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# Periodic Solutions of Some Infinite-Dimensional Hamiltonian Systems Associated with Non-Linear Partial Difference Equations I

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**Abstract.** We establish existence of a dense set of non-linear eigenvalues, $E$, and exponentially localized eigenfunctions, $u_E$, for some non-linear Schrödinger equations of the form

$$Eu_E(x) = [(-\Delta + V(x))u_E](x) + \lambda u_E(x)^3,$$

bifurcating off solutions of the linear equation with $\lambda = 0$. The points $x$ range over a lattice, $\mathbb{Z}^d$, $d = 1, 2, 3, \ldots$, $\Delta$ is the finite difference Laplacian, and $V(x)$ is a random potential. Such equations arise in localization theory and plasma physics. Our analysis is complicated by the circumstance that the linear operator $-\Delta + V(x)$ has dense point spectrum near the edges of its spectrum which leads to small divisor problems. We solve these problems by developing some novel bifurcation techniques. Our methods extend to non-linear wave equations with random coefficients.

## 0. Introduction. Motivation, Results, and Basic Ideas

The purpose of this paper is to construct infinitely many time-periodic solutions to some non-linear, partial difference equations which can be viewed as the equations of motion of Hamiltonian systems with infinitely many degrees of freedom. Physically, these systems describe infinite arrays of coupled anharmonic oscillators with the property that when the anharmonic (non-quadratic) terms in the Hamilton function are neglected the frequencies of the oscillators are non-resonant, in a suitably strong sense to be made precise later on. We propose to show that from infinitely many periodic solutions of the unperturbed system of harmonic oscillators periodic solutions of the perturbed system of coupled anharmonic oscillators bifurcate.

The main difficulty encountered in such an attempt is that the spectrum of frequencies of the unperturbed system is dense in some interval $I \subseteq \mathbb{R}$. This makes standard bifurcation techniques inapplicable and has motivated us to develop

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some novel techniques. The basic fact about the unperturbed system that enables us to carry out our construction successfully is that two or more different harmonic motions corresponding to nearly degenerate frequencies involve degrees of freedom in nearly disjoint regions of phase space.

We proceed to discuss some examples of non-linear equations that we are able to analyze. The first example is a non-linear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t}(x,t)=(-\Delta+V(x))\psi(x,t)+W(|\psi|)(x,t)\psi(x,t). \tag{0.1}$$

Here $x$ denotes a point in the simple, (hyper-) cubic lattice $\mathbb{Z}^\nu$, $\nu=1,2,3,\ldots$; $t$, the time variable, is real; $\Delta$ is the finite-difference Laplacian, i.e.

$$(\Delta\psi)(x)=\sum_{y:|y-x|=1}\psi(y). \tag{0.2}$$

$V(\cdot)$ is a random potential, more precisely $\{V(x)\}_{x\in\mathbb{Z}^\nu}$ are independent, identically distributed (i.i.d.) random variables. The distribution, $d\varrho(V)$, of $V\equiv V(x)$ is given e.g. by

$$d\varrho(V)=\sqrt{\frac{1}{\pi\zeta}}\,e^{-V^2/\zeta}dV \tag{0.3}$$

or by

$$d\varrho(V)=\frac{1}{\zeta}\chi_{[-\zeta/2,\zeta/2]}(V)dV, \tag{0.3$^1$}$$

etc., where $\zeta$ is a measure for the disorder in the system. Finally,

$$W(|\psi|)(x)=\sum_{y/\mathbb{Z}^\nu}W(|x-y|)\,|\psi(y)|^2, \tag{0.4}$$

where $W$ is a positive, non-zero, exponentially decreasing function, i.e.:

$$0\leqq W(|x-y|)\leqq \text{const}\,e^{-m_W|x-y|} \tag{0.4$^1$}$$

for a certain $m_W>0$.

The linear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}(x,t)=(H^0\psi)(x,t) \tag{0.5}$$

with

$$H^0=-\Delta+V$$

was introduced by Anderson [1] to model the dynamics of a quantum mechanical particle, the electron, moving in a disordered (random) background. This model is important in the theory of electrical conductivity in disordered metals. The long-time behaviour of the wave function of the electron may be characterized by its spread

$$r^2(t)=\sum_{x\in\mathbb{Z}^\nu}x^2|(e^{-itH^0}\psi)(x)|^2. \tag{0.6}$$

Anderson argued [1] that if the disorder, $\zeta$, in the distribution of the potential $V$ is large enough, or if the energy of $\psi$ is close to the edges of the spectrum of $H^0$, and if

$\psi(x)$ is rapidly decreasing, then

$$r^2(t) \leqq \text{const} < \infty \tag{0.7}$$

for all time. In one dimension ($\nu = 1$), (0.7) is known to hold for arbitrary disorder $\zeta > 0$ and all energies [2]. In higher dimensions ($\nu \geqq 2$), (0.7) was proven in [3], for $\zeta$ large, or for energy ranges close to the edges of the spectrum of $H^0$. In fact, much more is known: For $\nu = 1$, the spectrum, $\sigma(H^0)$, of $H^0$ is simple, dense and pure-point for arbitrary $\zeta > 0$, [2]. For $\nu \geqq 2$,

$$\sigma(H^0) \cap [(-\infty, -E_*(\zeta)) \cup (E_*(\zeta), \infty)] \tag{0.8}$$

consists of simple, dense and pure-point spectrum [4, 5]. Here $0 \leqq E_*(\zeta) < \infty$, and if $\zeta$ is large enough, the entire spectrum of $H^0$ is simple, dense and pure-point.

The eigenfunctions of $H^0$ associated with eigenvalues in the set (0.8) are decaying exponentially fast in $|x|$.

These properties can be used to prove (0.7). They are interpreted, physically as *localization*: If one prepares an electron with sufficiently small energy in some bounded region of a disordered background it will stay inside roughly the same region for all time.

The problem with Anderson's model is that it completely ignores the interactions between different electrons. Suppose $\{\psi_{z_i}(x)\}$ are eigenfunctions of $H^0$ localized near points $z_i \in \mathbb{Z}^\nu$, where $\{z_i\}$ is an infinite array of points of positive density, $\varrho^0$. (That such a family of eigenfunctions of $H^0$ exists, for almost every random potential $V$, is one of the basic results of [4].)

We recall that electrons are particles with spin $\frac{1}{2}$ obeying Pauli's exclusion principle. Thus, given a density, $\varrho < 2\varrho^0$, of electrons, we can fill every eigenfunction ("orbital") $\psi_{z_i}$ of $H^0$ with zero, one or two electrons, as long as electron-electron interactions are neglected. Suppose now that some orbital $\psi_{z_0}$ is filled with two electrons with anti-parallel spins. We now gradually turn on all electron-electron interactions and ask whether the electrons, initially in state $\psi_{z_0}$, remain localized near $z_0$. It is clear, intuitively, and has been verified in special situations in an approximate treatment [6], that the electrons in other orbitals $\psi_{z_i}$, $i \neq 0$, *enhance the localization* of the two electrons in $\psi_{z_0}$. However, the repulsion between the two electrons initially in $\psi_{z_0}$ tends to *delocalize* them. The electron-electron repulsion is described by a Coulomb potential. However, in a solid the Coulomb potential tends to be screened, and we therefore describe the electron-electron repulsion by a potential, $W$, of exponential decay. In order to study the delocalizing effect of the electron-electron repulsion on the electrons initially in $\psi_{z_0}$, we consider a Hartree approximation [7]. The state of the two electrons, after the electron-electron repulsion has been turned on, is described by a symmetric wave function, $u(x_1)u(x_2)$, where $u(x)$ solves the non-linear eigenvalue problem corresponding to (0.1),

$$(-\Delta + V(x) + \lambda W(u)(x))u(x) = Eu(x) \tag{0.9}$$

and $u$ can be chosen to be real, with

$$\|u\|_2^2 = \sum_{x \in \mathbb{Z}^\nu} (u(x))^2 = 1 \,. \tag{0.10}$$

$\lambda$ is proportional to the square of the electric charge of the electron.

One purpose of this paper is to construct solutions, $u_\lambda$, of (0.9) corresponding to some non-linear eigenvalues $E_\lambda$ and obeying (0.10), with the property that

$$\lim_{\lambda \downarrow 0} \|u_\lambda - \psi_{z_0}\|_z = 0 \tag{0.11}$$

and

$$\lim_{\lambda \downarrow 0} E_\lambda = E_0, \tag{0.12}$$

where $E_0$ is the eigenvalue of $H^0$ corresponding to the eigenfunction $\psi_{z_0}$. This result is interpreted, physically, as supporting the idea that Anderson localization, as described by (0.7) and (0.8), is stable under turning on electron-electron repulsion.

Another physical motivation for the study of a non-linear Schrödinger equation similar to (0.1), but with $\lambda < 0$, is found in plasma physics. If $\psi$ is replaced by the electric field **E**, and $W$ has finite range, e.g. $W(x-y) = \delta_{xy}$, then (0.1) is a limiting case of the *Zakharov equations* describing the propagation of the electric field through a plasma, in the presence of a disordered background [8]. Our methods can be used to construct an infinity of stationary (standing wave) solutions for that system.

The interpretation of (0.1) as a Hamiltonian mechanical system and further physical applications of the non-linear Schrödinger equation, e.g. to *classical spin wave theory*, have been discussed, for example, in [9].

We proceed to discussing a second example. Consider a non-linear wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - [(\Delta - \Omega_0^2)u](x,t) + W(u)(x,t) = 0, \tag{0.13}$$

where $u$ is now a *real* function on $\mathbb{Z}^\nu$, $\Delta$ is still given by (0.2), $\Omega_0^2 = \Omega_0^2(x)$ is a multiplication operator such that $\{\Omega_0^2(x)\}_{x \in \mathbb{Z}^\nu}$ are i.i.d. random variables, and $\Omega_0^2 = M + 2\nu$ has distribution

$$d\varrho(M) = N\theta(M) e^{-|M|^\alpha/\zeta} dM, \qquad \alpha \geqq 1, \tag{0.14}$$

where $N$ is chosen such that $\int d\varrho(M) = 1$ or a distribution similar to the one in $(0.3^1)$. [Here $\theta(M) = 0$, for $M < 0$, $\theta(M) = 1$, for $M \geqq 0$.

$$W(u)(x) = \sum_y W(|x-y|) u(y)^2, \tag{0.15}$$

where $W(|x|)$ decays exponentially in $|x|$.]

Clearly, (0.13) are the equations of motion for an infinite array of coupled, anharmonic oscillators. Our purpose is to construct time-periodic solutions to (0.13), using the following ansatz:

$$u(x,t) = (|\partial_t^2|^{-1/2} v)(x, \omega t), \tag{0.16}$$

where $\omega > 0$ and $v$ belongs to $L^2(\mathbb{Z}^\nu \times [-\pi, \pi])$, i.e.

$$\int_{-\pi}^{\pi} dt \left( \sum_{x \in \mathbb{Z}^\nu} v(x,t)^2 \right) = 1. \tag{0.17}$$

Moreover, $v$ is required to be *odd* and periodic in $t$, with period $2\pi$, i.e.

$$v(x,-t)=-v(x,t)\,,\qquad v(x,-\pi)=v(x,\pi)=0\,. \tag{0.18}$$

Finally, $\partial_t^2$ is the Laplacian with periodic boundary conditions on the space $L^2([-\pi,\pi])$. Its odd eigenfunctions are given by $\sin(nt)$, $n=1,2,3,\ldots$, corresponding to the eigenvalues $n^2$.

Our ansatz (0.16) will provide us with a periodic solution, $u(x,t)$, of (0.13) with period $\frac{2\pi}{\omega}$ if the function $v$ satisfies the equation

$$\tau[-\Delta+\Omega_0^2+\lambda W(\tau v)]\tau v=\omega^2 v\,, \tag{0.19}$$

for an arbitrary $\lambda>0$, where

$$\tau=|\partial_t^2|^{-1/2}\,. \tag{0.20}$$

Solutions of (0.19) satisfying (0.17) and (0.18) give rise to solutions of (0.13) of period $\frac{2\pi}{\omega}$ which are *odd* in $t$ and *localized* in space, i.e. square-summable over $\mathbb{Z}^\nu$. [We note that by rescaling $v$ in (0.19) we can fix the value of $\lambda$ at 1 at the price of varying

$$\int_{-\pi}^{\pi} dt\left(\sum_{x\in\mathbb{Z}^\nu} v(x,t)^2\right).$$

But we prefer to vary $\lambda$ and impose (0.17).]

Our ansatz (0.16)–(0.18) has converted the original problem of constructing time-periodic solutions to (0.13) into a non-linear eigenvalue problem (0.19) analogous to (0.9). We propose to cope with this problem as well as with (0.9) by using some novel bifurcation techniques. The main goal of this paper is to expose those techniques.

Next, we summarize our main results in the form of several theorems. We begin by recalling the main results for the linear eigenvalue problems

$$(-\Delta+V)u=Eu\,, \tag{0.21}$$

$$(-\Delta+\Omega_0^2)u=\omega^2 u\,, \tag{0.22}$$

underlying (0.9) and (0.19), respectively.

**Theorem L** [4, 5]. *Let $d\varrho$ be as in* (0.3), (0.14), *respectively. Then for every dimension $\nu=1,2,3,\ldots$ and arbitrary $\zeta>0$, there are constants $E_*(\nu,\zeta)$ and $\omega_*(\nu,\zeta)$ such that*

$$\sigma(-\Delta+V)\cap\{E:|E|>E_*(\nu,\zeta)\}$$

*and*

$$\sigma(-\Delta+\Omega_0^2)\cap\{\omega:\omega^2>\omega_*^2(\nu,\zeta)\}$$

*are simple, dense pure-point spectra, with probability* 1 (*w.p.*1) *with respect to $V$, $\Omega_0^2$, respectively.*

*If $\nu=1$, or if $\zeta$ is large enough, then the spectra of $-\Delta+V$ and $-\Delta+\Omega_0^2$ are simple, dense pure-point on* $[-2\nu,2\nu]+\operatorname{supp}d\varrho$, $[0,4\nu]+\operatorname{supp}d\varrho$, *respectively,* w.p.1.

*Remark.* If $d\varrho$ is as in (0.3[1]) Theorem L remains true for $\zeta$ large enough, and for arbitrary $\zeta$ when $v=1$.

For proofs and background material see [3–5]. (The simplicity of the pure-point spectrum of $-\Delta+V$ and of $-\Delta+\Omega_0^2$ has been proven in the second paper quoted in [5].)

It is expected, though not rigorously established, that, for $v\geqq 3$, the portions of the spectra of $-\Delta+V$ and $-\Delta+\Omega_0^2$ in the intervals $(-E_0(v,\zeta),E_0(v,\zeta))$, $[0,\omega_0^2(v,\zeta))$, respectively, are absolutely continuous with $0<E_0(v,\zeta)\leqq E_*(v,\zeta)$. $0<\omega_0(v,\zeta)\leqq\omega_*(v,\zeta)$, for $\zeta$ sufficiently small.

The purpose of this paper is to prove the following non-linear versions of Theorem L.

**Theorem NL 1.** *If $W(|x|)$ decays exponentially in $|x|$, then there exists some constant $E_1(v,\zeta)\geqq E_*(v,\zeta)$ such that for almost every $V$, for every (simple) eigenvalue, $E_0$, of $-\Delta+V$, with $|E_0|>E_1(v,\zeta)$, corresponding to an eigenfunction $u_0(x)$, there are a set $\Lambda\subset\mathbb{R}$ containing $0$ as an accumulation point and a family $(u_\lambda,E_\lambda)_{\lambda\in\Lambda}$ of solutions of the non-linear Schrödinger equation* (0.9), *with $\|u_\lambda\|_2=1$, for all $\lambda\in\Lambda$, such that*

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\Lambda}}\|u_\lambda-u_0\|_2=0\,. \tag{0.23}$$

*The set $\mathscr{E}=\{E_\lambda\}_{\lambda\in\Lambda}$ is a Cantor set with the property that*

$$2\varepsilon_k-l(\mathscr{E}\cap(E_0-\varepsilon_k,E_0+\varepsilon_k))$$

*tends to $0$ faster than any power of $\varepsilon_k$, as $k\to\infty$, for some sequence $\varepsilon_k$ which tends to $0$, as $k\to\infty$. Here $l$ denotes Lebesgue measure.*

*Remark.* The minimal and maximal elements of $\Lambda$ and the Lebesgue measure of $\Lambda$ are not controlled explicitly, because $\Lambda$ is obtained in a not fully constructive way (see Sect. 2).

We have, however, an alternative fully constructive result on the existence of solutions of the non-linear eigenvalue problem (0.9), (0.10) summarized in the following theorem.

**Theorem NL 2.** *Consider the distribution* (0.3), *and suppose that $W(|x|)$ is of finite range. Then there exists a constant $E_1'(v,\zeta)\geqq E_*(v,\zeta)$ such that for every $\lambda>0$ there is a set, $\Omega(\lambda)$, of potentials, $V$, of full measure with the property that for $V\in\Omega(\lambda)$, the non-linear Schrödinger equation* (0.9), (0.10) *has infinitely many solutions, and the corresponding eigenvalues form a dense subset of*

$$\{E:|E|>E_1'(v,\zeta)\}\,.$$

*Remarks.* (1) Our methods presumably enable us to show that the *density of states* of solutions to (0.9) and (0.10) is *strictly positive* on $\{E:|E|>E_1'(v,\zeta)\}$, but we have not checked all the details.

(2) While Theorem 1 shows that from every solution $(u_0,E_0)$ of (0.21) with $|E_0|>E_1(\zeta)$, a solution of (0.9) bifurcates, Theorem 2 shows that when one starts from *certain* solutions of (0.21) one can construct solutions of (0.9) for arbitrarily large values of $\lambda$.

Analogues of Theorem 1 and 2 can be proven for the non-linear eigenvalue problem (0.19), (0.17), with proofs very similar to those of Theorems 1, 2.

**Theorem NL 3.** *If $W(|x|)$ decays exponentially in $|x|$, then there exists a constant $\omega_1(v,\zeta) \geqq \omega_*(v,\zeta)$ such that, for almost every $\Omega_0^2$, for every eigenvalue $\omega_0^2 > \omega_1^2(v,\zeta)$ of $-\Delta + \Omega_0^2$ with the property that*

$$n^2\omega_0^2 \text{ is not an eigenvalue of } -\Delta + \Omega_0^2, \tag{0.24}$$

*for every $n = 2, 3, \ldots$, there are a set $\Lambda \subset \mathbb{R}$ containing 0 as an accumulation point and a family $(v_\lambda, \omega_\lambda)_{\lambda\in\Lambda}$ of solutions to* (0.19), (0.17) *such that*

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\Lambda}} \|v_\lambda - v_0\|_2 = 0, \tag{0.25}$$

*where*

$$v_0(x,t) = u_0(x)\tau^{-1}\sin t, \tag{0.26}$$

*and $u_0$ solves* (0.22) – *or, equivalently $v_0$ solves* (0.19), (0.17), *for $\lambda = 0$ – with $\omega^2 = \omega_0^2$.*

*The set $\mathcal{O} = \{\omega_\lambda\}_{\lambda\in\Lambda}$ is a Cantor set, and there exists a sequence $(\varepsilon_k)$ tending to* 0, *as $k\to\infty$, such that $2\varepsilon_k - l(\mathcal{O}\cap[\omega_0 - \varepsilon_k, \omega_0 + \varepsilon_k])$ tends to* 0 *faster than any power of $\varepsilon_k$.*

Theorem 3 is the exact analogue of Theorem 1. Similar to Theorem 2 there corresponds a result, *Theorem 4*, for the non-linear eigenvalue problem (0.19), (0.17) which holds for all $\lambda > 0$ and whose proof is completely constructive; see Part III.

*Remarks.* (1) Results analogous to Theorems 3 and 4 can also be proven for wave equations of the form

$$\frac{\partial^2 u}{\partial t^2}(x,t) - (\Delta_J u)(x,t) + W(u)(x,t)u(x,t) = 0, \tag{0.13$^1$}$$

where $W$ is as in (0.13), and

$$(\Delta_J u)(x) \equiv \sum_y J_{yx}(u(y) - u(x)), \tag{0.27}$$

where the $J_{yx}$ are positive, symmetric independent random variables, with $J_{yx} = 0$ if $|y-x| > l_0$, for some finite $l_0$. For $\Delta_J$ a result similar to Theorem L is available, and the methods of the present paper can be used to analyze ($0.13^1$).

The construction of a perturbation theory for periodic solutions of continuum equations of the form

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) + \lambda u^3(x,t) = 0, \tag{0.28}$$

where $x \in [-\pi, \pi]$, presents problems due to the density of the spectrum of the d'Alambertian which looks similar to ours. However, it appears that this problem requires an improvement of our methods, because after the transformation (0.16) the eigenvalues of the linear part are all of infinite multiplicity.

(2) We think it would be very interesting to generalize these results to quasi-periodic solutions. In fact, KAM methods tend to work only for finite dimensional hamiltonian systems and the radius of analyticity in the coupling constant of the perturbed tori tends very rapidly to zero as the dimension tends to infinity. On the basis of this work, our guess is that for some class of infinite dimensional Hamiltonian systems it may happen that though the radius of analyticity in the coupling is zero, nonetheless there is a "substantial" set of (real) coupling constants for which the tori still exist. In other words it may be that as we vary the coupling, tori cease to exist and are born again intermittently, an infinity of times. The problems involving quasi-periodic solutions are certainly harder than those analyzed in this paper, though not hopelessly, difficult. Some rather conventional steps in this direction have already made in [9, 10].

(3) The last open problem we would like to mention, would be to establish *non-linear localization* for an equation like the non-linear Schrödinger equation (0.1). The goal would be to show that, for $\zeta$ large enough, solutions $\psi(x,t)$ of (0.1) with initial conditions of compact support have the property that

$$r^2(t) = \sum_{x \in \mathbb{Z}^\nu} |x|^2 |\psi(x,t)|^2$$

grows less than linearly in $t$, as $|t| \to \infty$, corresponding to subdiffusive behaviour, or is bounded by $D(\zeta)|t|$, as $|t| \to \infty$, with $D(\zeta) \to 0$ rapidly, as $\zeta \to \infty$. See also [9] for some discussion of this problem, and [11] for a result in one dimension.

Our work is organized as follows:

*Part I.* Proof of Theorem NL1.

*Section 1.* Strategy for the proof of Theorem NL1.

*Section 2.* The gap set.

*Section 3.* Bounds on the pole-subtracted Green function.

*Section 4.* Proof of Theorem NL1.

*Part II.* Proof of Theorem NL2.

*Part III.* Anharmonic oscillators with random masses.

In the present publication only Part I is contained. The second and third parts will appear elsewhere.

## Part I. Proof of Theorem 1

### 1. Strategy for the Proof of Theorem 1

Let $(u_0, E_0)$ be a solution of the linear eigenvalue problem (0.21), as specified in Theorem 1 [i.e. with $|E_0| > E_1(\nu, \zeta)$]. We propose to prove that, for all $\varepsilon_0 > 0$, there exist a subset $\Lambda$ of the real line containing 0 as an accumulation point and a family $(u_\lambda, E_\lambda)_{\lambda \in \Lambda}$ of solutions of the non-linear eigenvalue problem (n.l.e.v.p.) (0.9), (0.10) such that $\mathscr{E}=\{E_\lambda\}_{\lambda\in A}$ is a Cantor set with the properties specified in Theorem 1, and

$$\|u_\lambda-u_0\|_2<\varepsilon_0, \qquad \forall\lambda\in A. \tag{1.1}$$

This is the contents of Theorem 1.

The formal proof of this result is begun in Sect. 2. Here we sketch the main ideas. Given $u_0$, we choose the origin $0\in\mathbb{Z}^\nu$ in such a way that $u_0$ is localized near 0, in the following sense.

We shall analyze the behaviour of $u_0$, and of $u_\lambda$, inside cubes $\tilde{\Lambda}_j$, $\Lambda_j$, defined by

$$\tilde{\Lambda}_j=\{x:|x|\leqq d_j\}, \qquad \Lambda_j=\{x:|x|\leqq 4d_j\}, \tag{1.2}$$

where $|x|=\max\limits_{1\leqq\alpha\leqq\nu}|x^\alpha|$, and

$$d_j=\exp(\beta(\tfrac{5}{4})^j), \qquad j=1,2,3,\ldots, \tag{1.3}$$

where $\beta>0$ some constant to be fixed later; see also [4]. We define annuli $A_j$ by

$$A_j=\Lambda_{j+1}\backslash\tilde{\Lambda}_j. \tag{1.4}$$

Let $G^0_\Lambda(z;x,y)$ be the Green function of the operator $H^0_\Lambda$ which is the restriction of $-\Delta+V$ to $l_2(\Lambda)$, $\Lambda$ a subset of $\mathbb{Z}^\nu$, with 0 Dirichlet data at the boundary $\partial\Lambda$ of $\Lambda$.

It is shown in Sect. 3 of [4] that, for $j>\bar{k}$, where $\bar{k}$ is a finite random integer depending on $(E_0=E_0(V),V)$, and for $x$ and $y$ in $A_j$, with $|x-y|\geqq\frac{d_{j-1}}{5}$,

$$|G^0_{\Lambda_j}(E_0;x,y)|\leqq\exp[-m'(E_0)|x-y|], \tag{1.5}$$

for some $m'(E_0)>0$; $[m'(E_0)\sim\ln|E_0|$, for $|E_0|$ large$]$. Inequality (1.5) – which is a rather deep fact about the operator $H^0=-\Delta+V$ – has two important consequences:

(i) The eigenfunction $u_0$ has uniform exponential decay outside $\Lambda_{\bar{k}}$,

$$|u_0(x)|\leqq\exp[-m(E_0)|x|], \qquad x\notin\Lambda_{\bar{k}}, \tag{1.6}$$

where $m(E_0)=cm'(E_0)$, for some purely geometrical constant $c>0$.

(ii) Eigenvalues of $H^0$ corresponding to eigenfunctions localized outside $\Lambda_{\bar{k}}$ do not resonate with $E_0$, in the following sense: If $(E,u)$ are such a pair of eigenvalues and eigenfunctions of $H^0$ and if $|E-E_0|\cong e^{-\sqrt{d_j}}$, $j>\bar{k}$, then $u$ is localized in an annulus $A_j$ separated from $\Lambda_{\bar{k}}$ by a distance $\cong d_j-4d_{\bar{k}}\cong d_j$, for $\beta$ large enough.

The dense set of eigenvalues of $H^0$ in an open interval around $E_0$ can thus be grouped in two subsets: Eigenvalues corresponding to eigenfunctions of $H^0$ localized inside $\Lambda_{\bar{k}}$, and eigenvalues corresponding to eigenfunctions localized outside $\Lambda_{\bar{k}}$. The first subset, $E_1,\ldots,E_{n(\bar{k})}$ is finite, and there is a gap $\Delta_{\bar{k}}$ such that

$$|E_i-E_0|\geqq\Delta_{\bar{k}}, \quad \text{for} \quad i=1,\ldots,n(\bar{k}). \tag{1.7}$$

The second subset is infinite, but its elements do not resonate with $E_0$, in the sense made precise in (ii), above.

In order to construct a solution of the n.l.e.v.p. (0.9), (0.10), we shall use a deformation technique in the parameter $\lambda$, yielding n.l. eigenvalues $E_\lambda$, $E_{i,\lambda}$, $i=1,\ldots,n(\bar{k})$, corresponding to $E_0,E_1,\ldots,E_{n(\bar{k})}$ respectively, and we shall keep $|\lambda|$ so small that $\mathrm{Re}\, E_{i,\lambda}$ does not cross $\mathrm{Re}\, E_\lambda$, for all $i$. The eigenvalues in the second subset are dense around $E_0$, but, thanks to property (ii) above, they do not collide with $E_\lambda$, if $\lambda$ is permitted to make excursions into the complex plane, as we shall describe. By taking $\mathrm{Im}\,\lambda$ to 0 at suitable values of $\mathrm{Re}\,\lambda$ (values constituting the set $\Lambda$ introduced in Theorem 1), we shall obtain solutions of the n.l.e.v.p. (0.9), (0.10).

We now make these ideas somewhat more precise: We shall construct a connected, open set

$$P(E_0, V) \subseteq \{(\lambda, \delta) \in \mathbb{R}^2 : \delta > 0\}, \tag{1.8}$$

(see Fig. 1), and a family $(E_{\lambda,\delta}, u_{\lambda,\delta})$ of solutions of the following n.l.e.v.p. which depend smoothly on $(\lambda, \delta) \in P(E_0, V)$:

$$\begin{gathered}[-\Delta + V + (\lambda + i\delta) W(|u_{\lambda,\delta}|)] u_{\lambda,\delta} = E_{\lambda,\delta} u_{\lambda,\delta}, \\ \|u_{\lambda,\delta}\|_2 = 1 .\end{gathered} \tag{1.9}$$

We shall further constrain the solutions $u_{\lambda,\delta}$ of (1.9) to belong to a subset $\mathscr{U}_{\bar k}$ of $l_2(\mathbb{Z}^\nu)$ defined as follows:

$$\mathscr{U}_{\bar k} = \left\{ u : \max_{x \in \Lambda_k} |u(x) - u_0(x)| \leqq \varepsilon_0,\ |u(y)| \leqq e^{-M|y|},\ \forall y \notin \Lambda_{\bar k} \right\}, \tag{1.10}$$

where $M = \min \left\{ \frac{m(E_0)}{2}, \frac{m_W}{2} \right\}$, with $m_W$ the decay rate of $W(|x-y|)$.

In Fig. 1, $\delta_{\bar k} \sim e^{-\sqrt{d_{\bar k}}}$.

We intend to construct solutions to (1.9) along any path $\gamma_1 \cup \gamma_2 \cup \gamma_3 \subset P(E_0, V)$ with the property that its image, $E_{\lambda,\delta}$, $(\lambda, \delta)$ $\gamma_1 \cup \gamma_2 \cup \gamma_3$, has piecewise constant real and imaginary parts, respectively. The important property will be that $\lim_{\substack{\delta \to 0 \\ (\lambda,\delta) \in \gamma_3}} E_{\lambda,\delta} \equiv \bar E$ must be contained in a certain Cantor set $\mathscr{E}$, which we call "*gap set*" and which depends on $(E_0, V)$. Energies in $\mathscr{E}$ will obey certain non-resonance conditions; see Sect. 2. The set $\mathscr{E}$ is the one described in Theorem 1. It will be shown that $\gamma_3 = \{(\lambda(\delta), \delta)\}_{\delta \in I}$, where $I$ is a closed interval on the positive $\delta$-axis containing 0, can be chosen such that $\mathrm{Re}\, E_{\lambda(\delta),\delta} = \bar E$, $\forall \delta \in I$, $\lim_{\delta \downarrow 0} \lambda(\delta) \equiv \bar\lambda \in \mathbb{R}$ exists, $\lim_{\delta \downarrow 0} \mathrm{Im}\, E_{\lambda(\delta),\delta} = 0$, $u_{\lambda(\delta),\delta} \in \mathscr{U}_{\bar k}$, $\forall \delta \in I$, and $\lim_{\delta \downarrow 0} u_{\lambda(\delta),\delta} \equiv u_{\bar\lambda}$ exists.
Finally, $(u_{\bar\lambda}, \bar E)$ is a real solution of (0.9), (0.10).

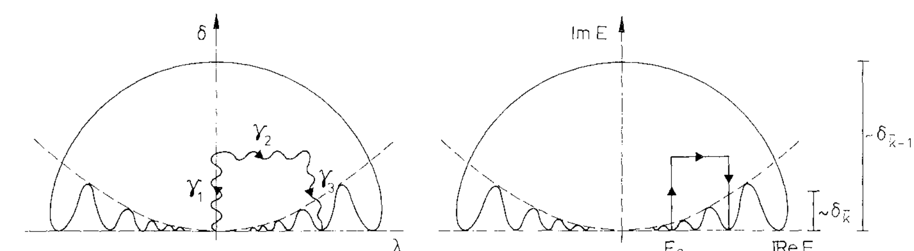


**Fig. 1**

Our construction of $P(E_0, V)$ will guarantee that, *for all* $(\lambda, \delta) \in P(E_0, V)$, $E_{\lambda,\delta}$ is a *simple and isolated eigenvalue* of the linear operator

$$-\Delta + V + (\lambda + i\delta)W(|u_{\lambda,\delta}|). \tag{1.11}$$

In order to construct $P(E_0, V)$ we shall integrate certain differential equations (deformation equations) for $(u_{\lambda,\delta}, E_{\lambda,\delta})$ which will guarantee that (1.11) is valid. We now describe, how these differential equations are obtained. We suppose that $P(E_0, V) \neq \emptyset$. Let $U$ be a sufficiently small open neighborhood of a point $(\lambda_0, \delta_0)$, with $U \subset P(E_0, V)$. The pair $(u_{\lambda,\delta}, E_{\lambda,\delta})$ then solves the n.l.e.v.p. (1.9), for all $(\lambda, \delta) \in U$ if and only if $u_{\lambda,\delta}$ is a fixed point of the following non-linear map:

$$T^{\lambda,\delta}(u) = \frac{c(u;\lambda,\delta)}{2\pi i} \oint_{\mathscr{C}} dz [z - (-\Delta + V + (\lambda + i\delta)W(|u|))]^{-1} u_{\lambda,\delta}. \tag{1.12}$$

Here $\mathscr{C}$ is a small circle in the complex plane which encloses precisely one eigenvalue $E_{\lambda,\delta}(u)$ of the linear operator

$$-\Delta + V + (\lambda + i\delta)W(|u|), \tag{1.13}$$

for every $(\lambda, \delta) \in U$ and every $u \in N$, where $N$ is a sufficiently small neighborhood of $u_{\lambda_0,\delta_0}$ in the sphere

$$S = \{u : \|u\|_2 = 1\}. \tag{1.14}$$

The existence of the neighborhoods $U$ and $N$ and the circle $\mathscr{C}$ follows from property (1.11), by means of analytic perturbation theory [12]. Furthermore $c(u;\lambda,\delta)$ is a normalization constant chosen such that $T^{\lambda,\delta}$ maps $S$ into itself.

The important point is now that $\mathscr{C}$ is independent of $(\lambda, \delta)$ if $U$ and $N$ are chosen small enough. Therefore, the fixed point equation

$$T^{\lambda,\delta}(u) = u \tag{1.15}$$

does *not* contain the n.l. eigenvalue $E_{\lambda,\delta}$ explicitly anymore, in contrast to (1.9).

There are different ways of trying to construct a solution of (1.15); (various fixed point theorems, implicit function theorems). We analyze (1.15) explicitly with the help of a differential equation. Differentiating (1.15) on a solution $u = u_{\lambda,\delta}$ with respect to $\delta$ yields

$$\begin{aligned} \frac{\partial u_{\lambda,\delta}}{\partial \delta} &= T^{\lambda,\delta}_{/\delta}(u_{\lambda,\delta}) + DT^{\lambda,\delta}(u_{\lambda,\delta}) \left[\frac{\partial u_{\lambda,\delta}}{\partial \delta}\right] \\ &= iQK^{\lambda,\delta}(E_{\lambda,\delta})W(|u_{\lambda,\delta}|)u_{\lambda,\delta} \\ &\quad + (\lambda + i\delta)QK^{\lambda,\delta}(E_{\lambda,\delta})DW(|u_{\lambda,\delta}|) \left[\frac{\partial u_{\lambda,\delta}}{\partial \delta}\right] u_{\lambda,\delta}, \end{aligned} \tag{1.16}$$

where

$$Q = 1 - \tfrac{1}{2}P^0_{u_{\lambda,\delta}} - \tfrac{1}{2}\tilde{P}^0_{u_{\lambda,\delta}} \tag{1.17}$$

with

$$P^0_u v = (u, v)u, \qquad \tilde{P}^0_u v = (v, u)u$$

and

$$(u, v) = \sum_{x \in \mathbb{Z}^\nu} \overline{u(x)} v(x) .$$

The operator $Q$ arises from the differentiation of the normalization constant $c(u_{\lambda,\delta}; \lambda, \delta)$ in $\delta$.

Moreover,

$$K^{\lambda,\delta}(z) \equiv [z - (-\Delta + V + (\lambda + i\delta) W(|u_{\lambda,\delta}|))]^{-1} \\ - (z - E_{\lambda,\delta})^{-1} P_{u_{\lambda,\delta}} . \tag{1.18}$$

This operator can be analytically continued to $z = E_{\lambda,\delta}$, for $U, N$ small enough, thanks to property (1.11).

The operator $K^{\lambda,\delta}(z)$ is called the "*(pole-)subtracted Green's function.*"

Finally, $DW$ denotes the ($\mathbb{R}$-linear!) derivative

$$DW(|u|)[v] = \frac{\partial W}{\partial \operatorname{Re} u} [\operatorname{Re} v] + \frac{\partial W}{\partial \operatorname{Im} u} [\operatorname{Im} v] , \tag{1.19}$$

and $DT$ is defined analogously.

All these objects are defined in detail in the Appendix, where the reader also finds a derivation of (1.16).

Suppose now that, for $(\lambda, \delta) \in P(E_0, V)$, the condition

$$\| Q K^{\lambda,\delta}(E_{\lambda,\delta}) DW[v] \| < |\lambda + i\delta|^{-1} \tag{1.20}$$

holds, for $v \perp u_{\lambda\delta}$ with $\|v\|_2 = 1$, where $\|\cdot\|$ is the operator norm of operators acting on $l_2(\mathbb{Z}^\nu)$. Let $K \equiv K^{\lambda,\delta}(E_{\lambda,\delta})$. Then we may rewrite the differential equation (1.16) in normal form

$$\frac{\partial u_{\lambda,\delta}}{\partial \delta} = [1 - (\lambda + i\delta) Q K D W(|u_{\lambda,\delta}|)[\cdot] u_{\lambda,\delta}]^{-1} i Q K W(|u_{\lambda,\delta}|) u_{\lambda,\delta} . \tag{1.21}$$

We remark that the operator which is inverted on the right-hand side is only $\mathbb{R}$-linear, and not $\mathbb{C}$-linear. A similar equation can be derived for $\frac{\partial u_{\lambda,\delta}}{\partial \lambda}$.

The basic problem in proving the bound (1.20) and thus deriving (1.21) is to find suitable bounds on the subtracted Green's function $K^{\lambda,\delta}(E_{\lambda,\delta})$. Such bounds will also be used to prove that $E_{\lambda,\delta}$ is a simple, isolated eigenvalue of

$$-\Delta + V + (\lambda + i\delta) W(|u_{\lambda,\delta}|) ,$$

i.e. to derive condition (1.11). In Sect. 3, we shall establish a bound on the integral kernel, $K^{\lambda,\delta}(x, y)$, of the operator $K^{\lambda,\delta}(E_{\lambda,\delta})$ for functions $u \in U_{\bar{k}}$ and for $|\lambda + i\delta|$ small enough. Our bounds will hold, provided

$$\operatorname{Re} E_{\lambda,\delta} \in (E_0 - \tfrac{1}{2} e^{-\sqrt{d_{\bar{k}-1}}}, E_0 + \tfrac{1}{2} e^{-\sqrt{d_{\bar{k}-1}}}) , \tag{1.22}$$

where $d_j$ has been defined in (1.3), and

$$\left. \begin{array}{ll} \textit{either} & \operatorname{Im} E_{\lambda,\delta} \geqq e^{-\sqrt{d_{\bar{k}}}} \\ \textit{or} & \operatorname{Re} E_{\lambda,\delta} \in \mathscr{G}(\bar{k}, V) , \end{array} \right\} \tag{1.23}$$

where $\mathscr{G}(\bar{k}, V)$ is a gap set to be defined more precisely in Sect. 2. Approximately, the gap set $\mathscr{G}(\bar{k}, V)$ is a Cantor set contained in $\{E:|E|>E_1(\nu,\zeta)\}$, obtained by deleting from $\{E:|E|>E_1(\nu,\zeta)\}$ an interval of length $e^{-d_j^{\gamma}}$ around *every* eigenvalue of $-\Delta+V$ corresponding to an eigenfunction localized in the annulus $A_j$, for all $j \geqq \bar{k}$. The constant $\gamma$ satisfies $0<\gamma\leqq\frac{1}{2}$ and will be defined in Sect. 2. The integer $\bar{k}$ is chosen such that $E_0 \in G(\bar{k}, V)$.

Our construction of solutions $u_{\lambda,\delta}$ to the n.l.e.v.p. (1.9) will proceed by integrating the differential equation (1.21) and a similar equation for $\frac{\partial u_{\lambda,\delta}}{\partial \lambda}$ along a suitable path in the $(\lambda,\delta)$-plane which is obtained imposing conditions on $\operatorname{Re} E_{\lambda,\delta}$, or on $\operatorname{Im} E_{\lambda,\delta}$ (see Fig. 1), respectively. Thereby we shall obtain the set $P(E_0, V)$ mentioned in (1.8). The equations along the first curve $\gamma_1=(\lambda(\delta),\delta)$ are:

$$\begin{cases} \dfrac{d}{d\delta} u_{\lambda(\delta),\delta} = [1-(\lambda(\delta)+i\delta)QKDW[\cdot]u_{\lambda(\delta),\delta}]^{-1} \\ \qquad\qquad \times QK\left(i+\dfrac{d\lambda}{d\delta}\right)Wu_{\lambda(\delta),\delta}, \\ \dfrac{d}{d\delta}\operatorname{Re} E_{\lambda(\delta),\delta}=0. \end{cases} \tag{1.24}$$

The initial conditions for (1.24) are:

$$\lambda(0)=0 \quad \text{and} \quad u_{0,0}=u_0. \tag{1.25}$$

By the choice of $\bar{k}$, and since $\operatorname{Re} E_{\lambda(\delta),\delta}=E_0$ [by (1.30)], condition (1.23) is satisfied along the curve $\gamma_1=(\lambda(\delta),\delta)$. This will enable us to prove bounds on the solution $u_{\lambda(\delta),\delta}$ using estimates on $|K^{\lambda(\delta),\delta}(x,y)|$ established in Sect. 3, provided $u_{\lambda(\delta),\delta}$ remains in the subset $\mathscr{U}_{\bar{k}}$ introduced in (1.10). For such values of $\delta$, the first alternative in (1.23) holds. This enables us to proceed in deforming $u_{\lambda,\delta}$ by varying $\lambda$ and adjusting $\delta$ in such a way that $\operatorname{Im} E_{\lambda(\delta),\delta}$ remains constant; see Fig. 1. As we vary $\lambda$ and $\delta$ in this fashion, $\operatorname{Re} E_{\lambda,\delta}$ turns out to sweep over an interval:

$$(E_0-\varepsilon, E_0+\varepsilon), \quad \text{with} \quad \varepsilon \geqq e^{-c\sqrt{d_{\bar{k}-1}}} \tag{1.26}$$

for some $c>1$.

Defining $\mathscr{E}$ as the intersection of $\mathscr{G}(\bar{k}, V)$ with the interval $(E_0-\varepsilon, E_0+\varepsilon)$, we pick a point $(\lambda,\delta)$ such that $\operatorname{Re} E_{\lambda,\delta} \in \mathscr{G}(\bar{k}, V)$. At this point, we may deform $u_{\lambda,\delta}$ by decreasing $\delta$ towards $\delta=0$ and adjusting $\lambda=\lambda(\delta)$ in such a way that $\operatorname{Re} E_{\lambda,\delta}$ remains constant.

A remark should be added on the initial condition (1.25): Although the right-hand side of Eq. (1.24) makes sense even at $\lambda=\delta=0$, the proof of it may, a priori, fail, because $E_0$ is not isolated. However, if we have a solution of (1.24) with initial conditions (1.25), then $u_{\lambda(\delta),\delta}$ has to be an eigenfunction of (1.9). In fact, if $\delta>0$ $E_{\lambda,\delta}$ turns out to be an isolated eigenvalue of the linear operator (1.11) and the difference

$$u_{\lambda(\delta),\delta}-T^{\lambda(\delta),\delta}(u_{\lambda(\delta),\delta}) \tag{1.27}$$

does not depend on $\delta$. By a limiting argument involving imposing Dirichlet boundary conditions on a box and letting the box $\uparrow \mathbb{Z}^{\nu}$, one can easily see that (1.27) has to vanish, because $u_0$ is an eigenfunction for $\delta=0$.

This concludes our outline of the strategy of the proof of Theorem 1. If the reader finds it somewhat complicated, he or she may find that the mist will lift in the following three sections.

## 2. The Gap Set $\mathscr{G}(\bar{k}, V)$

In this section we construct the gap set $\mathscr{G}(\bar{k}, V)$ introduced after (1.23). Our construction relies on the notions and results of [3, 4].

Following [3] we define a decreasing family of singular sets

$$S_0(E,V) \supseteqq S_1(E,V) \supseteqq S_2(E,V) \supseteqq \ldots, \tag{2.1}$$

where

$$S_0(E,V) = \{x \in \mathbb{Z}^\nu : |V(x) - \operatorname{Re} E| \leqq 2\nu + 2m'(E)\}, \tag{2.2}$$

where $m'(E)$ is a positive function, with $m'(E) \sim \ln|E|$, as $|E| \to \infty$. Simple probability estimates show that if $\zeta$ is large, or $\operatorname{Re} E$ is large, $S_0(E,V)$ is a subset of $\mathbb{Z}^\nu$ of very small density, and $S_0(E,V)$ does not contain an infinite connected cluster.

Let $\Lambda$ be an arbitrary subset of $\mathbb{Z}^\nu$ with the property that $\Lambda \cap S_0(E,V) = \emptyset$, and let $G^0_\Lambda(z; x, y)$ be the Green function of $H^0_\Lambda$. Then simple perturbation theory in $\Delta_\Lambda$ shows that

$$|G_\Lambda(E; x, y)| \leqq \exp(-\tfrac{3}{2} m'(E)\,|x-y|) \tag{2.3}$$

for all $x$ and $y$ in $\Lambda$.

The singular sets $S_k(E,V)$, with $k \geqq 1$, are defined inductively. Given $S_0, \ldots, S_k$, we define $S_{k+1}$ as follows:

$$S_{k+1}(E,V) = S_k(E,V) \setminus \bigcup_\alpha C^\alpha_k, \tag{2.4}$$

where $\{C^\alpha_k\}$ is a *maximal* family of *disjoint* subsets of $S_k(E,V)$ satisfying *Condition k*

(a) $$\operatorname{diam}(C^\alpha_k) \leqq d_k, \tag{2.5}$$

(b) $$\operatorname{dist}(C^\alpha_k, S_k \setminus C^\alpha_k) \geqq 2 d_k^{5/4} = 2 d_{k+1}, \tag{2.6}$$

(c) $$\operatorname{dist}(\sigma(H^0_{\overline{C^\alpha_k}}), \operatorname{Re} E) \geqq e^{-\sqrt{d_k}}, \tag{2.7}$$

where "diam" denotes the diameter of a lattice set,

$$\operatorname{dist}(A, b) = \min_{\substack{a \in A \\ b \in B}} |a - b|, \qquad d_k = \exp[\beta(\tfrac{5}{4})^k],$$

with $\beta > 0$, and $\overline{C^\alpha_k}$ is a lattice set such that

$$\operatorname{dist}(C^\alpha_k, \partial \overline{C^\alpha_k}) \geqq 4 d_k. \tag{2.8}$$

Let $\mathscr{C}_m$ be the collection of lattice cubes with sides parallel to the lattice axes and of length $2^m$ which are centered at the sites of $2^{m-1}\mathbb{Z}^\nu$. We require that

$$\overline{C^\alpha_k} \in C_{n(k)}, \quad \text{for all } \alpha, \tag{2.9}$$

for some $n(k)$ determined by

$$2^{n(k)} \geqq 10 d_k \geqq 2^{n(k)-1}. \tag{2.10}$$

If we must work in a finite region $B$ we may similarly define the sets $S_k(E, V, B)$, $k=0,1,2,\ldots$. See [4] for further details.

*Definition.* A set $A\subset\mathbb{Z}^\nu$ is called $(k,E)$-admissible iff

$$\partial A\cap C_j^\alpha=\phi \tag{2.11}$$

for all $j=0,1,\ldots,k$, and for all $\alpha$.

We recall from [3]

**Lemma 2.1.** *If $\beta$ is sufficiently large then, given arbitrary subsets $D_1$ and $D_2$ of $\mathbb{Z}^\nu$, with $D_2\subset D_1$ and $\operatorname{dist}(D_1,\partial D_2)\geqq 30d_k$, there exists a $(k,E)$-admissible set $R$ with*

$$D_1\subset R\subset D_2\,. \tag{2.12}$$

We will need the following probabilistic estimate proven in [4] (Lemma 3.3): For a given set $A\subset\mathbb{Z}^\nu$ and a finite interval $I$, with $I\cap[-E_*(\zeta),E_*(\zeta)]=\emptyset$, we define

$$\begin{aligned}\mathscr{B}_k(A,I)=\{V:\,&S_{k-1}(E,V,A)\neq\emptyset,\\ &\text{and }\forall c\in\mathscr{C}\ \operatorname{dist}(\sigma(H^0_{c\cap A},E)\\ &\geqq\exp[-d_{k-1}^\gamma],\text{ for some }E\in I\}\,.\end{aligned} \tag{2.13}$$

**Lemma 2.2.** *For $\gamma\in(0,\frac{1}{2}]$ sufficiently small and $|I|\leqq 1$,*

$$\operatorname{Prob}(\mathscr{B}_k(A,I))\leqq|A|d_k^{-\nu-1}\,. \tag{2.14}$$

*Definition 2.3.* The gap set, $\mathscr{G}(k,V)$, of order $k$ is defined as the following set of energies, $E$

$$\begin{aligned}\mathscr{G}(k,V)=\{E\in\mathbb{R}:&|E|>E_*(\zeta),\ \forall c\in\mathscr{C},\\ &\text{and }\forall j\geqq k\ \operatorname{dist}(\sigma(H^0_{c\cap A_j}),E)\geqq 2e^{-d_{j-1}^\gamma}\}\,.\end{aligned} \tag{2.15}$$

The constant $\gamma>0$ is the one introduced in (2.13).

We then have

**Lemma 2.4.** *For almost every $V$ there is a finite integer $k_1=k_1(V)$ such that if* $\operatorname{Re}E\in\mathscr{G}(k_1,V)$ *then*

$$S_{k-1}(E,V,A_{k-1})=\emptyset \tag{2.16}$$

*for every $k\geqq k_1(V)$. (Here $A_j=A_{j+1}\setminus\tilde{A}_j$, $j=1,2,3,\ldots$ are the annular regions introduced in* (1.4).*)*

*Proof.* If (2.16) was incorrect, for some $V$ and some $E$, with $\operatorname{Re}E\in\mathscr{G}(k,V)$, then by (2.13) $V\in\mathscr{B}_k(A_{k-1},I)$, for some $I$ containing $\operatorname{Re}E$. By Lemma 2.2, the probability for this event is

$$\operatorname{Prob}(\mathscr{B}_k(A_{k-1},I))\leqq d_k^{-1}\,.$$

Since

$$\sum_{k=1}^{\infty}\operatorname{Prob}(\mathscr{B}_k(A_{k-1},I))<\infty\,,$$

it follows from the first Borel-Cantelli lemma that, for almost every $V$, there exists some finite integer $k_1(V)$ such that

$$S_{k-1}(E,V,A_{k-1})=\emptyset\,,$$

for all $k\geqq k_1(V)$ if $\operatorname{Re}E\in\mathscr{G}(k_1,V)$.

**Lemma 2.5.** *For almost every V, for every $E_0$ in the pure-point spectrum of $H^0$, with $|E_0| > E_*(\zeta)$, there is a finite integer $k_2 = k_2(E_0, V)$ such that*

$$\operatorname{dist}(E_0, \sigma(H^0_{c\cap A_j})) \geqq 4e^{-d_{j-1}^\gamma} \tag{2.17}$$

*for all $j \geqq k_2(E_0, V)$ and all $c \in \mathscr{C}$.*

*Proof.* We know from Sect. 3 of [4] that w.p.1 there is a finite $k_2(E_0, V)$ such that, for all $j \geqq k_2(E_0, V)$ we have

$$\operatorname{dist}(\sigma(H^0_{c\cap \tilde{A}_j}), E_0) < e^{-d_{j-1}^\gamma} \tag{2.18}$$

for some $c \in \mathscr{C}$ (this follows directly from Lemma 2.2, and Lemma 3.2 of [4]) and

$$\operatorname{dist}(\sigma(H^0_{c\cap \tilde{A}_j}) \cap I,\ \sigma(H^0_{c'\cap A_j})) \geqq 5e^{-d_{j-1}^\gamma}, \tag{2.19}$$

for all $c, c'$ and an interval $I$ centered at $E_0$, with $|I| \leqq 1$. This is shown in (3.10) of [4]. Lemma 2.5 follows from (2.18) and (2.19).

From these lemmas we obtain the following Corollary:

**Corollary 2.6.** *For almost every V and every eigenvalue $E_0$ of $H^0$ with $|E_0| > E_*(\zeta)$ there is a finite integer $k_2 = k_2(E_0, V)$ such that*

$$E_0 \in \mathscr{G}(k_2, V). \tag{2.20}$$

**Lemma 2.7.** *Under the same hypothesis, we have that*

$$l(\mathscr{G}(k, V) \cap (E_0 - e^{-d_j^\gamma}, E_0 + e^{-d_j^\gamma})) = 2e^{-d_j^\gamma} - \text{const}\, d_{j+3}^{2\nu} e^{-d_{j+1}^\gamma},$$

*for some constant independent of j, for arbitrary $j \geqq k \geqq k_2$. (Here l denotes Lebesgue measure.)*

*Proof.* By definition, the gap set $\mathscr{G}(k, V)$ is obtained by excising an interval of length $4e^{-d_{j-1}^\gamma}$ symmetrically around every eigenvalue of $H^0_{c\cap A_j}$, for arbitrary $c \in \mathscr{C}$, with $c \not\supset A_j$, and all $j \geqq k$; see (2.15). Given $j$, the measure of the union of all those intervals is clearly bounded by

$$\sum_{\substack{c:c\cap A_j \neq \emptyset \\ c\not\supset A_j}} \#(\sigma(H^0_{c\cap A_j}))4e^{-d_{j-1}^\gamma} \leqq \sum_{\substack{c:c\cap A_j \neq \emptyset \\ c\not\supset A_j}} |c| 4e^{-d_{j-1}^\gamma} \leqq 4d_{j+1}^{2\nu} e^{-d_{j-1}^\gamma}. \tag{2.21}$$

By Lemma 2.5 we know that

$$\operatorname{dist}(E_0, \sigma(H^0_{c\cap A_m})) \geqq 4e^{-d_{m-1}^\gamma} \tag{2.22}$$

for all $m \geqq k_2$. Now, let $E \in (E_0 - e^{-d_j^\gamma}, E_0 + e^{-d_j^\gamma})$, for some $j \geqq k_2$, but

$$E \notin \mathscr{G}(k, V), \quad \text{for some } k \geqq k_2. \tag{2.23}$$

Then there is some $n \geqq k$ such that

$$\operatorname{dist}(E, \sigma(H^0_{c\cap A_n})) \leqq 2e^{-d_{n-1}^\gamma},$$

for some $c \in \mathscr{C}$, $c \not\supset A_n$. Hence

$$\operatorname{dist}(E_0, \sigma(H^0_{c\cap A_n})) \leqq 2e^{-d_{n-1}^\gamma} + e^{-d_j^\gamma}.$$

By (2.22) it follows that

$$n > \max(j+1, k-1). \tag{2.24}$$

By (2.21) and (2.24), the Lebesgue measure of all $E \in (E_0 - e^{-d_j^\gamma}, E_0 + e^{-d_j^\gamma})$ which do *not* belong to $\mathscr{G}(k, V)$ is thus bounded by

$$\sum_{n > \max(j+1, k-1)} 4d_{n+1}^{2\nu} e^{-d_{n-1}^\gamma} \leqq \mathrm{const}\, d_{j+3}^2 e^{-d_{j-1}^\gamma}, \tag{2.25}$$

for $j \geqq k$.

This completes the proof of Lemma 2.7.

**Lemma 2.8.** *For almost every $V$, for every eigenvalue $E_0$ of $-\Delta + V$, with $|E_0| \geqq E_*(\zeta)$, there is an integer $\bar{k} > 2$ and a sequence $R_n \subset \mathbb{Z}$ of $(\bar{k} - 2 + n)$-admissible sets such that:*

(i)
$$R_n \supset \tilde{\Lambda}_{\bar{k}-2+n} \tag{2.26}$$

*and*

$$\tfrac{1}{5} d_{\bar{k}-2+n} \leqq \mathrm{dist}(\partial R_n, \partial \tilde{\Lambda}_{\bar{k}-2+n}) \leqq \tfrac{2}{5}{}_{\bar{k}-2+n}, \tag{2.27}$$

(ii)
$$\#\{E : E \in \sigma(H_{R_n}^0), \mathrm{dist}(E, E_0) < e^{-\sqrt{d_{\bar{k}-3-n}}}\} = 1, \tag{2.28}$$

(iii)
$$\mathrm{dist}(E_0, \sigma(H_{R_n}^0)) \leqq e^{-\frac{m}{2} d_n}. \tag{2.29}$$

*Proof.* A sequence $R_n$ of $(n-2)$-admissible sets satisfying (2.26) and (2.2) exists, thanks to the Lemma in Appendix D of [3]. Moreover, for $\bar{k}$ large enough and for every $n \geqq 0$, the interval

$$\{E : |E - E_0| < e^{-\sqrt{d_{\bar{k}-3+n}}}\} \tag{2.30}$$

contains at least one eigenvalue of $H_{R_n}^0$ and (2.29) holds. This can be shown by noting that due to the exponential decay of $u_0$ we have

$$\|(H_{R_n}^0 - E_0) 1_{R_n} u_0\|^2 = \|\Gamma 1_{R_n} u_0\|^2 \leqq e^{-2m(E_0) d_n} d_n^\nu.$$

On the other hand, by using the spectral theorem we find

$$\|(H_{R_n}^0 - E_0) 1_{R_n} u_0\|^2 = \sum_{\lambda \in \sigma(H_{R_n}^0)} \varrho(\lambda)(\lambda - E_0)^2$$

with $\varrho(\lambda) \geqq 0$ and $\sum_\lambda \varrho(\lambda) = 1$. Thus we have

$$\mathrm{dist}(E, \sigma(H_{R_n}^0))^2 \leqq \|(H_{R_n}^0 - E_0) 1_{R_n} u_0\|^2 \leqq e^{-2m(E_0) d_n} d_n^\nu.$$

It now suffices to show that there is no subsequence $R_{n(j)}$, $j \in N$, such that the cardinality of the set in (2.28) is $\geqq 2$ for every $R_{n(j)}$. In fact, suppose that the contrary is true. Then there would exist two sequences $(u_{n(j)})$, $(v_{n(j)})$ of eigenfunctions of $H^0(R_{n(j)})$ with eigenvalues in the interval (2.30). By Lemmas 2.4 and 2.5 and the results of [4], $u_{n(j)}$ and $v_{n(j)}$ have uniform exponential decay outside $\tilde{\Lambda}_{\bar{k}-2}$; (1.15)-(1.18) in [4]. From this it follows easily that

$$l^2\text{-}\lim_{j \to \infty} u_{n(j)} = u, \quad l^2\text{-}\lim_{j \to \infty} v_{n(j)} = v$$

exist and are non-zero, and $(u, v) = 0$. But $u$ and $v$ are eigenfunctions of $H^0 = -\Delta + V$ with eigenvalue $E_0$, i.e. $E_0$ is not simple. This contradicts the simplicity of the point spectrum of $H^0$.

The results proven so far all hold for a set $\Omega'$ of potentials, $V$, of full measure, no matter how we choose the origin $0 \in \mathbb{Z}^\nu$; (countable union of sets of measure zero have measure zero!). This permits us to choose, for each $V\in\Omega'$ and each eigenvalue $E_0$ of $-\Delta+V$, with $|E_0|>E_*(\zeta)$, an origin $0\in\mathbb{Z}^\nu$ in such a way that

$$\bar{k}(E_0, V) \text{ is minimal}. \tag{2.31}$$

Finally, we note that the results discussed so far can be extended to linear operators of the form

$$-\Delta+V+(\lambda+i\delta)W(|u|), \tag{2.32}$$

where $u\in\mathcal{U}_{\bar{k}}$, and $\mathcal{U}_{\bar{k}}$ is the set introduced in (1.10), i.e.

$$\mathcal{U}_{\bar{k}}=\left\{u:\max_{x\in\Lambda_{\bar{k}}}|u(x)-u_0(x)|\leqq\varepsilon_0,\ |u(y)|\leqq e^{-M|y|}, \text{ for all } y\notin\Lambda_{\bar{k}}\right\},$$

with $\bar{k}$ as in (2.30), for a given eigenfunction $u_0$ of $-\Delta+V$ with eigenvalue $E_0$.

For the operator (2.32), we introduce sinuglar sets $S_k^{\lambda,\delta,u}(E,V)$, $S_k^{\lambda,\delta,u}(E,V,B)$ as above for $-\Delta+V$, with only one modification: Condition (2.7) is replaced by

$$\operatorname{dist}(\sigma(H^0_{C_k^\alpha}+(\lambda+i\delta)W(|u|)1_{C_k^\alpha}), \operatorname{Re}E)\geqq\tfrac{1}{2}e^{-\sqrt{d_k}}. \tag{2.33}$$

Since, for every $u\in\mathcal{U}_{\bar{k}}$, $W(|u|)(x)$ decays exponentially fast in $|x|$, we can prove the following lemma:

**Lemma 2.9.** *If*

$$|\lambda+i\delta|\leqq\frac{e^{-\sqrt{d_{\bar{k}-1}}}}{2\|W\|}, \tag{2.34}$$

*where* $\|W\|\equiv\sup_{v:\|v\|_2\leqq 1}\|W(|v|)\|$, *and* $\|A\|$ *is the norm of the operator $A$ on* $l_2(\mathbb{Z}^\nu)$, *and if the constant $\beta$ in definition* (1.3) *of the distance scales $d_j$ is chosen large enough, then, for all* $j\geqq\bar{k}(E_0,V)$,

$$S_j^{\lambda,\delta,u}(E,V,A_j)=\phi, \tag{2.35}$$

*for all $E$, with* $\operatorname{Re}E\in\mathcal{G}(\bar{k},V)$ *or* $\operatorname{Im}E\geqq e^{-\sqrt{d_{\bar{k}}}}$, *and every* $u\in\mathcal{U}_{\bar{k}}$.

*Remark.* This lemma permits us to extend Lemma 2.8 to the operator $-\Delta+V+(\lambda+i\delta)W(|u|)$, with

$$u\in\mathcal{U}_{\bar{k}},\qquad |\lambda+i\delta|\leqq(2\|W\|)^{-1}e^{-\sqrt{d_{\bar{k}-1}}},$$

provided we consider a simple eigenvalue $E$ of $-\Delta+V+(\lambda+i\delta)W(|u|)$ with

$$|\operatorname{Re}E-E_0|\leqq e^{-\sqrt{d_{\bar{k}-1}}}\quad\text{and}\quad \operatorname{Re}E\in\mathcal{G}(\bar{k},V)$$

or $\operatorname{Im}E\geqq e^{-\sqrt{d_{\bar{k}}}}$. This observation is the contents of the following lemma:

**Lemma 2.10.** *Let* $u\in\mathcal{U}_{\bar{k}}$ *and let $E$ be a simple eigenvalue of* $-\Delta+V+(\lambda+i\delta)W(|u|)$ *such that either* $\operatorname{Re}E\in\mathcal{G}(\bar{k},V)$, *or* $\operatorname{Im}E\geqq e^{-\sqrt{d_{\bar{k}}}}$. *Let* $H_R\equiv H_R^0+(\lambda+i\delta)W(|u|)1_R$.

*Then there exists a sequence of* $(\bar{k}-2+n)$*-admissible sets,* $(R_n)_{n=0,1,2,\dots}$ *such that*

(i) $R_n\supset\tilde{\Lambda}_{n+\bar{k}-2}$, *and*

$$\tfrac{1}{5}d_n\leqq\operatorname{dist}(\partial R_n,\partial\tilde{\Lambda}_{n+\bar{k}-2})\leqq\tfrac{2}{5}d_n,$$

(ii) $$\# \{E' : E' \in \sigma(H_{R_n}), \operatorname{dist}(E', E) \leqq e^{-\sqrt{d_{\bar{k}-2+n}}}\} = 1,$$

(iii) $$\operatorname{dist}(E, \sigma(H_{R_n})) \leqq e^{-\frac{m}{2} d_n}.$$

The proof of Lemma 2.10 is a straightforward variant of the proof of Lemma 2.8, using Lemma 2.9. It is therefore left to the reader. (Some familiarity with [3, 4] is understood, of course.)

## 3. Bounds on the Pole-Subtracted Green Functions

In this section we fix a pair $(u_0, E_0)$ solving

$$(-\Delta + V)u_0 = E_0 u_0 \tag{3.1}$$

with $V \in \Omega'$ and $|E_0| > E_1(\zeta, v)$. We choose some real $\lambda$ and some $\delta > 0$ such that

$$|\lambda + i\delta| \leqq e^{-\sqrt{d_{\bar{k}-1}}}(2\|W\|)^{-1}, \tag{3.2}$$

with $\bar{k}$ as in (2.30); see (2.34).

We now suppose to have a solution $(E, u)$ of the n.l.e.p. (1.9) with $u \in \mathscr{U}_{\bar{k}}$, where $\mathscr{U}_{\bar{k}}$ is defined in (1.10). Our purpose in this section is to prove an upper bound on the absolute value of the pole-subtracted Green function, $|K^{\lambda,\delta}(x, y)|$, where $K^{\lambda,\delta}(x, y)$ is the kernel of the operator

$$K^{\lambda,\delta}(z) = [z - (-\Delta + V + (\lambda + i\delta)W(|u|))]^{-1} - (z - E)^{-1} P_u, \tag{3.3}$$

($P_u$ is the spectral projection onto $u$) at $z = E$; see (1.19).

Next, we introduce our notations. Let $R_n$ ($n = 0, 1, 2, \ldots$) be the sequence of boxes constructed in Lemma 2.10. Let

$$H_{R_n} = -\Delta_{R_n} + V 1_{R_n} + (\lambda + i\delta)W(|u|)1_{R_n}, \tag{3.4}$$

where $\Delta_{R_n}$ is the finite difference Laplacian with zero Dirichlet data on $\partial R_n$. We define $\Gamma_{\partial R_n}$ by the equation

$$\Delta = \Delta_{R_n} \oplus \Delta_{\sim R_n} + \Gamma_{\partial R_n}. \tag{3.5}$$

If $E_{R_n}$ is the eigenvalue of the operator in (3.4) which is closest to $E$, let

$$\delta E_n \equiv E - E_{R_n} \tag{3.6}$$

and

$$\begin{aligned} \tilde{\Gamma}_n &= \Gamma_{\partial R_n} + \delta E_n 1_{R_n}, \\ \tilde{\Gamma}_n^{n+k} &= \Gamma_{\partial R_n} + \delta E_n 1_{R_n} - \delta E_{n+k} 1_{R_{n+k}}. \end{aligned} \tag{3.7}$$

By Lemma 2.10 we have

$$|\delta E_n| \leqq e^{-\frac{m}{2} d_{\bar{k}-2+n}}. \tag{3.8}$$

Finally, let us define

$$\tilde{H}_{R_n} \equiv H_{R_n} - \delta E_n \tag{3.9}$$

and

$$\tilde{G}_n(z) = (z - \tilde{H}_{R_n})^{-1}, \tag{3.10}$$

$$\tilde{K}_n(z) = \tilde{G}_k(z) - (z - E)^{-1} P_{k_n}, \tag{3.11}$$

where $u$ is the eigenvector of (3.4) which corresponds to $E$. The motivation for the introduction of these modified objects is that in the following it will be convenient to work with Green functions $\tilde{G}_{R_n}$ which have a fixed pole at $z = E$.

**Lemma 3.1.** *We assume that the complex eigenvalue $E$ satisfies the following conditions:*

(1) *$E$ is a simple eigenvalue of $-\Delta + V + (\lambda + i\delta)W(|u|)$ with*

$$|\mathrm{Re}\, E - E_0| \leqq e^{-\sqrt{d_{\bar{k}-1}}}, \tag{3.12}$$

*and*

(2) *either*

$$\mathrm{Re}\, E \in G(\bar{k}, V), \tag{3.13}$$

*or*

$$\mathrm{Im}\, E \geqq e^{-\sqrt{d_{\bar{k}}}}, \tag{3.14}$$

*then, for sufficiently large $\beta$, for every $k \geqq \bar{k} + 1$, and for arbitrary $x$ and $y$ in $\mathbb{Z}^\nu$, we have*

$$|K^{\lambda,\delta}(x, y)| \leqq \begin{cases} e^{2\sqrt{d_{\bar{k}-1}}}, & \text{if} \quad x, y \in \Lambda_{\bar{k}} \\ e^{2\sqrt{d_{k-1}}}, & \text{if} \quad \begin{array}{l} |x - y| \leqq \frac{1}{5} d_{k-1} \\ |y| \leqq d_{k-1} \end{array} \\ e^{-\bar{m}(E)|x-y|}, & \text{otherwise} \end{cases}, \tag{3.15}$$

*for some $\bar{m}(E) > (1/4) m(E_0) > 0$.*

*Proof.* Without loss of generality we may suppose that $|x| \leqq |y|$. Let us define

$$A_n \equiv R_{n+1} \sim R_n, \qquad A_{-1} \equiv R_0 \tag{3.16}$$

and assume that $y \in A_n$, while $x \in A_m$ with $m \in \{-1, 0, \ldots, n\}$.

The proof makes use of the following resolvent identities for the Green functions in (3.10):

$$G(z) = \tilde{G}_n \oplus G_{\sim R_n}(z) + (\tilde{G}_n \oplus G_{\sim R_n}) \tilde{\Gamma}_n G(z). \tag{3.17}$$

In order to derive an expansion for the pole subtracted Green functions $K(x, y; E)$ from (3.17), we use the following integral formula for the constant term of a Laurent series:

$$\tilde{K}_n(x, y; E) = \oint_{\mathscr{C}} \frac{1}{z - E} \tilde{G}_n(x, y; z) \frac{dz}{2\pi i}, \tag{3.18}$$

where $\mathscr{C}$ is a circle which encloses $E$ and no other point of the spectrum of $H+(\lambda+i\delta)W(|u|)$. Iterating (3.17) we find

$$\begin{aligned}\tilde{K}(x,y;E)=&\frac{1}{2\pi i}\oint_{\mathscr{C}_{n+1}}\frac{dz}{z-E}\tilde{G}_{n+1}(x,y;z)\\&+\frac{1}{2\pi i}\oint_{\mathscr{C}_{n+2}}\frac{dz}{z-E}\tilde{G}_{n+1}(x,\cdot)\\&\times\tilde{\Gamma}_{n+1}\tilde{G}_{n+2}(\cdot,y)+\ldots+\frac{1}{2\pi i}\oint_{\mathscr{C}_{n+k+1}}\frac{dz}{z-E}\tilde{G}_{n+1}(x,\cdot)\\&\times\tilde{\Gamma}_{n+1}\tilde{G}_{n+2}\tilde{\Gamma}_{n+2}\times\ldots\times\tilde{\Gamma}_{n+k-1}\tilde{G}_{n+k}\tilde{\Gamma}_{n+k}\tilde{G}_{n+k+1}(\cdot,y)+\ldots,\end{aligned}\tag{3.19}$$

where we used the freedom of deforming the integration path from $\mathscr{C}$ to the circles

$$\mathscr{C}_{n+k}=\left\{z\in\mathscr{C}:|z-E|=\bar{e}^{\sqrt{d_{n+k+1}}}\right\}.\tag{3.20}$$

We remark that thanks to Lemma 2.10 we have

$$\sup_{z\in\mathscr{C}_{n+k+1}}|\tilde{G}_{n+k}(z)|\leqq 2e^{\sqrt{d_{n+k}}}.\tag{3.21}$$

Let us begin by bounding the terms of (3.19) with $k\geqq 2$. Inserting the definition (3.7) of $\tilde{\Gamma}_{n+k}^{n+k+1}$ and $\tilde{\Gamma}_{n+k-1}^{n+k}$ into the $k^{\text{th}}$ term, we get the sum of three terms: the first ends with

$$\delta E_{n+k}1_{R_{n+k}}\tilde{G}_{n+k}\Gamma_{\partial R_{n+k}}\tilde{G}^{n+k+1},\tag{3.22}$$

the second with

$$\tilde{\Gamma}_{n+k-1}^{n+k}\tilde{G}_{n+k}(\delta E_{n+k}1_{R_{n+k}})\tilde{G}_{n+k+1},\tag{3.23}$$

and the third one with

$$\Gamma_{\partial R_{n+k-1}}\tilde{G}_{n+k}\Gamma_{\partial R_{n+k}}\tilde{G}_{n+k+1}.\tag{3.24}$$

Equations (3.22) and (3.23) are small, thanks to (3.21) and the fact that, due to (3.8) we have

$$|\delta E_{n+k-1}|\leqq e^{-\frac{m}{2}d_{n+k-2}},\tag{3.25}$$

and

$$d_{n+k-2}=(d_{n+k})^{(4/5)^2}\gg\sqrt{d_{n+k}}.\tag{3.26}$$

Moreover, (3.24) is small. This can be seen by noticing the following resolvent identities:

$$\tilde{G}_{n+k}=\tilde{G}_{n+k-2}\oplus G_{R_{n+k}\sim R_{n+k-2}}+(\tilde{G}_{n+k-2}\oplus G_{R_{n+k}\sim R_{n+k-2}})\tilde{\Gamma}_{n+k-2}\tilde{G}_{n+k},$$

and

$$\tilde{G}_{n+k}=\tilde{G}_{n+k-2}\oplus G_{R_{n+k}\sim R_{n+k-2}}+\tilde{G}_{n+k}\tilde{\Gamma}_{n+k-2}^{n+k}(\tilde{G}_{n+k-2}\oplus G_{R_{n+k}\sim R_{n+k-2}}).\tag{3.28}$$

If $z \in \mathscr{C}_{n+k+1}$, $v \in \partial R_{n+k-1}$ and $w \in \partial R_{n+k}$, we have

$$\begin{aligned}\tilde{G}_{n+k}(v,w;z) &= G_{R_{n+k} \sim R_{n+k-2}}(v,w;z) \\ &\quad \times [G_{R_{n+k} \sim R_{n+k-2}} \tilde{\Gamma}^{n+k}_{n+k-2} G_{R_{n+k} \sim R_{n+k-2}}](v,w;z) \\ &\quad + [G_{R_{n+k} \sim R_{n+k-2}} \tilde{\Gamma}^{n+k}_{n+k-2} \tilde{G}_{n+k} \tilde{\Gamma}_{n+k-2} G_{R_{n+k} \sim R_{n+k-2}}](v,w;z). \qquad (3.29)\end{aligned}$$

This term is exponentially small thanks to (3.21) and to the exponential decay of $G_{R_{n+k} \sim R_{n+k-2}}(z)$. This last fact is a consequence of Lemma 2.9 and of Theorem 1.2 in [3].

Thus, the sum over all terms with $k \geqq 2$ appearing in the right-hand side of (3.19) can be uniformly bounded by

$$\sum_{k=2}^{\infty} e^{-\frac{m}{2} d_{n+k-2} e^{\sqrt{d_{n+k}}} k} \leqq e^{-\bar{m} d_n} \qquad (3.30)$$

for a suitably defined $\bar{m}$. Moreover, for the term with $k=1$ we have

$$\begin{aligned}|\tilde{\Gamma}^{n+2}_{n+1} \tilde{G}_{n+2}(\cdot, y)| &= |(\Gamma_{\partial R_{n+1}} - \delta E_{n+1} 1_{R_{n+1}} + \delta E_{n+2} 1_{R_{n-2}}) \tilde{G}_{n+2}(\cdot, y)| \\ &\leqq e^{-\frac{m}{2} d_{n-1}} + 2e^{-\frac{m}{2} d_n} e^{\sqrt{d_{n+1}}} \leqq e^{-\bar{m} d_n}. \qquad (3.31)\end{aligned}$$

Hence it is sufficient to control the first term of the expansion (3.19). Now, the first two inequalities in (3.15) are readily proven on the basis of the bound (3.21) on the norm of $\tilde{G}_{n+1}(z)$ for $z \in \mathscr{C}_{k+1}$. Moreover, the third inequality in (3.15) can be verified by using the resolvent identity (3.29) with $k=0$ and, again, the exponential decay estimates contained in [3].

## 4. Proof of Theorem 1

In this section we present our proof of Theorem 1, using the technical estimates collected in Sects. 2 and 3. Our strategy is the one explained in Sect. 1. Thus we start from the differential equations

$$\frac{d}{d\delta} u_{\lambda(\delta),\delta} = [1 - (\lambda(\delta) + i\delta) QKDW[\cdot]u]^{-1} QK \left( i + \frac{d\lambda}{d\delta}(\delta) \right) W u_{\lambda(\delta),\delta}, \qquad (4.1)$$

$$\frac{d}{d\delta} \operatorname{Re} E_{\lambda(\delta),\delta} = 0; \qquad (4.2)$$

see (1.30). Our initial conditions are

$$\lambda(0) = 0, \qquad u_{0,0} = u_0. \qquad (4.3)$$

with $E_{0,0} = E_0$, where $(u_0, E_0)$ is a solution of the linear eigenvalue problem

$$(-\Delta + V) u_0 = E_0 u_0. \qquad (4.4)$$

$E_0$ belongs to the gap set $\mathscr{G}(\bar{k}, V)$ (see Sect. 2), and $u_0$ has uniform exponential decay outside a finite box $\Lambda_k$, as described in (1.6). According to (1.23), we must integrate the system (4.1), (4.2) with initial conditions (4.3) up to values of $\delta$ such that

$$\operatorname{Im} E_{\lambda(\delta),\delta} \geqq e^{-\sqrt{d_{\bar{k}}}}. \qquad (4.5)$$

From that point on, we shall treat $\lambda$ as the independent variable and integrate equations for $\frac{d}{d\lambda} u_{\lambda,\delta(\lambda)}$, with $\frac{d}{d\lambda} \operatorname{Im} E_\lambda = 0$, up to values of $\lambda$ such that $\operatorname{Re} E_{\lambda,\delta(\lambda)} \in \mathscr{G}(\bar{k}, V)$. At that point, $\delta$ will be treated as the independent variable again and will approach 0, with $\operatorname{Re} E_{\lambda(\delta)\delta}$ kept fixed; see Fig. 1, Sect. 1.

We now describe the first of these three steps in detail. The other two steps are completed by very similar arguments. We start by expanding the right-hand side of (4.1) in a geometric series. Dropping subscripts $\lambda$ and $\delta$, we get

$$\frac{du}{d\delta} = \left(i + \frac{d\lambda}{d\delta}\right)\Bigg(QKWu + (\lambda + i\delta) QKDW[\cdot]u \times \left[\sum_{r=0}^{\infty} ((\lambda + i\delta) QKDW[\cdot]u)^r QKWu\right] u\Bigg). \tag{4.6}$$

In order to prove convergence of the series on the right-hand side of (4.6). we bound the operator norm of the operator $QKDW[\cdot]u$ acting on $l^2(\mathbb{Z}^\nu)$. We assume that

$$u \in \mathscr{U}_{\bar{k}} = \Big\{ u : \max_{x \in \Lambda_{\bar{k}}} |u(x) - u_0(x)| \leqq \varepsilon_0, \quad |u(y)| \leqq e^{-\frac{M}{2}|y|} \text{ for all } y \notin \Lambda_{\bar{k}} \Big\},$$

see (1.10). Clearly

$$\|QKDW[\cdot]u\| = \sup_{\|v\|_2 = 1} \|QKDW[v]u\|_2, \tag{4.7}$$

and

$$\begin{aligned} \|QKDW[v]u\|^2 &\leqq 2\|KDW[v]u\|^2 \\ &= 2\|DW[v]\|_\infty^2 \sum_{x,y,\tilde{y}} |K(x,y)K(x,y')u(y)u(y')| \\ &= 2\|DW[v]\|_\infty^2 \sum_x \Big[\sum_y |K(x,y)u(y)|\Big]^2 \\ &\leqq 2\|DW[v]\|_\infty^2 \Big[\sum_{x,y} |K(x,y)u(y))\Big]^2 . \end{aligned} \tag{4.8}$$

Let us write and estimate the sum in the brackets as follows:

$$\begin{aligned} &\left(\sum_{x \in \Lambda_k} + \sum_{k_1 = \bar{k}}^{\infty} \sum_{x \in A_k \setminus \Lambda_k}\right)\left(\sum_{y \in \tilde{\Lambda}_k} + \sum_{k_2 = \bar{k}}^{\infty} e^{-\frac{M}{2} d_k} \sum_{y \in \Lambda_{\bar{k}_2}}\right) |K(x,y)| \\ &\leqq |\Lambda_{\bar{k}}|\,|\tilde{\Lambda}_{\bar{k}}| e^{2\sqrt{d_{\bar{k}} - 1}} + |\Lambda_{\bar{k}}| \sum_{k_2 = \bar{k}} e^{-\frac{M}{2} d_k} |A_k| e^{-\frac{M}{2} d_{\bar{k}}} \\ &\quad + |\tilde{\Lambda}_{\bar{k}}| \sum_{k_1 = \bar{k}} \sum_{x \in \Lambda_{k_1} \setminus \Lambda_{k_1}} e^{-\frac{M}{2}(|x| - d_{\bar{k}})} \\ &\quad + \sum_{k_1 = \bar{k}} \sum_{k_2 = k_1, k_1 \pm 1} e^{-\frac{M}{2} d_{k_2}} |A_{k_1}|\,|A_{k_2}| e^{2\sqrt{d_{k_1}}} \\ &\quad + \sum_{k_1 = \bar{k}} \sum_{k_2 = k_1, k_1 \pm 1} e^{-\frac{m}{2} d_{k_2}} |A_{k_1}|\,|A_{k_2}| e^{-\frac{m}{2}|x - y|} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} C_1(\bar{k}) . \end{aligned} \tag{4.9}$$

Let us remark that $C_1(\bar{k}) \approx d_{\bar{k}}^{2\nu} e^{2\sqrt{d_{\bar{k}-1}}} \ll e^{\sqrt{d_{\bar{k}}}}$. Hence we got the bound

$$\|QKDW[\cdot]u\| \leqq \|DW\| C_1(\bar{k}), \tag{4.10}$$

where

$$\|DW\| \stackrel{\text{def}}{=} \sup_{\|u\|_2 = \|v\|_2 = 1} \|DW(u)[v]\|_\infty . \tag{4.11}$$

Similarly, we can show that for $u \in \mathscr{U}_{\bar{k}}$

$$\|QKWu\|_2 \leqq \|W\| C_1(\bar{k}), \tag{4.12}$$

where

$$\|W\| = \sup_{\|u\|_2 = 1} \|W(u)\|_\infty . \tag{4.13}$$

Next we wish to bound the vector $(QKWu)(x)$ pointwise. By the definition of $Q$,

$$|(QKWu)(x)| \leqq \sum_y |K(x,y)W(y)u(y)| + u(x) \|KWu\|_2 . \tag{4.19}$$

First, if $x \in \Lambda_{\bar{k}}$ and $u \in \mathscr{U}_{\bar{k}}$, we get from (4.18) and (4.19),

$$\begin{aligned} |(QKWu)(x)| \leqq & \sum_{y \in \Lambda_{\bar{k}}} \|W\| \, |K(x,y)| + \sum_{k = \bar{k}} e^{-\frac{m}{2} d_k} |\Lambda_k| \, \|W\| \\ & \times \max_{\substack{x \in \Lambda_{\bar{k}} \\ y \in \Lambda_k \setminus \Lambda_{\bar{k}}}} |K(x,y)| + \|W\| C_1(\bar{k}) \stackrel{\text{def}}{=} \|W\| C_2(\bar{k}) . \end{aligned}$$

Second, if $\frac{5}{3} d_k \leqq |x| \leqq \frac{5}{3} d_{k+1}$, with $k \geqq \bar{k}$, $u \in \mathscr{U}_{\bar{k}}$, we get from our bounds on $|K(x,y)|$, proven in Sect. 3,

$$\begin{aligned} |QKWu(x)| \leqq & \sum_{y \in \Lambda_k} e^{2\sqrt{d_{k-1}}} |W(y)| \, |u(y)| \\ & + \sum_{y \notin \Lambda_k} e^{-\frac{m}{2}|x-y|} |W(y)u(y)| + \|W\| C_1(\bar{k}) e^{-\frac{M}{2}|x|} \\ & \leqq C_3(\bar{k}, W) e^{-\frac{M}{2}|x|} \end{aligned}$$

where

$$C_3(\bar{k}, W) \approx d_{\bar{k}}^{2\nu} e^{2\sqrt{d_{\bar{k}-1}}} \ll e^{\sqrt{d_{\bar{k}}}} .$$

Similarly, using (4.10) one can get a bound of the following form:

$$\begin{aligned} & \left| (\lambda + i\delta) QKDW \left[ \sum_{r=0}^{\infty} ((\lambda + i\delta)) (QKDW[\cdot]u))^r \, QKWu \right] u(x) \right| \\ & \leqq \begin{cases} \|DW\| C_4(\bar{k}) & \text{if} \quad x \in \Lambda_{\bar{k}} \\ C_5(\bar{k}, W) e^{-\frac{M}{2}|x|} & \text{otherwise} \end{cases} \end{aligned} \tag{4.20}$$

for every $\lambda, \delta$ such that

$$|\lambda + i\delta| \leqq \tfrac{1}{2} \min(\|W\|^{-1}, \|DW\|^{-1}) C_1(\bar{k})^{-1} . \tag{4.21}$$

Finally, let us bound $\frac{d\lambda}{d\delta}$. In Appendix 1 we show that the second equation in (1.24) can be written in the following way:

$$\frac{d\lambda}{d\delta}=\left[2\delta(u,W(u)u)\,\mathrm{Im}\left(\frac{\partial u}{\partial\delta},u\right)+2\delta\,\mathrm{Im}\left(\frac{\partial u}{\partial\delta},W(u)u\right)\right.$$
$$\left.+\lambda\left(u,DW\left[\frac{\partial u}{\partial\delta}\right]u\right)\right]\left[(u,W(u)u)-2\delta(u,W(u)u)\right.$$
$$\left.\times\,\mathrm{Im}\left(\frac{\partial u}{\partial\lambda},u\right)+2\delta\,\mathrm{Im}\left(\frac{\partial u}{\partial\lambda},W(u)u\right)+\lambda\left(u,DW\left[\frac{\partial u}{\partial\lambda}\right]u\right)\right]^{-1}. \quad (4.22)$$

One can readapt the estimates which led to the inequalities above, to find

$$\left\|\frac{\partial u}{\partial\delta}\right\|\leqq C_6(\bar{k}),\qquad \left\|\frac{\partial u}{\partial\lambda}\right\|\leqq C_6(\bar{k}), \quad (4.23)$$

where $C_6(\bar{k})\approx d_{\bar{k}-1}^{2\nu}e^{2\sqrt{d_{\bar{k}-1}}}\ll e^{\sqrt{d_{\bar{k}}}}$. Thus if we change our definition of $\lambda$, $\delta$ and $W$ by a constant factor so that:

$$(u_0,W(u_0)u_0)=2, \quad (4.24)$$

and we replace (4.21) by the stronger condition

$$|\lambda+i\delta|\leqq\tfrac{1}{8}\min(\|W\|^{-1},\|DW\|^{-1})\min(C_1(\bar{k})^{-1},C_6(\bar{k})^{-1}), \quad (4.25)$$

we get

$$\left|\frac{d\lambda}{d\delta}\right|\leqq 1. \quad (4.26)$$

We have thus proven the following lemma:

**Lemma 4.1.** *There is a constant* $C_7(\bar{k},W)\approx d_{\bar{k}}^{2\nu}e^{\sqrt{d_{\bar{k}-1}}}\ll e^{\sqrt{d_{\bar{k}-1}}}$ *such that as far as* $u_{\lambda(\delta),\delta}\varepsilon_{\bar{k}}$ *and solves Eq. (1.24), we have*

$$\left|\frac{du_{\lambda(\delta),\delta}}{d\delta}(x)\right|\leqq\begin{cases}C_7 & \text{if } x\in\Lambda_{\bar{k}}\\ C_7e^{-\frac{m}{2}|x|} & \text{otherwise}.\end{cases} \quad (4.27)$$

*In particular, Eq. (1.24) admits a solution* $(\lambda(\delta),u_{\lambda(\delta)\delta})$ *with* $u_{\lambda(\delta),\delta}$ *parametrized by* $\delta\varepsilon[0,\varepsilon_0C_7^{-1}]$. *Finally, if* $\delta\geqq e^{-\sqrt{d_{\bar{k}}}}$ *we have* $\mathrm{Im}\,E_{\lambda(\delta),\delta}\leqq e^{-\sqrt{d_{\bar{k}}}}$.

For what concerns the proof of the last statement, it follows from (4.24), (4.25) and the following calculation:

$$\frac{d}{d\delta}\mathrm{Im}\,E_{\lambda(\delta),\delta}=(u,W(u)u)+2\delta\,\mathrm{Re}\left(\frac{du}{d\delta},W(u)u\right)$$
$$+2\delta\left(u,DW\left[\frac{du}{d\delta}\right]u\right).$$

This completes our analysis. In particular, Theorem NL 1 is now proven. The proofs of Theorems NL 2 through NL 4 will appear in a forthcoming article.

## Appendix

In this appendix, we derive the deformation equations (1.16) and (4.22). We refer to [13] for a general discussion of deformation equations in an abstract setting.

As explained in Sect. 1, our aim is to find solutions of the following fixed point equation:

$$\mathscr{T}^{\lambda,\delta}(u_{\lambda,\delta}) = u_{\lambda,\delta} \tag{A1}$$

on $S = \{u : \|u\|_2 = 1\}$. If $u_{\lambda,\delta}$ is a solution, we can define the map $\mathscr{T}^{\lambda,\delta}(u)$ for $u$ near $u_{\lambda,\delta}$ by

$$\mathscr{T}^{\lambda,\delta}(u) = \frac{c(u,\lambda,\delta;u_0)}{2\pi i} \oint_{\mathscr{C}} \frac{1}{z + \Delta - V - (\lambda + i\delta)W(|u|)} u_0 . \tag{A2}$$

where $u_0$ is a vector not orthogonal to $u_{\lambda,\delta}$ and $c(u,\lambda,\delta;u_0)$ is a vector not orthogonal to $u_{\lambda,\delta}$ and $c(u,\lambda,\delta;u_0)$ is a normalization constant. Note that the operator $\mathscr{T}^{\lambda,\delta}$ is not analytic in $u$. Hence it has derivatives only in the real sense, i.e. in the decomplexified space $l^2(\mathbb{Z};\mathbb{C})^{\mathbb{R}}$. The ($\mathbb{R}$-linear!) operator $D$ of differentiation can be defined as in (1.19). Having to work on the decomplexified space $l^2(\mathbb{Z};\mathbb{C})^{\mathbb{R}}$, let us remark that the imaginary unit $i$ is no more a number in the field of our vector space, but has to be interpreted as the matrix

$$i = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{A3}$$

acting on $l^2(\mathbb{Z},\mathbb{C})^{\mathbb{R}} = l^2(\mathbb{Z};\mathbb{R}) \oplus i l^2(\mathbb{Z};\mathbb{R})$.

To derive (1.16) we have to differentiate $\mathscr{T}^{\lambda,\delta}(u_{\lambda,\delta})$ with respect to $\delta$. We remark that the integrand in (A2) is still analytic with respect to $z$, and so are its derivatives with respect to $\delta$ and $u$.

We have

$$\frac{d}{d\delta}\mathscr{T}^{\lambda,\delta}(u_{\lambda,\delta}) = \mathscr{T}^{\lambda,\delta}_{/\delta}(u_{\lambda,\delta}) + D\mathscr{T}^{\lambda,\delta}(u_{\lambda,\delta})\left[\frac{du_{\lambda,\delta}}{d\delta}\right].$$

The two partial derivatives appearing here can be computed in the same way; hence we shall compute only the second one. If $v$ is a vector we have

$$\begin{aligned} D\mathscr{T}^{\lambda,\delta}(u_{\lambda,\delta})[v] &= (Dc)(u_{\lambda,\delta},\lambda,\delta;u_0)[v] \oint_{\mathscr{C}} \frac{dz}{2\pi i} \frac{1}{z + \Delta - V - (\lambda + i\delta)W(|u_{\lambda,\delta}|)} u_0 \\ &\quad + c(u_{\lambda,\delta},\lambda,\delta;u_0) \oint \frac{dz}{2\pi i} \frac{1}{z + \Delta - V - (\lambda + i\delta)W(|u_{\lambda,\delta}|)} DW[v] \\ &\quad \times \frac{1}{z + \Delta - V - (\lambda + i\delta)W(|u_{\lambda,\delta}|)} u_0 . \end{aligned}$$

One can now make the choice $u_0 = u_{\lambda,\delta}$, so that $c=1$, and

$$(z+\Delta - V - (\lambda + i\delta)W(|u_{\lambda,\delta}|)^{-1}u_0 = (z - E_{\lambda,\delta})u_0 \,.$$

By using the residue theorem, we can calculate the second integral and obtain

$$\oint_{\mathscr{C}} \frac{dz}{2\pi i}\left(K(z) + \frac{P_u}{z - E_{\lambda,\delta}}\right) DW[v] \frac{1}{z - E_{\lambda,\delta}} u_{\lambda,\delta} = K(E_{\lambda,\delta}) DW[v] u_{\lambda,\delta} \,.$$

Finally, it is easy to calculate the derivative of $c(u_{\lambda,\delta}, \lambda, \delta)$ at $u = u_{\lambda,\delta}$. We set $\tilde{\mathscr{T}}^{\lambda,\delta}(n) = c^{-1}\mathscr{T}^{\lambda,\delta}(u)$. Then

$$\begin{aligned} Dc(u,\lambda,\delta)[v] &= D[(\mathrm{Re}\,\tilde{\mathscr{T}}^{\lambda,\delta}(u), \mathrm{Re}\,\tilde{\mathscr{T}}^{\lambda,\delta}(u)) + (\mathrm{Im}\,\tilde{\mathscr{T}}^{\lambda,\delta}(u), \mathrm{Im}\,\tilde{\mathscr{T}}^{\lambda,\delta}(u))]^{-1/2} \\ &= -\frac{1}{2c^{3/2}}[2(\mathrm{Re}\,\tilde{\mathscr{T}}^{\lambda,\delta}(u), \mathrm{Re}\,D\tilde{\mathscr{T}}^{\lambda,\delta}(u)[v]) + 2(\mathrm{Im}\,\tilde{\mathscr{T}}^{\lambda,\delta}(u), \\ &\qquad \mathrm{Im}\,D\tilde{\mathscr{T}}^{\lambda,\delta}(u)[v])] \,, \end{aligned}$$

and, at $u = u_{\lambda,\delta}$, we have

$$Dc(u_{\lambda,\delta}, \lambda, \delta)u_{\lambda,\delta} = -\tfrac{1}{2}[P^0_{u_{\lambda,\delta}} + \tilde{P}^0_{u_{\lambda,\delta}}] D\tilde{\mathscr{T}}^{\lambda,\delta}(u)[v]|_{u=u_{\lambda,\delta}} \,.$$

This proves (1.16).

Let us now prove that the second equation in (1.24) can be written in the form (4.22). We have

$$\begin{aligned} \frac{d}{d\delta}\mathrm{Re}\,E_{\lambda(\delta),\delta} &= \frac{d}{d\delta}(u_{\lambda(\delta),\delta}, [-\Delta + V + \lambda(\delta)W(|u_{\lambda(\delta),\delta}|)]u_{\lambda(\delta),\delta}) \\ &= \frac{d\lambda}{d\delta}\Bigg[(u_{\lambda,\delta}, W(|u_{\lambda,\delta}|)u_{\lambda,\delta}) \\ &\qquad + 2\,\mathrm{Re}\left(\frac{\partial u_{\lambda,\delta}}{\partial\delta}, \left[-\Delta + V + \lambda W(|u_{\lambda,\delta}|)\right] u_{\lambda,\delta}\right) \\ &\qquad + \lambda\left(u_{\lambda,\delta}, DW\left[\frac{\partial u_{\lambda,\delta}}{\partial\lambda}\right] u_{\lambda,\delta}\right)\Bigg] \\ &\qquad + 2\,\mathrm{Re}\left(\frac{\partial u_{\lambda,\delta}}{\partial\delta}, [-\Delta + V + \lambda W(|u_{\lambda,\delta}|)u_{\lambda,\delta}\right) \\ &\qquad + \lambda\left(u_{\lambda,\delta}, DW\left[\frac{\partial u_{\lambda,\delta}}{\partial\delta}\right] u_{\lambda,\delta}\right). \end{aligned}$$

Omitting the subscripts and using the eigenvalue equation fulfilled by $u_{\lambda,\delta}$ as well as the equations

$$\mathrm{Re}\left(\frac{\partial u}{\partial\lambda}, u\right) = \mathrm{Re}\left(\frac{\partial u}{\partial\delta}, u\right) = 0 \,,$$

we get

$$\frac{d}{d\delta}\operatorname{Re}E=\frac{d\lambda}{d\delta}\left\{(u,Wu)+2\left[-i\delta\left(\frac{\partial u}{\partial\lambda},Wu\right)\right.\right.$$
$$\left.\left.+E\left(\frac{\partial u}{\partial\lambda},u\right)\right]+\lambda\left(u,DW\left[\frac{\partial u}{\partial\lambda}\right]u\right)\right\}$$
$$+2\left[-i\delta\left(\frac{\partial u}{\partial\delta},Wu\right)+E\left(\frac{\partial u}{\partial\delta},u\right)\right]+\lambda\left(u,DW\left[\frac{\partial u}{\partial\delta}\right]u\right)$$
$$=\frac{d\lambda}{\delta\delta}\left\{(u,Wu)+2\delta\operatorname{Im}\left(\frac{\partial u}{\partial\lambda},Wu\right)\right.$$
$$\left.-2\delta(u,Wu)\operatorname{Im}\left(\frac{\partial u}{\partial\lambda},u\right)+\lambda\left(u,DW\left[\frac{\partial u}{\partial\lambda}\right]u\right)\right\}$$
$$+2\delta\operatorname{Im}\left(\frac{\partial u}{\partial\delta},Wu\right)-2\delta(u,Wu)\operatorname{Im}\left(\frac{\partial u}{\partial\delta},u\right)$$
$$+\lambda\left(u,DW\left[\frac{\partial u}{\partial\delta}\right]u\right).\quad \text{Q.E.D.}$$

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## References

1. Anderson, P.W.: Absence of diffusion in certain random lattices. Phys. Rev. **109**, 1492 (1985)
2. Goldsheid, I., Molchanov, S., Pastur, L.: Pure point spectrum of stochastic one-dimensional Schrödinger operators. Funct. Anal. Appl. **11**, 1 (1977)
3. Fröhlich, J., Spencer, T.: Absence of diffusion in the Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. **88**, 151–184 (1983)
4. Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Commun. Math. Phys. **101**, 21–46 (1985)
5. Simon, B., Wolff, O.: Some rigorous results for the Anderson model, Caltech preprint 1985 Delyon, F., Levy, Y., Souillard, B.: Anderson localization for multidimensional systems at large disorder or low energy. Commun. Math. Phys. **100**, 463–470 (1985)
6. Albanese, C.: Hartree equations for narrow band crystals. E.T.H. preprint 1987
7. Slater, J.: Quantum theory of atomic structure, Vols. I and II, New York: McGraw-Hill 1960
8. Zakharov, V.E.: Collapse of Langmuir waves. Sov. Phys. JETP **35**, 908–912
9. Fröhlich, J., Spencer, T., Wayne, E.: Localization in disordered nonlinear dynamical systems. J. Stat. Phys. **42**, 247
10. Belissard, J., Vittot, M.: Preprint, Marseille 1985
11. Devillard, J., Souillard, B.: Polynomially decaying transmission for the nonlinear Schrödinger equation in a Random medium. J. Stat. Phys. **43**, 423
12. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966
13. Albanese, C.: A continuation method for nonlinear eigenvalue problems. J. Funct. Anal. (to appear)

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