An advanced version GIFEES (Graphical Interface Framework for Educational and Engineering Support) extending beyond the scope of education that can be employed by practitioners to address and solve real engineering problems is currently under construction.

Conclusion

A perturbation solution has been developed for a one-dimensional model of solid-layer thickness oscillation due to periodic duct flow modulations arising from various plant work operations. The perturbation solution demonstrates that the relevant parameters are S (the Stefan number based on the cooling temperature) and ω (the non-dimensional angular frequency based on the diffusion time in the solid layer). A numerical code for computing dynamic solidification processes based on a boundary tracking model has been also developed. The good agreement of the results obtained by the two methods for a wide range of parameters proves that the presented perturbation solution is capable of capturing and reliably representing the in-depth mechanisms of dynamic solidification processes. Results obtained in the scope of this research are employed in the design and development of an advanced version of the Graphical Interface Framework for Educational and Engineering Support, GIFEES.

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References

1. Macdonald, K. C. Mid-Ocean Ridges: Fine Scale Tectonic, Volcanic and Hydrothermal Processes within the Plate Boundary Zone. – *Ann. Rev. Earth Planet. Sci.*, 10, 1982, 155-190.

2. Huppert, H. E., R. S. J. Sparks. Double-Diffusive Convection due to Crystallization in Magmas. – *Ann. Rev. Earth Planet. Sci.*, 12, 1984, 11-37.

3. Hibler, III D., J. E. Walsh. On Modeling Seasonal and Interannual Fluctuations of Arctic Sea Ice. – *J. Physical Oceanography*, 12, 1982, 1514-1523.

4. Freeze R. A., J. A. Cherry. Groundwater. Prentice Hall, NJ, 1979.

5. Flemings, M. C. Solidification Processing. McGraw-Hill, New York, 1974.

6. Zerkle, R. D., J. E. Sunderland. The Effect of Liquid Solidification in a Tube Upon Laminar-Flow Heat Transfer and Pressure Drop. – *Journal of Heat Transfer*, 90, 1968, 183-190.

7. Weigand, B., M. Henze. The Time Dependent Growth of a Solid Crust and the Freeze-shut Inside a Cooled Cylindrical Nozzle Subjected to Laminar Internal Liquid Flow. – *Heat and Mass Transfer*, 40, 2004, 347-354.

8. Kimura, S., K. Kanev. e-Learning of Phase Change Processes under Vigorous Convection Heat Transfer. – *Information Technology and Control*, (8) 3, 2010, 12-18.

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