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PERIODIC SOLUTIONS OF LIENARD TYPE SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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The first purpose of this paper is to prove the existence of solutions to the problem

$$x'' + f(x)x' + g(t, x, x') = e(t)\,, \tag{1}$$

$$x(0) = x(2\pi)\,, \quad x'(0) = x'(2\pi)\,. \tag{2}$$

Here $f: R \to R$, $g: [0, 2\pi]\times R\times R \to R$ and $e: [0, 2\pi] \to R$ are continuous.

This is a well-studied problem. In his survey [1], Cesari outlines a branch of research followed by Lefschetz, Levinson, Graffi, Cesari, and Cesari and Kannan. A related branch may be followed in the papers by Lazer [6], Lazer and Leech [7], Mawhin [9], Reissig [10]-[12], Chang [2] and Martelli [8].

Hypotheses which insure a solution to (1), (2) have gradually been refined to something like the following:

(A) Almost no restrictions on f.

(B) There exist constants k, R, positive, and A, B (with $A > B$) such that (i) $|x| \geqq R \Rightarrow |g(x)| < k|x|$; and (ii) $x \geqq R \Rightarrow g(x) \geqq A$, $x \leqq -R \Rightarrow g(x) \leqq B$, and $B < e_m < A$ where $e_m = (2\pi)^{-1}\int_0^{2\pi} e(t)dt$. (For simplicity we have let $g = g(x)$.)

In elaboration we note: (a) If g has the form $g(x) = m^2x + h(x)$, m an integer, then (i) becomes $|g(x) - m^2(x)| < l|x|$. (b) The results can be extended to vector equations with $f(x)x'$ becoming $(d/dt)[\nabla f(x(t))]$. (c) The best results seem to relate k to the eigenvalues of the problem (1), (2), which in this case is $k = 1$.

In this paper we use the Alternative or Lyapunov-Schmitt Method to solve the problem. We develop further a technique begun in [13] and we use a splitting of the operator $Lx = -x''$ into $T^*x = -x'$, $Tx = x'$. (See Kannan and Locker [3], or Cesari [1].) We can then (a) eliminate the term $f(x)x'$ in a natural way; (b) introduce an x' into $g(t, x, x')$; (c) have a scheme which can be applied to higher order problems.

Additionally we note: (a) Half of the work is showing that our version of hypotheses (A) and (B) make this abstract scheme work. (Related results for a fourth order problem are presented in [14]). (b) With the inclusion of an x' term we have had to sacrifice in the choice of k and take $k < 1/\sqrt{6}$.

THEOREM. *Let* (1), (2) *be given and assume*

(h_1) $|g(t,x,y)| \leqq k(x^2+y^2)^{1/2}$, *where* $0<k<1/\sqrt{6}$ *for all* $(t,x,y) \in [0,2\pi] \times R \times R$; *and*

(h_2) *there exist constants* $R > 0$, A, B (*with* $A > B$) *such that* $x \geqq R \Rightarrow g(t, x, y) \geqq A$, $x \leqq -R \Rightarrow g(t, x, y) \leqq B$, *for all* $(t, y) \in [0, 2\pi] \times R$, *and* $B < e_m < A$ *where* $e_m = (2\pi)^{-1}\int_0^{2\pi} e(t)dt$.

Then (1), (2) *has at least one solution.*

PROOF. 1°. We write the problem as an operator equation in a Banach space and employ the Alternative Method. (For more details see [1].)

Let $X = \{x \in C^2[0, 2\pi]: x(0) = x(2\pi), x'(0) = x'(2\pi)\}$ and, for $x \in X$, let $Px = (1/2\pi)\int_0^{2\pi} x$. Then P is a projection. Let $X_0 = PX$, $X_1 = (I - P)X$ (so $X = X_1 \oplus X_0$). Let $Z = C[0, 2\pi]$ and, for $z \in Z$, let $Qz = (1/2\pi)\int_0^{2\pi} z$. Let $Z_0 = QZ$ and $Z_1 = (I - Q)Z$ (so $Z = Z_1 \oplus Z_0$). Define L, N and H by $D(L)$ (domain of L) $= X$, $Lx = -x''$; $D(N) = C^1[0, 2\pi]$, $Nx = \{f(x)x'\} + \{g(\cdot, x, x') - e\} = \{N_1x\} + \{N_2x\}$; and $H = [L|X_1]^{-1}$. Note that $K(L)$ (the kernel of L) $= [1]$ (the constant functions) $= X_0$; that $R(L)$ (the range of L) $= Z_1$; and that $Z_0 = [1]$.

Now (1), (2) can be written as

$$Lx = Nx \tag{3}$$

and (3) is equivalent to the pair of equations

$$x = Px + H(I - Q)Nx \tag{4}$$

$$0 = QNx\ . \tag{5}$$

2°. We "split" the operator H into J^*J. (For more details see [3] or [1].)

Let $Y = \{y \in C^1[0, 2\pi]: y(0) = y(2\pi)\}$ and, for $y \in Y$, let $Py = (1/2\pi)\int_0^{2\pi} y$. Let $PY = Y_0$ $(=[1])$ and $(I - P)Y = Y_1$ (so $Y = Y_1 \oplus Y_0$). Define T^* and T by

$$D(T^*) = X\ , \quad T^*x = -x' \quad (\text{so } K(T^*) = X_0\ , \quad R(T^*) = Y_1)\ ;$$

and

$$D(T) = Y\ , \quad Ty = y' \quad (\text{so } K(T) = Y_0\ , \quad R(T) = Z_1)\ .$$

Now let $L = TT^*$. If we let $J^* = [T^*|X_1]^{-1}$ and $J = [T|Y_1]^{-1}$, then $H = J^*J$. If $x = x_1 + x_0 \in X_1 \oplus X_0$, then $x_1 = J^*y_1$ for some $y_1 \in Y_1$ and $x_0 = Px$. Hence (4) may be written as $J^*y_1 = J^*J(I - Q)N(J^*y_1 + x_0)$. Now J^* is one-to-one so we may cancel it and $Y_0 = [1]$, $X_0 = [1]$ so we may write y_0 in place of x_0.

Thus (4), (5) is equivalent to

$$y_1 = J(I - Q)N(J^*y_1 + y_0) \tag{6}$$

$$y_0 = y_0 + QN(J^*y_1 + y_0) \, . \tag{7}$$

To be precise we should write UQN in (7) where $U: Z_0 \to Y_0$ is a bijection. But since $Z_0 = [1]$, $Y_0 = [1]$, we can omit the U.

3° Continuity and compactness of operators. Now we change notation and let $Y = \{y \in L^2[0, 2\pi]: y(0) = y(2\pi)\}$ (the periodic, square-integrable functions) with the usual norm $||\cdot||$ and inner product $\langle \cdot, \cdot \rangle$. Again for $y \in Y$ we let $Py = (1/2\pi)\int_0^{2\pi} y$, $PY = Y_0$, $(I - P)Y = Y_1$, and $Y = Y_1 \oplus Y_0$. A solution y_1 of (6) will be in the range of J, i.e., $y_1 \in C^1[0, 2\pi]$, and hence the solution of (4), (5), $x = J^*y_1 + y_0$ is in $C^2[0, 2\pi]$.

To show the compactness of the operator appearing in (6) we introduce

$H^1 = \{x(t): x' \in Y$ (so x is absolutely continuous)$\}$

with the norm $||x||_H = |x|_0 + ||x'||$ $(|x|_0 = \sup_{[0,2\pi]} |x(t)|)$ and

$$L^1 = \left\{x(t): x \text{ is Lebesgue integrable on } [0, 2\pi] \text{ and } \int_0^{2\pi} x = 0\right\}$$

with the usual norm $||\cdot||_1$. We will show that the composition of the following sequence of operators is compact and continuous:

$$Y_1 \xrightarrow{J^*} H^1 \xrightarrow{(I-Q)N} L^1 \xrightarrow{J} Y_1 \, .$$

This has been discussed in detail in [4] so we shall only outline the proof here.

(a) Since J can be represented as $Jz = \int_0^t z - (1/2\pi)\int_0^{2\pi}\int_0^t z$ we see that its domain can be extended to include L^1. And, as shown above, the solution y_1 will still be in $C^1[0, 2\pi]$.

(b) $N: H^1 \to L^1$ is bounded: $||N_1 x||_1 = \int_0^{2\pi} |f(x(t))x'(t)|\,dt$ so by the continuity of f, N_1 takes sets bounded in H^1 into sets bounded in L^1. That N_2 is bounded follows along similar lines.

(c) $N: H^1 \to L^1$ is continuous:

$$||N_1 x - N_1 x_0||_1 \leqq \int_0^{2\pi} |f(x(t))x'(t) - f(x_0(t))x'(t)|\,dt$$
$$+ \int_0^{2\pi} |f(x_0(t))x'(t) - f(x_0(t))x_0'(t)|\,dt \, .$$

The first integral: with $x' \in L^2[0, 2\pi]$, $f(c)x'(t) = g(c, t)$ is continuous in c and measurable in t. By Krasnosel'skii's version of Lusin's theorem [5] we may divide $[0, 2\pi]$ into disjoint subsets I_1 and I_2 such that I_1 is closed and $g|R^n \times I_1$ is continuous in (c, t) and I_2 has arbitrarily small measure.

We have just seen that the integrands are bounded so the integral over I_2 can be made small. On $\{(c, t): |c - x_0(t)| \leqq 1, t \in I_1\}$, $g(c, t)$ is uniformly continuous, so the integral over I_1 can be made small by making $|x - x_0|_0$ small.

In the second integral we have $|f(x_0(t))x'(t) - f(x_0(t))x_0'(t)| \leqq |f(x_0(t))| |x'(t) - x_0'(t)|$ and the proof is straightforward. That N_2 is continuous follows along similar lines.

(c) J and J^* are integral operators and are known to be continuous and compact and the projection $(I - Q)$ is continuous. Hence $J(I - Q)N(J^*y_1 + y_0)$ is a continuous, compact mapping from $Y_1 \oplus Y_0$ into Y_1.

(d) The projection Q is continuous and its range is finite dimensional. Hence $QN(J^*y_1 + y_0)$ is a continuous, compact mapping from $Y_1 \oplus Y_0$ into Y_0.

(e) Since $(I-Q)u = u - (1/2\pi)\int_0^{2\pi} u$ and $Jv = \int_0^t v - (1/2\pi)\int_0^{2\pi}\int_0^t v$, $J(I-Q)u = \int_0^t u - (1/2\pi)\int_0^t\left(\int_0^{2\pi} u\right) - (1/2\pi)\int_0^{2\pi}\int_0^t u + (1/2\pi)^2\int_0^{2\pi}\int_0^t\left(\int_0^{2\pi} u\right) = Ju$ and hence $||J(I - Q)|| = ||J||$.

4°. A theorem from the Leray-Schauder theory of degree. Returning to (6), (7) let

$$T_1(y_1, y_0) = J(I - Q)N(J^*y_1 + y_0)\ ,$$
$$T_0(y_1, y_0) = y_0 + QN(J^*y_1 + y_0)$$

and I_k be the identity operator on Y_k $(k = 1, 0)$. Let $I =$ column (I_1, I_0) and $T =$ column (T_1, T_0). Then (6), (7) can be written as

$$(I - T)(y_1, y_0) = 0\ . \tag{8}$$

This is of the form of identity plus a compact operator from $Y_1 \oplus Y_0$ into $Y_1 \oplus Y_0$ and the theory of degree may be applied. We shall use the following variant of the Borsuk theorem:

Let $\lambda(I - T)(y_1, y_0) \neq (1 - \lambda)(I - T)(-y_1, -y_0)$ for $1/2 \leqq \lambda < 1$ and $(y_1, y_0) \in \partial B(R_1, R_0)$ where $B(R_1, R_0) = \{(y_1, y_0) \in Y_1 \oplus Y_0: ||y_1|| \leqq R_1, ||y_0|| \leqq R_0\}$. Then (8) has a solution in $B(R_1, R_0)$.

We will show that $\lambda(I_1 - T_1)(y_1, y_0) \neq (1 - \lambda)(I_1 - T_1)(-y_1, -y_0)$ on $S^1 = \{(y_1, y_0): ||y_1|| = R_1, ||y_0|| \leqq R_0\}$ and $\lambda(I_0 - T_0)(y_1, y_0) \neq (1 - \lambda)(I_0 - T_0)(-y_1, -y_0)$ on $S^0 = \{(y_1, y_0): ||y_1|| \leqq R_1, ||y_0|| = R_0\}$.

Now $\lambda(I_1 - T_1)(y_1, y_0) = (1-\lambda)(I_1 - T_1)(-y_1, -y_0)$ implies $y_1 - \lambda J(I - Q)N(J^*y_1 + y_0) + (1-\lambda)J(I-Q)N(J^*(-y_1)-y_0) = 0$ which implies

$$(9) \qquad \|y_1\|^2 - \lambda\langle J(I-Q)N(J^*y_1 + y_0), y_1\rangle + (1-\lambda)\langle J(I-Q)N(J^*(-y_1)-y_0), y_1\rangle = 0 \quad \text{for} \quad (y_1, y_0) \in S^1 .$$

And $\lambda(I_0 - T_0)(y_1, y_0) = (1-\lambda)(I_0 - T_0)(-y_1, -y_0)$ implies

$$(10) \qquad \lambda QN(J^*y_1 + y_0) = (1-\lambda)QN(J^*(-y_1)-y_0) \quad \text{for} \quad 1/2 \leqq \lambda < 1 , \quad (y_1, y_0) \in S^0 .$$

We will show that (9) and (10) do not hold under the hypotheses of the theorem.

5°. Hypothesis (h_1) implies that (9) does not hold.

(a) Let $x = J^*y_1 + y_0$, so $x' = -y_1$ and $x(0) = x(2\pi)$. Then $2\pi(QN_1x) = \int_0^{2\pi} f(x(t))x'(t)dt = F(x(2\pi)) - F(x(0)) = 0 \left(F(u) = \int_0^u f\right)$; $J(I - Q)N_1x =$ (see 3°, (a)); $\int_0^t f(x(s))x'(s)ds - (1/2\pi)\int_0^{2\pi}\int_0^t f(x(s))x'(s)dsdt = F(x(t)) - c)$ (a constant); and $\langle J(I-Q)N_1x, y_1\rangle = \int_0^{2\pi} (F(x(t)) - c)x'(t)dt = G(x(2\pi)) - G(x(0)) - c(x(2\pi) - x(0)) = 0 \left(G(u) = \int_0^u F\right)$. Likewise $\langle J(I-Q)N_1(-x), y_1\rangle = 0$.

(b) From hypothesis (h_1) it follows that $|g(t, x(t), y(t))|^2 \leqq k^2(|x(t)|^2 + |y(t)|^2)$. This, together with the continuity of g, implies that N_2 takes H_1 into L^2. So in this estimate we can work in L^2.

If $(I-Q)y = y_1 = \sum_1^\infty (a_k\varphi_k + b_k\psi_k)$ $(\varphi_k(t) = (1/\sqrt{\pi})\cos kt,\ \psi_k(t) = (1/\sqrt{\pi})\sin kt)$, then $Jy_1 = \sum_1^\infty k^{-1}(a_k\psi_k - b_k\varphi_k)$ and $\|Jy_1\|^2 = \sum_1^\infty k^{-2}(a_k^2 + b_k^2) \leqq \sum_1^\infty (a_k^2 + b_k^2) = \|y_1\|^2$, so $\|J\| \leqq 1$. And $\|J\varphi_1\| = \|\varphi_1\|$ so $\|J\| = 1$.

(c) With k, $0 < k < 1/\sqrt{6}$, given in hypothesis (h_1) let ε, $0 < \varepsilon < 1$, be such that $k = (1/\sqrt{6})(1-\varepsilon)(1+\varepsilon)^{-1/2}$. In the definition of $B(R_1, R_0)$ (see 4°) let $R_1 = \max(2\|J\|\,\|e\|/\varepsilon, R/2\eta\sqrt{2\pi}, 1)$ where $\eta > 0$ is such that $(1+\eta)^2 = (1 + 6\varepsilon/4)$, i.e., $\eta = (1 + 6\varepsilon/4)^{1/2} - 1$. (Here $e = e(t)$ is from (1), R given in hypothesis (h_2).)

Let $R_0 = 2R_1 + R/\sqrt{2\pi}$. Then $|\langle J(I-Q)(-e), y_1\rangle| \leqq \|J\|\,\|e\|\,\|y_1\| \leqq \varepsilon\|y_1\|^2/2$ for $\|y_1\| = R_1$. Next $\|x\|^2 = \|J^*y_1\|^2 + \|y_0\|^2 \leqq \|y_1\|^2(1 + 4 + 6\varepsilon)$ if $\|y_0\|^2 \leqq R_0^2 = (2R_1 + R/\sqrt{2\pi})^2 \leqq 4R_1^2(1 + 6\varepsilon/4)$ and $\|y_1\| = R_1$. And $\|g(\cdot, x, x')\|^2 \leqq k^2(\|x\|^2 + \|y_1\|^2) \leqq (1/6)(1-\varepsilon)^2(1+\varepsilon)^{-1}\|y_1\|^2(6 + 6\varepsilon) = (1-\varepsilon)^2\|y_1\|^2$. So $|\langle J(I-Q)g(\cdot, x, x'), y_1\rangle| \leqq (1-\varepsilon)\|y_1\|^2$ and $|\langle J(I-Q)N_2(J^*y_1 + y_0), y_1\rangle| < \|y_1\|^2$ for $(y_1, y_0) \in S^1$. The same estimates hold for $|\langle J(I - Q)N_2(J^*(-y_1)-y_0), y_1\rangle|$ and hence (9) does not hold.

6°. Hypothesis (h_2) implies that (10) does not hold. Let $(y_1, y_0) \in S^0$ and $x(t) = x_1(t) + x_0 = J^*y_1 + y_0$. From $x_1(t) = -\int_0^t y_1(t)dt + (1/2\pi)\int_0^{2\pi}$

$\int_0^t y_1(s)dsdt$, it follows that $\sup_{[0,2\pi]}|x_1(t)| \leqq 2\int_0^{2\pi} |y_1(t)|dt \leqq 2\sqrt{2\pi}\,||y_1||$. Hence $|x_1(t) + x_0| \geqq |y_0| - 2\sqrt{2\pi}\,||y_1|| = \sqrt{2\pi}(||y_0|| - 2||y_1||) \geqq R$. Thus, for $(y_1, y_0) \in S^0$ either $g(t, x(t), x'(t)) \geqq A$ or $\leqq B$.

In 5° we showed that $QN_1x = 0$. So $QN(J^*y_1 + y_0) = (1/2\pi) \left[\int_0^{2\pi} g(t, x(t), x'(t))dt - \int_0^{2\pi} e(t)dt\right]$. If (10) holds, then

$$\int_0^{2\pi} \{g(t, x(t), x'(t)) - \mu g(t, -x(t), -x'(t))\}dt = (1 - \mu)(2\pi e_m)\,,$$
$$0 < \mu < 1\,,$$

and this is impossible by (h_2) since the integral is $\geqq A - \mu B$ and $A - \mu B > (1 - \mu)e_m$, $0 < \mu < 1$.

To consider equation (1) with $x = \mathrm{col}(x_1, \cdots, x_n)$, a vector, we change $f(x)x'$ to $(d/dt)[\nabla f(x(t))]$ where $f: R^n \to R$ is of class C^2 and ∇ is the gradient operation. Also g and e are assumed to be vector-valued. Hypothesis (h_1) remains the same while (h_2) is most simply stated as

(h_2') $\int_0^{2\pi} e(t)dt = 0$ and there exists a constant $R > 0$ such that $\sum_1^n x_i^2 \geqq R^2$ and $x_i \geqq R/\sqrt{n}$ implies $x_i g_i(t, x, y) > 0$ for all (t, y) (or < 0 for all (t, y)), $i = 1, \cdots, n$.

The proof is much the same.

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