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PERIODIC SOLUTIONS FOR ELASTIC PRISMS*

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1. Introduction. If one looks at sketches of the shapes of prisms which are predicted by Euler's theory of the elastica, one sees curves which, more often than not, have a periodic character. Some solutions of Kirchhoff's equations for rods also exhibit this quality. It seems reasonable to expect that some solutions of three-dimensional nonlinear elasticity theory will, in their grosser features, closely resemble solutions obtained from these old theories, so we might expect to see such periodicity. For a fair comparison of rod theory with three-dimensional theory, the length of the prism should be large compared to its diameter. We might expect the best comparison to involve prisms of infinite length, and it is here where the basic idea of periodicity seems to make most sense. Of course, we can take a restriction, to get a solution for a prism of finite length. In elastica theory, restrictions of periodic solutions are what we use to describe buckling. The thought that this might be a fruitful, if unorthodox way of thinking about buckling prompted this little study. Our purpose is to propose a definition of periodic solutions, and to derive the most elementary properties of these. Although we shall not do so, it should be feasible to calculate approximate solutions of this type, using formal asymptotic theory of the kind discussed by Parker [1, 2].

2. Formulation. Let X^α ($\alpha = 1, 2$) denote rectangular Cartesian coordinates to which a plane generating cross-section of a prism is referred, and let X^3 be the third such coordinate, varying along the length of the prism. For an infinite prism, we are then concerned with an infinite domain D which is the Cartesian product of a finite plane domain D',

$$X^\alpha \varepsilon \in D', \tag{2.1}$$

and the line

$$X^3 \in (-\infty, \infty), \tag{2.2}$$

viz.

$$D = D' \times (-\infty, \infty). \tag{2.3}$$

We are concerned with elastic prisms, characterized by a strain energy per unit reference volume, of the form

$$W = W(\mathbf{x}_{,k}, X^\alpha), \tag{2.4}$$

where latin indices take on all three values, and $\mathbf{x}$ refers to the triplet of spatial coordinates. It should be Galilean invariant; W should be invariant under transformations of

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the type induced by

$$\bar{\mathbf{x}} = \mathbf{R}(\mathbf{x} + \mathbf{a}), \tag{2.5}$$

where **R** is a rotation matrix, it and **a** being constants. Our definition will lean on the idea that this induces an equivalence relation on point sets. We are concerned with classical or possibly weak solutions of the equilibrium equations

$$\left(\frac{\partial W}{\partial \mathbf{x}_{,k}}\right)_{,k} = 0 \quad \text{in} \quad D \tag{2.6}$$

that leave the lateral boundary free. That is,

$$\frac{\partial W}{\partial \mathbf{x}_{,\alpha}} N_\alpha = 0 \quad \text{on} \quad \partial D' \times (-\infty, \infty), \tag{2.7}$$

where N_α is the usual normal vector. Actually, we are interested in a broader issue, which involves exploring a rather drastically revised version of St.-Venant's principle. Roughly, we wish to know what kinds of solutions of (2.6) and (2.7) are likely to occur, and whether it is feasible to use restrictions of these decently to approximate solutions for prisms of finite length, D' and the function W being fixed. If the old rod theories are a guide, some solutions should be in the small-strain, large-rotation category, and such theory gives as some picture of what they look like, how we can decide which restriction to select, etc. Elastica theory has suggested that there should be the periodic solutions, so we will concentrate on this smaller issue.

As is well known, the equilibrium conditions imply that the resultant force **F** and moment **M** for a cross-section X^3 = const. are independent of the cross-section. Specifically, we have that the following integrals are independent of X^3:

$$\mathbf{F} = \int_{D'} \frac{\partial W}{\partial \mathbf{x}_{,3}} = \text{const.}, \tag{2.8}$$

$$\mathbf{M} = \int_{D'} (\mathbf{x} - \mathbf{k})_\wedge \frac{\partial W}{\partial \mathbf{x}_{,3}} = \text{const.} \tag{2.9}$$

Here, we take moments about $\mathbf{x} = \mathbf{k}$, an arbitrary point. If St. Venant's principle were correct, for finite deformations, any two restrictions of a solution of (2.6) and (2.7) would differ only by a transformation of the type (2.5), plus a deformation which is small, except near ends. This is hardly plausible, given the nature of elastica solutions. On the other hand, if some generalization does not apply, it is hard to understand how rod theory can be successful, since it ignores the fine details of loadings.

What might first occur to one, as a definition of periodic solution, is that, for some constant b, the period

$$\mathbf{x}(X^\alpha, X^3 + b) = \mathbf{x}(X^\alpha, X^3).$$

This would have the cross-sections $X^3 = 0$ and $X^3 = b$ coincident. Even a glance at sketches of elastica solutions suffices to indicate that this is not what we want. What we propose is not to take this so literally, but to interpret the equals sign as reading "is equivalent to", where (2.5) serves to indicate what is to be meant by equivalence. We have not found fault with this, and it is easy to adapt the idea to rod theories, *etc.* Formalizing this is

DEFINITION 1: A solution of (2.6) and (2.7) is called *periodic* if there exist constants **a**, b, and **R**, the latter a rotation matrix, such that, for all $X^k \in D$,

$$\mathbf{x}(X^\alpha, X^3 + b) = \mathbf{R}[\mathbf{x}(X^\alpha, X^3) + \mathbf{a}]. \tag{2.10}$$

Actually, a special type has been studied extensively in connection with theories of torsion, bending, etc. This serves to provide some assurance that we are not talking about the null set. The type is covered by

DEFINITION 2: A periodic solution is said to be of the *St.-Venant type* provided (2.10) holds for every choice of b, with **R** and **a** depending on the choice of b.

If we view (2.10) as a purely kinematical statement, the additional requirement indicated in Definition 2 is, according to Ericksen [3, p. 206], equivalent to the statement that a deformation measure appropriate for finite deformations is independent of X^3, a result which he attributes to Muncaster. The analogue for linear theory is the assumption that the strain (or stress) is independent of X^3. Most of the book by Lekhnitskii [4] is devoted to such solutions. With approximations which are generally accepted in linear theory, it can be shown that, again for such linear theory, every periodic solution is of St.-Venant type. If it were otherwise, linear theory might hope to treat some buckling problems.

3. Properties. It is convenient to rewrite (2.10) in an equivalent form. Suppose first that $\mathbf{R} \neq \mathbf{1}$. Then, **R** uniquely defines a direction, except for its sense, the axis of rotation **e**, with

$$\mathbf{Re} = \mathbf{e}. \tag{3.1}$$

It is easily seen that any vector perpendicular to **e** is representable in the form $(\mathbf{1} - \mathbf{R})\mathbf{l}$, where **l** is some vector. Adding to **l** a vector parallel to **e** has no effect on this combination, which indicates the extent to which it is not uniquely determined. Thus, we can decompose **a** into parts parallel and perpendicular to **e**, writing

$$\mathbf{a} = \mathbf{b} + (\mathbf{1} - \mathbf{R})\mathbf{l}, \qquad \mathbf{Rb} = \mathbf{b}, \tag{3.2}$$

b being proportional to **e**, and **l** being any one of the infinitely many vectors which are compatible with (3.2). Then (2.10) becomes

$$\mathbf{x}(X^3 + b - \mathbf{l}) = \mathbf{R}[\mathbf{x}(X^3) - \mathbf{l} + \mathbf{b}] = \mathbf{R}[\mathbf{x}(X^3) - \mathbf{l}] + \mathbf{b}, \tag{3.3}$$

where we have suppressed dependence on X^α. Now, from (2.4), (2.5) and (3.3), it follows that

$$W|_{X^3+b} = W|_{X^3}, \tag{3.4}$$

$$\left.\frac{\partial W}{\partial \mathbf{x}_{,3}}\right|_{X^3+b} = \mathbf{R}\left.\frac{\partial W}{\partial \mathbf{x}_{,3}}\right|_{X^3}. \tag{3.5}$$

Then (2.8) yields

$$\mathbf{F} = \mathbf{RF}. \tag{3.6}$$

Further, taking $\mathbf{k} = \mathbf{l}$, we obtain from (2.9), (3.3), (3.5) and (3.6)

$$\mathbf{M} = \mathbf{RM} + \mathbf{b}_\wedge \mathbf{F}. \tag{3.7}$$

However, (3.6) implies that **F** is parallel to **e**, as is **b**, so (3.7) reduces to

$$\mathbf{M} = \mathbf{RM}, \tag{3.8}$$

implying that **M** is also parallel to **e**. In summary, then, we have

THEOREM 1: If $\mathbf{R} \neq \mathbf{1}$, and we take $\mathbf{k} = \mathbf{l}$, then **e**, **b**, **F** and **M** are all parallel.

Here, we count **F**, say, as parallel to **e** when $\mathbf{F} = 0$, or when it is anti-parallel, with $\mathbf{F} \cdot \mathbf{e} < 0$. Such correlations are, perhaps, familiar, for the solutions of St.-Venant type for torsion and bending, at least.

Now suppose that $\mathbf{R} = \mathbf{1}$, but $\mathbf{F} \neq 0$. Then, however we choose **k**, (2.9) and (2.10) yield

$$\mathbf{M} = \mathbf{M} + \mathbf{a}_\wedge \mathbf{F}, \tag{3.9}$$

so **a** must be parallel to **F**. As is well known, and easily shown, we can choose **k** so that **M** is also parallel to **F**, so we have a result which is much the same as Theorem 1, *viz.*

THEOREM 2: If $\mathbf{R} = \mathbf{1}$ and $\mathbf{F} \neq 0$, one can choose **k** so that **F**, **M** and **a** are all parallel.

Most of the physical problems which can be treated by elastica theory are such that $\mathbf{F} \neq 0$ and this $\mathbf{M} = 0$. For these, the most obvious periodicities have $\mathbf{R} = \mathbf{1}$. If $\mathbf{R} = \mathbf{1}$ and $\mathbf{F} = 0$, a natural extrapolation of Theorem 2 would be that **M**, which is then independent of **k**, is parallel to **a**. I do not know of a proof of this. Neither do I know of a counter-example to disprove it, although I have not made a diligent search.

According to elastica theory, every solution for a prism of finite length is uniquely continuable to give a solution for an infinite prism, if we tolerate solutions which have the prism intersecting itself. In this sense, every solution is a restriction of a solution for an infinite prism. It seems that little or nothing is known about continuability of analogous solutions in three-dimensional theory. For the elastica, most, but not all solutions are periodic. Those that are not have the curve asymptotic to a straight line parallel to **F**. This suggests that, for three-dimensional theory, there should also be solutions with $\mathbf{x}_{,k}$ approaching limits as $X^3 \to \pm\infty$. This is almost the same as assuming that, for $X^3 \to \infty$,

$$\mathbf{x} = \mathbf{f}(X^\alpha) + \mathbf{a}X^3 + O(1/X^3), \tag{3.10}$$

where $\mathbf{a} = \text{const.}$, with a similar assumption holding for $X^3 \to -\infty$. Consideration of **M** then indicates that **a** must be parallel to **F**. In elastica theory, such solutions can also be viewed as limits of periodic solutions, as the period approaches infinity.

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