Czechoslovak Mathematical Journal

Hana Petzeltová

Periodic solutions of the equation $u_{tt} + u_{xxxx} = \varepsilon f(\cdot, \cdot, u, u_t)$

Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 2, 269–285

Persistent URL: http://dml.cz/dmlcz/101166

Czechoslovak Mathematical Journal 23 (98) 1973, Praha

PERIODIC SOLUTIONS OF THE EQUATION $u_{tt} + u_{xxxx} = \varepsilon f(\cdot, \cdot, u, u_t)$

HANA PETZELTOVÁ, Praha

(Received February 4, 1972)

INTRODUCTION

The purpose of this paper is to prove the existence of 2π-periodic solutions of the equation

$$Lu \equiv u_{tt} + u_{xxxx} = \varepsilon f(\cdot, \cdot, u, u_t) \tag{0.1}$$

$$u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0 \tag{0.2}$$

under the assumption that f is 2π-periodic in t.

The main point in the method of the proof is that the problem is solved in a Banach space A, which can be decomposed into two complementary subspaces B and C, where B is the null space of the operator L and L is boundedly invertible only on C, the complement in A of B. If we denote by P_1, P_2 respectively the projectors of A onto B, C and seek the solution in the form $u = v + w$, where $v \in B$, $w \in C$, then the equation (0.1) is equivalent to the system

$$P_1 F(v + w) = 0 \tag{0.3}$$

$$Lw = \varepsilon P_2 F(v + w) \tag{0.4}$$

where $F(u)(t, x) = f(t, x, u(t, x), u_t(t, x))$.

This method is used in several papers, e.g. in [1], [2], [3], [7], [8], [9], [10] to prove the existence of a solution to the equation (0.1) or to the wave equation. The essential assumption for solving the bifurcation equation (0.3) is

$$f_{u_t} \geqq \gamma > 0 \tag{0.5}$$

or

$$f_u \geqq \gamma > 0 \tag{0.6}$$

in the case that f depends only on t, x, u. The assumption (0.6) is used for solving the bifurcation equation for the wave equation in [2], [3], [7], [9]. HALL [1] and TORELLI [10] found weak solutions of the wave equation under a weaker monotonicity condition on f, which permitted $f_u = 0$ to occur but at the expense of a growth condition on f. In a later paper [8] RABINOWITZ extended the results of [7]. He found classical solutions of the wave equation under the monotonicity condition which permitted $f_u = 0$ and which required no growth condition on f. The existence of periodic solutions of a class of equations

$$u_{tt} + (-1)^p \frac{\partial^{2p}}{\partial x^{2p}} u = \varepsilon f(t, x, u)$$

is proved in [1], [2].

In this paper the problem is solved in a slightly different way than in [2] which allows the function f to be dependent also on u_t under weaker conditions on the smoothness of f.

The general sufficient conditions for the existence of periodic solutions to the equation $Lu = g + \varepsilon f(\cdot, \cdot, u, u_x, u_{xx}, u_t, \varepsilon)$ have been investigated by KRYLOVÁ, VEJVODA [5]. KRYLOVÁ [4] proved the existence of periodic solutions to the equation

$$u_{tt} + \Delta^2 u + cu + u_t + u_t|u_t| = f$$

for n-dimensional Laplace operator.

In Section 1 some properties of the used spaces are established. The necessary and sufficient condition for the existence of a solution to the linear equation is given in Section 2. In Section 3 the nonlinear equation is treated. The special case of the equation (0.1) is investigated in Section 4. The case when f depends only on t, x, u is solved in Section 5.

1. PRELIMINARY

We begin with some notations. Let $I = \langle 0, 2\pi \rangle \times \langle 0, \pi \rangle$, $G = R \times \langle 0, \pi \rangle$. Let D be the set of real valued, 2π-periodic, infinitely differentiable functions on G such that $(\partial^{2k}/\partial x^{2k})\, \varphi(t, 0) = (\partial^{2k}/\partial x^{2k})\, \varphi(t, \pi) = 0$, $k = 0, 1, \ldots$ for $\varphi \in D$.

Denote by A_n the completion of D under the norm

$$\|u\|_n = \left(\int_0^{2\pi} \int_0^{\pi} \left(\left| \frac{\partial^n}{\partial t^n} u(t, x) \right|^2 + \left| \frac{\partial^{2n}}{\partial x^{2n}} u(t, x) \right|^2 \right) \mathrm{d}x \, \mathrm{d}t \right)^{1/2}. \tag{1.1}$$

A_n are Hilbert spaces with the inner product

$$(u, v)_n = \left(\frac{\partial^n}{\partial t^n} u, \frac{\partial^n}{\partial t^n} v \right) + \left(\frac{\partial^{2n}}{\partial x^{2n}} u, \frac{\partial^{2n}}{\partial x^{2n}} v \right)$$

where $(u, v) = \int_0^{2\pi} \int_0^{\pi} u(t, x)\, v(t, x)\, dx\, dt$. For $n = 0$ we shall simply write A, $\|\cdot\|$, $(\cdot, \cdot)$. If k, l are integers, $l > 0$, define the functions e_{kl} by

$$(1.2) \qquad e_{kl}(t, x) = e^{ikt} \sin lx .$$

The functions e_{kl}/π form a complete orthonormal system in the space A. Denote by $\{u_{kl}\}$ the sequence of Fourier coefficients of the function $u \in A$.

$$(1.3) \qquad u_{kl} = (u, e_{kl})$$

By means of integration by parts we get the following lemma.

Lemma 1.1. *The functions* $e^n_{kl} = e_{kl}/\pi(k^{2n} + l^{4n})^{1/2}$ *form a complete orthonormal system in* A_n.

By Parseval's equality and Lemma 1.1 we get using the integration by parts

$$(1.4) \quad \pi^2 \|u\|_n^2 = \sum_{\substack{k=-\infty \\ l=1}}^{\infty} |(u, e^n_{kl})_n|^2 = \sum_{\substack{k=-\infty \\ l=1}}^{\infty} (k^{2n} + l^{4n})\, |(u, e_{kl})|^2 = \sum_{\substack{k=-\infty \\ l=1}}^{\infty} (k^{2n} + l^{4n}) |u_{kl}|^2 .$$

This norm is equivalent to that used in [2] for $n \geqq 0$

$$\|u\|'_n = \sum_{\substack{k=-\infty \\ l=1}}^{\infty} (k^2 + l^4)^n\, |u_{kl}|^2$$

because

$$k^{2n} + l^{4n} \leqq (k^2 + l^4)^n \leqq 2^{n-1}(k^{2n} + l^{4n}) .$$

In the sequel, the following two theorems from [2] will be used.

Theorem 1.1. *If* $u \in A_k$, *then* $D^m_t D^n_x u$ *are continuous when* $k \geqq m + \frac{1}{2}n + 1$ *and*

$$(1.5) \qquad \|D^m_t D^n_x u\|_{\infty} = \sup_I |D^m_t D^n_x u(t, x)| \leqq \text{const}\, \|u\|_k .$$

Let c_0 denote the upper bound of the embedding operator $A_1 \ni u \to u \in C^0$, C^0 being the space of continuous functions.

$$(1.6) \qquad \|u\|_{\infty} \leqq c_0 \|u\|_1 .$$

Let f depend on t, x, u_1, u_2. For $u \in A_1$ denote by $F(u)$

$$(1.7) \qquad F(u)\,(t, x) = f(t, x, u(t, x), u_t(t, x)) .$$

Theorem 1.2. *Let* u *and* $v \in A_n$ *for* $n \geqq 2$ *with* $\|u\|_n$ *and* $\|v\|_n \leqq b$. *Suppose* f *and its derivatives up to order* $2n$ *are continuous and bounded whenever the arguments*

u_i *are bounded. Then there are constants* c_1, c_2 *such that*

$$\|F(u)\|_{n-1} \leqq c_1 \tag{1.8}$$

$$\|f_{u_i}(\cdot, \cdot, u, u_t)\, v_i\|_{n-1} \leqq c_2 \|v\|_n\,, \quad i = 1, 2\,, \quad v_1 = v\,, \quad v_2 = v_t\,. \tag{1.9}$$

Remark 1.1. To prove the condition (1.8) it suffices to suppose $2(n - 1)$ continuous derivatives of f.

Remark 1.2. The assertion (1.9) and the mean value theorem give immediately the relation

$$\|F(u) - F(v)\|_{n-1} \leqq 2c_2 \|u - v\|_n\,. \tag{1.10}$$

Lemma 1.2. *The set* $A_n^r = \{u \in A_n, \|u\|_n \leqq r\}$ *is a closed subset of* A.

Proof. Let $\{u_j\}$ be a sequence of the elements of A_n^r and let $u_j \to u$ in the space A. We prove that $u \in A_n^r$. By the Banach-Saks theorem there is a subsequence $\{v_j\}$ of arithmetic means of $\{u_j\}$ converging strongly to an element $v \in A_n^r$. But $u_j \to u$ in A. Thus $v_j \to u$ in A and so $u = v$. Hence $u \in A_n^r$.

2. THE LINEAR EQUATION

We turn our attention to the existence in A_n of solutions to the equation

$$Lu \equiv u_{tt} + u_{xxxx} = g \tag{2.1}$$

$$u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0\,. \tag{2.2}$$

Definition. We say that $u \in A_n$ is a solution to the problem (2.1), (2.2) for $g \in A_n$, if $(u, L\varphi)_n = (g, \varphi)_n$ for all $\varphi \in D$.

Remark 2.1. The choice of the space D implies immediately that the boundary conditions (2.2) are fulfilled in the weak sense.

Let $u \in A_n$ be a solution of the equation

$$Lu = 0\,. \tag{2.3}$$

It is easily seen that u has the representation

$$u \sim \sum_{k^2 = l^4} u_{kl} e_{kl}\,. \tag{2.4}$$

Denote the set of such solutions by B_n. The coefficients of an element in B_n are zero when $|k| \neq l^2$ and hence for such u the norm reduces to

$$(2.5) \qquad |u|_n^2 = \sum_{l=1}^{\infty} 2l^{4n}|u_{\pm l^2,l}|^2 = 2\int_0^{2\pi}\int_0^{\pi}\left|\frac{\partial^n}{\partial t^n} u(t, x)\right|^2 \mathrm{d}x\, \mathrm{d}t\,.$$

In [2] the following lemma is proved:

Lemma 2.1. *If $u \in B_n$ for $n \geqq 1$ and $2a_1 + a_2 \leqq 2(n-1)$, then $D_t^{a_1}D_x^{a_2}u$ is Hölder continuous with the exponent $\frac{1}{2}$ and*

$$(2.6) \qquad \|D_t^{a_1}D_x^{a_2}u\|_{1/2} \leqq \text{const}\ \|u\|_n$$

where

$$\|u\|_{1/2} = \sup\left\{|u(t+h,\ x+k) - u(t,x)|\,.\,|h^2 + k^2|^{-1/4},\ (t, x, h, k) \in I\right\}.$$

Denote by C_n the complement of B_n in A_n. Then the function $u \in C_n$ has the representation $u \sim \sum_{k^2 \neq l^4} u_{kl}e_{kl}$. Clearly, $(u, v)_n = 0$ for $u \in B_n$, $v \in C_n$.

Lemma 2.2. *The problem* (2.1), (2.2) *has a solution in A_{n+1} if and only if $g \in C_n$. The solution is unique if its component in B_{n+1} is zero. The unique solution is given by $u = Kg$ where K is defined by*

$$(2.7) \qquad K\,g(t, x) \sim \sum_{k^2 \neq l^4} \frac{g_{kl}}{l^4 - k^2}\, e_{kl}(t, x)\,.$$

Finally, $K : C_n \to C_{n+1}$ and

$$(2.8) \qquad \|Kg\|_{n+1} \leqq \|g\|_n\,.$$

Proof. For $g \in C_n$ we have

$$(Kg, L\varphi)_n = \lim_{N\to\infty}\left(\sum_{\substack{l=1\\k=-N}}^{N} \frac{g_{kl}}{l^4 - k^2}\, e_{kl},\ L\varphi\right)_n = \lim_{N\to\infty}\ \Big(\sum_{\substack{l=1\\k=-N}}^{N} g_{kl}e_{kl},\ \varphi\Big)_n = (g, \varphi)_n$$

and so Kg is a solution of (2.1). On the other hand, if u is a solution of (2.1), then for $\varphi \in D \cap B_n$ $(g, \varphi)_n = (u, L\varphi)_n = 0$. This implies that $g \in C_n$. Let $u \in C_n$, $v \in C_n$ be two solutions of (2.1). Then $u - v \in C_n$ and $(u - v, L\varphi) = 0$ for all $\varphi \in D$. Hence $u - v \in B_n$. But $C_n \cap B_n = \emptyset$. Hence $u = v$. Further

$$\|Kg\|_{n+1}^2 = \sum_{k^2 \neq l^4} |g_{kl}|^2\, \frac{k^{2(n+1)} + l^{4(n+1)}}{(l^4 - k^2)^2} \leqq \max \frac{k^2 + l^4}{(l^4 - k^2)^2} \sum_{k^2 \neq l^4} |g_{kl}|^2\,(k^{2n} + l^{4n}) =$$
$$= \max \frac{k^2 + l^4}{(l^4 - k^2)^2}\, \|g\|_n\,.$$

Since $|k| \neq l^2$, we have $(l^4 - k^2)^2 = (l^2 - k)^2 (l^2 + k)^2 \geqq (l^2 + |k|)^2 \geqq l^4 + k^2$. Hence $(k^2 + l^4)(l^4 - k^2)^{-2} \leqq 1$ and (2.8) holds.

We conclude this section with lemma which will be useful in the sequel.

Lemma 2.3. *Let $\varphi \in A$ and $\iint_I \varphi(t, x)\,dx\,dt = 0$. Then for $\psi \in A$ such that $m \leqq \psi \leqq M$ the following estimate holds.*

$$(2.9) \qquad -\tfrac{1}{2}(M - m) \iint_I |\varphi(t, x)|\,dx\,dt \leqq \iint_I (\varphi\psi)(t, x)\,dx\,dt \leqq \leqq \tfrac{1}{2}(M - m) \iint_I |\varphi(t, x)|\,dx\,dt\,.$$

Proof. Denote by $I_1 = \{(t, x) \in I,\ \varphi(t, x) \geqq 0\}$, $I_2 = I - I_1$. Then

$$\iint_I (\varphi\psi)(t, x)\,dx\,dt = \iint_{I_1} + \iint_{I_2} \leqq M \iint_{I_1} \varphi(t, x)\,dx\,dt + m \iint_{I_2} \varphi(t, x)\,dx\,dt =$$

$$= (M - m) \iint_{I_1} \varphi(t, x)\,dx\,dt + m \iint_I \varphi(t, x)\,dx\,dt = \tfrac{1}{2}(M - m) \iint_I |\varphi(t, x)|\,dx\,dt\,.$$

On the other hand

$$\iint_I (\varphi\psi)(t, x)\,dx\,dt = \iint_{I_1} + \iint_{I_2} \geqq m \iint_{I_1} \varphi(t, x)\,dx\,dt + M \iint_{I_2} \varphi(t, x)\,dx\,dt =$$

$$= m \iint_I \varphi(t, x)\,dx\,dt + (M - m) \iint_{I_2} \varphi(t, x)\,dx\,dt =$$

$$= -\tfrac{1}{2}(M - m) \iint_I |\varphi(t, x)|\,dx\,dt\,.$$

Remark 2.2. For $u \in B_1$ we get that $uu_t = \frac{1}{2}(\partial/\partial t)\,u^2$ fulfils the assumptions of Lemma 2.3. In this case we have the estimate

$$-\tfrac{1}{2}(M - m)\,\|u\|\,\|u_t\| \leqq \iint_I (\psi u u_t)(t, x)\,dx\,dt \leqq \tfrac{1}{2}(M - m)\,\|u\|\,\|u_t\|\,.$$

In the space B_1 we have $\|u\| \leqq \|u_t\|$ and $\|u_t\| = \frac{1}{2}\|u\|_1$. Hence

$$(2.10) \qquad -\tfrac{1}{2}(M - m)\,\|u\|_1^2 \leqq 2 \iint_I (\psi u u_t)(t, x)\,dx\,dt \leqq \tfrac{1}{2}(M - m)\,\|u\|_1^2\,.$$

This estimate will be used in the next section.

Now we turn to the problem

$$Lu = \varepsilon f(\cdot, \cdot, u, u_t) \tag{3.1}$$

with the boundary conditions (2.2).

Let P_1, P_2 respectively denote the projectors of A onto B, C. Then the equation (3.1) is equivalent to the system

$$P_1 F(v + w) = 0\,, \tag{3.2}$$

$$Lw = \varepsilon P_2 F(v + w) \tag{3.3}$$

where $v = P_1 u$, $w = P_2 u$. Let us denote

$$B_n^R = \{u \in B_n,\ \|u\|_n \leqq R\}\,, \quad R > 0\,, \tag{3.4}$$

$$C_n^r = \{u \in C_n,\ \|u\|_n \leqq r\}\,, \quad r > 0\,.$$

Theorem 3.1. *Let the function f have continuous derivatives up to the order $2(n-1)$, $n \geqq 2$, which are bounded when the argument is bounded. If the equation* (3.2) *has a unique solution $v \in B_n^R$ for each $w \in C_n^r$ such that $v(w)$ is Lipschitz continuous in w in the norm of the space A_1, then the system* (3.2), (3.3) *has a unique solution in $B_n^R \times C_n^r$ for $\varepsilon \neq 0$ and small.*

Proof. By Lemma 2.2 we can write the equation (3.3) as

$$w = \varepsilon K P_2 F(v(w) + w)\,. \tag{3.5}$$

The operator $Tw = KP_2F(v(w) + w)$ is Lipschitz continuous and maps the set C_n^r into $C_n^{r'}$. If we choose $\varepsilon \neq 0$ small enough, we get that εT is a contraction in the norm of the space A_1 of C_n^r into itself. Hence there is a fixed point of the operator εT in $\bar{C}_n^r$, the closure of C_n^r in the space A_1. But C_n^r is a closed subset of A_n^r and by Lemma 1.2 $\bar{C}_n^r = C_n^r$. Thus there is $w_0 \in C_n^r$ such that $v(w_0) + w_0$ is a solution of the system (3.2), (3.3).

By Theorem 1.2 and (3.5) we get for w_0

$$\|w_0\|_n \leqq \varepsilon c_1\,. \tag{3.6}$$

For $w \in C_n^r$ we prove the existence of the solution of the equation (3.2) in B_n^R. This will be done following Hall [2]. Let $B(j)$, $j = 1, 2, \ldots$ be the spaces

$$B(j) = \{u \in B,\ u(t, x) = \sum_{i=1}^{j} u_{\pm i^2, i} e_{\pm i^2, i}(t, x)\} \tag{3.7}$$

equipped with the norm $\|\cdot\|_2$. Let P_1^j be the projector from A on $B(j)$ and J be the operator on B defined by

$$Ju = \sum_{l=1}^{\infty} \frac{1}{il^2} u_{\pm l_2, l} e_{\pm l^2 l}, \cdot \tag{3.8}$$

For $w \in C_n^r$ we define the operators

$$S_w^j v = JP_1^j F(v + w) \,, \tag{3.9}$$
$$S_w v = JP_1 F(v + w)$$

and we shall prove that there is such $v^j \in B(j)$ that

$$S_w^j v^j = 0. \tag{3.10}$$

Then also $P_1^j F(v + w) = 0$.

Lemma 3.1. *Let f be as in Theorem* 3.1 *and moreover let the following assumptions be fulfilled*:

(i) $$f_{u_t} \geqq \gamma > 0 \quad \text{on} \quad G_1 = G \times \langle -c_0(R + r), c_0(R + r)\rangle^2 \,,$$

(ii) $$\gamma - \tfrac{1}{2}(\sup_{G_1} f_u(t, x, u_1, u_2) - \inf_{G_1} f_u(t, x, u_1, u_2)) = \alpha > 0 \,,$$

(iii) $$\sup_{G_1} |f_t(t, x, u_1, u_2)| < \alpha R \,,$$

then the equations (3.10) *have unique solutions in* $B^R(j) = B(j) \cap B_2^R$.

Proof. For $v \in \partial B^R(j) = \{u \in B(j), \|u\|_2 = R\}$ we have (as $\|v_t\| \leqq R$ and by (2.10))

$$(S_w^j v, v)_2 = 2\left(\frac{\partial}{\partial t} F(v + w), v_{tt}\right) =$$
$$= 2[(f_t, v_{tt}) + (f_u(v_t + w_t), v_{tt}) + (f_{u_t}(v_{tt} + w_{tt}), v_{tt})] \geqq$$
$$\geqq R^2\alpha - R[\sup_{G_1} |f_t(t, x, u_1, u_2)| +$$
$$+ r(\sup_{G_1} |f_u(t, x, u_1, u_2)| + \sup_{G_1} |f_{u_t}(t, x, u_1, u_2)|)] \geqq 0$$

if we choose r small enough. Hence, by the known theorem (see e.g. [11]) there is such $v^j \in B^R(j)$, that (3.10) is fulfilled. As for the uniqueness, if $v_1, v_2 \in B^R(j)$ are two solutions of (3.10), then by the mean value theorem

$$0 = (S_w^j v_1 - S_w^j v_2, v_{1_{tt}} - v_{2_{tt}}) =$$
$$= -(F(v_1 + w) - F(v_2 + w), v_{1_t} - v_{2_t}) \leqq -\frac{\alpha}{2} \|v_1 - v_2\|_1^2 \,.$$

Hence $v_1 = v_2$.

Lemma 3.2. *Let f and v^j be as in Lemma* 3.1. *Then $\|v^j\|_n$ is bounded independently of j.*

Proof. Lemma is proved for $n = 2$. For $n = 3$ we have

$$0 = \left(S_w^j v^j, \frac{\partial^6}{\partial t^6} v^j\right) = -\left(\frac{\partial^2}{\partial t^2} F(v^j + w), \frac{\partial^3}{\partial t^3} v^j\right) =$$

$$= (f_{tt} + 2f_{tu}u_t^j + 2f_{tu_t}u_{tt}^j + 2f_{uu_t}u_t^j u_{tt}^j + f_{uu}u_t^{j2} + f_{u_t u_t}u_{tt}^{j2} + f_u u_{tt}^j + f_{u_t}u_{ttt}^j, v_{ttt}^j)$$

where $u^j = v^j + w$. Since we assume the continuity of the second derivatives of f and by Lemma 3.1 and Theorem 1.1 $\|u_{tt}\| \leqq R$ and $\|u_t\|_\infty \leqq c_0 R$, we have for $w \in C_3^r$

$$\gamma\|v_{ttt}\| \leqq \text{const}\,(1 + \|v_{tt}^2\|)\,. \tag{3.11}$$

Rabinowitz has shown that if $\varphi \in D$, then

$$\|\varphi_t^2\| \leqq a\delta^{1/4}\|\varphi\|_{1/2}\,\|\varphi_{tt}\| + b(\delta) \tag{3.12}$$

provided $\delta^{1/2}\|\varphi\|_{1/2} < \frac{1}{12}$. Here a is a constant and b depends on δ, which itself can be chosen as small as needed. By Lemma 2.1 $\|v_t^j\|_{1/2} \leqq c\|v_t^j\|_1 = c\|v^j\|_2$. Thus if δ is sufficiently small, (3.11), (3.12) combine to prove that $\|v^j\|_3 \leqq$ const. Estimates of $\|v^j\|_n$ for $n > 3$ are now quite evident.

Lemma 3.3. *Let f be as in Lemma* 3.1. *Then there is a unique element $v_0 \in B_n^R$ such that $S_w v_0 = 0$ for $n = 2, 3, \ldots$*

Proof. By Lemma 3.1 and 3.2 there is a unique $v^j \in B(j)$ for each $j = 1, 2, \ldots$ such that $S_w^j v^j = 0$. Further $\{v^j\}$ is bounded in B_n for $n \geqq 2$. By Lemma 2.1 and Theorem 1.1 the assumptions of Arzela's theorem for v_t^j are fulfilled. Hence there is a subsequence, also denoted by $\{v^j\}$, and $v_0 \in A_1$ such that

$$\|v_t^j - v_{0t}\|_\infty \to 0\,. \tag{3.13}$$

In the same way as in Lemma 1.2, with help of Banach-Saks theorem one proves that $v_0 \in B_n^R$.

$$S_w v_0 = S_w v_0 - S_w^j v_0 + S_w^j v_0 - S_w^j v^j \to 0$$

for $j \to \infty$, because $F(v^j + w) \to F(v_0 + w)$ when (3.13) holds.

Lemma 3.4. *Let f be as in Lemma* 3.1. *Then $v(w)$ is Lipschitz continuous in w in the norm of the space B_1.*

Proof. Let $w_1, w_2 \in C_n^r$ and $v_1, v_2 \in B_n^R$ be the corresponding solutions of the equation $S_{w_i} v_i = 0$. Then

$$0 = (S_{w_1}v_1 - S_{w_2}v_2, v_1 - v_2)_1 = 2(F(v_1 + w_1) - F(v_2 + w_2), v_{1t} - v_{2t}) =$$
$$= 2[(f(\cdot, \cdot, v_1 + w_1, v_{1t} + w_{1t}) - f(\cdot, \cdot, v_2 + w_1, v_{1t} + w_{1t}), v_{1t} - v_{2t}) +$$
$$+ (f(\cdot, \cdot, v_2 + w_1, v_{1t} + w_{1t}) - f(\cdot, \cdot, v_2 + w_2, v_{1t} + w_{1t}), v_{1t} - v_{2t}) +$$
$$+ (f(\cdot, \cdot, v_2 + w_2, v_{1t} + w_{1t}) - f(\cdot, \cdot, v_2 + w_2, v_{2t} + w_{1t}), v_{1t} - v_{2t}) +$$
$$+ (f(\cdot, \cdot, v_2 + w_2, v_{2t} + w_{1t}) - f(\cdot, \cdot, v_2 + w_2, v_{2t} + w_{2t}), v_{1t} - v_{2t})] =$$
$$= 2[(f_u(\text{int. pt.})(v_1 - v_2), v_{1t} - v_{2t}) + (f_u(\text{int. pt.})(w_1 - w_2), v_{1t} - v_{2t}) +$$
$$+ (f_{u_t}(\text{int. pt.})(v_{1t} - v_{2t}), v_{1t} - v_{2t}) + (f_{u_t}(\text{int. pt.})(w_{1t} - w_{2t}), v_{1t} - v_{2t})] .$$

By Lemma 2.3

$$2(f_u(\text{int. pt.})(v_1 - v_2), v_{1t} - v_{2t}) \geqq$$
$$\geqq -\tfrac{1}{2}(\sup_{G_1} f_u(t, x, u_1, u_2) - \inf_{G_1} f_u(t, x, u_1, u_2)) \|v_1 - v_2\|_1^2 .$$

Thus

$$\alpha\|v_1 - v_2\|_1^2 \leqq \sup_{G_1} |f_u(t, x, u_1, u_2)| \|w_1 - w_2\| \|v_1 - v_2\|_1 +$$
$$+ \sup_{G_1} |f_{u_t}(t, x, u_1, u_2)| \|w_{1t} - w_{2t}\| \|v_1 - v_2\|_1 .$$

Hence

$$\|v_1 - v_2\|_1 \leqq \frac{1}{\alpha} (\sup_{G_1} |f_u(t, x, u_1, u_2)| + \sup_{G_1} |f_{u_t}(t, x, u_1, u_2)|) \|w_1 - w_2\|_1 .$$

We summarize our results in the next theorem.

Theorem 3.2. *Let the function f fulfil the following assumptions. There are $R > 0$, $r > 0$ such that*

(i) *f has continuous derivatives up to the order $2(n - 1)$ on $G_1 = G \times \langle -c_0(R + r), c_0(R + r)\rangle^2$, c_0 is given by (1.6);*

(ii) $$\frac{\partial^{2k}}{\partial x^{2k}} F(u)(t, 0) = \frac{\partial^{2k}}{\partial x^{2k}} F(u)(t, \pi) = 0 , \quad k = 0, 1, \ldots, n - 1 , \quad u \in A_n ;$$

(iii) $f_{u_t} \geqq \gamma > 0$ *on* G_1;

(iv) $\gamma - \frac{1}{2}(\sup_{G_1} f_u(t, x, u_1, u_2) - \inf_{G_1} f_u(t, x, u_1, u_2)) = \alpha > 0$;

(v) $\sup_{G_1} |f_t(t, x, u_1, u_2)| < \alpha R$.

Then if ε is sufficiently small, (3.1) has a unique 2π-periodic solution in $B_n^R \times C_n^r$ satisfying the conditions (2.2).

Remark 3.1. If f fulfils the assumptions of Theorem 3.2 for $n = 3$, then by Theorem 1.1 we get a classical solution of the problem.

Remark 3.2. Let f depend on $t, x, u, u_t, u_x, u_{xx}$. Then if we substitute the condition (iv) in Theorem 3.2 by the condition

$$\gamma - \tfrac{1}{2}(\sup_{G_2} f_u - \inf_{G_2} f_u) - \sup_{G_2} |f_{u_x}| - \sup_{G_2} |f_{u_{xx}}| = \alpha > 0$$

where $G_2 = G \times \langle -c_0(R + r), c_0(R + r)\rangle^4$, we can prove in the same way the existence of a 2π-periodic solution of the equation

$$Lu = \varepsilon f(\cdot, \cdot, u, u_t, u_x, u_{xx})$$

in the space A_2.

4. THE EQUATION $Lu = \varepsilon[\alpha u_t + f(.,.,u)]$

This problem will be solved in a slightly different way. No assumptions on the behavior of f_u will be needed. We shall prove the following lemma first.

Lemma 4.1. *Let f be continuously differentiable up to the order $2(n - 1)$, $n \geqq 2$. Then the equation*

$$(4.1) \qquad w = \varepsilon K P_2 F(v + w) \quad (F(u) = \alpha u_t + f(\cdot, \cdot, u))$$

has a unique solution $w \in C_n^r$ for each $v \in B_n^R$ such that

$$(4.2) \qquad \|w\|_n \leqq \varepsilon c_1 ,$$

$$(4.3) \qquad \|w(v_1) - w(v_2)\|_1 \leqq \varepsilon K_1 \|v_1 - v_2\|_1 ,$$

$$(4.4) \qquad \|w\|_n \leqq \frac{\varepsilon}{1 - \varepsilon\alpha} (\alpha\|v\|_n + \|f(\cdot, \cdot, v + w)\|_{n-1}) \quad \textit{for} \quad \varepsilon < \frac{1}{\alpha} .$$

Proof. The existence of the solution of (4.1) and the relation (4.2) is proved in the same way as in Theorem 3.1. (4.3) is established by means of the method of successive approximations and Theorem 1.2. Let $v_1, v_2 \in B_1^R$, $u_0 = w(v_2)$, $u_{n+1} = \varepsilon K P_2 F(v_1 + u_n)$. Then if $K_2 < 1$ is the Lipschitz constant of the operator εT, $Tu = KP_2F(v_1 + u)$

$$\|w(v_1) - w(v_2)\|_1 = \lim_{j \to \infty} \|u_j - u_0\|_1 \leqq \frac{1}{1 - K_2} \|u_1 - u_0\|_1 =$$

$$= \frac{1}{1 - K_2} \|\varepsilon K P_2(F(v_1 + w(v_2)) - F(v_2 + w(v_2)))\|_1 \leqq$$

$$\leqq \frac{\varepsilon}{1 - K_2} \|F(v_1 + w(v_2)) - F(v_2 + w(v_2))\| \leqq \varepsilon K_1 \|v_1 - v_2\|_1$$

$$\|w\|_n = \varepsilon\|KP_2F(v + w)\|_n \leqq \varepsilon\|F(v + w)\|_{n-1} \leqq$$
$$\leqq \varepsilon\alpha(\|w\|_n + \|v\|_n) + \varepsilon\|f(\cdot, \cdot, v + w)\|_{n-1}$$
$$\|w\|_n \leqq \frac{\varepsilon\alpha}{1 - \varepsilon\alpha} \|v\|_n + \frac{\varepsilon}{1 - \varepsilon\alpha} \|f(\cdot, \cdot, v + w)\|_{n-1} .$$

This proves (4.4).

Now, it suffices to prove the existence of a solution to the equation $P_2F(v + w(v)) = F(v + w(v))$, or, which is the same, $P_1F(v + w(v)) = 0$. We shall investigate this problem again in the spaces $B(j)$ defined by (3.7), equipped in this case with the norm $\|\cdot\|_1$. As in Section 3, we define the operators S^j on $B(j)$ by

$$S^jv = JP_1^jF(v + w(v)), \quad Sv = JP_1F(v + w(v)) .$$

Then for $v \in B^R(j)$

$$(S^jv, v)_1 = 2(\alpha(v_t + w_t(v)) + f(\cdot, \cdot, v + w(v)), v_t) \geqq$$
$$\geqq \alpha\|v\|_1^2 - \|v\|_1(\alpha\varepsilon c_1 + \sup_{G_3} |f(t, x, u)|)$$

where $G_3 = G \times \langle -c_0(R + r), c_0(R + r)\rangle$, c_0 given by (1.6). If we suppose $\sup\{|f(t, x, u)|, (t, x, u) \in G_3\} < \alpha R$ and choose ε small enough, we get $(S^jv, v) \geqq 0$ on $\partial B^R(j)$. Hence there is such $v^j \in B^R(j)$ that $S^jv^j = 0$.We proceed as in Section 3 and we show that $\|v^j\|_n \leqq$ const independently of j. We have proved that $\|v^j\|_1 \leqq R$. If $\|v^j\|_{n-1} \leqq$ const, then

$$0 = \left(F(v^j + w), \frac{\partial^{2n-1}}{\partial t^{2n-1}} v^j\right) = (-1)^{n-1} \left(\frac{\partial^{n-1}}{\partial t^{n-1}} F(v^j + w), \frac{\partial^n}{\partial t^n} v^j\right) =$$
$$= (-1)^{n-1} \left(\alpha \frac{\partial^n}{\partial t^n} v^j + \alpha \frac{\partial^n}{\partial t^n} w + \frac{\partial^{n-1}}{\partial t^{n-1}} f(\cdot, \cdot, v^j + w), \frac{\partial^n}{\partial t^n} v^j\right)$$
$$\|v^j\|_n \leqq \|w\|_n + \frac{1}{\alpha} \|f(\cdot, \cdot, v^j + w)\|_{n-1} \Rightarrow \left(1 - \frac{\varepsilon\alpha}{1 - \varepsilon\alpha}\right) \|v^j\|_n \leqq$$
$$\leqq c\|f(\cdot, \cdot, v^j + w)\|_{n-1} \leqq \text{const}$$

by Theorem 1.2. Hence $\|v^j\|_n \leqq$ const, if ε is small enough. In the same way as in section 3, with help of (4.3) one proves that for $n \geqq 2$ there is a unique $v_0 \in B_n^R$ such that $Sv_0 = 0$.

Theorem 4.1. *Let us suppose that there are $R > 0, r > 0$ such that f is continuously differentiable up to the order $2(n - 1)$ on G_3,*

$$\frac{\partial^{2k}}{\partial x^{2k}} F(u)(t,0) = \frac{\partial^{2k}}{\partial x^{2k}} F(u) = (t, \pi) = 0 , \quad k = 0, 1, \ldots, n - 1, \ u \in A_n$$

and is 2π-periodic in t. Further let $\sup\{|f(t, x, u)|, (t, x, u) \in G_3\} < \alpha R$. Then there is a unique solution of the problem $Lu = \varepsilon[\alpha u_t + f(\cdot, \cdot, u)]$ with the boundary conditions (2.2) in $B_n^R \times C_n^r$, provided that ε is small enough.

We shall treat the equation

$$(5.1)\qquad L_k u \equiv u_{tt} + \Delta^2 u = \varepsilon f(\cdot, \cdot, u)$$

for the k-dimensional Laplace operator, $k = 1, 2, 3$, on the domain

$$(5.2)\qquad Q_k = R \times \Omega_k\,, \quad \Omega_k = \langle 0, \pi \rangle^k$$

with the boundary conditions

$$(5.3)\qquad u\big|_{\delta Q_k} = 0\,, \quad \frac{\partial^2 u}{\partial x_i^2}\bigg|_{\substack{x_i=0\\ x_i=\pi}} = 0\,, \quad i = 1, \dots, k\,.$$

Let $A_{n,k}$ denote the completion of the set D_k of infinitely differentiable 2π-periodic functions on Q_k, such that

$$\frac{\partial^{2m} u}{\partial x_i^{2m}}\bigg|_{\substack{x_i=0\\ x_i=\pi}} = 0\,, \quad m = 0, 1, 2, \dots$$

in the normes

$$(5.4)\quad \|u\|_{n,k} = \int_{\bar{Q}_k} \left(\left| \frac{\partial^n u(t, x)}{\partial t^n} \right|^2 + \sum_1^k \left| \frac{\partial^{2n} u(t, x)}{\partial x_i^{2n}} \right|^2 \right) \mathrm{d}x\, \mathrm{d}t\,, \quad \bar{Q}_k = \langle 0, 2\pi \rangle \times \Omega_k\,,$$
$$x = (x_1, \dots, x_k)\,.$$

$A_{n,k}$ are Hilbert spaces with the inner products

$$(5.5)\qquad (u, v)_{n,k} = \int_{\bar{Q}_k} \left(\frac{\partial^n u(t, x)}{\partial t^n} \cdot \frac{\partial^n v(t, x)}{\partial t^n} + \sum_1^k \frac{\partial^{2n} u(t, x)}{\partial x_i^{2n}} \cdot \frac{\partial^{2n} v(t, x)}{\partial x_i^{2n}} \right) \mathrm{d}x\, \mathrm{d}t\,.$$

In the same way as in Section 1 one proves that the functions

$$(5.6)\quad e^k_{rs}(t, x) = \left[\frac{\pi^{k+1}}{2^{k-1}} \left(r^{2n} + \sum_1^k s_i^{4n} \right) \right]^{-1/2} e^{irt} \sin s_1 x_1 \dots \sin s_k x_k\,, \quad r \text{ integers}\,,$$
$$s = (s_1, \dots, s_k)\,, \quad s_i > 0 \quad \text{integers}\,,$$

form a complete orthonormal systems in $A_{n,k}$.

We say that $u \in A_{n,k}$ is a solution to the problem $L_k u = g$, (5.3), $g \in A_{n,k}$, if $(u, L\varphi)_{n,k} = (g, \varphi)_{n,k}$ for all $\varphi \in D_k$.

As in Section 3 we shall write $A_{n,k} = B_{n,k} + C_{n,k}$, where $B_{n,k}$ is the null space of the operator L_k and $C_{n,k}$ is the orthogonal complement of $B_{n,k}$ in $A_{n,k}$. Clearly,

$$(5.7)\qquad B_{n,k} = \{ u \in A_{n,k},\ u = \sum_{|r| = \sum_1^k s_i^2} u_{rs} e^k_{rs} \}\,,$$

$$(5.8)\qquad u \in C_{n,k}\,, \quad u = \sum u_{rs} e^k_{rs} \Rightarrow u_{rs} = 0 \quad \text{for} \quad |r| = \sum_1^k s_i^2\,.$$

The equation $L_k u = g$ has a unique solution $u = Kg \in C_{n,k}$ for each $g \in C_{n,k}$, $g = \sum g_{rs} e^k_{rs}$ and for Kg we have

$$Kg = \sum \frac{g_{rs}}{r^2 - (\sum_1^k s_i^2)^2} e^k_{rs}, \tag{5.9}$$

$$\|Kg\|_{n,k} \leqq \|g\|_{n,k}. \tag{5.10}$$

Indeed, $|g_{rs}(r^2 - (\sum_1^k s_i^2)^2)^{-1}| \leqq |g_{rs}|$ for $g \in C_{n,k}$.

Lemma 5.1. *$u \in A_{n,k} \Rightarrow u \in C^0$ and*

$$\|u\|_\infty \leqq b_k \|u\|_{n,k} \quad \text{for} \quad n \geqq 2. \tag{5.11}$$

Proof. Let $u \in A_{n,k}$, $u = \sum u_{rs} e_{rs}$. Then

$$\sum |u_{rs}| = \sum (r^{2n} + \sum_1^k s_i^{4n})^{1/2} |u_{rs}| (r^{2n} + \sum_1^k s_i^{4n})^{-1/2} \leqq$$
$$\leqq \|u\|_{n,k} (\sum (r^{2n} + \sum_1^k s_i^{4n})^{-1})^{1/2} \leqq b_k \|u\|_{n,k}$$

because $r^{2n} + \sum_1^k s_i^{4n} \geqq r^n \prod_1^k s_i^{2n/k}$ and the series $\sum (r^n \prod_1^k s_i^{2n/k})^{-1}$ converges for $n \geqq 2$, $k = 1, 2, 3$.

Lemma 5.2. *$u \in B_{1,k} \Rightarrow u \in C^0$ and*

$$\|u\|_\infty \leqq d_k \|u\|_{1,k}. \tag{5.12}$$

Proof. For functions from $B_{1,k}$ only the coefficients u_{rs}, where $|r| = \sum_1^k s_i^2$ are different from zero. Hence

$$\sum^k |u_{rs}| \leqq \sum^k \frac{1}{r^2} \|u\|_{1,k}, \quad \sum^k \ldots = \sum_{\substack{r=-\infty \\ s_i=1 \\ |r| = \sum_1^k s_i^2}}^{\infty} \ldots,$$

$$\sum^k \frac{1}{r^2} = 2 \sum_{r=1}^{\infty} \frac{D_k(r)}{r^2},$$

where $D_k(r) = \sum_{\sum_1^k s_i^2 = N} 1$, s_i integers. By Randol [13] the following estimate holds:

$A_k(x) = Cx^{k/2} + O(x^{k(k-1)/2(k+1)})$ for $k \geqq 2$, where $A_k(x) = \sum_{\sum s_i^2 \leqq x} 1$, s_i integers. Then

$$(5.13) \qquad D_k(r) = A_k(r) - A_k(r-1) = O(r^{k/2-1}) + O(r^{k(k-1)/2(k+1)})$$

$$D_2(r) = O(r^{1/3}), \quad D_3(r) = O(r^{3/4}).$$

Thus the series $\sum^k 1/r^2$ converge for $k = 1, 2, 3$.

We solve the nonlinear equation under the assumptions

$$(5.14) \qquad f_u \geqq \gamma > 0 \quad \text{on} \quad G_{4,k} = Q_k \times \langle -(d_k R + b_k r), d_k R + b_k r \rangle$$

$$\text{for some } R, \quad r > 0$$

$$(5.15) \qquad \sup_{G_{4,k}} |f_t(t, x, u)| < \gamma R$$

in the same way as in Section 3. We prove that there is a unique $v(w) \in B_{n,k}$ to each $w \in C_{n,k}$, $n \geqq 2$ such that $P_1^k F(v + w) = 0$ and this v is Lipschitz continuous in w in this case in the norm of the space $A_{0,k}$, because f does not depend on the derivatives of u. We work again in the spaces of finite dimension $B_k(j)$

$$(5.16) \qquad B_k(j) = \{u \in B_{0,k}, \ u = \sum_{|r| \leqq j} u_{rs} e_{rs}\}$$

equipped with the norm $\|\cdot\|_{1,k}$. We define the operators $S_w^{j,k} = P_1^{j,k} F(v + w)$, where $P_1^{j,k}$ is the projector of $A_{0,k}$ on $B_k(j)$ and under the assumptions (5.14), (5.15) and $\varepsilon > 0$ small enough we get $(S_w^{j,k} v, v)_{1,k} \geqq 0$ on $\partial B_k^R(j) = \{u \in B_k(j), \|u\|_{1,k} = R\}$. With help of the following two lemmas we prove in the same way as in Section 3 the existence of such $v_0 \in B_{n,k}^R$ that

$$(5.17) \qquad S_w^k v_0 = 0.$$

Lemma 5.3. *If we define $\omega(\delta)$ by*

$$(5.18) \qquad \omega(\delta) = \sup_{v \in B_k^R(j)} \sup_{\substack{(t,x) \in Q_k \\ |h| \leqq \delta}} |u(t + h, x) - u(t, x)|$$

we get $\omega(\delta) \to 0$ for $\delta \to 0$.

Proof.

$$|u(t + h, x) - u(t, x)| = |\textstyle\sum^k u_{rs}(e_{rs}^k(t + h, x) - e_{rs}^k(t, x))| \leqq$$

$$\leqq \sum^k |u_{rs}| |e^{irh} - 1| \leqq 2\|u\|_{1,k} \sum_{r=1}^{\infty} \frac{D_k(r)}{r^2} |e^{irh} - 1| \leqq 2R \sum_{r=1}^{\infty} \frac{D_k(r)}{r^2} |e^{irh} - 1|.$$

The series converges uniformly and its value for $h = 0$ is zero.

Lemma 5.4. *Let u have two continuous derivatives. Then*

$$(5.19)\qquad \int_{Q_k} v_t^4(t,x)\,\mathrm{d}x\,\mathrm{d}t\,(1-6\omega(\delta)) \leqq 3\omega(\delta)\int_{Q_k} v_{tt}^2(t,x)\,\mathrm{d}x\,\mathrm{d}t + C(\delta)$$

where $\omega(\delta)$ is given by (5,18) and $\delta > 0$ can be chosen as small as needed.

Remark. Integrating (5.19) over Ω_k we get an estimate which we use instead of (3.12).

Proof is due to Rabinowitz [7]. Let $\{\eta_i^2\}$ be a finite partition of the unity of $\langle 0, 2\pi \rangle$ by 2π-periodic differentiable functions. Let the norm of the partition be $\leqq \delta$. Integrating by parts and using the periodicity of η_i and v_t, we have

$$\int_0^{2\pi} \eta_i^2 v_t^4\,\mathrm{d}t = -3\int_0^{2\pi} \eta_i^2[v(x,t) - v(x,\tau_i)]\, v_t^2 v_{tt}\,\mathrm{d}t - $$
$$- 2\int_0^{2\pi} \eta_i \eta_{it}[v(x,t) - v(x,\tau_i)]\, v_t^3\,\mathrm{d}t$$

where $\tau_i \in \operatorname{supp} \eta_i^2$. Thus

$$\int_0^{2\pi} \eta_i^2 v_t^4\,\mathrm{d}t \leqq 3\omega(\delta)\int_0^{2\pi}\left(|\eta_i^2 v_t^2 v_{tt}| + |\eta_i \eta_{it} v_t^3|\right)\mathrm{d}t \leqq$$
$$\leqq 3\omega(\delta)\int_0^{2\pi}\left(\eta_i^2 v_t^4 + \eta_i^2 v_{tt}^2 + \frac{(\eta_i\eta_{it})^4}{4\alpha^4} + \alpha^{4/3}\,\frac{v_t^4}{\frac{4}{3}}\right)\mathrm{d}t\,.$$

Here we used the Hölder inequality and $\alpha > 0$ can be chosen arbitrarily. We sum over i. We can assume $\sum 1 \leqq 2/\delta$. Taking $\alpha = (\delta/2)^{3/4}$ we find

$$\int_0^{2\pi} v_t^4\,\mathrm{d}t \leqq 3\omega(\delta)\int_0^{2\pi}(v_t^4 + v_{tt}^2)\,\mathrm{d}t + 3\omega(\delta)\left[2\delta^{-3}\sum_i\int_0^{2\pi}(\eta_i\eta_{it})^4\,\mathrm{d}t + \int_0^{2\pi} v_t^4\,\mathrm{d}t\right].$$

Hence

$$\int_0^{2\pi} v_t^4\,\mathrm{d}t(1 - 6\omega(\delta)) \leqq 3\omega(\delta)\int_0^{2\pi} v_{tt}^2\,\mathrm{d}t + C(\delta)$$

where

$$C(\delta) = 6\omega(\delta)\,\delta^{-3}\sum_i\int_0^{2\pi}(\eta_i\eta_i)^4\,\mathrm{d}t\,.$$

Now we prove that $\|v(w_1) - v(w_2)\| \leqq K\|w_1 - w_2\|$ where $v(w_i)$ are solutions of the equations (5.17) in $B_{n,k}^R$ corresponding to $w_i \in C_{n,k}^r$.

$$0 = (S_{w_1}v_1 - S_{w_2}v_2,\ v_1 - v_2) =$$
$$= (F(v_1 + w_1) - F(v_1 + w_2) + F(v_1 + w_2) - F(v_2 + w_2),\ v_1 - v_2) =$$
$$= (f_u(\text{int. pt})\,(w_1 - w_2),\ v_1 - v_2) + (f_u(\text{int. pt})\,(v_1 - v_2),\ v_1 - v_2)\,,$$
$$v_i = v(w_i)\,.$$

Thus
$$\gamma\|v_1 - v_2\|^2 \leqq \sup_{G_{4,k}} |f_u(t, x, u)| \, \|v_1 - v_2\| \, \|w_1 - w_2\|$$
$$\|v_1 - v_2\| \leqq \frac{1}{\gamma} \sup_{G_{4,k}} |f_u(t, x, u)| \, \|w_1 - w_2\| \, .$$

Thus the following theorem holds.

Theorem 5.1. *Let $n \geqq 2$ and assume that there are $R > 0$, $r > 0$ such that*

(i) *f is continuously differentiable up to the order $2n$ on $G_{4,k}$*
$$\left.\frac{\partial^{2m} F(u)}{\partial x_i^{2m}}\right|_{\substack{x_i = 0 \\ x_i = \pi}} = 0 \, , \quad m = 0, 1, \ldots, \quad n - 1, \quad i = 1, \ldots, \ k, \ u \in A_{n,k}$$

(ii) $$f_u \geqq \gamma > 0 \quad on \quad G_{4,k} \, ,$$

(iii) $$\sup \{|f_t(t, x, u)|, \ (t, x, u) \in G_{4,k}\} < \gamma R \, .$$

Then there is a unique solution to the problem (5.1), (5.3) in $B^R_{n,k} \times C^r_{n,k}$ provided that $\varepsilon > 0$ is sufficiently small, $k = 1, 2, 3$.

References

[1] *Hall, W. S.*, On the existence of periodic solutions for the equations $D_{tt}u + (-1)^p D_x^{2p}u = \varepsilon f(., ., u)$ Jour. of Diff. Equations, *7*, 509—526, 1970.

[2] *Hall, W. S.*, Periodic solutions of a class of weakly nonlinear evolution equations, Arch. Rat. Mech. Anal., Vol. 39, *4*, 294—322.

[3] *Lovicarová H.*, Periodic solutions of a weakly nonlinear wave equation in one dimension, Czech. Math. J. *19*, 1969, 324—342.

[4] *Krylová N.*, Periodic solutions of hyperbolic partial differential equation with quadratic dissipative term, Czech. Math. J. *20*, 1970, 375—405.

[5] *Krylová N., Vejvoda O.*, A linear and weakly nonlinear equation of a beam: The boundary-value problem for free extremities and its periodic solutions, Czech. Math. J. *21*, 1971, 535—566.

[6] *Moser J.*, A rapidly convergent iteration method and nonlinear partial differential equations, Ann. Scuola Norm. Super. Pisa, Ser. 3, *20*, 1956, 265—315.

[7] *Rabinowitz P. H.*, Periodic solutions of nonlinear hyperbolic partial differential equations, Comm. Pure Appl. Math., *20*, 1967, 145—205.

[8] *Rabinowitz P. H.*, Time periodic solutions of nonlinear wave equations, Manuscripta math., *5*, 1971, 165—194.

[9] *De Simon L., Torelli* C., Soluzioni periodiche di equazioni a derivate parziali di tipo iperbolico non lineari, Rend. Sem. Mat. Univ. Padova XL, 1968, 380—401.

[10] *Torelli G.*, Soluzioni periodiche dell'equazione non lineare $u_{tt} - u_{xx} + f(x, t, u) = 0$, Rend. Ist. di Matem. Univ. Trieste *1*, 1969, 123—137.

[11] *Petryshyn W. V.*, On nonlinear *P*-compact operators in Banach space with applications to constructive fixed-point theorems, J. Math. Anal. Appl. *15*, 1966, 228—242.

[12] *Bers L., John F., Schechter W.*, Partial Differential Equations, New York, Interscience 1964.

[13] *Randol B.*, A lattice-point problem II, TAMS *125*, 1966, 101—113.

Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).