Periodic solutions of a nonlinear n-th order vector differential equation.

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Summary. - *We investigate a non-autonomous n-th order differential equation for a function $x(t) \in R^m$ supposing that the equation contains one nonlinear term only depending on x. Our aim is to prove the existence of at least one periodic solution (with the same period as the external forcing). The conditions developed for the nonlinear term are rather general and do not imply the global boundedness of the solutions.*

1. Introduction

Let us consider the n-th order differential equation ($n \geqq 2$)

(1) $$\boldsymbol{x}^{(n)} + \boldsymbol{A}_1 \boldsymbol{x}^{(n-1)} + \dots + \boldsymbol{A}_{n-1} \boldsymbol{x}' + \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}(t)$$

where $\boldsymbol{x} \in R^m$ (R^m denotes the m-dimensional Euclidean space with norm $|\boldsymbol{x}|$), the m,m-matrices $\boldsymbol{A}_i$ are constant and the functions $\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{p}(t)$ are continuous for all $\boldsymbol{x} \in R^m$ respectively $t \in R$. Sedziwy [5] proves the following theorem concerning with the asymptotic behavior of solutions of equation (1):

These solutions are globally bounded if

a) the polynomial

$$p(\lambda) = \mathrm{Det}\,(\lambda^{n-1} \boldsymbol{E}_m + \lambda^{n-2} \boldsymbol{A}_1 + \dots + \boldsymbol{A}_{n-1})$$

$$(\boldsymbol{E}_m - m,\ m\text{-unit matrix})$$

has only roots with negative real parts;

b) $\boldsymbol{A}_{n-1}$ *is a symmetric and positive-definite matrix;*

c) $|\boldsymbol{p}(t)| \leqq \boldsymbol{M}_0$, $|\boldsymbol{P}(t)| = \left|\int_0^t \boldsymbol{p}(\tau)\, d\tau\right| \leqq M_1$ *for* $t \geqq 0$;

d) $|\boldsymbol{f}(\boldsymbol{x})| \leqq F$ *for all* $\boldsymbol{x}$;

e) $\lim\limits_{|\boldsymbol{x}| \to \infty} (\boldsymbol{f}(\boldsymbol{x}) . \boldsymbol{x}) = +\infty$.

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Applying the Brouwer Fixed Point Theorem in the special case

$$\boldsymbol{p}(t+\omega)\equiv\boldsymbol{p}(t) \tag{2}$$

Sedziwy still establishes the existence of at least one ω-periodic solution. But for this an additional assumption is needed: uniqueness of the solution of the initial value problem and continuous dependence of the solutions on the initial conditions.

Omitting the latter restrictions and replacing conditions *d*), *e*) by weaker ones we prove an existence theorem for periodic solutions with the aid of the Leray-Schauder fixed point technique (see [2], [3]).

THEOREM 1. - *Equation* (1) *with a* ω*-periodic forcing term admits at least one* ω*-periodic solution if apart from a*) - *c*) *the following conditions are fulfilled*:

d) $\lim\limits_{|\boldsymbol{x}|\to\infty}\dfrac{|\boldsymbol{f}(\boldsymbol{x})|}{|\boldsymbol{x}|}=0$;

e) *either* $f_i(\boldsymbol{x})x_i\geqq 0$ *for* $|x_i|\geqq h_i$ $(1\leqq i\leqq m)$

or $f_i(\boldsymbol{x})x_i\leqq 0$ *for* $|x_i|\geqq h_i$ $(1\leqq i\leqq m)$.

REMARK - Defining

$$\max_{|\boldsymbol{x}|\leqq R}|\boldsymbol{f}(\boldsymbol{x})|=F(R)\quad\text{(monotonely increasing)} \tag{3}$$

we can easily show:

$$\lim_{R\to\infty}\frac{F(R)}{R}=0. \tag{4}$$

Equation (4) is evident if $|\boldsymbol{f}(\boldsymbol{x})|$ is bounded.

Now let $|\boldsymbol{f}(\boldsymbol{x})|$ be unbounded; hence we have

$$\lim_{R\to\infty}F(R)=\infty.$$

Supposing that equation (4) is not true we choose a divergent set $\{R_n\}$ with the properties

$$\frac{F(R_n)}{R_n}\geqq\eta\quad\text{(for a certain positive value }\eta),\quad F(R_{n+1})>F(R_n).$$

From

$$F(R_{n+1})=|\boldsymbol{f}(\boldsymbol{x}_{n+1})|$$

for a vector $\boldsymbol{x}_{n+1}(R_n < |\boldsymbol{x}_{n+1}| \leqq R_{n+1})$ we conclude

$$\frac{F(R_{n+1})}{R_{n+1}} = \frac{|\boldsymbol{f}(\boldsymbol{x}_{n+1})|}{|\boldsymbol{x}_{n+1}|}\frac{|\boldsymbol{x}_{n+1}|}{R_{n+1}} \leqq \frac{|\boldsymbol{f}(\boldsymbol{x}_{n+1})|}{|\boldsymbol{x}_{n+1}|} \to 0 \;\; (n\to\infty),$$

on the contrary to (4).

2. - Equivalent system of first order, characteristic equation

Denoting

$$\boldsymbol{x}' = \boldsymbol{y}_1, \; \boldsymbol{y}_1' = \boldsymbol{y}_2, \; \dots, \; \boldsymbol{y}_{n-2}' = \boldsymbol{y}_{n-1}$$

we obtain according to (1) the n-th differential equation

$$\boldsymbol{y}_{n-1}' = -\boldsymbol{A}_{n-1}\boldsymbol{y}_1 - \dots - \boldsymbol{A}_1 \;\boldsymbol{y}_{n-1} - [\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{p}(t)].$$

Introducing the $m(n-1)$, $m(n-1)$-matrix

$$\boldsymbol{A} = \begin{bmatrix} 0, & \boldsymbol{E}_m, & 0, & \dots, & 0 \\ 0, & 0, & \boldsymbol{E}_m, & \dots, & 0 \\ & & \dots & & \\ 0, & 0, & 0, & \dots, & \boldsymbol{E}_m \\ -\boldsymbol{A}_{n-1}, & -\boldsymbol{A}_{n-2}, & -\boldsymbol{A}_{n-3}, & \dots, & -\boldsymbol{A}_1 \end{bmatrix},$$

the $m(n-1)$, m-matrices

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\boldsymbol{E}_m \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{E}_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the $m(n-1)$-vector of derivatives

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_{n-1} \end{pmatrix}$$

we deduce the differential system (see [5])

$$(5) \qquad \boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{p}(t)], \; \boldsymbol{x}' = \boldsymbol{C}'\boldsymbol{y}.$$

Regarding $\boldsymbol{x} = \boldsymbol{x}(t)$ as a known continuous function of t and the differential equation for $\boldsymbol{y}$ as a non-homogeneous linear one we determine the corresponding characteristic equation (of degree $m(n-1)$)

$$p(\lambda) = \operatorname{Det}(\lambda \boldsymbol{E}_{m(n-1)} - \boldsymbol{A}) = \begin{vmatrix} \lambda \boldsymbol{E}_m, & -\boldsymbol{E}_m, & 0, & \dots, & 0 \\ 0, & \lambda \boldsymbol{E}_m, & -\boldsymbol{E}_m, & \dots, & 0 \\ & & \dots & & \\ \boldsymbol{A}_{n-1}, & \boldsymbol{A}_{n-2}, & \boldsymbol{A}_{n-3}, & \dots, & \lambda \boldsymbol{E}_m + \boldsymbol{A}_1 \end{vmatrix}$$

$$= \operatorname{Det}(\lambda^{n-1} \boldsymbol{E}_m + \lambda^{n-2} \boldsymbol{A}_1 + \dots + \lambda \boldsymbol{A}_{n-2} + \boldsymbol{A}_{n-1}) = 0 .$$

The real parts of the roots $\lambda = \lambda_i$, $1 \leqq i \leqq m(n-1)$ are assumed to be negative; consequently

$$Re\, \lambda_i < -\sigma < 0 \text{ for } 1 \leqq i \leqq m(n-1). \tag{6}$$

Furthermore we study the homogeneous linear system

$$\boldsymbol{x}' = \boldsymbol{C}^t \boldsymbol{y}, \ \boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + h \boldsymbol{B}\boldsymbol{x} \tag{7}$$

(h an adequate real constant)

or

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}' = \boldsymbol{D} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}, \tag{7'}$$

$$\boldsymbol{D} = \begin{bmatrix} 0 & , \boldsymbol{E}_m, 0, \dots, 0 \\ \cdot & \\ \cdot & \\ \cdot & \boldsymbol{A} \\ 0 & \\ -h \boldsymbol{E}_m & \end{bmatrix}$$

a nm,nm-matrix.

The corresponding characteristic polynomial (of degree nm) is

$$q_h(\lambda) = \operatorname{Det}(\lambda \boldsymbol{E}_{nm} - \boldsymbol{D}) = \operatorname{Det}(\lambda^n \boldsymbol{E}_m + \lambda^{n-1} \boldsymbol{A}_1 + \dots + \lambda \boldsymbol{A}_{n-1} + h \boldsymbol{E}_m),$$

where

$$q_0(\lambda) = \lambda^m p(\lambda) = 0$$

possesses m roots $\lambda = 0$ and the above-mentioned $m(n-1)$ roots $\lambda = \lambda_i$ with negative real parts.

Choosing $h \neq 0$ we find

$$q_h(0) = h^m \neq 0,$$

all roots of $q_h(\lambda) = 0$ are different from zero.

In order to investigate these roots for a positive but sufficiently small $|h|$ we consider the matrix

$$(\xi \boldsymbol{E}_m + \boldsymbol{A}_{n-1}) + (\boldsymbol{R} + i\,\boldsymbol{S}), \tag{8}$$

$\xi = \alpha + i\beta$ a complex constant, $\boldsymbol{R}$ and $\boldsymbol{S}$ real m, m-matrices. We look for conditions on which the determinant of this matrix is unequal zero. Hence we establish the equation ($\boldsymbol{z} = \boldsymbol{u} + i\boldsymbol{v}$ being a complex m-vector)

$$(\xi \boldsymbol{E}_m + \boldsymbol{A}_{n-1} + \boldsymbol{R} + i\,\boldsymbol{S})\,\boldsymbol{z} = 0, \tag{9}$$

that is

$$\left\{\begin{aligned} &\alpha\boldsymbol{u} - \beta\boldsymbol{v} + \boldsymbol{A}_{n-1}\,\boldsymbol{u} + \boldsymbol{R}\boldsymbol{u} - \boldsymbol{S}\boldsymbol{v} = 0 \\ &\beta\boldsymbol{u} + \alpha\boldsymbol{v} + \boldsymbol{A}_{n-1}\,\boldsymbol{v} + \boldsymbol{S}\boldsymbol{u} + \boldsymbol{R}\boldsymbol{v} = 0. \end{aligned}\right. \tag{9'}$$

Forming the inner product with $\boldsymbol{u}$ respectively $\boldsymbol{v}$ and summing up we obtain

$$\begin{aligned} \Phi(\boldsymbol{u}, \boldsymbol{v}) \equiv \alpha(|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2) + (\boldsymbol{u}^t\boldsymbol{A}_{n-1}\,\boldsymbol{u} + \boldsymbol{v}^t\boldsymbol{A}_{n-1}\,\boldsymbol{v}) \\ + (\boldsymbol{u}^t\boldsymbol{R}\boldsymbol{u} + \boldsymbol{v}^t\boldsymbol{R}\boldsymbol{v}) + (\boldsymbol{v}^t\boldsymbol{S}\boldsymbol{u} - \boldsymbol{u}^t\boldsymbol{S}\boldsymbol{v}) = 0. \end{aligned}$$

Because $\boldsymbol{A}_{n-1}$ is a symmetric positive-definite matrix we can estimate

$$\boldsymbol{u}^t\boldsymbol{A}_{n-1}\,\boldsymbol{u} + \boldsymbol{v}^t\boldsymbol{A}_{n-1}\,\boldsymbol{v} \geqq \rho\,(|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2)$$

($\rho > 0$ being the smallest eigenvalue of $\boldsymbol{A}_{n-1}$).

Provided that

$$\alpha \geqq -\frac{\rho}{4}, \quad |\boldsymbol{R}| \leqq \frac{\rho}{4}, \quad |\boldsymbol{S}| \leqq \frac{\rho}{4} \tag{10}$$

the following inequalities are valid:

$$\Phi(\boldsymbol{u}, \boldsymbol{v}) \geqq (\rho + \alpha)\,|\boldsymbol{z}|^2 - \frac{\rho}{4}|\boldsymbol{z}|^2 - \frac{\rho}{2}|\boldsymbol{u}|\,|\boldsymbol{v}|$$

$$\geqq\left(\frac{\rho}{2}+\alpha\right)|\boldsymbol{z}|^2$$

$$\geqq\frac{\rho}{4}|\boldsymbol{z}|^2>0 \quad (\text{if } \boldsymbol{z}\neq 0).$$

For this reason equation (9) does not admit a non-zero solution, matrix

$$(\xi\boldsymbol{E}_m+\boldsymbol{A}_{n-1})+(\boldsymbol{R}+i\boldsymbol{S});\ \operatorname{Re}\xi\geqq-\frac{\rho}{4},\ |\boldsymbol{R}|\leqq\frac{\rho}{4},\ |\boldsymbol{S}|\leqq\frac{\rho}{4} \tag{11}$$

is non-singular.

Let λ be a complex number and

$$\boldsymbol{R}+i\boldsymbol{S}=\lambda^{n-1}\boldsymbol{E}_m+\lambda^{n-2}\boldsymbol{A}_1+\dots+\lambda\boldsymbol{A}_{n-2};$$

the restrictions contained in (11) are realised if

$$|\lambda|\leqq l=\frac{\rho}{\rho+4(|\boldsymbol{E}_m|+|\boldsymbol{A}_1|+\dots+|\boldsymbol{A}_{n-2}|)}. \tag{12}$$

The algebraic equation $q_h(\lambda)=0$ $(h\neq 0)$ being satisfied by a value λ $|\lambda|\leqq l$, can be rewritten as

$$\operatorname{Det}\left(\xi\boldsymbol{E}_m+\boldsymbol{A}_{n-1}+[\lambda^{n-1}\boldsymbol{E}_m+\dots+\lambda\boldsymbol{A}_{n-2}]\right)=0,\ \xi=\frac{h}{\lambda};$$

an immediate consequence of the above statement [see (11)] is

$$\operatorname{Re}\xi=\frac{h\operatorname{Re}\lambda}{|\lambda|^2}<-\frac{\rho}{4},$$

that is

$$\operatorname{Re}\lambda<0 \text{ for } h>0,\ \operatorname{Re}\lambda>0 \text{ for } h<0. \tag{13}$$

The roots of equation $q_h(\lambda)=0$ are continuously depending on the parameter h. Let $\varepsilon>0$ be a prescribed number; then there exists a number $\delta(\varepsilon)>0$ such that $|h|\leqq\delta(\varepsilon)$ implies

$$|\lambda_i'-\lambda_i|\leqq\varepsilon \text{ for } m(n-1) \text{ roots } \lambda_i',$$

$$|\lambda_j''|\leqq\varepsilon \text{ for } m \text{ roots } \lambda_j''.$$

Choosing

$$\varepsilon \leqq \varepsilon^* = \mathrm{Min}\left(l\,,\ \frac{\sigma}{2}\right),\ 0 < |h| \leqq \delta^* = \delta(\varepsilon^*)$$

we obtain

$$\mathrm{Re}\ \lambda_i' \leqq -\frac{\sigma}{2}\,,\ |\lambda_j''| \leqq l$$

and by virtue of (13)

$$\mathrm{Re}\ (h\lambda_j'') < 0\,.$$

Therefore system (7) admits no ω-periodic solution apart from the trivial one.

REMARK - Condition *b)* can be replaced by the weaker one

b') $\boldsymbol{A}_{n-1}$ has no purely imaginary eigenvalue.

Denoting the eigenvalues of $\boldsymbol{A}_{n-1}$ by $\rho_1, \dots, \rho_m$ and the eigenvalues of the «perturbed» matrix $\boldsymbol{A}_{n-1} + (\boldsymbol{R} + i\,\boldsymbol{S})$ by $\sigma_1, \dots, \sigma_m$ we conclude from

$$|\mathrm{Re}\ \rho_j| \geqq \rho > 0\ \ (1 \leqq j \leqq m)$$

by virtue of continuity:

$$|\mathrm{Re}\ \sigma_j| \geqq \frac{\rho}{2}\ \text{if}\ |\boldsymbol{R} + i\boldsymbol{S}| \leqq \eta\ \text{(small enough)}.$$

Identifying (like above)

$$\boldsymbol{R} + i\,\boldsymbol{S} = \lambda^{n-1}\boldsymbol{E}_m + \dots + \lambda \boldsymbol{A}_{n-2}\,,\ \sigma_j = -\frac{h}{\lambda_j''}$$

we satisfy the restriction for the «perturbation matrix» by the choice $|\lambda| \leqq l$. Then we obtain

$$\frac{|\mathrm{Re}\ \lambda_j''|}{|\lambda_j''|^2} \geqq \frac{\rho}{2|h|}\,.$$

Hence the linear system (7) does not permit any free oscillation.

3. - Boundedness of derivatives

Let $(\boldsymbol{x}(t), \boldsymbol{y}(t))$ be a solution of system (5) the $\boldsymbol{x}$-component of which is bounded for $t \geqq 0$. We determine

$$F = \sup_{t \geqq 0} |\boldsymbol{f}(\boldsymbol{x}(t))|\,.$$

The differential equation for the $\boldsymbol{y}$-component

$$y' = Ay + B[f(x(t)) - p(t)] \tag{14}$$

may be considered as a nonhomogeneous linear system the general solution of which is

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-\tau)A} B[f(x(\tau)) - p(\tau)]\, d\tau, \quad t \geqq 0.$$

Matrix $\boldsymbol{A}$ being stable the following estimation is possible:

$$|e^{tA}| \leqq \varkappa\, e^{-\sigma t} \; (\sigma > 0).$$

With regard to this we obtain

$$\begin{aligned} |y(t)| &\leqq \varkappa |y_0| e^{-\sigma t} + \frac{\varkappa}{\sigma} |B| (F + M_0) \\ &\leqq k(F + M_0) \text{ for } t \geqq t_0 \\ &\left(k > \frac{\varkappa}{\sigma} |B|\right). \end{aligned} \tag{15}$$

4. - Uniform boundedness of periodic solutions

Following to the Leray-Schauder procedure we replace system (5) by a more general system depending on a parameter μ:

$$x' = C'y, \; y' = h\,Bx + Ay + \mu B[f(x) - hx - p(t)] \tag{16}$$
$$(0 \leqq \mu \leqq 1, \; 0 < |h| \leqq \delta^*).$$

For $\mu = 1$ systems (5) and (16) are identical, for $\mu = 0$ systems (7) and (16) are identical; in the latter case the zero solution is the only ω-periodic one.

If the possible ω-periodic solutions of system (16) are a priori bounded for $0 < \mu < 1$ then the existence of at least one ω-periodic solution for $\mu = 1$ is ensured (see [4]).

Let $(x(t), y(t))$ be such a ω-periodic solution of (16), $0 < \mu < 1$, and let be

$$R = \operatorname*{Max}_{[0,\omega]} |x(t)| > 0.$$

Equations (16) are equivalent to the n-th order equation

$$\frac{d}{dt}\left(\boldsymbol{x}^{(n-1)}(t)+\boldsymbol{A}_1\boldsymbol{x}^{(n-2)}(t)+\dots+\boldsymbol{A}_{n-1}\boldsymbol{x}(t)-\mu\boldsymbol{P}(t)\right) = -\varphi(\boldsymbol{x}(t)), \tag{17}$$

$$\varphi(\boldsymbol{x})=\mu\boldsymbol{f}(\boldsymbol{x})+(1-\mu)\,h\boldsymbol{x}.$$

Let us suppose

$$h>0 \text{ if } f_i(\boldsymbol{x})\,x_i \geqq 0 \;(|x_i|\geqq h_i)$$

and

$$h<0 \text{ if } f_i(\boldsymbol{x})\,x_i \leqq 0 \;\;(|x_i|\geqq h_i).$$

Then it results for $0<\mu<1$, $1\leqq i\leqq m$, $|x_i|\geqq h_i$

$$\varphi_i(\boldsymbol{x})\,x_i>0 \text{ [respectively } \varphi_i(\boldsymbol{x})\,x_i<0]. \tag{18}$$

Apparently we have for all t

$$|\varphi(\boldsymbol{x}(t))|<F_0=F(R)+|h|\,R \text{ [see (3)]};$$

hence the vector $\boldsymbol{y}(t)$ of derivatives can be estimated like in section 3.,

$$|\boldsymbol{y}(t)|\leqq k\,(F_0+M_0) \text{ for sufficiently large values } t \tag{19}$$

(that is for all t because of periodicity).

Let be $x_i(t)>h_i$ on an interval (t', t''); then we conclude from (17):

$$\left(\boldsymbol{x}^{(n-1)}(t)+\dots+\boldsymbol{A}_{n-1}\,\boldsymbol{x}(t)-\mu\boldsymbol{P}(t)\right)_i\Big|_{t'}^{t''}=-\int_{t'}^{t''}\varphi_i(\boldsymbol{x}(t))\;dt$$

$$<0 \text{ (respectively } >0).$$

The left hand member vanishes for $t''-t'=\omega$; consequently the length of the considered interval is smaller then ω. There must exist a value τ such that

$$x_i(\tau)\leqq h_i.$$

Applying (19) we obtain on the interval $\tau\leqq t\leqq\tau+\omega$

$$x_i(t)=x_i(\tau)+(t-\tau)\,(\boldsymbol{y}_1(\xi))_i,$$

$$x_i(t)\leqq h_i+\omega k\,(F_0+M_0).$$

The component $x_i(t)$ being ω-periodic the derived estimation is true for all t.

An analogous result can be deduced for $-x_i(t)$. From

$$|x_i(t)| \leqq h_i + \omega k(F_0 + M_0), \quad 1 \leqq i \leqq m$$

we conclude that

$$\begin{aligned} |\boldsymbol{x}(t)| &\leqq H + m\,\omega k\, F_0 \\ &= H + m\,\omega k(F(R) + |h|\,R) \end{aligned}$$

$$(H = h_1 + \ldots + h_m + m\omega k M_0);$$

particularly the amplitude R can be bounded by the last quantity. Choosing the amount of parameter h sufficiently small,

$$|h| \leqq \frac{1}{2m\omega k},$$

we can solve the inequality

$$R \leqq H + m\omega k\,(F(R) + |h|\,R):$$

$$\frac{1}{2} R \leqq (1 - m\omega k\,|h|)R \leqq H + m\omega k\, F(R),$$

$$1 \leqq \frac{2H}{R} + 2m\omega k \frac{F(R)}{R}$$

and hence

$$R \leqq 4H + R_0 \tag{20}$$

when

$$\frac{F(R)}{R} < \frac{1}{4m\omega k} \text{ for } R > R_0 \text{ [compare (4)]}.$$

From (20) we see that the ω-periodic solutions of the system (16), $0 < \mu < 1$, are uniformly bounded. This completes our proof.

5. - Special case $n = 1$

At last we turn to the simple equation

$$\boldsymbol{x}' + \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}(t), \quad \boldsymbol{p}(t + \omega) \equiv \boldsymbol{p}(t), \tag{21}$$

$\boldsymbol{x}$ a m-vector, $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{p}(t)$ continuous for all $\boldsymbol{x} \in R^m$ respectively $t \in R$.

THEOREM 2. - *The conditions*

a) $\int_0^\omega \boldsymbol{p}(t)dt = 0$;

b) for each index $i (1 \leqq i \leqq m)$
either $f_i(\boldsymbol{x}) x_i \geqq 0$ *or* $f_i(\boldsymbol{x}) x_i \leqq 0$ $(|x_i| \geqq h_i)$

ensure the existence of at least one ω*-periodic solution.*

This solution can be established by the Leray-Schauder technique like above.

Let $\boldsymbol{L}$ be a nonsingular real m,m-matrix of diagonal form,

$$\boldsymbol{L} = \begin{pmatrix} \lambda_1 & & & \\ & \cdot & 0 & \\ & & \cdot & \\ & 0 & \cdot & \\ & & & \lambda_m \end{pmatrix};$$

then we introduce the auxiliary system

$$\boldsymbol{x}' + \boldsymbol{L}\boldsymbol{x} = \mu[-\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{L}\boldsymbol{x} + \boldsymbol{p}(t)] \tag{22}$$

$$(0 \leqq \mu \leqq 1)$$

containing the original system (21) for $\mu = 1$.

The characteristic equation corresponding to the homogeneous linear system $(\mu = 0)$ is

$$\mathrm{Det}\,(\lambda \boldsymbol{E}_m + \boldsymbol{L}) = (\lambda + \lambda_1) \cdot \ldots \cdot (\lambda + \lambda_m) = 0;$$

the roots are

$$\lambda = -\lambda_i (1 \leqq i \leqq m).$$

Let us choose

$$\lambda_i > 0 \text{ if } f_i(\boldsymbol{x}) x_i \geqq 0 \ (|x_i| \geqq h_i) \tag{23'}$$

and

$$\lambda_i < 0 \text{ if } f_i(\boldsymbol{x}) x_i \leqq 0 \ (|x_i| \geqq h_i). \tag{23''}$$

In order to prove the a priori boundedness of ω-periodic solutions of (22) for $0<\mu<1$ we consider such a solution $\boldsymbol{x}(t)$ and write instead of (22)

$$\frac{d}{dt}[\boldsymbol{x}(t)-\mu \boldsymbol{P}(t)]=-\varphi(\boldsymbol{x}), \tag{24}$$

$$\varphi(\boldsymbol{x})=\mu \boldsymbol{f}(\boldsymbol{x})+(1-\mu)\boldsymbol{L}\boldsymbol{x}.$$

Evidently we have for $|x_i|\gtreqless h_i$

$$\varphi_i(\boldsymbol{x})x_i>0 \text{ respectively } \varphi_i(\boldsymbol{x})x_i<0.$$

By virtue of this condition we conclude like above that

$$x_i(\tau')\leqq h_i \text{ and } x_i(\tau'')\geqq -h_i$$

for certain values τ', τ''.

At first we study the case (23'). Let t' be a moment when $x_i(t')>h_i$, and let be

$$\tau'=\sup\{\tau<t':x_i(\tau)=h_i\}$$

$$(t'-\tau'<\omega).$$

After integrating the i-th component of equation (24) from τ' to t' we obtain

$$x_i(t')=x_i(\tau')+\mu[P_i(t')-P_i(\tau')]-\int_{\tau'}^{t'}\varphi_i(\boldsymbol{x}(t))\,dt \tag{25}$$

$$<h_i+2M_1;$$

consequently the estimation

$$x_i(t)\leqq h_i+2M_1$$

is valid for all t. This result can be deduced too for $-x_i(t)$.
Now we turn to the case (23''). Let t' be a moment when $x_i(t')>h_i$, and let be

$$\tau'=\inf\{\tau>t':x_i(\tau)=h_i\}.$$

Because of

$$\varphi_i(\boldsymbol{x}(t))<0 \text{ for } t'\leqq t\leqq\tau',\ t'<\tau'$$

we have again

$$\int_{\tau'}^{t'} \varphi_i(\boldsymbol{x}(t))\,dt > 0$$

and therefore

$$x_i(t') < h_i + 2M_1 .$$

As a consequence of the m inequalities

$$|x_i(t)| \leqq h_i + 2M_1 \quad (\text{for all } t)$$

we obtain the boundedness result

$$|\boldsymbol{x}(t)| \leqq H + 2mM_1 , \quad H = h_1 + \ldots + h_m$$

which completes the proof.

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