Periodic Solutions of Hamiltonian Systems with Superquadratic Potential (*) (**).

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Sunto. – *In questo lavoro si dimostra un teorema astratto di punti critici per funzionali fortemente indefiniti. Si applica poi tale teorema alla ricerca di soluzioni T-periodiche, con periodo T prefissato, del sistema Hamiltoniano*

$$\dot{p} = -H_q(p, q)\,, \quad \dot{q} = H_p(p, q)$$

dove $p, q \in \boldsymbol{R}^n$, *e l'Hamiltoniano* $H \in C^1(\boldsymbol{R}^{2n}, \boldsymbol{R})$ *è del tipo*

$$H(p, q) = \sum_{i,j} a_{i,j}(q) p_i p_j + \sum_i b_i(q) p_i + V(q)$$

con $V(q)/|q|^2 \to +\infty$ *per* $|q| \to +\infty$.

0. – Introduction and statements of the main results.

Consider the Hamiltonian system of $2n$ ordinary differential equations

$$\dot{p} = -H_q(t, p, q)\,, \quad \dot{q} = H_p(t, p, q)\,, \quad p, q \in \boldsymbol{R}^n,\ t \in \boldsymbol{R}\,, \tag{0.1}$$

where $H \in C^1(\boldsymbol{R}^{2n+1}, \boldsymbol{R})$, $\cdot$ denotes d/dt, $H_q = \partial H/\partial q$, $H_p = \partial H/\partial p$. The system (0.1) can be represented more concisely as

$$-J\dot{z} = H_z(t, z)\,, \tag{0.2}$$

where $z = (p, q)$, $H_z = \partial H/\partial z$ and J is the simplectic matrix in $\boldsymbol{R}^{2n}$, i.e.

$$J = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

Id being the identity matrix in $\boldsymbol{R}^n$.

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There are many types of questions, both local and global, in the study of periodic solutions of (0.2) (cf. e.g. the review article of RABINOWITZ [35] and its references). We suppose in the sequel that $H(t, z)$ is T-periodic in t.

Here we are concerned about the existence of T-periodic solutions of (0.2). RABINOWITZ, in a pioneering work [34], has proved that if $H(t, p, q)$ is «superquadratic» in both the variables p and q, i.e.

(0.3) there exist $r > 0$ and $\mu > 2$ such that

$$(H_z(t, z)|z)_{\mathbf{R}^{2n}} \geqslant \mu H(t, z) > 0 \quad \text{for } |z| > r \text{ and } t \in [0, T]$$

and it satisfies other assumptions, then (0.2) has a T-periodic solution. If $\partial H/\partial t = 0$ and $H(t, z)$ satisfies (0.3), then RABINOWITZ has proved that (0.2) has a nonconstant T-periodic solution for every prescribed period T [36]. Later many other papers appeared dealing with (0.2) when $H(t, z)$ is «superquadratic» ([4], [6], [11], [17], [19], [25], [26]).

Unfortunately the above results on superquadratic Hamiltonians do not cover the classical mechanical problems. In fact, consider a mechanical system with holonomous constraints imbedded in a conservative field of forces. The Hamiltonian of such a system has the form

$$H(t, p, q) = \sum_{i,j=1}^{n} a_{ij}(t, q) p_i p_j + \sum_{i=1}^{n} b_i(t, q) p_i + V(t, q) , \tag{0.4}$$

where $\{a_{ij}(t, q)\}$ is a positive definite matrix for every t and q. The Hamiltonian (0.4) is quadratic in p, then it does not satisfy (0.3).

If

$$\begin{cases} a_{ij} \quad \text{do not depend on } q & (i, j = 1, \ldots, n) \\ b_i = 0 & (i = 1, \ldots, n) \end{cases} \tag{0.5}$$

(0.1) can be reduced to a second order system of n equations of the form

$$\ddot{x} = -\frac{\partial U}{\partial x}, \quad U = U(t, x), \ x \in \mathbf{R}^n \tag{0.6}$$

which is more easy to study then (0.1) (cf. discussion in [18]). In this case, for example, it is known that if $\partial U/\partial t = 0$ and U grows more than quadratically at infinity, in the sense of (0.3), then (0.6) has a non-constant T-periodic solution for each fixed $T > 0$ (cf. [16], [33] and references in [35]).

In this paper we consider Hamiltonians with the form (0.4) without the restrictions (0.5) and with «superquadratic» growth in q. We make the following assumptions on the Hamiltonian (0.4):

Assumptions (H_0):

(V_1) There exist constants $R > 0$, $\alpha > 2$ such that

$$0 < \alpha V(t, q) \leqslant (V_q(q, t), q) \quad \text{for } |q| > R \text{ and every } t \in R .$$

(V_2) There exist constants $C_1, C_2, s, R > 0$ such that

$$|V_q(q, t)| \leqslant C_1 V(q, t) \leqslant C_2 |q|^s \quad \text{for } |q| > R \text{ and every } t \in R.$$

(A_1) There exists a real, continuous function $\nu(q) > 0$ such that

$$\sum_{ij} a_{ij}(q, t) p_i p_j \geqslant \nu(q) |p|^2 \quad \text{for every } p, q \in R^n \text{ and } t \in R .$$

(A_2) There exist constants $\beta \in]0, \alpha - 2[$ and $\mu > 0$ such that

$$\sum_{ij} M_{ij}(q, t) p_i p_j \geqslant \mu |p|^2 \quad \text{where} \quad \{M_{ij}(q, t)\} = \left\{ \beta a_{ij} + \Sigma_k \frac{\partial a_{ij}}{\partial q_k} q_k \right\} .$$

(A_3) There exists a constant $C_3 > 0$ such that

$$\left| \sum_{ij} \frac{\partial a_{ij}}{\partial q_k} (q, t) p_i p_j \right| \leqslant C_3 \sum_{ij} a_{ij}(q, t) p_i p_j \quad \text{for every} \quad k = 1, \ldots, n;\ q \in R^n,\ t \in R .$$

(A_4) There exists a constant $C_4 > 0$ such that

$$|a_{ij}(q, t)| \leqslant C_4 V(q, t) \quad \text{for } |q| \text{ large and every } t \in R .$$

(B_1)
$$\lim_{|q| \to \infty} \frac{b_i(q, t)^2}{\nu(q) V(q, t)} = 0 \quad \text{for every } i = 1, \ldots, n$$

(B_2)
$$\lim_{|q| \to \infty} \frac{|\partial b_i / \partial q_k(q, t) q_k|^2}{\nu(q) V(q, t)} = 0 \quad \text{for every } i, k = 1, \ldots, n .$$

Assumption (V_1) implies that V grows more than $|q|^\alpha$ at infinity. It replaces assumption (0.3) of other papers.

(A_1) is a physical assumption which depends on the fact that the « kinetic energy » is positive. Observe that it is allowed that $\nu(q) \to 0$ as $|q| \to \infty$.

(A_2) is a technical assumption which is deeply related to the nature of our results. Probably it has some meaning which we have not fully understood.

(V_2), (A_3), (A_4), (B_1), (B_2) are growth conditions on the coefficients of (0.4). Probably they can be weakened using a cut-off technique as in [33], [19] or [36].

We have the following results for Hamiltonians of the form (0.4).

Theorem 0.1. – *Suppose that H satisfies the assumptions (H_0) and*

(H_1) *the system is autonomous i.e.* $\partial H/\partial t = 0$.

Then (0.2) *has infinitely many nonconstant T-periodic solutions for every prescribed period T.*

Theorem 0.2. – *Suppose that H satisfies the assumptions (H_0) and*

(H_2) $H(t, z)$ *is T-periodic in t*

(H_3) $z = 0$ *is the minimum point of H for every* $t \in R$

(H_4) H *is twice differentiable for* $z = 0$

(H_5) *there exists a constant* $\gamma \in]0, 1[$ *such that*

$$\sum_{i,j} \frac{\partial^2 H(t, 0)}{\partial z_i \, \partial z_j} \zeta_i \zeta_j \leqslant \frac{2\pi}{T} \gamma |\zeta|^2 \quad \text{for every } t \in R \text{ and } \zeta \in R^{2n} .$$

Then (0.2) *has at least a nonconstant T-periodic solution.*

Remark 0.3. – Theorem 0.1 just states the existence of periodic solutions but not of prime periodic solutions, i.e. solutions for which T is the minimal period.

Remark 0.4. – If H does not depend on t and it is twice differentiable for $z = 0$, Theorem 0.1 can be deduced from Theorem 0.2. In fact by virtue of the assumption (H_0), H has a minimum in R^{2n}. It is not restrictive to suppose that the minimum point is $z = 0$. Given any period T, there is a period $T_1 = T/k_1$ ($k_1 \in N$) such that (H_5) is satisfied. Since a T_1-periodic solution is also a T-periodic solution, we can deduce from Theorem 0.2 that for any period $T > 0$ we have a nonconstant T-periodic solution $z_1(t)$. Also there exists a number h_1 such that z_1 has the minimal period equal to $T/h_1 k_1$. If we take $k_2 > h_1 k_1$ we can find, using Theorem 0.2 a (T/k_2)-periodic solution z_2 which is of course a T-periodic solution and $z_2 \neq z_1$. In this way we can find infinitely many nonconstant T-periodic solutions. We finally observe that, if $b_i = 0$ ($i = 1, \ldots, n$), and $\partial H/\partial t = 0$, variants of Theorem 0.1 can be found in [12], [29].

This paper consists of five sections and two appendices. Sections 1-2-3 and A_1, A_2 are devoted to some abstract critical point theorems. Sections 4-5 contain the proofs of Theorems 0.1-0.2.

I. SOME ABSTRACT CRITICAL POINTS THEOREMS

1. – Statements of the abstract results.

Before stating the main results of this section we shall introduce some notations and definitions. We denote by E a real Hilbert space, by $(\cdot, \cdot)$ the scalar product in E, by $\|\cdot\|$ the norm in E. By $C^1(E, R)$ we denote the space of Frechét differentiable maps from E to R and, if $f \in C^1(E, R)$, by $f'(u)$ its derivative at $u \in E$. We shall identify E with its dual E' so that $f' \in C^0(E, E)$. For $u \in E$ and $R > 0$ we set $B(u, R) = \{v \in E: \|v - u\| < R\}$, $B_R = B(0, R)$, $S_R = \partial B_R$. Let G be a compact Lie group and let $r: G \to U(E)$ be a representation of G on the group of the unitary linear transformations on E. We set $\mathcal{G} = r(G)$.

Definition 1.1. – A functional f on E is called $\mathcal{G}$-*invariant* if $f \circ T = f$ for every $T \in \mathcal{G}$.

Definition 1.2. – A map h from E to E is called $\mathcal{G}$-*equivariant* if $h \circ T = T \circ h$ for every $T \in \mathcal{G}$.

Definition 1.3. – A subset $A \subset E$ is called $\mathcal{G}$-*invariant* if $T(A) = A$ for every $T \in \mathcal{G}$.

Sometimes, when no ambiguity is possible, we will write « G-invariant », and « G-equivariant », etc. instead of « $\mathcal{G}$-invariant », etc. We set Fix $G = \{u \in E: T(u) = u$ for every $T \in G\}$. If $u \in E$ the « orbit » of u is the set $\{T(u): T \in G\}$. In the sequel we shall consider $G = S^1 = R/2\pi Z$. Moreover if L is a linear operator on E we denote by $\sigma(L)$ the spectrum of L.

In the sequel we will be concerned with functionals $f \in C^1(E, R)$ satisfying the following assumptions:

(f_1) $f(u) = \frac{1}{2}(Lu, u) - \psi(u)$, where

(i) L is a continuous self-adjoint operator on E,

(ii) $\psi \in C^1(E, R)$, $\psi(0) = 0$ and ψ' is a compact operator.

(f_2) (i) $E = \oplus M_\lambda$ where the M_λ's are eigenspaces of L (which might be infinite dimensional),

(ii) 0 is a regular value for L or it is an isolated eigenvalue of finite multiplicity of L;

(f_3) given $c \in]0, +\infty[$, every sequence $\{u_r\}$, for which $\{f(u_r)\} \to c$ and $\|f'(u_r)\| \cdot \|u_r\| \to 0$, possesses a bounded subsequence.

We set

$$E^+ = \overline{\bigoplus_{\lambda>0} M_\lambda}\,, \qquad E^- = \overline{\bigoplus_{\lambda<0} M_\lambda}\,, \qquad E^0 = \ker L$$

and let P_+, P_- and P_0 be the relative orthogonal projections. Then

$$E = E^+ \oplus E^0 \oplus E^- . \tag{1.1}$$

In the case in which E^+ (resp. E^-) is finite-dimensional, f is bounded from above (resp. from below) modulo weakly continuous perturbations. In fact we can write

$f(u) = \frac{1}{2}(LP_+u, P_+u) + \frac{1}{2}(LP_-u, P_-u) - \psi(u)$ and if, for example, $\dim E^- < +\infty$ then $\Phi(u) = \frac{1}{2}(LP_-u, P_-u) - \psi(u)$ has compact derivative. We shall consider the case in which f can be « strongly indefinite », i.e. E^+ and E^- are both infinite-dimensional, as it occurs in the study of periodic solutions of Hamiltonian systems.

THEOREM 1.4. – *Let $f \in C^1(E, R)$ be a functional satisfying (f_1), (f_2) and (f_3). Moreover we suppose that a unitary representation of the group S^1 acts on E such that*

(f_4) *L and ψ' are S^1-equivariant;*

(f_5) *there exist two closed linear subspaces $V, W \subset E$ such that*

(i) *V and W are S^1 invariant,*

(ii) $\dim (V \cap W) < +\infty$, $\operatorname{codim} (V + W) < +\infty$,

(iii) $\operatorname{Fix}(S^1) \subset V$ *and/or* $\operatorname{Fix}(S^1) \subset W$,

(iv) *there exist positive constants C_0 and ϱ such that*

$$f(u) > C_0 \quad \text{for every} \quad u \in V \cap S_\varrho\,,$$

(v) *there exist $C_\infty \in R$ such that $f(u) < C_\infty$ for every $u \in W$,*

(vi) *$f(u) < C_0$ for $u \in \operatorname{Fix}(S^1)$ such that $f'(u) = 0$. Under the above assumptions there exist at least*

$$\tfrac{1}{2}(\dim (V \cap W) - \operatorname{codim} (V + W))$$

orbits of critical points, with critical values in $[C_0, C_\infty]$.

REMARK 1.5. – In the Theorem 1.4 the assumptions (f_2) and (f_3) replace the well known conditions of Palais and Smale (P.S.) used in similar theorems. They do not imply (P.S.), but a weaker condition (i.e. (i) and (ii) of Lemma 1 of App. 1, which has been introduced by G. CERAMI (cf. [22]; cf. also [8]). The conditions (f_5) are geometrical assumptions, which allow us to give a lower bound to the number of orbits of critical points of the functional f.

REMARK 1.6. – Theorem 1.4 generalizes Theorem 4.1 of [11] in two points. The assumptions (f_2) and (f_3) are easier to verify than (P.S.). This fact allows to treat Hamiltonians of the form (0.4). Moreover in [11] the assumption (f_5) (iii) is replaced by the stronger assumption

$$\operatorname{Fix} S^1 \subset W\,.$$

This generalization permits us to obtain the multiplicity results for asymptotically quadratic Hamiltonian systems contained in [14] (for the proof we refer to [15]).

In the case in which the functional f does not exibit any symmetry, we have the following theorem:

THEOREM 1.7. – *Let $f \in C^1(E, \mathbf{R})$ be a functional satisfying (f_1), (f_2) and (f_3). Moreover suppose that there exist a L-invariant subspace $V \subset E$, an eigenvector $e \in V$ of L, and positive constants $R_1, R_2, C_\infty, C_0, \varrho$ with $0 < C_0 < C_\infty$ and $\varrho < R_1$ such that*

(i) $\sup f(Q) = C_\infty$,

(ii) $\inf f(S_\varrho \cap V) = C_0$,

(iii) $\sup f(\partial Q) < 0$,

where $Q = \{m + v: m \in V^\perp \cap B_{R_2}, v \in T\}$, $T = \{te: t \in [0, R_1]\}$.

Under the above assumptions f has at least one critical value $c \in [C_0, C_\infty]$.

REMARK 1.8. – Theorem 1.7 generalizes Theorem 0.1 of BENCI-RABINOWITZ [19], because (f_1), (f_2) and (f_3) are weaker assumptions than the respective assumptions in [19]. This fact allows us to obtain the Theorem 0.2, which applies to Hamiltonians of the form (0.4).

REMARK 1.9. – Using the techniques developed in this paper it is possible to generalize also Theorems of [19] (cf. [20], [21]).

REMARK 1.10. – The assumption (f_2) (i) is not necessary. In fact, if it does not hold, we can replace the inner product of E with a new inner product such that (f_2) (i) is satisfied.

The new inner product is defined as follows

$$(u, v)_N = (LP^+u, v) - (LP^-u, v) + (P_0u, v) .$$

We observe that every $T \in G$ is a unitary transformation also with respect to the new inner product. If we define a linear operator $\tilde{L}: E \to E$ as follows:

$$\begin{aligned} \tilde{L}u &= u && \text{if } u \in E^+ \\ \tilde{L}u &= -u && \text{if } u \in E^- \\ \tilde{L}u &= 0 && \text{if } u \in E^0 \end{aligned}$$

then we have

$$(\tilde{L}u, v)_N = (Lu, v)$$

and

$$f(u) = \tfrac{1}{2}(\tilde{L}u, u)_N - \psi(u) .$$

So the function f satisfies (f_1), (f_2) and (f_4) in E equipped with the new inner product. Since (f_3) and (f_5) essentially are topological properties, they are as well satisfied (of course minor changes are necessary). Then Theorem 1.4 holds without assumptions (f_2) (ii). A similar remark can be done about Theorem 1.7. However, in the applications which we consider in this paper, assumption (f_2) (ii) is satisfied.

REMARK 1.11. – An analogous version of Theorem 1.4 can be obtained if f is even, i.e. it is Z_2-invariant. Then if V (respectively W) is finite-dimensional, we get a variant of a theorem of CLARK [23] (respectively AMBROSETTI-RABINOWITZ [5]).

REMARK 1.12. – Applications of Theorem 1.4 are contained in [9], [32], [37].

2. – Index and pseudoindex theory.

We recall some notions (as the notion of index theory) and some theorems which are often used in the critical point theory.

First, some notation is necessary. We set

$$N_\delta(A) = \{u \in E\colon \operatorname{dist}(u, A) \leqslant \delta\}$$

where dist (u, A) denotes the distance from u to A. For $f \in C^1(E, R)$ and $c \in R$, we set

$$K_c = \{u \in E\colon f'(u) = 0,\ f(u) = c\}$$
$$A_c = \{u \in E\colon f(u) \leqslant c\}\,.$$

DEFINITION 2.1. – Let E be a Hilbert space on which a representation $r\colon G \to r(G) \subset U(E)$ of a compact Lie group G acts. An index theory is a triplet $\{\Pi, H, i\}$ where

Π is the family of G-invariant closed subsets of E;

H is the set of G-equivariant continuous mappings;

$i\colon \Pi \to N \cup \{+\infty\}$ is a mapping, which satisfies the following properties:

(2.1)
- (a) $i(A) = 0$ if and only if $A = \emptyset$;
- (b) if $A \subset B$ then $i(A) \leqslant i(B)$ for all $A, B \in \Pi$;
- (c) $i(A \cup B) \leqslant i(A) + i(B)$ for all $A, B \in \Pi$;
- (d) if $A \in \Pi$ is a compact set, then there exists $\delta > 0$ such that $$i(N_\delta(A)) = i(A);$$
- (e) $i(A) \leqslant i\big(\overline{h(A)}\big)$ for every $A \in \Pi$ and for every $h \in H$.

DEFINITION 2.2. – We say that an index theory satisfies the d-dimensional property ($d \in N$) if

$$i(\partial\Omega \cap V) = \frac{\dim V}{d} \tag{2.2}$$

where V is a finite dimensional, G-invariant subspace of E such that $V \cap \mathrm{Fix}\,(G) = \{0\}$ and Ω is a bounded invariant neighborhood of the origin.

The Definition 2.2 makes sense, because, in the examples which we know, if V is as before, then the dimension of V is a multiple of some integer number d.

REMARK 2.3. – Since in the sequel we shall consider the case in which $G = S^1$, then we shall consider the homological index defined in [28] or the geometrical index defined in [10]. These index theories satisfy the 2-dimension property for any representation $r\colon G \to U(E)$.

We refer to [7] for an abstract construction of an index theory.

In the following theorem we shall list some property of the index which will be used in this paper.

THEOREM 2 4 – *Let $\{\Pi, H, i\}$ be an index theory which satisfies the dimension property. Then we have*

(i) *if $[\mathrm{Fix}\,(G)]^\perp$ is infinite dimensional, and $A \cap \mathrm{Fix}\,(G) \neq \emptyset$, then $i(A) = +\infty$;*

(ii) *if $V \in \Pi$ is a finite dimensional space and $A \subset V - \mathrm{Fix}\,(G)$ then $i(A) \leqslant \dim V/d$;*

(iii) *if $A \cap \mathrm{Fix}\,(G) = \emptyset$ and $i(A) \geqslant 2$ then A contains infinitely many distinct G-orbits;*

(iv) *if $h \in H$ is a homeomorphism, then $i(h(A)) = i(A)$.*

For the proof of this theorem we refer to [10] and [11].

DEFINITION 2.5. – Given an index theory $\{\Pi, H, i\}$ and a group of homeomorphisms $H^* \subset H$, for every $A, B \in \Pi$ we set

$$i^*(A, B, H^*) = \min_{h \in H^*} i(h(A) \cap B)\,.$$

The triplet $\{\Pi, H^*, i^*\}$ will be called pseudoindex theory (cf. [11] or [8]). When no ambiguity is possible we shall write $i^*(\cdot,\cdot)$ instead of $i^*(\cdot,\cdot, H^*)$.

DEFINITION 2.6. – Given a G-invariant functional $f \in C^1(E, R)$ and a group of

G-equivariant homeomorphism H^*, we say that f satisfies the condition (B) in $]\alpha, \beta[(-\infty < \alpha < \beta < +\infty)$ with respect to H^* if for every $c \in]\alpha, \beta[$

(i) K_c is compact;

(ii) for every $N = N_\delta(K_c)$ there exists $\eta \in H^*$ and a constant $\varepsilon > 0$ such that

(*a*) $[c - \varepsilon, c + \varepsilon] \subset]\alpha, \beta[$,

(*b*) $\eta(A_{c+\varepsilon} - N) \subset A_{c-\varepsilon}$.

The concept of pseudoindex and the property (B) are related to the critical point theory by means of the following theorem.

THEOREM 2.7. – *Let $f \in C^1(E, R)$ be a G-invariant functional satisfying the condition (B) in $]\alpha, \beta[$ with respect to H^*. Given $D, F \in \Pi$, we suppose that*

$$\text{(2.3)} \qquad \begin{aligned} &\text{(i)} \quad \sup f(D) = c_\infty < \beta\,; \\ &\text{(ii)} \quad \inf f(F) = c_0 > \alpha\,; \\ &\text{(iii)} \quad i^*(D, F, H^*) = \bar{k}\,. \end{aligned}$$

If we set

$$\Gamma_k = \{A \in \Pi\colon i^*(A, F, H^*) \geq k\}$$

then, for $k = 1, \ldots, \bar{k}$, the numbers

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} f(u)$$

are well defined, are critical values of f and

$$c_0 \leq c_1 \leq \ldots \leq c_{\bar{k}} \leq c_\infty\,.$$

Moreover if $c = c_k = \ldots = c_{k+r}$ $(k \geq 1,\ k + r \leq \bar{k})$, then $i(K_c) \geq r + 1$.

The proof of this theorem follows standard arguments of the critical point theory and it will not be given here (see e.g. [8]).

REMARK 2.8. – If Theorem 2.7 holds we cannot deduce that f has at least $\bar{k}$ distinct orbits of critical points. In fact it might happen that

$$c_1 = \ldots = c_{\bar{k}} = c$$

and $K_c = \{\bar{u}\}$ where $\bar{u} \in \operatorname{Fix}(G)$.

Then in this case, by Theorem 2.4 (i), we have $i(K_c) = +\infty$, but we have only one orbit of critical points i.e. $\{\bar{u}\}$. However if $i(K_c) \geq 2$ and $K_c \cap \operatorname{Fix}(G) = \emptyset$, by

Theorem 2.4 (iii) deduce that K_c contains infinitely many distinct orbits. Therefore if the assumptions of Theorem 2.7 hold, we can deduce that one of the following alternatives follows

(a) there exists at least one critical point $\bar{u} \in \mathrm{Fix}\,(G)$;

(b) there exist at least $\bar{k}$ distinct orbits of critical points.

Now we shall enounce the analogous of Theorem 2.7 in the case in which the functional has no symmetry. In this case we can suppose that the functional is G-invariant with respect to the trivial group $G = \{\mathrm{Id}\}$. Then the property (B) makes sense (cf. def. 2.6).

DEFINITION 2.9. – Given two sets D and F and a group of homeomorphisms K we say that « D and F K-intersect » if

$$h(D) \cap F \neq \emptyset \quad \text{for every } h \in K .$$

THEOREM 2.10. – *Let* $f \in C^1(E, R)$ *be a functional satisfying the property* (B) *in* $]\alpha, \beta[$ *with respect to* K *and let* C_0, $C_\infty \in R$ *be two constants such that*

(i) $\sup f(D) = C_\infty < \beta$;

(ii) $\inf f(F) = C_0 > \alpha$;

(iii) F *and* D K*-intersect.*

Then f *has at least a critical value* $c \in [C_0, C^\infty]$.

The proof follows standard arguments and it will not be given here (cf. e.g. [8]).

3. – Proof of the abstract theorems.

PROOF OF THEOREM 1.4. – In order to prove Theorem 1.4 we want to use Theorem 2.7. The crucial point is to determine a class of equivariant homeomorphisms H^* such that

(i) if (f_1), (f_2), (f_3) and (f_4) hold, f satisfies the property (B) with respect to H^*;

(ii) if (f_5) holds, then
the pseudoindex $i(\cdot, \cdot, H^*)$ can be estimated by means of $\dim\,(V \cap W)$ and $\mathrm{codim}\,(V + W)$.

In order to define H^* we need the following lemma:

LEMMA 3.1. – *Suppose that* L *satisfies* (f_2). *Moreover suppose that* L *is* G*-inva-*

riant, where G *is a unitary representation of a compact Lie group* G. *Then*

$$E = \overline{\bigoplus_{j \in Z} E_j} \tag{3.1}$$

where the E_j's *are* G*-invariant and* L*-invariant finite dimensional subspaces, orthogonal with each other.*

PROOF. – If $u \in M_\lambda$, then $LTu = TLu = T\lambda u = \lambda Tu$ for every $T \in G$. So every eigenspace of L is G-invariant.

Then by Peter-Weyl theorem M_λ can be decomposed in finite dimensional G-invariant subspaces orthogonal with each other

$$M_\lambda = \bigoplus_j E_j\,.$$

Of course, the spaces E_j's constructed in this way, are L-invariant because they are subspaces of an eigenspace of L. □

Now we define the class H^* as follows:

DEFINITION 3.2_j – Let $\mathcal{U}$ be a class of continuous maps $U\colon E \to E$ such that

(V_1) U is bounded;

(V_2) $U(u) = \exp[\alpha(u)L][u]$ where $\alpha\colon E \to R$ is a G-invariant functional.

Clearly every $U \in \mathcal{U}$ is G-equivariant.
Let $\mathcal{B}$ be a class of continuous maps $b\colon E \to E$ such that

(b_1) b is G-equivariant and bounded;

(b_2) for every $R > 0$, there exists a finite set of indexes $I(R) \subset Z$ such that

$$b(B_R) \subset \bigoplus_{j \in I(R)} E_j\,.$$

Finally we define H^* as the class of all maps h such that

(H_1^*) h is a homeomorphism;

(H_2^*) $h = U_0 + b_0$ where $U_0 \in \mathcal{U}$, $b_0 \in \mathcal{B}$;

(H_3^*) $h^{-1} = U_1 + b_1$ where $U_1 \in \mathcal{U}$, $b_1 \in \mathcal{B}$;

(H_4^*) $h(0) = 0$.

Obviously H^* is a nonempty class of bounded G-equivariant homeomorphisms. It is not difficult to prove that H^* is a group (cf. [13]).

From now on H^* will denote the class of homeomorphisms just defined and $i^*(\cdot,\cdot) = i^*(\cdot,\cdot, H^*)$.

The following « deformation theorem » holds:

THEOREM 3.3. – *Suppose that $f \in C^1(E, \mathbf{R})$ satisfies (f_1), (f_2) and (f_3) and that it is G-invariant. Given $c > 0$ and a neighborhood N of K_c, there exist constants $\bar{\varepsilon} > \varepsilon > 0$ (with $\bar{\varepsilon} < c$) and an operator $\eta\colon E \to E$ such that*

(*a*) $\eta(A_{c+\varepsilon} - N) \subset A_{c-\varepsilon}$;

(*b*) $\eta = U + b \in H^*$;

(*c*) $U(u) = u$, $b(u) = 0$ *for every* $u \notin f^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}])$.

In particular f satisfies the condition (B) *in* $]0, +\infty[$ *with respect to* H^* (*cf. Definition* 2.6).

The proof of this theorem is quite technical and it will be given in Appendix 1.

The following theorem permits us to estimate the pseudoindex of suitable sets of E:

THEOREM 3.4. – *Consider two G-invariant closed linear subspaces, V, $W \subset E$ and a bounded G-invariant neighborhood of the origin Ω. Suppose that*

(i) $\operatorname{Fix} G \subset W$ (*or* $\operatorname{Fix} G \subset V$);

(ii) $\dim(V \cap W) < +\infty$, $\operatorname{codim}(V + W) < +\infty$;

(iii) *the index theory i satisfies the d-dimension property* (*cf. Definition* 2.2).

Then

$$i^*(\Omega \cap V, W) \geqslant \frac{\dim(V \cap W) - \operatorname{codim}(V + W)}{d}.$$

Also the proof of this theorem is quite technical and it will be given in Appendix 2.

Now we are ready to prove Theorem 1.4. The proof is based on Theorem 2.7. We have to check that all the assumptions of Theorem 2.7 are fulfilled.

We choose $G = S^1$ and $G = r(G)$ where r is a unitary representation of S^1. By virtue of Theorem 3.3, f satisfies the condition (B) in $]0, +\infty[$. We set $D = W$ and $F = S_\varrho \cap V$. Then (2.7) (i) and (ii) follow from (f_5) (iv) and (v).

By virtue of (f_5) (i), (ii), (iii), the assumptions of Theorem 3.4 are satisfied.

Moreover, $G = r(S^1)$ satisfies the 2-dimension property (cf. [10]). Then

$$\bar{k} = \tfrac{1}{2}[\dim(V \cap W) - \operatorname{codim}(V + W)].$$

Therefore $c_1, \ldots, c_{\bar{k}}$ are critical values of f.

By (f_5) (vi), it follows that $K_{c_i} \cap \mathrm{Fix}\,(S^1) = \emptyset$; then the second alternative of Remark 2.8 (*b*) holds. □

PROOF OF THEOREM 1.7. – In order to prove Theorem 1.7, we shall apply Theorem 2.10.

First, we define the class of homeomorphism K as follows: Set

$$K = \{h = U + b \in H^*\colon h(u) = u \text{ for every } u \in f^{-1}(]-\infty, 0[)\}\,. \tag{3.2}$$

In this case H^* is given by the Definition 3.2 with $G = \{\mathrm{Id}\}$, i.e. no invariancy property is required for $h \in H^*$.

Now we need a lemma which is a variant of other similar results (cf. e.g. [19], [8]).

LEMMA 3.5. – *Q and $S_\varrho \cap V$, as defined in Theorem 1.7, K-intersect (cf. Definition 2.9).*

PROOF. – We have to show that

$$h(Q) \cap (S_\varrho \cap V) \neq \emptyset\,, \qquad \forall h \in K\,.$$

The above formula holds provided that for each $h \in K$ the following equations have at least one solution:

$$\begin{cases} s \in [0, R_1]; \quad u \in B_{R_2} \cap V^\perp \\ \|P_V \cdot h(u + se)\| = \varrho \\ P_{V^\perp} \cdot h(u + se) = 0 \end{cases} \tag{3.3}$$

where P_V and $P_{V^\perp}$ denote the projection on V and $V^\perp$ respectively. Let $h = U + b \in K$, $U = \exp[\alpha(\cdot)L][\cdot]$, then the second equation in (3.3) can be written

$$P_{V^\perp}[\exp[\alpha(u + se)L](u + se)] + P_{V^\perp} b(u + se) = 0\,. \tag{3.4}$$

Since $se \in V$, we have

$$\exp[\alpha(u + se)L](se) \in V\,.$$

Then (3.4) can be written as follows

$$P_{V^\perp}[\exp[\alpha(u + se)L](u)] + P_{V^\perp} b(u + se) = 0\,. \tag{3.5}$$

Moreover, since $u \in V^\perp$, we have

$$\exp[\alpha(u + se)L](u) \in V^\perp\,.$$

Then (3.5) can be written

$$\exp[\alpha(u+se)L]u + P_{V^\perp}b(u+se) = 0 . \tag{3.6}$$

(3.6) is equivalent to the following equation

$$u + \exp[-\alpha(u+se)L][P_{V^\perp}b(u+se)] = 0 . \tag{3.7}$$

Then (3.3) can be written as follows

$$\begin{cases} s \in [0, R_1], \quad u \in B_{R_2} \cap V^\perp \\ \|P_V h(u+se)\| = \varrho \\ u + \exp[-\alpha(u+se)L][P_{V^\perp}b(u+se)] = 0 . \end{cases} \tag{3.8}$$

Using a Leray-Schauder degree argument as in [19] (cf. also [8] and [16]) it can be proved that equation (3.8) has at least one solution. □

PROOF OF THEOREM 1.7. - If K is the class of homeomorphisms (3.2), then by virtue of Theorem 3.3, f satisfies the property (B) in $]0, +\infty[$. We now set $D = Q$ and $F = S_\varrho \cap V$. Then by virtue of Lemma 3.5, F and D K-intersect.

Therefore the conclusion follows from Theorem 2.10.

II. APPLICATIONS TO HAMILTONIAN SYSTEMS

4. - Some estimates for the action functional.

We initially introduce some functional spaces we shall need in the following. If $m \in N$ and $t > 1$ we set

$$L^t = L^t(S^1, R^m) .$$

If $s \in R$ we set

$$W^s = \left\{ u \in L^2(S^1, R^{2n}) : \sum_{\substack{j \in Z \\ k=1,\dots,2n}} (1+|j|^2)^s |u_{jk}|^2 < +\infty \right\}$$

where u_{jk} $(j \in Z, k = 1, \dots, 2n)$ are the Fourier components of u with respect to the basis (in $L^2(S^1, R^{2n})$)

$$\psi_{jk} = \exp[jtJ]\Phi_k = \cos(jt)\Phi_k + J\,\mathrm{sen}(jt)\Phi_k \tag{4.1}$$

where $\{\Phi_k\}$ ($k = 1, \ldots, 2n$) is the standard basis in R^{2n}. W^s equipped with the inner product

$$(u|v)_{W^s} = \sum_{j,k} (1 + |j|^2)^s u_{jk} v_{jk} \tag{4.2}$$

is a Hilbert space. We recall that the embedding $W^s \to L^t$ is compact if $1/t > \frac{1}{2} - s$. So in particular $W^{\frac{1}{2}}$ is compactly embedded in L^t for any $t \geqslant 1$.

Now we consider the Hamiltonian system (0.2) where $H(t, z)$ is T-periodic in t. Making the change of variable $t \to 2\pi t/T$, (0.2) becomes

$$-J\dot{z} = \omega H_z(\omega t, z) \quad \text{where} \quad \omega = \frac{T}{2\pi}. \tag{4.3}$$

Obviously the 2π-periodic solutions of (4.3) correspond to the T-periodic solutions of (0.2).

In order to construct the action functional whose critical points are the 2π-periodic solutions of (4.3) we introduce the following bilinear form

$$a(u, v) = \sum_{j \in Z} \sum_{k=1}^{2n} j u_{jk} v_{jk}, \qquad u, v \in W^{\frac{1}{2}}$$

where u_{jk}, v_{jk} are the Fourier-components of u, v with respect to the basis (4.1). The bilinear form $a(\cdot,\cdot)$ is symmetric and continuous in $W^{\frac{1}{2}}$. Let $L\colon W^{\frac{1}{2}} \to W^{\frac{1}{2}}$ be the self-adjoint, continuous operator defined by

$$(Lu|v)_{W^{\frac{1}{2}}} = a(u, v), \qquad u, v \in W^{\frac{1}{2}}. \tag{4.4}$$

Observe that if $u, v \in C^1(S^1, R^{2n})$

$$(Lu|v)_{W^{\frac{1}{2}}} = \int_0^{2\pi} (-J\dot{u}, v)\, dt.$$

Suppose now that there are positive constants c_1, c_2, s such that

$$|H_z(t, z)| \leqslant c_1 + c_2|z|^s \quad \text{for any } t \text{ and } z. \tag{4.5}$$

Standard arguments show that the functional

$$f(z) = \tfrac{1}{2}(Lz|z)_{W^{\frac{1}{2}}} - \omega\int_0^{2\pi} H(\omega t, z)\, dt, \qquad z \in W^{\frac{1}{2}} \tag{4.6}$$

is Frechét-differentiable and that its critical points correspond to the 2π-periodic solutions of (4.3). For simplicity in the sequel we shall take $\omega = 1$ and suppose

$H(t, z)$ 2π-periodic in t, so (4.6) becomes

$$f(z) = \tfrac{1}{2}(Lz|z)_{W^{\frac{1}{2}}} - \psi(z) \tag{4.7}$$

where $\psi(z) = \int_0^{2\pi} H(t, z)\, dt$.

Since $W^{\frac{1}{2}}$ is compactly embedded in L^t for any $t \geq 1$, by (4.5) we have that the map $z \to H_z(t, z)$ is compact from $W^{\frac{1}{2}}$ on $W^{-\frac{1}{2}}$, then ψ' is compact.

Now it is easy to verify (cf. [17], sec. 3) that the spectrum of L consists of the limit points $-1, 1$ and of the eigenvalues

$$\lambda_j = \frac{j}{(1 + j^2)^{\frac{1}{2}}}, \qquad j \in Z,$$

and that each eigenvalue λ_j has multiplicity $2n$. Then the functional (4.7) is « strongly indefinite » in the sense used in Section 1, moreover it satisfies the assumptions (f_1) and (f_2) of § 1, because we can suppose $H(t, 0) = 0$.

Let M_{λ_j} denote the eigenspace corresponding to the eigenvalue λ_j. We set

$$W^+ = \overline{\bigoplus_{j>0} M_{\lambda_j}}, \qquad W^- = \overline{\bigoplus_{j<0} M_{\lambda_j}}, \qquad W^0 = \operatorname{Ker} L.$$

Every $z \in W^{\frac{1}{2}}$ can be decomposed as follows

$$z = z^+ + z^- + z^0 \qquad \text{with } z^+ \in W^+,\ z^- \in W^-,\ z^0 \in W^0.$$

So we have

$$\begin{aligned}
&(a) \quad \langle Lz, z\rangle = \langle Lz^+, z^+\rangle + \langle Lz^-, z^-\rangle \\
(4.8) \qquad &(b) \quad (1/\sqrt{2})\,\|z^+\|^2 \leq \langle Lz^+, z^+\rangle \leq \|z^+\|^2 \\
&(c) \quad (1/\sqrt{2})\,\|z^-\|^2 \leq -\langle Lz^-, z^-\rangle \leq \|z^-\|^2 .
\end{aligned}$$

Now our aim is to find conditions on the Hamiltonian H which guarantee that also the assumption (f_3) is satisfied. We consider a sequence $\{z_n\} \subset W^{\frac{1}{2}}$, $z_n = (p_n, q_n)$ such that

$$f(z_n) \to c \in]0, +\infty[\tag{4.9}$$

$$\|f'(z_n)\| \cdot \|z_n\| \to 0 . \tag{4.10}$$

Let us initially prove the following lemma.

Lemma 4.1. – *Let* $\{z_n\} \subset W^{\frac{1}{2}}$, $z_n = (p_n, q_n)$, *be a sequence satisfying* (4.9) *and* (4.10), *then the following sequences*

$$\int_0^{2\pi} \Big(H(t, z_n) - \big(H_p(t, z_n)|p_n\big)\Big)\, dt \tag{4.11}$$

$$\int_0^{2\pi} \Big(H(t, z_n) - \big(H_q(t, z_n)|q_n\big)\Big)\, dt \tag{4.12}$$

are bounded.

Proof. – Easy computations show that

$$\begin{aligned} &(a)\quad \langle f'(z_n), (p_n, 0)\rangle = \int_0^{2\pi} \Big(\big(\dot q_n|p_n\big) - \big(H_p(t, z_n)|p_n\big)\Big)\, dt \\ &(b)\quad \langle f'(z_n), (0, q_n)\rangle = \int_0^{2\pi} \Big(\big(\dot q_n|p_n\big) - \big(H_q(t, z_n)|q_n\big)\Big)\, dt \\ &(c)\quad f(z_n) = \int_0^{2\pi} \Big(\big(\dot q_n|p_n\big) - H(t, z_n)\Big)\, dt\,. \end{aligned} \tag{4.13}$$

By (4.9) and (4.10) the sequences

$$\langle f'(z_n), (p_n, 0)\rangle\,, \qquad \langle f'(z_n), (0, q_n)\rangle\,, \qquad f(z_n)$$

are bounded. Then also right hand sides of the (4.13)'s are bounded. Subtracting (4.13) (*c*) from (4.13) (*a*) we get that (4.11) is bounded. Subtracting (4.13) (*c*) from (4.13) (*b*) we get that (4.12) is bounded. □

Now we consider the case in which H has the form (0.4) with a_{ij}, b_i and V of class C^1.

In the sequel we shall use the following shortened notation:

(4.14) $a(q)$, $A(q)$, $a^k(q)$ $(k = 1, \dots, n)$ will denote respectively the matrices

$$\{a_{ij}(t, q)\}\,, \qquad \{(\text{grad}\, a_{ij}(t, q)|q)\}\,, \qquad \left\{\frac{\partial a_{ij}}{\partial q_k}(t, q)\right\}, \qquad (k = 1, \dots, n)\,.$$

Moreover

(4.15) $b(q)$, $B(q)$, $b^k(q)$ $(k = 1, \dots, n)$ will denote respectively the vectors in R^n

$$\{b_i(t, q)\}\,, \qquad \{(\text{grad}\, b_i(t, q)|q)\}\,, \qquad \left\{\frac{\partial b_i}{\partial q_k}(t, q)\right\}, \qquad (k = 1, \dots, n)\,.$$

Moreover, if v is a vector in R^n or R^{2n}, $|v|$ will denote its norm.

LEMMA 4.2. – *Assume that the Hamiltonian H has the form (0.4) with a_{ij}, b_i $(i, j = 1, \ldots, n)$ and V of class C^1. Assume moreover that (V_1), (A_1), (A_2), (B_1), (B_2) hold. Then, if $\{z_n\}$ $\big(z_n = (p_n, q_n)\big)$ is a sequence in $W^{\frac{1}{2}}$ satisfying (4.9) and (4.10), the following sequences*

$$\int_0^{2\pi} V(t, q_n)\, dt\,, \qquad \int_0^{2\pi} \big(a(q_n)p_n|p_n\big)\, dt$$

are bounded.

PROOF. – Let $\delta > 0$ be a constant such that

$$\alpha - \beta - 2\delta = 2\,. \tag{4.16}$$

(α and β are the constants of assumptions (V_1) and (A_2)).

By Lemma 4.1 we have that the sequences

$$(1 + \beta + \delta)\int_0^{2\pi} \big[\big(a(q_n)p_n|p_n\big) - V(t, q_n)\big]\, dt \tag{4.17}$$

and

$$\int_0^{2\pi} \big[\big(A(q_n)p_n|p_n\big) + \big(B(q_n)|p_n\big) + \big(V_q(t, q_n)|q_n\big) - H(t, z_n)\big]\, dt \tag{4.18}$$

are bounded.

Adding (4.17) to (4.18) we obtain that the sequence

$$\begin{aligned}\int_0^{2\pi} \big[\delta\big(a(q_n)p_n|p_n\big) &+ \big(A(q_n)p_n|p_n\big) + \beta\big(a(q_n)p_n|p_n\big) + \\ &+ \big(V_q(t, q_n)|q_n\big) + (-\beta - 2 - \delta)\,V(t, q_n) + \big(B(q_n)|p_n\big) - \big(b(q_n)|p_n\big)\big]\, dt\end{aligned} \tag{4.19}$$

is bounded.

By (V_1), (A_2), (4.16) and (4.19) there exists $M_1 > 0$ such that

$$M_1 > \int_0^{2\pi} \big[\delta\big(a(q_n)p_n|p_n\big) + \delta V(t, q_n) + \big(B(q_n)|p_n\big) - \big(b(q_n)|p_n\big)\big]\, dt \tag{4.20}$$

for every $n \in N$.

Now, by (B_1) and (B_2)

$$\frac{|B(q)|^2 + |b(q)|^2}{\delta \nu(q)} \leqslant \frac{\delta}{2}\, V(t, q) + M_2 \qquad \text{for every } t \in R \text{ and } q \in R^n \tag{4.21}$$

where M_2 is a positive constant. Then, using (4.21), we get

$$\int_0^{2\pi}[(B(q_n)|p_n)-(b(q_n)|p_n)]\,dt\leqslant\int_0^{2\pi}[|B(q_n)||p_n|+|b(q_n)||p_n|]\,dt\leqslant \tag{4.22}$$
$$\leqslant\int_0^{2\pi}\left[\frac{|B(q_n)|^2}{\delta\nu(q_n)}+|p_n|^2\cdot\frac{\delta}{4}\nu(q_n)+\frac{|b(q_n)|^2}{\delta\nu(q_n)}+\frac{\delta}{4}\nu(q_n)|p_n|^2\right]dt\leqslant$$
$$\leqslant\int_0^{2\pi}\left[\frac{\delta}{2}V(t,q_n)+\frac{\delta}{2}\nu(q_n)|p_n|^2\right]dt+M_3\qquad\text{for every } n\in N$$

where M_3 is a positive constant. By (4.20), (4.22) and (A_1) we deduce that

$$M_1\geqslant\int_0^{2\pi}\left[\delta(a(q_n)p_n|p_n)+\delta V(t,q_n)-\frac{\delta}{2}V(t,q_n)-\frac{\delta}{2}\nu(q_n)|p_n|^2\right]dt-M_3\geqslant$$
$$\geqslant\int_0^{2\pi}\left[\frac{\delta}{2}(a(q_n)p_n|p_n)+\frac{\delta}{2}V(t,q_n)\right]dt-M_3\qquad\text{for every } n\in N\,.$$

From the above inequality, the conclusion follows. □

Lemma 4.3. – *Let the assumptions of Lemma 4.2 hold. Moreover assume that* (V_2), (A_3) *and* (A_4) *hold. Then, if* $\{z_n\}$, $(z_n=(p_n,q_n))$, *is a sequence in* $W^{\frac{1}{2}}$ *satisfying* (4.9) *and* (4.10), *the sequence*

$$\int_0^{2\pi}|H_z(t,z_n)|\,dt$$

is bounded.

Proof. – Just computing $H_z(t,z)$, we get

$$|H_z(t,z_n)|\leqslant 2|a(q_n)p_n|+|b(q_n)|+\sum_k|(a^k(q_n)p_n|p_n)|+ \tag{4.23}$$
$$+\sum_k|(b^k(q_n)|p_n)|+|V_q(t,q_n)|\qquad\text{for every } n\in N\,.$$

Observe that

$$\text{for every } q,p\in R^n\,,\qquad |a(q)p|\leqslant\|a(q)\|+(a(q)p|p)\,. \tag{4.24}$$

By (4.24), (A_4) and Lemma 4.2, it follows that

$$\text{for every } n\in N\,,\qquad \int_0^{2\pi}|a(q_n)p_n|dt\leqslant\int_0^{2\pi}[\|a(q_n)\|+(a(q_n)p_n|p_n)]\,dt\leqslant M_4 \tag{4.25}$$

where M_4 is a positive constant. By (A_1), we get that

$$\|a(q)\| \geqslant \nu(q) \quad \text{for every } q \in R^n . \tag{4.26}$$

Then, from (B_1), the above formula and (A_4) we get:

$$\int_0^{2\pi} |b(q_n)|\, dt \leqslant \int_0^{2\pi} \nu(q_n)^{\frac{1}{2}} \cdot |V(t, q_n)|^{\frac{1}{2}}\, dt + M_5 \leqslant \left(\int_0^{2\pi} \nu(q_n)\, dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} |V(t, q_n)|^{\frac{1}{2}}\, dt\right) + M_5 \leqslant$$

$$\leqslant \left(\int_0^{2\pi} \|a(q_n)\|\, dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} |V(t, q_n)| dt\right)^{\frac{1}{2}} + M_5 \leqslant M_6 \int_0^{2\pi} |V(t, q_n)|\, dt + M_7 \quad \text{for every } n \in N .$$

Then, by Lemma 4.2 and the above inequality, it follows that

$$\forall n \in N , \quad \int_0^{2\pi} |b(q_n)|\, dt \leqslant M_8 . \tag{4.27}$$

Now, by (A_3) and Lemma 4.2, we have

$$\forall n \in N \qquad \sum_k \int_0^{2\pi} |(a^k(q_n) p_n | p_n)|\, dt \leqslant M_9 \int_0^{2\pi} (a(q_n) p_n | p_n)\, dt \leqslant M_{10} . \tag{4.28}$$

Moreover, using (B_2) and (4.26), we have

$$\forall n \in N \qquad \sum_k \int_0^{2\pi} |(b^k(q_n)|p_n)|\, dt \leqslant \sum_k \left(\int_0^{2\pi} \frac{|b^k(q_n)|^2}{\nu(q_n)}\right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} \nu(q_n) |p_n|^2\, dt\right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \left(M_{11} + M_{12} \int_0^{2\pi} V(t, q_n)\, dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{2\pi} (a(q_n) p_n | p_n)\, dt\right)^{\frac{1}{2}} .$$

Then, from Lemma 4.2, we get

$$\forall n \in N \qquad \sum_k \int_0^{2\pi} |(b^k(q_n)|p_n)|\, dt \leqslant M_{13} . \tag{4.29}$$

At last we observe that by Lemma 4.2 and (V_2)

$$\forall n \in N , \quad \int_0^{2\pi} |V(t, q_n)|\, dt \leqslant M_{14} . \tag{4.30}$$

So, by (4.23), (4.25), (4.27), (4.28), (4.29) and (4.30), we deduce that the sequence $\int_0^{2\pi} |H_z(t, z_n)|\, dt$ is bounded. $\square$

LEMMA 4.4. – *Let the assumption of Lemma 4.3 hold. Let $\{z_n\} \subset W^{\frac{1}{2}}$ be a sequence which satisfies* (4.9) *and* (4.10). *Then we can select from $\{z_n\}$ a subsequence which is bounded in $W^{\frac{1}{2}}$.*

PROOF. – Suppose that $\{z_n\} \subset W^{\frac{1}{2}}$ satisfies (4.9) and (4.10). Then by Lemma 4.3 $\{H_z(t, z_n)\}$ is bounded in L^1. L^1 is continuously embedded into $W^{-\frac{1}{2}-\eta/2}$, for any $\eta > 0$. Then

$$\|H_z(t, z_n)\|_{W^{-\frac{1}{2}-\eta/2}} \quad \text{is bounded}. \tag{4.31}$$

By (4.10) we have:

$$Lz_n - H_z(t, z_n) \to 0 \quad \text{in } W^{-\frac{1}{2}}. \tag{4.32}$$

So by (4.31) and (4.32) we have

$$Lz_n \quad \text{is bounded in} \quad W^{-\frac{1}{2}-\eta/2}. \tag{4.33}$$

By the definition of the space $W^{\frac{1}{2}}$ and easy computation, we get

$$\text{for each } z \in W^{\frac{1}{2}}, \quad \|\tilde{z}\|_{W^{\frac{1}{2}-\eta/2}} \leqslant \text{const}\, \|Lz\|_{W^{-\frac{1}{2}-\eta/2}} \tag{4.34}$$

where $\tilde{z} = z - z^0 = z^+ + z^-$. By (4.33) and (4.34) we have that

$$\|\tilde{z}_n\|_{W^{\frac{1}{2}-\eta/2}} \quad \text{is bounded}. \tag{4.35}$$

Then, since $\eta > 0$ is arbitrary, by the Sobolev embedding theorems,

$$\|\tilde{z}_n\|_{L^t} \quad \text{is bounded for any } t \geqslant 1. \tag{4.36}$$

Then next step is to prove that

$$\{z_n^0\} \quad \text{is bounded in } L^1. \tag{4.37}$$

We set

$$(p_n^0, q_n^0) = z_n^0, \quad \forall n \in N.$$

By (V_1) we have

$$\int_0^{2\pi} |q_n^0|^\alpha \, dt \leqslant c_1 \int_0^{2\pi} V(t, q_n^0)\, dt + c_2, \quad \forall n \in N \tag{4.38}$$

where c_1, c_2 are positive constants.

Then, by (4.38) and Lemma 4.2,

$$\{q_n^0\} \quad \text{is bounded in } L^\alpha \text{ and then in } L^1 . \tag{4.39}$$

Now we have to show that also $\{p_n^0\}$ is bounded in L^1.

To this end we initially show that there exists $\mu > 0$ s.t.

$$\forall n \in N , \quad \int_0^{2\pi} \nu(q_n) > \mu . \tag{4.40}$$

By (4.36) and (4.39) there exists $M > 0$ s.t.

$$\forall n \in N , \quad \|q_n\|_{L^1} \leqslant M . \tag{4.41}$$

We now set

$$\nu_0 = \inf_{|q| \leqslant M/\pi} \nu(q) \quad \text{and} \quad \Omega_n = \{t \in [0, 2\pi] \mid |q_n(t)| \leqslant M/\pi\} .$$

Then

$$\forall n \in N , \quad M \geqslant \|q_n\|_{L^1} \geqslant \int_{[0,2\pi]\setminus\Omega_n} |q_n|\, dt \geqslant M/\pi(2\pi - \text{meas } \Omega_n) .$$

From which we get

$$\forall n \in N , \quad \text{meas } \Omega_n \geqslant \pi .$$

Therefore we have

$$\forall n \in N , \quad \int_0^{2\pi} \nu(q_n)\, dt \geqslant \int_{\Omega_n} \nu(q_n)\, dt \geqslant \nu_0 \cdot \text{meas } \Omega_n \geqslant \nu_0 \pi .$$

Then (4.40) holds with $\mu = \nu_0 \pi$.

Now, by Lemma 4.2 and (A_1) there exists $c > 0$ s.t.

$$\begin{aligned} \forall n \in N , \quad c \geqslant \int_0^{2\pi} (a(q_n) p_n | p_n)\, dt \geqslant \int_0^{2\pi} \nu(q_n) |p_n|^2 = \int_0^{2\pi} \nu(q_n) |p_n^0 + \tilde{p}_n|^2\, dt \geqslant \\ = |p_n^0|^2 \int_0^{2\pi} \nu(q_n)\, dt - 2|p_n^0| \int_0^{2\pi} \nu(q_n) |\tilde{p}_n|\, dt + \int_0^{2\pi} \nu(q_n) |\tilde{p}_n|^2\, dt \geqslant \\ \geqslant \|p_n^0\|^2 \int_0^{2\pi} \nu(q_n)\, dt - 2|p_0| \int_0^{2\pi} \nu(q_n) |\tilde{p}_n|\, dt . \end{aligned} \tag{4.42}$$

Now

$$\int_0^{2\pi} \nu(q_n) |\tilde{p}_n|\, dt \leqslant \|\nu(q_n)\|_{L^2} \cdot \|\tilde{p}_n\|_{L^2} . \tag{4.43}$$

By (A_4) and (V_2) we get

$$(4.44)\qquad \forall n \in N\,, \quad \|\nu(q_n)\|^2_{L^2} \leqslant c_1\int_0^{2\pi} V(t, q_n)^2\, dt + c_2 \leqslant c_3\int_0^{2\pi} |q_n|^{2s}\, dt + c_4$$

where c_1, c_2, c_3, c_4 are positive constants.

Moreover, because $\ker L$ is finite dimensional, from (4.39) and (4.36) we deduce that

$$(4.45)\qquad \|q_n\|_{L^{2s}} \quad \text{is bounded}\,.$$

Then from (4.43), (4.44), (4.45) it follows that

$$(4.46)\qquad \forall n \in N\,, \quad \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|\, dt \leqslant \|\nu(q_n)\|_{L^2}\|\tilde{p}_n\|_{L^2} \leqslant c_5\|\tilde{p}_n\|_{L^2}\,.$$

Using (4.36) and (4.46) we get

$$(4.47)\qquad \forall n \in N\,, \quad \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|\, dt \leqslant c_6$$

where c_6 is a positive constant. So from (4.42), (4.40) and (4.47) we get

$$(4.48)\qquad \forall n \in N\,, \quad c \geqslant \mu|p_n^0|^2 - c_7|p_n^0|\,.$$

Then

$$(4.49)\qquad |p_n^0| \quad \text{is bounded}\,.$$

Finally, because $\dim\ker L < +\infty$, from (4.39), (4.49) and (4.36) we deduce that

$$(4.50)\qquad \text{for any } t > 1\,, \quad \|z_n\|_{L^t} \text{ is bounded}\,.$$

Let us now show that $\|z_n\|_{W^{\frac{1}{2}}}$ is bounded.

By (4.10) we have

$$(4.51)\qquad \forall n \in N\,, \quad \|z_n^+\|^2_{W^{\frac{1}{2}}} \leqslant c_7\Big(1 + \int_0^{2\pi} |H_z(t, z_n)|\,|z_n^+|\, dt\Big)$$

where c_7 is a positive constant.

By (4.23) and the assumptions (H_0) there exists $\gamma > 0$ s.t.

$$\forall z \in R^{2n},\ \forall t \in R\,, \quad |H_z(t, z)| \leqslant \text{const}\,(1 + |z|^\gamma)\,.$$

Then from (4.51) we get

$$\forall n \in N\,, \quad \|z_n^+\|_{W^{\frac12}}^2 \leqslant \text{const}\,(1 + \|z_n\|_{L^{2\nu}}^{\gamma} \cdot \|z_n^+\|_{W^{\frac12}})\,. \tag{4.52}$$

Then from (4.50) and (4.52) it follows that

$$\|z_n^+\| \quad \text{is bounded}\,.$$

Analogously it can be proved that

$$\|z_n^-\|_{W^{\frac12}} \quad \text{is bounded}\,.$$

Finally, because $\ker L$ is finite dimensional, we deduce that also

$$\|z_n^0\|_{W^{\frac12}} \quad \text{is bounded}\,. \quad \square$$

We conclude this section with the following lemma.

LEMMA 4.5. – *If* (H_0) *hold, the functional* (4.7) *satisfies* (f_1), (f_2) *and* (f_3) *in the space* $W^{\frac12}$.

PROOF. – (f_1) (i) and (f_2) follow from the construction of L.
By assumptions (V_2), (A_3), (A_4), (B_1), (B_2) and standard majorizations, it follows that H satisfies (4.5). Then (f_1) (ii) is satisfied. (f_3) follows from Lemma 4.4. $\square$

5. – Proof of theorems 0.1 and 0.2.

In this section we shall prove Theorem 0.1 and 0.2. It will be useful to introduce the following notation

$$W_j^+ = \overline{\bigoplus_{k>j} M_{\lambda_k}}\,, \qquad W_j^- = \overline{\bigoplus_{k<j} M_{\lambda_k}}\,. \tag{5.1}$$

If $j > 0$, then $W_j^+ \subset W^+$ so that, for every $z \in W_j^+$, (4.8) (b) holds. The following lemmas provide estimates which shall be used in the proof of the theorems.

LEMMA 5.1. – *For every* $c_0 > 0$, *there exist* $j \in Z$ *and* $R > 0$ *such that*

$$f(z) \geqslant c_0 \quad \textit{for every } z \in W_j^+,\ \|z\| = R$$

where f *is the functional defined by* (4.7).

PROOF. – Since H grows polynomially, there are constants $r, c_1, c_2 > 0$ such that

$$|H(t, z)| \leq c_1 + c_2 |z|^r .$$

Then

$$|\psi(z)| \leq 2\pi c_1 + c_2 \|z\|_{L^r}^r . \tag{5.2}$$

Now, by the Sobolev embedding theorem, there are constants $c_3, s > 0$ such that

$$\|z\|_{L^r} \leq c_3 \|z\|_{W^{\frac{1}{2}-s}} . \tag{5.3}$$

If $z \in W_j^+$, $j > 1$, we have

$$\|z\|^2_{W^{\frac{1}{2}-s}} = \sum_{k>j} (1 + k^2)^{\frac{1}{2}-s} |z_k|^2 \leq (1 + j^2)^{-s} \sum_{k>j} (1 + k^2)^{\frac{1}{2}} |z_k|^2 = (1 + j^2)^{-s} \|z\|^2 \leq j^{-2s} \|z\|^2 .$$

Then by the above formula, (5.2) and (5.3) we get

$$|\psi(z)| \leq c_4 j^{-\varrho} \|z\|^r + c_5 \quad \text{for every } z \in W_j^+$$

where c_4 and c_5 are suitable positive constants and $\varrho = sr > 0$.

Then, by (4.8) and the above formula, for $z \in W_j^+$, $\|z\| = R$ we have

$$f(z) = \tfrac{1}{2} \langle Lz, z \rangle - \psi(z) \geq \tfrac{1}{4} R^2 - c_4 j^{-\varrho} R^r - c_5 = [\tfrac{1}{4} - c_4 j^{-\varrho} \cdot R^{r-2}] R^2 - c_5 .$$

The above formula proves the lemma, in fact, it is sufficient to choose R such that $\frac{1}{8} R^2 > c_5 + c_0$ and j such that

$$c_4 j^{-\varrho} R^{r-2} < \tfrac{1}{8} . \quad \square$$

LEMMA 5.2. – *Suppose that H satisfies assumptions (H_0). Then there exist constants a_1 and $a_2 > 0$ such that*

$$H(z, t) > a_1 |q|^\alpha - a_2 \tag{5.4}$$

and

$$\beta H(z, t) + \big(H_z(z, t) | z\big) > a_1 |q|^\alpha + \mu |p|^2 - a_2 \tag{5.5}$$

where $z = (p, q)$ and μ is the constant in (A_2).

PROOF. – *We prove* (5.5):

We shall use the notations introduced in Section 4 (cf. (4.14), (4.15)), moreover $c_1, \ldots$ will denote positive constants.

By (A_1), (A_2) and (V_1) we have

$$\begin{aligned}(5.6)\qquad \beta H(z,t) + (H_z(z,t)|z) &= \big([\beta a(q) + 2a(q) + A(q)]p|p\big) + \\ &+ \big((\beta+1)b(q) + B(q)|p\big) + \beta V(q,t) + \big(V_q(q,t)|q\big) \geqslant \\ &\geqslant \mu|p|^2 + 2\nu(q)|p|^2 - |(\beta+1)b(q) + B(q)||p| + \beta V(q,t) - c_1 .\end{aligned}$$

Using (B_1), (B_2) we have

$$\begin{aligned}(5.7)\qquad |(\beta+1)b(q) + B(q)||p| &\leqslant \frac{|(\beta+1)b(q) + B(q)|^2}{2\nu(q)} + \frac{\nu(q)}{2}|p|^2 \leqslant \\ &\leqslant \frac{\beta}{2} V(q,t) + \nu(q)|p|^2 + c_2 .\end{aligned}$$

Then, by (5.6), (5.7) we have

$$\beta H(z,t) + (H_z(z,t)|z) \geqslant \mu|p|^2 + \nu(q)|p|^2 + \frac{\beta}{2} V(q,t) - c_3 .$$

Then, using again assumption (V_1), we get (5.5). Similar arguments can be used to prove (5.4). □

Lemma 5.3. – *Let φ a Frechét differentiable functional on a Hilbert space E, with $\varphi(0) = 0$. Suppose that φ satisfies the following assumption: there exist R, M, $\lambda > 0$ such that*

$$(5.8)\qquad \lambda\varphi(x) + \langle\varphi'(x), x\rangle \leqslant \begin{cases} M & \text{if } \|x\| \leqslant R \\ -1 & \text{if } \|x\| \geqslant R . \end{cases}$$

Then there exist $\bar{R} > 0$ such that

$$\varphi(x) < 0 \qquad \text{for } \|x\| > \bar{R} .$$

Proof. – Let $v_0 \in H$, $\|v_0\| = 1$ and set

$$g(t) = \lambda\varphi(tv_0) , \qquad t \geqslant 0 .$$

We shall initially prove that

$$(5.9)\qquad g(t) \leqslant M \qquad \text{for any } t \geqslant 0 .$$

We argue by contradiction and suppose that there exists $t_1 > 0$ s.t.

$$g(t_1) > M .$$

Then, since $g(0) = 0$, there exists $t_0 < t_1$ such that

$$g(t) > M\,, \quad \forall t \in]t_0, t_1[\quad \text{and} \quad g(t_0) = M\,.$$

Obviously there is $\bar{t} \in]t_0, t_1[$ s.t.

$$g'(\bar{t}) > 0\,.$$

Then

$$g(\bar{t}) + \frac{\bar{t}}{\lambda}\, g'(\bar{t}) > M$$

which means that

$$\lambda\varphi(\bar{t}v_0) + \langle \varphi'(\bar{t}v_0), \bar{t}v_0 \rangle > M$$

and this contradicts (5.8).

Now consider

$$\bar{R} > 0 \quad \text{s.t.} \quad M - \lambda \ln \bar{R}/R < 0\,.$$

Let us now show that

(5.10) $$\text{there exists} \quad t_2 \in [R, \bar{R}] \quad \text{s.t.} \quad g(t_2) < 0\,.$$

By (5.8) we have

(5.11) $$g(t) + \frac{1}{\lambda}\, g'(t)\cdot t \leqslant -1 \quad \text{if } t > R\,.$$

Then, since $g(R) \leqslant M$ (cf. 5.9), we have:

$$g(\bar{R}) \leqslant \int_R^{\bar{R}} g'(s) + M \leqslant -\int_R^{\bar{R}} \frac{\lambda}{s}\, ds - \int_R^{\bar{R}} \frac{g(s)}{s}\, ds + M = M - \lambda \ln \bar{R}/R - \int_R^{\bar{R}} \frac{g(s)}{s}\, ds \leqslant -\int_R^{\bar{R}} \frac{g(s)}{s}\, ds\,.$$

From this inequality it is easy to deduce that (5.10) holds.

Now we prove that

$$\varphi(x) < 0 \quad \text{for } \|x\| > \bar{R}\,.$$

Obviously it is sufficient to show that

(5.12) $$g(t) < 0 \quad \text{for } t > t_2\,.$$

Arguing by contradiction suppose that there exists $t_4 > t_2$ s.t. $g(t_4) > 0$. Then obviously there exists $t_3 \in (t_2, t_4)$ such that

$$g(t_3) = 0 \quad \text{and} \quad g'(t_3) \geqslant 0 . \tag{5.13}$$

Since $t_3 > R$, by (5.8) we get

$$g(t_3) + \frac{g'(t_3)}{\lambda} t_3 \leqslant -1 . \tag{5.14}$$

Obviously (5.14) contradicts (5.13). □

LEMMA 5.4. – *Suppose that H satisfies (H_0) then for any $j \in Z$, there exists $R > 0$ such that*

$$f(z) < 0 \quad \text{for} \quad \|z\| > R , \quad z \in W_j^- = \overline{\bigoplus_{k \leqslant j} M_{\lambda_k}} .$$

PROOF. – The interesting case occurs when $j > 0$, otherwise it is trivial.

By virtue of Lemma 5.3 it is enough to prove that

$$\beta f(z) + \langle f'(z), z\rangle \to -\infty \quad \text{as } \|z\| \to \infty . \tag{5.15}$$

In the following $c_1, \ldots, c_6$ will denote positive constants.

Let $z = \begin{pmatrix} p \\ q \end{pmatrix} \in W_j^-$ and set

$$z = z^* + z_0 + \hat{z}$$

where

$$z^* = \begin{pmatrix} p^* \\ q^* \end{pmatrix} \in M_{\lambda_{-j}} \oplus M_{\lambda_{-j+1}} \oplus \ldots \oplus M_{\lambda_{-1}} \oplus M_{\lambda_1} \oplus \ldots \oplus M_{\lambda_j}$$

$$z_0 = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \in \operatorname{Ker} L , \qquad \hat{z} = \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} \in W_{-j-1}^- = \overline{\bigoplus_{k \leqslant -j-1} M_{\lambda_k}} .$$

Then, by using Lemma (5.2), it is easy to see that

$$\begin{aligned} \beta f(z) + \langle f'(z), z\rangle &\leqslant \left(\frac{\beta}{2} + 1\right)(\langle Lz^*, z^*\rangle + \langle L\hat{z}, \hat{z}\rangle) - \\ &- \mu(\|p^*\|_{L^2}^2 + \|\hat{p}\|_{L^2}^2 + \|p_0\|_{L^2}^2) - c_1(\|q^*\|_{L^2}^\alpha + \|\hat{q}\|_{L^2}^\alpha + \|q_0\|_{L^2}^\alpha) + c_2 \leqslant \\ &\leqslant \left(\frac{\beta}{2} + 1\right)\left(\langle Lz^*, z^*\rangle - \frac{1+j}{2+j}\|\hat{z}\|^2\right) - \\ &- \mu\|p^*\|_{L^2}^2 - c_1\|q^*\|_{L^2}^\alpha - c_3(\|\hat{z}\|_{L^2}^2 + \|z_0\|_{L^2}^2) + c_2 \leqslant h(z^*) - c_4(\|\hat{z}\|^2 + \|z_0\|_{L^2}^2) + c_2 \end{aligned} \tag{5.16}$$

where

$$h(z^*) = \left(\frac{\beta}{2} + 1\right)\langle Lz^*, z^*\rangle - \mu\|p^*\|^2_{L^2} - c_1\|q^*\|^{\alpha}_{L^2}.$$

The above formula shows that (5.15) is verified once we prove that

$$h(z^*) \to -\infty \quad \text{as } \|z^*\|_{L^2} \to +\infty. \tag{5.17}$$

In order to prove (5.17) we need to find a more « explicit » form of $\langle Lz^*, z^*\rangle$, $\|p^*\|_{L^2}$, $\|q^*\|_{L^2}$. We set

$$z^* = \sum_{l=1}^{j} (z_l + z_{-l}), \qquad z_l = \begin{pmatrix} p_l \\ q_l \end{pmatrix} \in M_{\lambda_l}.$$

It is not difficult to verify that for any l we have

$$p_l = \sum_{k=1}^{n} a_{lk} \cos lte_k - b_{lk} \sin lte_k$$

$$q_l = \sum_{k=1}^{n} a_{lk} \sin lte_k + b_{lk} \cos lte_k$$

where e_k $(k = 1, \ldots, n)$ is the standard basis in R^n and a_{lk}, b_{lk} are real coefficients.
By straight computations we obtain

$$\langle Lz^*, z^*\rangle = \sum_{l=1}^{j} l(\|z_l\|^2_{L^2} - \|z_{-l}\|^2_{L^2}) = \sum_{l=1}^{j}\sum_{k=1}^{n} 2l(a^2_{lk} + b^2_{lk} - a^2_{-lk} - b^2_{-lk}). \tag{5.18}$$

Moreover

$$\|p^*\|^2_{L^2} = \sum_{l=1}^{j}\sum_{k=1}^{n} (a_{lk} + a_{-lk})^2 + (b_{lk} - b_{-lk})^2 \tag{5.19}$$

and

$$\|q^*\|^2_{L^2} = \sum_{l=1}^{j}\sum_{k=1}^{n} (a_{lk} - a_{-lk})^2 + (b_{lk} + b_{-lk})^2. \tag{5.20}$$

Then

$$h(z^*) \leqslant q(z^*)$$

where

$$q(z^*) = \sum_{l=1}^{j}\sum_{k=1}^{n} \left(\frac{\beta}{2} + 1\right)2l(a^2_{lk} - a^2_{-lk}) - \mu(a_{lk} + a_{-lk})^2 - c_5|a_{lk} - a_{-lk}|^{\alpha} + \\ + \left(\frac{\beta}{2} + 1\right)(b^2_{lk} - b^2_{-lk}) - \mu(b_{lk} - b_{-lk})^2 - c_5|b_{lk} + b_{-lk}|^{\alpha}.$$

Since $\alpha > 2$ it can be verified that

$$q(z^*) \to -\infty \quad \text{as} \quad \|z^*\|_{L^2}^2 = \sum_{l=1}^{j} \sum_{k=1}^{n} a_{lk}^2 + a_{-lk}^2 + b_{lk}^2 + b_{-lk}^2 \to \infty .$$

Then (5.17) easily follows. □

PROOF OF THEOREM 0.1. – We will apply Theorem 1.4. We consider the unitary representation of S^1 on $W^{\frac{1}{2}}$ given by the time-translations (i.e. if $g \in S^1$, $r(g)$ is the unitary map in $W^{\frac{1}{2}}$ defined by $z(t) \to z(t+g)$).

By Lemma 4.5, (f_1), (f_2) and (f_3) follow. Since the Hamiltonian H does not depend on t, also (f_4) is satisfied. It remain to verify the geometrical assumptions (f_5).

We set

$$c_0 = \max \{1, -2\pi \cdot \inf_{z \in R^{2n}} H(z)\} + 1 .$$

The constant c_0 is well defined because by Lemma 5.2, H is bounded from below.

By virtue of Lemma 5.1, it is possible to choose $R > 0$ and $j \in Z$ such that

$$f(z) \geqslant c_0 \quad \text{for every } z \in W_j^+; \ \|z\| = R .$$

Now set

$$V = W_j^+$$

and, chosen n arbitrarily, set

$$W = W_{j+n}^- = (W_{j+n}^+)^\perp .$$

With such a choice of V and W, the assumptions (f_5) (i), (ii), (iii) and (iv) are trivially satisfied. Moreover (f_5) (v) is satisfied by virtue of Lemma 5.4 and (f_5) (vi) is satisfied by our choice of c_0.

Then the conclusion of Theorem 1.4 applies and we get the existence of at least

$$\tfrac{1}{2}\big(\dim (V \cap W) - \operatorname{codim} (V + W)\big) = n$$

critical values with critical points $z_1, \ldots, z_n$ such that

$$f(z_k) \geqslant c_0 . \tag{5.21}$$

It remains to show that the corresponding critical points are not constants.

Suppose that one of them is a constant function $\bar{z}$. Then we have

$$f(\bar{z}) = -2\pi H(z) < c_0 .$$

This contradicts (5.21).

By the arbitrariness of n the conclusion follows. □

PROOF OF THEOREM 0.2. – We shall apply Theorem 1.7.
We can assume without loss of generality that

$$H(t, 0) = 0 \quad \text{for every } t \in R .$$

It is not difficult to prove that f is twice Fréchet differentiable for $z = 0$. Then by (H_3), we have:

$$(5.22)\qquad f(z) = f(0) + \langle f'(0), z\rangle + \frac{1}{2} f''(0)[z, z] + o(\|z\|^2) = \\ = \frac{1}{2}\langle Lz, z\rangle - \frac{\omega}{2}\int_0^{2\pi} (H_{zz}(\omega t, 0)z|z)\, dt + o(\|z\|^2)$$

where $\omega = T/2\pi$, and $z \in W^{\frac{1}{2}}$. By (H_5), it follows that

$$\omega\int_0^{2\pi}(H_{zz}(\omega t, 0)z, z)\, dt \leqslant \gamma\int_0^{2\pi}|z|^2\, dt .$$

Then by the above inequality and (5.22)

$$(5.23)\qquad f(z) \geqslant \frac{1}{2}\langle Lz, z\rangle - \frac{\gamma}{2}\|z\|^2_{L^2} + o(\|z\|^2) .$$

By the definition of $\langle Lz, z\rangle$, we have that

$$\langle Lz, z\rangle \geqslant \|z\|^2_{L^2} \quad \text{for every } z \in W^+ .$$

Then by the above inequality, (5.23) and (4.8) (*b*) we get

$$f(z) \geqslant \frac{1}{2}(1-\gamma)\langle Lz, z\rangle + \frac{\gamma}{2}\langle Lz, z\rangle - \frac{\gamma}{2}\|z\|^2_{L^2} + o(\|z\|^2) \geqslant \\ \geqslant \frac{1}{4}(1-\gamma)\|z\|^2 + o(\|z\|^2) \qquad \text{for every } z \in W^+ .$$

So there exist ϱ, $c_0 > 0$ such that

$$(5.24)\qquad f(z) \geqslant c_0 \quad \text{for every } z \in W^+,\ \|z\| = \varrho .$$

Now let $e \in W^+$ be the eigenfunction corresponding to the first positive eigenvalue λ_1 of L and let R_1, R_2 be two positive constants. We set

$$T = \{se\colon s \in [0, R_1]\} , \qquad Q = \{u + v\colon u \in W^- \oplus \ker L,\ \|u\| \leqslant R_2 \text{ and } v \in T\} .$$

Observe that $Q \subset W_1^-$. Then by Lemma 5.4

$$\sup_{z \in Q} f(z) < +\infty .$$

Moreover, by Lemma 5.4, if R_1 and R_2 are large enough, we get that

$$f(z) \leqslant 0 \quad \text{for every } z \in \partial Q .$$

Thus all the assumptions of Theorem 1.7 are satisfied with $V = W^+$. Then f has a critical value c

$$c \geqslant c_0 > 0 . \tag{5.25}$$

The corresponding critical point $\bar{z} \in W^{\frac{1}{2}}$ cannot be constant because in this case we would have

$$c = f(\bar{z}) = -\int_0^{8\pi} H(\omega t, z) \leqslant 0$$

and this inequality contradicts (5.25). □

Appendix 1.

The proof of Theorem 3.3 is based on the following lemmas:

LEMMA 1. – *If f satisfies (f_1), (f_2) and (f_3) then we have:*

(i) *every bounded sequence $\{u_k\} \subset f^{-1}(]0, \infty[)$ such that $f'(u_k) \to 0$, admits a convergent subsequence;*

(ii) *for every $c > 0$, there exist constants $\bar{\varepsilon}$, $\bar{R}$, b, $\mu > 0$ such that*

  (*a*) $[c - \bar{\varepsilon}, c + \bar{\varepsilon}] \subset [0, +\infty[$,

  (*b*) $\|f'(u)\| \cdot \|u\| \geqslant \mu$ *for every* $u \in f^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]) \cap (E - B_{\bar{R}})$;

(iii) *for every $c > 0$, K_c is compact;*

(iv) *for every c and $R > 0$ and for every neighborhood N of K_c, there exist positive constants $\bar{\varepsilon}$, b such that*

$$\|f'(u)\| > b \quad \text{for every } u \in (A_{c+\bar{\varepsilon}} - A_{c-\bar{\varepsilon}}) \cap (B_R - N) .$$

PROOF. – (i) We put

$$S = L + \lambda P_0$$

where $\lambda \neq 0$ and P_0 is the orthogonal projector on $\ker L$. Clearly S is a bounded invertible operator. Now let u_k be a bounded sequence such that $f'(u_k) \to 0$.

Then we can write

$$Lu_k - \psi'(u_k) = v_k$$

with $v_k \to 0$. Then we have

$$Su_k - \lambda P_0 u_k - \psi'(u_k) = v_k$$

or

$$Su_k = \lambda P_0 u_k + \psi'(u_k) + v_k .$$

Since P_0 and ψ' are compact operators, there is a subsequence u'_k such that Pu'_k and $\psi'(u'_k)$ converge. Thus Su'_k converges. Since S is invertible u'_k converges.

(ii), (iii) and (iv) follow by using standard arguments. □

The condition (i) and (ii) of the above lemma can be considered as a weakened version of the well known condition of Palais and Smale (cf. Remark 1.5).

Lemma 2. – *Let $k: E \to E$ be a compact operator. For every $\varepsilon > 0$ there exists a compact operator $\tilde{k}: E \to E$ such that:*

(*a*) *$\tilde{k}$ is locally Lipschitz continuous;*

(*b*) *$\|k(u) - \tilde{k}(u)\| \cdot (1 + \|u\|) \leq \varepsilon$ for every $u \in E$.*

Moreover, if k is G-equivariant, $\tilde{k}$ can be chosen G-equivariant.

Proof. – The proof follows the same argument as Lemma 3.2 in [11].

Lemma 3. – *Let $\tilde{k}: E \to E$ be a locally Lipschitz continuous, G-equivariant, compact operator. For every $R > 0$ and $\varepsilon > 0$ there exists an operator $\tilde{b} \in B$ (cf. Definition 3.2) such that*

(*a*) *$\|\tilde{k}(u) - \tilde{b}(u)\| < \varepsilon$ for every $u \in B_R$;*

(*b*) *$\tilde{b}$ is locally Lipschitz continuous.*

Proof. – Since $\tilde{k}(B_R)$ is relatively compact, for every $\varepsilon > 0$ there exists a finite set of points $y_1, \ldots, y_s$ such that $\tilde{k}(B_R) \subset \bigcup_{i=1}^{s} B(y_i, \varepsilon/2)$. Let $n \in N$ and set P_n the projector on $\bigoplus_{i=-n}^{n} E_i$. If n is big enough, we have

$$\|y_i - P_n y_i\| < \frac{\varepsilon}{2}, \quad \forall i \in \{1, \ldots, s\} .$$

Consider now the operator

$$\tilde{b}: B_R \to \bigoplus_{i=-n}^{n} E_i\,, \qquad \tilde{b}(u) = \frac{\sum_{i=1}^{s} \mu_i(u)\, P_n y_i}{\sum_{i=1}^{s} \mu_i(u)}$$

where $\mu_i(u) = \operatorname{dist}\big(\tilde{k}(u), E - (B(y_i, \varepsilon/2))\big)$. It is easy to check that $\tilde{b}$ is a bounded, Lip. continuous operator and that for every $u \in B_R$, $\|\tilde{k}(u) - \tilde{b}(u)\| < \varepsilon$. To prove that b can be chosen G-equivariant it is sufficient to repeat the arguments of Lemma 3.2 in [11].

LEMMA 4. – *Let $\tilde{k}: E \to E$ be as in Lemma 3.6; given $\varepsilon > 0$ there exists an operator $b \in B$ such that*

(*a*) $\|\tilde{k}(u) - b(u)\| \cdot (1 + \|u\|) < \varepsilon$ *for every* $u \in E$;

(*b*) b *is locally Lipschitz continuous.*

PROOF. – Given $\varepsilon > 0$, by Lemma 3 for every $n \in N$ there exists a locally Lipschitz continuous operator $\tilde{b}_n: B_{n+1} \to V_{n+1}$ such that

$$V_{n+1} = \bigoplus_{i \in I(n)} E_i \quad \text{for a finite set } I(n) \subset Z \tag{A1.1}$$

$$\|\tilde{k}(u) - \tilde{b}_n(u)\| < \frac{\varepsilon}{2(n+1)} \qquad \text{for every } u \in B_{n+1}\,. \tag{A1.2}$$

For every $n \in N$ we consider a non-increasing map $X_n(t) \in C^1(R, [0, 1])$ such that

$$X_n(t) = \begin{cases} 1 & \text{if } t \in [0, n] \\ 0 & \text{if } t \in [n + \frac{1}{2}, +\infty]\,. \end{cases}$$

we set

$$b_n(u) = \begin{cases} \tilde{b}_n(u) & \text{if } u \in B_{n+1} \\ 0 & \text{if } u \notin B_{n+1}\,. \end{cases}$$

We define a sequence of operators $c_n: E \to E$ as follows:

$$\begin{aligned} &c_1(u) = b_1(u) \\ &c_2(u) = X_1(\|u\|)\,c_1(u) + \big(1 - X_1(\|u\|)\big)\,b_2(u) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &c_{n+1}(u) = X_n(\|u\|)\,c_n(u) + \big(1 - X_n(\|u\|)\big)\,b_{n+1}(u)\,. \end{aligned} \tag{A1.3}$$

We observe that if $u \in B_n$, $c_n(u) = c_{n+1}(u) = \dots$. We set for $u \in E$

$$b(u) = \lim_{n\to\infty} c_n(u) . \tag{A1.4}$$

Clearly $b \in B$ and satisfies (b). Let us prove (a). If $u \in B_{n+1}$ we have

$$\begin{aligned} \|b(u) - \tilde{k}(u)\| &= \|c_{n+1}(u) - \tilde{k}(u)\| = \\ &= \|X_n(\|u\|)c_n(u) + \big(1 - X_n(\|u\|)\big)b_{n+1}(u) - \tilde{k}(u)\| = \\ &= \|X_n(\|u\|)(c_n(u) - \tilde{k}(u)) + \big(1 - X_n(\|u\|)\big)(b_{n+1}(u) - \tilde{k}(u))\| \leqslant \\ &\leqslant X_n(\|u\|)\|c_n(u) - \tilde{k}(u)\| + \big(1 - X_n(\|u\|)\big)\|b_{n+1}(u) - \tilde{k}(u))\| . \end{aligned} \tag{A1.5}$$

Since if $u \in B_{n+1}$, $\tilde{b}_{n+1}(u) = b_{n+1}(u)$, then by (A1.2) we have

$$\|b_{n+1}(u) - \tilde{k}(u)\| < \frac{\varepsilon}{2(n+2)} \qquad \text{if } u \in B_{n+1} . \tag{A1.6}$$

To prove (a) it is sufficient to prove that, for every $n \in N$, if $u \in B_n$

$$\|b(u) - \tilde{k}(u)\| < \frac{\varepsilon}{1 + \|u\|} . \tag{A1.7}$$

In order to prove (A1.7) we argue by induction: if $n = 1$ by (A1.3), (A1.5) and (A1.6) we get

$$\|b(u) - \tilde{k}(u)\| = \|c_1(u) - \tilde{k}(u)\| = \|b_1(u) - \tilde{k}(u)\| < \frac{\varepsilon}{4} < \frac{\varepsilon}{1 + \|u\|} .$$

Now suppose that

$$\|b(u) - \tilde{k}(u)\| \leqslant \frac{\varepsilon}{1 + \|u\|} \qquad \text{for every } u \in B_n . \tag{A1.8}$$

We have to verify (A1.8) for $u \in B_{n+1} - B_n$.

We observe that for $u \in B_{n+1} - B_n$, $c_n(u) = b_n(u)$. Then by (A1.2)

$$\|c_n(u) - \tilde{k}(u)\| = \|b_n(u) - \tilde{k}(u)\| = \|\tilde{b}_n(u) - \tilde{k}(u)\| < \frac{\varepsilon}{2(n+1)} . \tag{A1.9}$$

Then for $u \in B_{n+1} - B_n$ by (A1.5), (A1.6) and (A1.9) we get

$$\begin{aligned} \|b(u) - \tilde{k}(u)\| &\leqslant X_n(\|u\|)\frac{\varepsilon}{2(n+1)} + \\ &+ \big(1 - X_n(\|u\|)\big)\frac{\varepsilon}{2(n+2)} < \frac{\varepsilon}{2(n+1)} < \frac{\varepsilon}{1 + (n+1)} < \frac{\varepsilon}{1 + \|u\|} . \end{aligned} \tag{A1.10}$$

Finally by (A1.8) and (A1.10) we have that

$$\|b(u)-\tilde{k}(u)\|<\frac{\varepsilon}{1+\|u\|} \qquad \text{for every } u\in B_{n+1} \text{ and (A1.7) is proved.} \tag{A1.11}$$ □

By Lemma 2 and 4, we get the following lemma:

LEMMA 5. – *Let $k\colon E\to E$ be a G-equivariant, compact operator. Given $\varepsilon>0$ there exists a bounded operator $b\in B$ such that*

(*a*) $\|k(u)-b(u)\|\cdot(1+\|u\|)<\varepsilon$ *for every* $u\in E$;

(*b*) b *is locally Lipschitz continuous.*

Now we can prove the Theorem 3.3.

PROOF. – Given $c\in\,]\alpha,\beta[$, by Lemma 1 (iii), K_c is compact, hence there exists $\delta>0$ such that $N\supset M_\delta\supset K_c$, where $M_\delta=N_\delta(K_c)$. Moreover, by Lemma 1 (iv) there exist $\bar\varepsilon>0$, and $b>0$ such that

$$\|f'(u)\|>b\,, \qquad \forall u\in(A_{c+\bar\varepsilon}-A_{c-\bar\varepsilon})\cap(B_{\bar R}-M_{\delta/8})\,. \tag{A1.12}$$

We can assume that $\bar R$ is big enough such that $B_{\bar R}\supset M_\delta$. Also we can assume that

$$\bar\varepsilon<\frac{\delta b}{12}\,. \tag{A1.13}$$

Let $\gamma>0$ be such that

$$\gamma<\min\left[\frac{\bar\varepsilon}{4},\frac{b}{4}\right]. \tag{A1.14}$$

By Lemma 5 there exists a locally Lipschitz continuous operator $b\in B$ such that

$$\|k(u)-b(u)\|\leqslant\frac{\gamma}{1+\|u\|} \qquad \text{for every } u\in E\,. \tag{A1.15}$$

We set $S=(A_{c+\bar\varepsilon}-A_{c-\bar\varepsilon})\backslash M_{\delta/8}$, $S_1=S\cap B_R$, $S_2=S-B_R$. By (A1.14) and (A1.12) we have

$$\frac{\gamma}{1+\|u\|}<\frac{b}{4}<\frac{\|f'(u)\|}{4} \qquad \text{for every } u\in S_1\,, \tag{A1.16}$$

and by (A1.14) and Lemma 1 (ii) we have

$$\frac{\gamma}{1+\|u\|}\leqslant\frac{\|f'(u)\|}{4} \qquad \text{for every } u\in S_2\,. \tag{A1.17}$$

Thus, by (A1.10), (A1.16) and (A1.17),

$$\|k(u) - b(u)\| \leqslant \tfrac{1}{4}\|f'(u)\| \quad \text{for every } u \in S\,. \tag{A1.18}$$

We observe that if $u \in S$

$$\|Lu + b(u)\| = \|f'(u) - (k(u) - b(u))\| \geqslant \|f'(u)\| - \|k(u) - b(u)\|\,,$$

Then by the above inequality and (A1.18)

$$\|Lu + b(u)\| \geqslant \tfrac{3}{4}\|f'(u)\| > 0 \quad \text{for every } u \in S\,. \tag{A1.19}$$

Now we set

$$V(u) = 2\,\frac{Lu + b(u)}{\|Lu + b(u)\|^2} \quad \text{for every } u \in S\,. \tag{A1.20}$$

By (A1.19) we have

$$\|V(u)\| \leqslant \frac{8}{3}\,\frac{1}{\|f'(u)\|} \quad \text{for every } u \in S\,, \tag{A1.21}$$

then by Lemma 1 (ii), (A1.12) and (A1.21)

$$\|V(u)\| < K_1 + K_2\|u\| \quad \text{for every } u \in S\,, \tag{A1.22}$$

where K_1 and K_2 are positive constants.

Now we observe that if $u \in S$, by virtue of (A1.18)

$$\|k(u) - b(u)\| \leqslant \tfrac{1}{4}\|f'(u)\| = \tfrac{1}{4}\|Lu + k(u)\| \leqslant \tfrac{1}{4}\|Lu + b(u)\| + \tfrac{1}{4}\|k(u) - b(u)\|\,,$$

then

$$\|k(u) - b(u)\| \leqslant \tfrac{1}{3}\|Lu + b(u)\|\,.$$

From the above inequality, we get

$$\begin{aligned}\langle V(u), f'(u)\rangle &= 2\left\langle \frac{Lu + b(u)}{\|Lu + b(u)\|^2}, Lu + k(u)\right\rangle = \\ &= \frac{2}{\|Lu + b(u)\|^2}\langle Lu + b(u), Lu + b(u) - b(u) + k(u)\rangle = \\ &= \frac{2}{\|Lu + b(u)\|^2}\left[\|Lu + b(u)\|^2 + \langle Lu + b(u), k(u) - b(u)\rangle\right] \geqslant \\ &\geqslant 2 - 2\,\frac{\|Lu + b(u)\|\cdot\|k(u) - b(u)\|}{\|Lu + b(u)\|^2} \geqslant 2 - \frac{2}{3} > 1 \quad \text{for every } u \in S\,.\end{aligned} \tag{A1.23}$$

Now we consider a Lipschitz continuous functional $\varphi: E \to R$ such that

$$\varphi(u) = \begin{cases} 0 & \text{if } u \notin f^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]) \text{ or } u \in M_{\delta/8} \\ 1 & \text{if } u \in f^{-1}([c - \varepsilon, c + \varepsilon]) - M_{\delta/4} \end{cases} \tag{A1.24}$$

where $\varepsilon = \bar{\varepsilon}/2$. We can assume that φ is G-invariant. We set

$$\bar{V}(u) = \begin{cases} -\varphi(u)V(u) & \text{if } u \in S \\ 0 & \text{if } u \notin S\,. \end{cases} \tag{A1.25}$$

Consider now the following initial value problem

$$\begin{cases} \dfrac{d\eta}{dt} = \bar{V}(\eta) \\ \eta(0) = u \end{cases} \qquad u \in E\,. \tag{A1.26}$$

Since $\bar{V}$ is loc. Lipschitz continuous, by (A1.22) and standard arguments (cf.: also [11], [8], [13]) it can be proved (*a*).

In order to prove (*b*), we set

$$\bar{\phi}(u) = \frac{-2\varphi(u)}{\|Lu + b(u)\|^2}$$

so the equation (A1.26) becomes

$$\frac{d\eta}{dt} = \bar{\phi}(\eta)[L\eta + b(\eta)]\,, \qquad \eta(0) = u\,. \tag{A1.27}$$

Following an idea of HOFER [30] we set:

$$\alpha(t, s, u) = \int_0^{t-s} \bar{\varphi}(\eta(t + s, u))\, dt\,. \tag{A1.28}$$

It can be proved that the Cauchy problem (A1.27) is equivalent to the following integral equation:

$$\eta(t, u) = \exp[\alpha(t, 0, u)L][u] + \int_0^t \alpha^{\alpha(t,s,u)L}[\bar{\varphi}(\eta(s, u)b(\eta(s, u))]\, ds\,.$$

and it is not difficult (cf. [13]) to verify that (*b*) and (*c*) are satisfied with

$$\begin{aligned} U(u) &= \exp[\alpha(t, 0, u)L][u] \\ b(u) &= \int_0^t \exp[\alpha(t, s, u)L[\bar{\varphi}(\eta(t, u))b(\eta(t, u))]\, ds\,. \end{aligned}$$

Appendix 2.

The proof of Theorem 3.4 is based on two lemmas.

Lemma 1. – *Let $V, W, Z \subset E$ be G-invariant, finite dimensional subspaces (V, $W \subset Z$), and Ω be a bounded G-invariant neighborhood of* 0. *Given a G-equivariant bounded continuous map $h: E \to E$, we suppose that*

(i) $\operatorname{Fix} G \subset W$;

(ii) *the index theory i satisfies the d-dimension property*;

(iii) $h(\partial\Omega \cap V) \subset Z$

then

$$i(h(\partial\Omega \cap V) \cap W) \geqslant \frac{\dim (V \cap W) - \operatorname{codim}_Z (V + W)}{d}. \tag{A2.1}$$

Proof. – We set $S = \partial\Omega$. We distinguish two cases

Case I $\quad V \cap \operatorname{Fix} G \underset{\neq}{\supset} \{0\}$

Case II $\quad V \cap \operatorname{Fix} G = \{0\}$.

In the Case I we have that

$$V \cap S \cap \operatorname{Fix} G \neq \emptyset .$$

Since $h(\operatorname{Fix} G) \subset \operatorname{Fix} G$,

$$h(S \cap V) \cap \operatorname{Fix} G \supset h(V \cap S \cap \operatorname{Fix} G) \cap \operatorname{Fix} G \neq \emptyset .$$

Using assumption (i) and the above formula we have

$$h(S \cap V) \cap \operatorname{Fix} G \cap W \neq \emptyset .$$

Then by Theorem 2.4 (i), it follows that

$$i(h(V \cap S) \cap W) = +\infty .$$

Therefore, in the Case I, (A2.1) holds.

We now consider the Case II. Since W is finite dimensional, $h(S \cap V) \cap W \in \Pi$ is compact. Then, by (2.1) (d), there exists $N = N_\varepsilon(h(S \cap V) \cap W)$ such that

$$i(N) = i(h(S \cap V) \cap W) . \tag{A2.2}$$

We set

$$A_1 = h(S \cap V) \cap N\,, \qquad A_2 = \overline{h(S \cap V) - N}\,. \tag{A2.3}$$

Obviously $A_1, A_2 \in \Pi$ and

$$h(S \cap V) = A_1 \cup A_2\,. \tag{A2.4}$$

Since $V \cap \text{Fix}\,(G) = \{0\}$, then

$$\begin{aligned} \frac{\dim V}{d} &= i(S \cap V) && \text{(by the dimension property, cf. Definition 2.2)} \\ &\leqslant i(h(S \cap V)) && \text{(by (2.1) } (e)) \\ &= i(A_1 \cup A_2) && \text{(by (A2.4))} \\ &\leqslant i(A_1) + i(A_2) && \text{(by (2.1) } (c))\,. \end{aligned} \tag{A2.5}$$

By (A2.3), (2.1) (b) and (A2.2) we have

$$i(A_1) \leqslant i(N) = i\big(h(S \cap V) \cap W\big)\,. \tag{A2.6}$$

Let $W^\perp$ denote the orthogonal complement of W is Z and let $P_W^\perp$ denote the relative orthogonal projection. $P_W^\perp$ is a G-equivariant map, then, by (2.1) (c)

$$i(A_2) \leqslant i(P_W^\perp A_2)\,. \tag{A2.7}$$

By the construction of N, $(P_W^\perp A_2) \subset W^\perp - \{0\}$, then since $\text{Fix}\, G \subset W$,

$$(P_W^\perp A_2) \subset W^\perp - \{0\} = W^\perp - \text{Fix}\,(G)\,.$$

Therefore, by Theorem 2.4 (ii)

$$i(P_W^\perp A_2) \leqslant \frac{\dim W^\perp}{d}\,. \tag{A2.8}$$

By (A2.5), (A2.6), (A2.7) and (A2.8), we get

$$\frac{\dim V}{d} \leqslant i\big(h(S \cap V) \cap W\big) + \frac{\dim W^\perp}{d}\,.$$

By the above formula we have:

$$i\big(h(S \cap V) \cap W\big) \geqslant \frac{\dim V - \dim W^\perp}{d} = \frac{\dim V - \text{cod}_Z\, W}{d}\,. \qquad \square$$

LEMMA 2. – *Let the hypotheses of Lemma 4.2 be satisfied with* (i) *and* (iii) *replaced by*

(i′) $\mathrm{Fix}\, G \subset V \oplus Z^{\perp}$;

(iii′) (*a*) *h is a bounded homeomorphism,*

(*b*) $h(\Omega \cap Z) \subset Z$,

(*c*) $h(0) = 0$.

Then

$$i(h(\partial\Omega \cap V) \cap W) \geqslant \frac{\dim (V \cap W) - \mathrm{codim}_Z (V + W)}{d}. \tag{A2.9}$$

PROOF. – To shorten the notation, we set $S = \partial\Omega$. Since $h(S \cap V) \cap W \in \Pi$ is compact, by (2.1) (*d*) there exists $N = N_{\varepsilon_1}(h(S \cap V) \cap W)$ such that

$$i(N) = i(h(S \cap V) \cap W). \tag{A2.10}$$

There exist constants ε_2, ε_3, $\varepsilon > 0$ such that

$$N \supset N_{\varepsilon_2}(h(S \cap V) \cap W) \supset h(N_{\varepsilon_3}(S \cap V)) \cap W \supset h(S \cap V_\varepsilon) \cap \cap W \supset h(S \cap V) \cap W \tag{A2.11}$$

where $V_\varepsilon = N_\varepsilon(V) \cap Z$. By the above formula and (2.1) (*b*) it follows that

$$i(N) \geqslant i(h(S \cap V_\varepsilon) \cap W) \geqslant i(h(S \cap V) \cap W).$$

Then, by (A2.10),

$$i(h(S \cap V_\varepsilon) \cap W) = i(h(S \cap V) \cap W). \tag{A2.12}$$

We now set

$$R = \overline{Z - V_\varepsilon}.$$

Then $Z = V_\varepsilon \cup R$ and

$$h(S \cap Z) \cap W = [h(S \cap V_\varepsilon) \cap W] \cup [h(S \cap R) \cap W].$$

By the above formula and (2.1) (*c*), we have:

$$i(h(S \cap Z) \cap W) \leqslant i(h(S \cap V_\varepsilon) \cap W) + i(h(S \cap R) \cap W).$$

Comparing this inequality with (A2.12), we get

$$i(h(S \cap V) \cap W) \geqslant i(h(S \cap Z) \cap W) - i(h(S \cap R) \cap W)\,. \tag{A2.13}$$

Now we shall give an estimate to the terms on the right hand side of (A2.13). Let $V^\perp$ denote the orthogonal complement of V in Z and $P_V^\perp$ the relative projection. Obviously $P_V^\perp$ is equivariant. Moreover, by (i′), $P_V^\perp R \subset V^\perp - \operatorname{Fix}(G)$. Then by (2.1) (*e*) and Theorem 2.4 (ii), we have

$$i(R) \leqslant i(P_V^\perp R) \leqslant \frac{\dim V^\perp}{d}\,. \tag{A2.14}$$

Now

$$\begin{aligned} i(h(S \cap R) \cap W) &\leqslant i(h(S \cap R)) && \text{(by (2.1) } (b)) \\ &= i(S \cap R) && \text{(by Theorem 2.4 (iv) and (iii′) } (a)) \\ &\leqslant i(R) && \text{(by (2.1) } (b)) \\ &\leqslant \frac{\dim V^\perp}{d} && \text{(by (A2.14))}\,. \end{aligned} \tag{A2.15}$$

By (iii′) (*b*) and (*c*), $h(\Omega \cap Z)$ is a bounded neighborhood of 0 in Z. Then the set

$$\tilde{\Omega} = \{z + \tilde{z} \colon z \in h(\Omega \cap Z),\ \tilde{z} \in Z^\perp,\ |z| < 1\}$$

is a neighborhood of 0 in E. It is easy to check that

$$h(\partial\Omega \cap Z) = \partial\tilde{\Omega} \cap Z\,.$$

Then

$$h(S \cap Z) \cap W = h(\partial\Omega \cap Z) \cap W = \partial\tilde{\Omega} \cap Z \cap W = \partial\tilde{\Omega} \cap W\,.$$

So, by the above inequality and the dimension property it follows that

$$i(h(S \cap Z) \cap W) = i(\partial\tilde{\Omega} \cap W) \geqslant \frac{\dim W}{d}\,. \tag{A2.16}$$

(In the above formula we have to use the inequality because it might happen that $\partial\tilde{\Omega} \cap W \cap \operatorname{Fix} G \neq \emptyset$; cf. Theorem 2.4 (ii).)

Finally, by (A2.13), (A2.16) and (A2.15) we conclude the proof:

$$i(h(S \cap V) \cap W) \geqslant \frac{\dim W}{d} - \frac{\dim V^\perp}{d} = \frac{\dim W}{d} - \frac{\operatorname{cod}_Z V}{d}. \quad \square$$

PROOF OF THEOREM 3.4. – We set $S = \partial\Omega$ and

$$\text{(A2.17)}\qquad \begin{aligned} &E_2 = V \cap W \\ &E_1 = \text{orthogonal complement of } E_2 \text{ in } V \\ &E_3 = \text{orthogonal complement of } E_2 \text{ in } W \\ &E_4 = \text{orthogonal complement of } E_1 \oplus E_2 \oplus E_3 \text{ in } E\,. \end{aligned}$$

We have, obviously, that $V = E_1 \oplus E_2$, $W = E_2 \oplus E_3$, $E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$. We observe, also, that the subspaces E_1, E_2, E_3, E_4, defined by (A2.17) are G-invariant. Let $h = U + b \in H^*$ and $Z \subset E$ be a G-invariant, finite-dimensional subspace such that

$$E_2 \subset Z\,, \qquad E_4 \subset Z\,, \qquad b(\Omega) \subset Z\,.$$

Then

$$\text{(A2.18)}\qquad h(\Omega \cap Z) \subset Z\,.$$

If we set $Z_1 = E_1 \cap Z$, $Z_3 = E_3 \cap Z$, we have that

$$\text{(A2.19)}\quad h(S \cap V) \cap W \supset h(S \cap V \cap Z) \cap W \cap Z = h\big(S \cap (Z_1 \oplus E_2)\big) \cap (E_2 \oplus Z_3)\,.$$

If we set $\tilde{V} = Z_1 \oplus E_2$, $\tilde{W} = E_2 \oplus Z_3$, we have that $\tilde{V}$ and $\tilde{W}$ satisfy the assumption of Lemma 1 or Lemma 2 depending on the fact that Fix $G \subset V$ or Fix $G \subset W$. Then by (A2.18), (A2.19), Lemma 1 and Lemma 2 we have that

$$i(h(S \cap V) \cap W) \geqslant \frac{\dim E_2 - \dim E_4}{d} = \frac{\dim (V \cap W) - \operatorname{codim} (V + W)}{d}\,.$$

By the above formula it easily follows that

$$i^*(S \cap V, W) \geqslant \frac{\dim (V \cap W) - \operatorname{codim} (V + W)}{d}\,. \qquad \square$$

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