

PERIODIC SOLUTIONS OF FINITE DIFFERENCE EQUATIONS*

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Introduction. It is well known that the qualitative properties of solutions of finite difference approximations of an ordinary differential equation may differ greatly from the qualitative properties of the solutions of the differential equation. A classic example is the differential equation

$$\ddot{x} + x = 0. \tag{1}$$

If we set $y_1 = x$ and $y_2 = \dot{x}$, then the Euler approximation of (1) is

$$y_{k+1} = Ay_k = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix} y_k. \tag{2}$$

The eigenvalues of the matrix A are $1 \pm hi$. Hence all the iterates y_1, y_2, \dots , of (2) spiral away from the origin. On the other hand, every non-trivial solution of (1) is periodic with period 2π .

The essential difficulty with the differential equation (1) is that the equilibrium solution $x = \dot{x} = 0$ is a center. Equivalently, the periodic solutions of (1) are not orbitally asymptotically stable, and thus they may be destroyed under an arbitrary small perturbation. The situation is very different, however, when $x = \phi(t)$ ($x = (x_1, \dots, x_n)$) is an orbitally asymptotically stable (unstable) solution of the differential equation

$$\dot{x} = f(x). \tag{3}$$

Let

$$x_{k+1} = F(x_k, h) \tag{4}$$

be a finite difference approximation of (3). While we cannot expect the difference equation (4) to possess a closed invariant curve of periodic points, we would expect that the difference equation (4) has a closed invariant curve Γ near $x = \phi(t)$, and that the solutions x_k of (4) spiral into Γ as $k \rightarrow \infty$ ($-\infty$). This was verified recently for the special case of the Van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0. \tag{5}$$

It is well known (see Hale [4]) that this equation has a periodic solution $x = \phi(t)$ with $\phi^2 + \dot{\phi}^2 \simeq 2$, for ϵ sufficiently small, and positive. Using theorems of Halanay [2], [3] on the existence of periodic solutions of difference equations, and a suitable integral

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manifold method, Hurt [5] was able to show that the central difference analogue

$$\frac{1}{h^2} [x(t_{k+1}) - 2x(t_k) + x(t_{k-1})] + \epsilon(x^2(t_k) - 1) \frac{(x(t_{k+1}) - x(t_{k-1}))}{2h} + x(t_k) = 0$$

to (5) has a closed invariant curve near $x = \phi(t)$ for $\epsilon > 0$ and h sufficiently small.

In this paper we generalize Hurt's result to the case of any orbitally asymptotically stable (unstable) periodic solution of any autonomous differential equation. In particular, we prove the following theorem:

THEOREM 1: Let $x = \phi(t)$ be a periodic solution of (3), with characteristic multipliers $\mu_1, \dots, \mu_{n-1}, \mu_n = 1$ and let

$$x_{k+1} = F(x_k, h) \tag{6}$$

be a finite difference approximation of (3) which is at least as accurate as Euler's method. Assume that either $|\mu_j| < 1, j = 1, \dots, n - 1$ or $|\mu_j| > 1, j = 1, \dots, n - 1$. Then the difference equation (6) has a closed invariant curve Γ near $x = \phi(t)$, for h small. Furthermore, the iterates x_k of x_0 spiral into Γ as $k \rightarrow \infty$, or $-\infty$, for x_0 sufficiently close to Γ .

As a corollary to this theorem, we show that the second characteristic multiplier of the periodic solution $x = \phi(t)$ of the Van der Pol equation (5) has modulus less than one. Hence, any finite difference approximation of (5) which is at least as accurate as Euler's method has a closed invariant curve Γ near $x = \phi(t)$, for h small, and all the iterates x_k of x_0 spiral into Γ , for x_0 sufficiently close to Γ . This result answers all of the questions raised in [5].

The mapping M^N . For simplicity of writing and notational convenience, we will prove Theorem 1 for the case $n = 2$, and then show how the proof generalizes to arbitrary n . Thus we assume that the differential equation

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y) \tag{7}$$

has a periodic solution $x = \phi(t), y = \psi(t)$, with characteristic multipliers $\mu_1 = 1$ and μ_2 , with $|\mu_2| \neq 1$. The functions f and g have three continuous derivatives with respect to x and y . Then, any finite difference approximation of (7) which is at least as accurate as Euler's method can be written in the form

$$\begin{aligned} x_{k+1} &= x_k + hf(x_k, y_k) + h^2\hat{f}(x_k, y_k, h) \\ y_{k+1} &= y_k + hg(x_k, y_k) + h^2\hat{g}(x_k, y_k, h) \end{aligned} \tag{8}$$

The key step in proving Theorem 1 is to introduce "normal coordinates" r, θ near the periodic solution $x = \phi(t), y = \psi(t)$ of (7). Following Hale [4], we transform this periodic solution to the circle $r = 0$ by a suitable transformation $r = u(x, y), \theta = v(x, y)$. This can be done in such a way that

$$\begin{aligned} \dot{r} &= u_x f + u_y g = a(\theta)r + f_1(r, \theta) \\ \dot{\theta} &= v_x f + v_y g = 1 + f_2(r, \theta) \end{aligned} \tag{9}$$

where

$$f_2(0, \theta) = f_1(0, \theta) = (\partial f_1 / \partial r)(0, \theta) = 0 \tag{10}$$

and $\exp(\int_0^{2\pi} a(\theta) d\theta)$ is the characteristic multiplier μ_2 . Applying this transformation to the difference equation (8) gives

$$\begin{aligned} r_{k+1} &= u(x_{k+1}, y_{k+1}) \\ &= u(x_k + hf(x_k, y_k) + h^2\hat{f}, y_k + hg(x_k, y_k) + h^2\hat{g}) \\ &= u(x_k, y_k) + h(u_x f + u_y g) + h^2\hat{F}_1 \\ &= r_k + ha(\theta_k)r_k + hf_1(r_k, \theta_k) + h^2F_1(r_k, \theta_k, h). \end{aligned}$$

Similarly,

$$\begin{aligned} \theta_{k+1} &= v(x_k + hf(x_k, y_k) + h^2\hat{f}, y_k + hg(x_k, y_k) + h^2\hat{g}) \\ &= \theta_k + h[1 + f_2(r_k, \theta_k)] + h^2G_1(r_k, \theta_k, h). \end{aligned}$$

Thus, we consider the mapping M

$$\begin{aligned} M: \quad r_1 &= [1 + ha(\theta)]r + hf_1(r, \theta) + h^2F_1(r, \theta, h) \\ \theta_1 &= \theta + hf_2(r, \theta) + h^2G_1(r, \theta, h). \end{aligned} \tag{11}$$

Together with M , we include the N th iterate M^N of M , where $N = [2\pi/h]$, and we write M^N in the form

$$\begin{aligned} M^N: \quad r_N &= \lambda(\theta, r, h)r + F(r, \theta, h) \\ \theta_N &= \theta + 2\pi + G(r, \theta, h). \end{aligned} \tag{12}$$

We will show that the mapping M^N has a closed invariant curve Γ near the circle $r = 0$, and then we will show that Γ is also invariant under M . It is very natural to consider the mapping M^N and its iterates, because this is the section map that figures very prominently in the proof of the existence and stability of periodic solutions of differential equations.

Estimates of λ , F and G . We first derive explicit representations for λ , F and G . To this end, set

$$a_k = 1 + ha(\theta_k), \quad b_k = hf_1(r_k, \theta_k) + h^2F_1(r_k, \theta_k, h).$$

Then (see Braun [1]),

$$\begin{aligned} r_N &= a_0 \cdots a_{N-1}r + \sum_{j=0}^{N-1} a_0 \cdots a_{N-1} \frac{b_j}{a_0 \cdots a_j} \\ &= \prod_{j=0}^{N-1} [1 + ha(\theta_j)]r + \left[\prod_{j=0}^{N-1} (1 + ha(\theta_j)) \right] \sum_{j=0}^{N-1} \frac{hf_1(r_j, \theta_j) + h^2F_1(r_j, \theta_j, h)}{\prod_{\nu=0}^j [1 + ha(\theta_\nu)]}. \end{aligned}$$

Hence,

$$\lambda(\theta, r, h) = \prod_{j=0}^{N-1} [1 + ha(\theta_j)], \quad F(r, \theta, h) = \lambda \sum_{j=0}^{N-1} \frac{hf_1(r_j, \theta_j) + h^2F_1(r_j, \theta_j, h)}{\prod_{\nu=0}^j [1 + ha(\theta_\nu)]}. \tag{13}$$

Our first step is to estimate the iterates r_j , $j = 1, \dots, N - 1$. To this end, choose a

positive constant c such that the functions $a(\theta)$, $a'(\theta)$, $f_1(r, \theta)/r^2$, $f_2(r, \theta)$, $F_1(r, \theta, h)$ and $G_i(r, \theta, h)$ are all less than or equal to c in absolute value for $0 \leq \theta \leq 2\pi$, $|r| \leq 1$, and $0 \leq h \leq 1$. (If necessary, we can rescale r so that all these functions are defined for $|r| \leq R$, with $R > 1$.) Then, we have the following lemma:

LEMMA 1: Choose r and h sufficiently small that $|r| + h \leq \exp(-4\pi c)$. Then,

$$|r_j| \leq (1 + 2hc)^j(|r| + h) \leq (\exp(4\pi c))(|r| + h), \quad j = 1, \dots, N. \quad (14)$$

Proof: From (11),

$$\begin{aligned} |r_{k+1}| &\leq (1 + hc) |r_k| + hc |r_k|^2 + ch^2 \\ &\leq (1 + 2hc) |r_k| + ch^2 \quad (\text{as long as } |r_k| \leq 1). \end{aligned}$$

These inequalities imply that

$$\begin{aligned} |r_j| &\leq (1 + 2hc)^j |r| + \frac{ch^2}{2hc} [(1 + 2hc)^j - 1] \\ &\leq (1 + 2hc)^j(|r| + h) \leq (\exp(2hcj))(|r| + h) \\ &\leq (\exp(4\pi c))(|r| + h) \end{aligned}$$

as long as $|r_k| \leq 1$, $k = 1, \dots, j$, and this in turn follows immediately if $|r| + h \leq \exp(-4\pi c)$.

Our next step is to show that $\lambda(r, \theta, h) = \prod_{i=0}^{N-1} (1 + ha(\theta_i))$ is approximately the second characteristic multiplier μ_2 of the periodic solution $r = 0$ of (9). In particular, if $\mu_2 < 1$, then $0 < \lambda(r, \theta, h) \leq \rho < 1$ uniformly for $0 \leq \theta \leq 2\pi$, and $|r|$ and h sufficiently small. This is the content of Lemmas 2 and 3.

LEMMA 2:

$$\text{Limit}_{h \rightarrow 0} \frac{\prod_{i=0}^{N-1} (1 + ha(\theta_i))}{\exp\left(h \sum_{i=0}^{N-1} a(\theta_i)\right)} = 1.$$

Proof: Observe that

$$\exp(ha(\theta_i)) = 1 + ha(\theta_i) + e_i$$

where

$$|e_i| \leq \frac{h^2 c^2}{2} \exp(hc).$$

Hence,

$$\prod_{i=0}^{N-1} (1 + ha(\theta_i)) = \exp\left(h \sum_{i=0}^{N-1} a(\theta_i)\right) \prod_{i=0}^{N-1} (1 - \exp((-ha(\theta_i))e_i))$$

so that

$$\prod_{i=0}^{N-1} \left(1 - \frac{h^2 c^2}{2} \exp(2hc)\right) \leq \frac{\prod_{i=0}^{N-1} (1 + ha(\theta_i))}{\exp\left(h \sum_{i=0}^{N-1} (a\theta_i)\right)} \leq \prod_{i=0}^{N-1} \left(1 + \frac{h^2 c^2}{2} \exp(2hc)\right).$$

Lemma 2 now follows immediately from the trivial observation that

$$\lim_{h \rightarrow 0} \left[1 \pm \frac{h^2 c^2}{2} \exp(2hc) \right]^{[2\pi/h]} = 1.$$

LEMMA 3:

$$\text{Limit}_{|r|+h \rightarrow 0} \sum_{j=0}^{N-1} h[a(\theta + jh) - a(\theta_j)] = 0$$

if $N = [2\pi/h]$.

Proof. From (11),

$$\theta_j = \theta + jh + \sum_{k=0}^{j-1} hf_2(r_k, \theta_k) + h^2 F_2(r_k, \theta_k, h).$$

Therefore,

$$a(\theta_j) = a(\theta + jh) + a'(\phi_j) \sum_{k=0}^{j-1} hf_2(r_k, \theta_k) + h^2 F_2(r_k, \theta_k, h)$$

for some ϕ_j between θ and $\theta + jh$. This implies that

$$\begin{aligned} |ha(\theta_j) - ha(\theta + jh)| &\leq hc \sum_{k=0}^{j-1} [hc(1 + 2hc)^k (|r| + h) + ch^2] \\ &\leq h^2 c^2 \left[jh + \frac{(1 + 2hc)^j}{2hc} (|r| + h) \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=0}^{N-1} |ha(\theta_j) - ha(\theta + jh)| &\leq h^3 c^2 \frac{N(N-1)}{2} + \frac{hc}{2} (|r| + h) \left[\frac{(1 + 2hc)^N - 1}{2hc} \right] \\ &\leq 2\pi^2 c^2 h + \frac{e^{4\pi c}}{4} (|r| + h), \end{aligned}$$

and this quantity approaches zero as $|r|$ and h approach zero.

Finally, observe that

$$\lim_{h \rightarrow 0} h \sum_{j=0}^{N-1} a(\theta + jh) = \int_0^{2\pi} a(\theta) d\theta.$$

It now follows immediately from Lemmas 2 and 3 that we can find a number ρ with $0 < \rho < 1$ such that

$$|\lambda(\theta, r, h)| \leq \rho$$

if $|\mu_2| < 1$.

Proof of Theorem 1. Let B be the Banach space of continuous curves $r = g(\theta)$ with $g(\theta + 2\pi) = g(\theta)$, $\|g\| = \max |g(\theta)|$, and $|g(\theta)| \leq \delta$, for some appropriate δ . We will show that the mapping M^N takes this space into itself, and is a contraction on this space. It will then follow immediately that the mapping M^N has a closed invariant curve Γ near the circle $r = 0$.

LEMMA 4: Let $l = 4\pi\rho c \exp(12\pi c)$, and let δ satisfy the inequality

$$\frac{\rho\delta}{1-hc} + l(h + \delta^2 + h^2) \leq \delta. \tag{15}$$

Then, both M^N and M^{N-1} take B into itself.

Proof: Suppose that $|r| \leq \delta$. Then, from (12) and (13),

$$\begin{aligned} |r_N| &\leq \rho\delta + \rho \sum_{i=0}^{N-1} \frac{ch^2 + chr_i^2}{(1-hc)^{i+1}} \\ &\leq \rho\delta + \frac{2\pi\rho ch}{(1-hc)^{2\pi/h}} + \frac{2\pi\rho c \exp(8\pi c)}{(1-hc)^{2\pi/h}} (\delta + h)^2. \end{aligned}$$

Now, for h small, $1 - hc \geq \exp(-2hc)$. Hence,

$$\begin{aligned} |r_N| &\leq \rho\delta + \frac{2\pi\rho ch}{\exp(-4\pi c)} + 2\pi\rho c \exp(12\pi c)(\delta + h)^2 \\ &\leq \rho\delta + l(h + \delta^2 + h^2). \end{aligned}$$

Thus M^N takes B into itself if

$$\rho\delta + l(h + \delta^2 + h^2) \leq \delta.$$

However, in proving that M also has a closed invariant curve, we will need that M^{N-1} also takes B into itself. Now it is easily verified that

$$|r_{N-1}| \leq \frac{\rho}{1-hc} \delta + l(h + \delta^2 + h^2)$$

if $|r| \leq \delta$. Hence, if δ satisfies (15), then B is invariant under both M^N and M^{N-1} . Finally we observe that δ must lie between δ_- and δ_+ , where

$$\delta_{\pm} = \frac{1 - \frac{\rho}{1-hc} \pm \left[\left(\frac{\rho}{1-hc} - 1 \right)^2 - 4l(lh + lh^2) \right]^{1/2}}{2l}.$$

Hence we can find a suitable δ if we choose h so small so that

$$4l(lh + lh^2) \leq \left(\frac{\rho}{1-hc} - 1 \right)^2.$$

LEMMA 5: The mapping M^N is a contraction on the space B , for h and δ suitably small.

Proof: Let $r = g(\theta)$ be a curve in B . The image of this curve is the curve $r = g_1(\theta)$, where (see Eq. (12))

$$\lambda(\theta, g(\theta), h)g(\theta) + F(g(\theta), \theta, h) = g_1(\theta + G(g(\theta), \theta, h)).$$

Equivalently,

$$g_1(\theta) = \lambda(\theta_{-N}, g(\theta_{-N}), h)g(\theta_{-N}) + F(g(\theta_{-N}), \theta_{-N}, h).$$

If $r = \hat{g}(\theta)$ is a second curve in B , then its image under M^N is $r = \hat{g}_1(\theta)$ where

$$\hat{g}_1(\theta) = \lambda(\hat{\theta}_{-N}, \hat{g}(\hat{\theta}_{-N}), h)\hat{g}(\hat{\theta}_{-N}) + F(\hat{g}(\hat{\theta}_{-N}), \hat{\theta}_{-N}, h).$$

(Notice that $\hat{\theta}_{-N}$ depends on $\hat{g}(\theta)$.) We want to show that

$$\|g_1(\theta) - \hat{g}_1(\theta)\| \leq \sigma \|g(\theta) - \hat{g}(\theta)\| \tag{16}$$

for some constant σ , with $0 \leq \sigma < 1$. To this end we first prove that $\partial r_k/\partial r$ and $\partial \theta_k/\partial r$ are bounded, for $1 \leq k \leq N - 1$.

LEMMA 6: The partial derivatives $\partial r_k/\partial r$ and $\partial \theta_k/\partial r$, $k = 1, \dots, N - 1$, are uniformly bounded for r and h small.

Proof: Observe that

$$\begin{pmatrix} \partial r_k/\partial r \\ \partial \theta_k/\partial r \end{pmatrix} = \begin{pmatrix} \partial r_k/\partial r_{k-1} & \partial r_k/\partial \theta_{k-1} \\ \partial \theta_k/\partial r_{k-1} & \partial \theta_k/\partial \theta_{k-1} \end{pmatrix} \begin{pmatrix} \partial r_{k-1}/\partial r \\ \partial \theta_{k-1}/\partial r \end{pmatrix}$$

so that

$$\begin{pmatrix} \partial r_k/\partial r \\ \partial \theta_k/\partial r \end{pmatrix} = \prod_{i=2}^k \begin{pmatrix} \partial r_i/\partial r_{i-1} & \partial r_i/\partial \theta_{i-1} \\ \partial \theta_i/\partial r_{i-1} & \partial \theta_i/\partial \theta_{i-1} \end{pmatrix} \begin{pmatrix} \partial r_1/\partial r \\ \partial \theta_1/\partial r \end{pmatrix}.$$

Choose α so large that

$$\begin{cases} |\partial f_1/\partial r| \leq \alpha |r|, & |\partial F_1/\partial r| \leq \alpha, & |\partial f_2/\partial r| \leq \alpha, & |\partial G_1/\partial r| \leq \alpha \\ |\partial f_1/\partial \theta| \leq \alpha |r|^2, & |\partial F_1/\partial \theta| \leq \alpha, & |\partial f_2/\partial \theta| \leq \alpha |r|, & |\partial G_1/\partial \theta| \leq \alpha. \end{cases}$$

Then,

$$\begin{aligned} |\partial r_i/\partial r_{i-1}| &\leq 1 + hc + h\alpha \exp(4\pi c)(|r| + h) + \alpha h^2, \\ |\partial r_i/\partial \theta_{i-1}| &\leq hc \exp(4\pi c)(|r| + h) + h\alpha \exp(8\pi c)(|r| + h)^2 + \alpha h^2, \\ |\partial \theta_i/\partial r_{i-1}| &\leq h\alpha + \alpha h^2, \\ |\partial \theta_i/\partial \theta_{i-1}| &\leq 1 + h\alpha \exp(4\pi c)(|r| + h) + \alpha h^2. \end{aligned}$$

Consequently,

$$\begin{pmatrix} |\partial r_k/\partial r| \\ |\partial \theta_k/\partial r| \end{pmatrix} \leq \begin{pmatrix} 1 + \beta h & \beta h \\ \beta h & 1 + \beta h \end{pmatrix}^{k-1} \begin{pmatrix} 1 + \beta h \\ \beta h \end{pmatrix}$$

for some suitable constant β . Using the fact that $\|A^i\| \leq \|A\|^i$ for any matrix A , and

$$\left\| \begin{pmatrix} 1 + \beta h & \beta h \\ \beta h & 1 + \beta h \end{pmatrix} \right\| = 1 + 2\beta h.$$

we see that

$$\begin{aligned} \max \{ |\partial r_k/\partial r|, |\partial \theta_k/\partial r| \} &\leq (1 + 2\beta h)^{k-1} (1 + \beta h) \\ &\leq (1 + 2\beta h)^k \leq \exp(2\beta hk) \leq \exp(4\pi\beta) \end{aligned}$$

and this proves Lemma 6.

Lemma 6, together with the estimate $|\lambda(\theta, r, h)| \leq \rho < 1$ and the observation that $|\theta_{-N} - \hat{\theta}_{-N}|$ is very small compared to $|g(\theta) - \hat{g}(\theta)|$, gives us the estimate (16), and this completes the proof of Lemma 5.

It follows immediately from Lemma 5 that M^N has a closed invariant curve Γ near the circle $r = 0$ if $\mu_2 < 1$. This in turn implies that the curves $M\Gamma, M^2\Gamma, \dots, M^{N-1}\Gamma$ are also invariant under M^N . As remarked previously, the curve $M^{N-1}\Gamma$ lies in B . However, since M^N is a contraction on B , the curves Γ and $M^{N-1}\Gamma$ must be the same, and this can only be the case if $M\Gamma = \Gamma$. Thus Γ is also invariant under M . Finally, if $\mu_2 > 1$,

then we work with M^{-1} instead of M , and the proof goes through exactly as before. This completes the proof of Theorem 1 in the special case $n = 2$.

Remarks: 1. Our proof of Theorem 1 generalizes quite easily to the case $n > 2$. Let $x = \phi(t)$ be a periodic solution of $\dot{x} = f(x)$ with characteristic multipliers $\mu_1, \dots, \mu_{n-1}, \mu_n = 1$. Then, we can find new coordinates r, θ such that

$$\dot{r} = A(\theta)r + f_1(r, \theta), \quad \dot{\theta} = 1 + f_2(r, \theta)$$

and, moreover, the periodic system $\dot{r} = A(t)r$ has characteristic multipliers μ_1, \dots, μ_{r-1} . The only change in the proof of Theorem 1 is in estimating the quantity $\lambda = \prod_{i=1}^{N-1} [1 + hA(\theta_i)]$. Here, we must appeal to our knowledge that the solutions of the finite difference scheme converge to the solutions of the differential equation as $h \rightarrow 0$.

2. Using our knowledge that the solutions of the finite difference scheme converge to the solutions of the differential equations, we can greatly sharpen the estimates in this proof. Thus, the mapping M has a closed invariant curve for much larger values of h than our proof implies.

Application to the Van der Pol equation. It is well known (Hale [4]) that the periodic solution of Van der Pol's equation (5) is orbitally asymptotically stable. This follows from the Poincaré-Bendixson Theorem. To apply Theorem 1 to this equation, we need the additional fact that $\mu_2 < 1$. We prove this as follows: Set

$$y = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}.$$

Then, the variational equation of (5) about the periodic solution

$$y = \begin{pmatrix} 2 \cos t \\ -2 \sin t \end{pmatrix} + O(\epsilon)$$

is

$$\dot{\zeta} = \begin{pmatrix} 0 & 1 \\ -1 - 2\epsilon r \dot{x} & -\epsilon(x^2 - 1) \end{pmatrix} \zeta. \quad (17)$$

To first order in ϵ , the determinant of the monodromy matrix Ω of (17) is

$$\begin{aligned} \det \Omega &= \exp \left(-\epsilon \int_0^{2\pi} \left[4 \left(\frac{1}{2} + \frac{\cos 2t}{2} \right) - 1 \right] dt + O(\epsilon^2) \right) \\ &= \exp (-2\pi\epsilon + O(\epsilon^2)). \end{aligned}$$

Hence, $\det \Omega < 1$ for $\epsilon > 0$ sufficiently small, and this implies that $\mu_2 < 1$. Consequently, every finite difference approximation of the Van der Pol equation which is at least as accurate as Euler's method has a closed invariant curve near the circle of radius 2 for $\epsilon > 0$ and h sufficiently small.

The inverse problem. In this section we show that the existence of a closed invariant curve Γ of a finite difference approximation implies the existence of a periodic solution of the differential equation. Indeed, we reverse our steps. Assume that the difference

approximation

$$x_1 = x + hf(x) + h^2F(x, h)$$

of the differential equation $\dot{x} = f(x)$ has a closed invariant curve Γ for some value h , and that we can find new coordinates r, θ such that $r = 0$ is Γ and

$$r_1 = r + hA(\theta)r + hf_1(r, \theta, h)$$

$$\theta_1 = \theta + hg(\theta) + hg_1(r, \theta, h)$$

with $g(\theta) > 0$ and f_1 and g_1 small compared to h . If we write the differential equation in terms of r and θ , we find that

$$\dot{r} = A(\theta)r + hF_1(r, \theta, h), \quad \dot{\theta} = g(\theta) + hG_1(r, \theta, h).$$

Define a new time scale τ by the relation $dt/d\tau = 1/g(\theta)$. Then,

$$dr/d\tau = (A(\theta)/g(\theta))r + hF_1/g(\theta) \tag{18}$$

$$d\theta/d\tau = 1 + hG_1/g(\theta)$$

If all the characteristic multipliers of the system

$$\dot{\zeta} = (A(t)/g(t))\zeta$$

have modulus $\neq 1$, then the system (18) has a periodic solution if h is sufficiently small. This follows immediately from Poincaré's continuation theorem.

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