Tôhoku Math. J.
48 (1996), 33–50

PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN UNBOUNDED DOMAINS

Dedicated to Professor Kôji Kubota on his sixtieth birthday

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(Received August 29, 1994, revised February 21, 1995)

Abstract. We shall construct a periodic strong solution of the Navier-Stokes equations for the prescribed external force in unbounded domains.

Introduction. The purpose of this paper is to show that if the incompressible fluid in *unbounded* domains is governed by the periodic external force, the Navier-Stokes equations have a *periodic strong* solution with the same period as the external force. Let Ω be a domain in $\boldsymbol{R}^n$ ($n \geq 3$), not necessarily bounded, with smooth boundary $\partial\Omega$. Consider the following Navier-Stokes equations in Ω:

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f, & x \in \Omega, \quad t \in \boldsymbol{R}, \\ \operatorname{div} u = 0, \quad x \in \Omega, \quad t \in \boldsymbol{R}, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $u = u(x, t) = (u^1(x, t), \ldots, u^n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and pressure of the fluid at point $(x, t) \in \Omega \times \boldsymbol{R}$, respectively; while $f = f(x, t) = (f^1(x, t), \ldots, f^n(x, t))$ is the given periodic external force.

Under some restrictive conditions, Serrin [20] gave a criterion for the existence of periodic solutions of (N-S) when Ω is a three-dimensional bounded domain whose boundary moves periodically in time. Kaniel-Shinbrot [11] considered a simpler case such as bounded domains whose boundary is fixed in time and realized the criterion of Serrin. Having introduced the notion of reproductive property, they showed the existence of periodic strong solutions with periodic small forces f. In two-dimensional bounded domains, Takeshita [23] obtained the same result as Kaniel-Shinbrot [11] without assuming the smallness of f. The original problem posed by Serrin had been treated by Morimoto [19] and Miyakawa-Teramoto [18] who showed the existence of periodic weak solutions. Later on, Teramoto [25] constructed periodic strong solutions in a situation such that the boundary moves slowly in time.

All of these results are obtained in two- or three-dimensional *bounded* domains. On the other hand, few results are known in *unbounded* domains. Recently, Maremonti

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1991 *Mathematics Subject Classification*. Primary 35Q30.

[15], [16] showed the existence of periodic strong solutions in the three-dimensional whole space $\boldsymbol{R}^3$ and the half space $\boldsymbol{R}^3_+$, respectively. However, the result corresponding to exterior domains has not been obtained up to the present. The main difficulty in unbounded domains stems from the lack of exponential decay in time for solutions to the initial value problem of (N-S). Indeed, Serrin [20] and Kaniel-Shinbrot [11] made full use of the fact that $\|u(t)\|_2$ and $\|\nabla u(t)\|_2$ decay exponentially in t provided the initial data at $t=0$ are prescribed. Such a decay property is due to the Poincaré inequality in bounded domains, and invertibility of the Stokes operator in L^2 makes it easy to obtain better asymptotic behaviour of solutions at $t\to\infty$.

To overcome this difficulty, Maremonti [15], [16] first showed the algebraic decay rates in time of strong solutions for initial value problem of (N-S) in $\boldsymbol{R}^3$ and in $\boldsymbol{R}^3_+$. As a by-product, he constructed periodic strong solutions for periodic small external forces. His method is based on the skillful energy estimates in L^2 for higher derivatives of solutions. Although our results are not altogether new, our approach is different and gives more results than those by Maremonti [15], [16]. We do not employ the energy estimates in L^2 but the L^p-thoery of the Stokes operator. Making use of L^p-L^r estimates for the semigroup generated by the Stokes operator, we shall show the existence and uniqueness of *periodic strong* solutions more directly than Maremonti [15], [16]. Compared with the energy estimates in L^2, our L^p method can cover also the higher dimensional cases. Unfortunately, we cannot obtain the same result in three-dimensional exterior domains because the corresponding L^p-L^r estimate is still an open problem.

We shall first reduce our problem to an integral equation, the solution of which is necessarily periodic with the same period as the external force. The solution will be constructed in the class of functions defined on the whole interval $\boldsymbol{R}$ with values in $L^n(\Omega)$. Then by a regularity criterion similar to Serrin's [21], we shall show that our solution is actually a strong solution. For that purpose, we shall estimate a time-interval of the existence of local strong solutions for the initial-boundary value problem to (N-S) in terms of the given data. Our estimate extends the result obtained by Giga [7, Theorem 4]. The stability of periodic solutions will be discussed in a forthcoming paper.

1. Results. Before stating our results, we need to impose the following assumption on the domain Ω:

ASSUMPTION 1. (Case I) Ω is the whole space $\boldsymbol{R}^n$ or the half-space $\boldsymbol{R}^n_+$, where $n\geq 3$.

(Case II) Ω is an exterior domain in $\boldsymbol{R}^n$ with $C^{2+\mu}(\mu>0)$-boundary $\partial\Omega$, where $n\geq 4$.

The reason why we exclude three-dimensional exterior domains in (Case II) is due to the restriction on gradient bounds for the Stokes semigroup in L^p (see Lemma 2.1 (2) below).

We shall next introduce some notation and function spaces. Let $C^\infty_{0,\sigma}$ denote the set of all real vector C^∞-functions $\phi=(\phi^1,\dots,\phi^n)$ with compact support in Ω such that $\operatorname{div}\phi=0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\ \|_r$; $(\cdot,\ \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r+1/r'=1$. L^r stands for the usual (vector-valued) L^r-space over Ω, $1<r<\infty$. When X is a Banach space, its norm is denoted by $\|\cdot\|_X$. Then $C^m([t_1, t_2); X)$ is the usual Banach space, where $m=0, 1, 2, \dots$ and t_1 and t_2 are real numbers such that $t_1<t_2$. $BC^m([t_1, t_2); X)$ is the set of all functions $u\in C^m([t_1, t_2); X)$ such that $\sup_{t_1<t<t_2}\|d^m u(t)/dt^m\|_X<\infty$.

Let us recall the Helmholtz decomposition:

$$L^r=L^r_\sigma\oplus G^r(\text{direct sum})\,,\qquad 1<r<\infty\,,$$

where $G^r=\{\nabla p\in L^r\,;p\in L^r_{\mathrm{loc}}(\bar{\Omega})\}$. For the proof, see Fujiwara-Morimoto [4], Miyakawa [17] and Simader-Sohr [22]. P_r denotes the projection operator from L^r onto L^r_σ along G^r. The Stokes operator A_r on L^r_σ is then defined by $A_r=-P_r\Delta$ with domain $D(A_r)=\{u\in H^{2,r}(\Omega);\ u|_{\partial\Omega}=0\}\cap L^r_\sigma$. It is known that the dual space $(L^r_\sigma)^*$ of L^r_σ and the adjoint operator A_r^* of A_r are respectively

$$(L^r_\sigma)^*=L^{r'}_\sigma\,,\qquad A_r^*=A_{r'}\,,$$

where $1/r+1/r'=1$. Moreover, we have:

PROPOSITION 1 (Giga [5], Giga-Sohr [9]). *Let $1<r<\infty$. Then $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}\}_{t\geq 0}$ of class C^0 in L^r_σ.*

Applying the projection operator P_r to both sides of the first equation of (N-S), we have

$$\text{(E)}\qquad \frac{du}{dt}+A_r u+P_r(u\cdot\nabla u)=P_r f\quad\text{on } L^r_\sigma\,,\quad t\in\boldsymbol{R}\,.$$

The above (E) can be further transformed to the following integral equation:

$$\text{(I.E.)}\qquad u(t)=\int_{-\infty}^{t} e^{-(t-s)A_r}P_r f(s)ds-\int_{-\infty}^{t} e^{-(t-s)A_r}P_r(u\cdot\nabla u)(s)ds\,.$$

Concerning the external force f, we impose the following assumption:

ASSUMPTION 2. Let the exponents r and q be according to the (Case I) and (Case II) of Assumption 1 as

(Case I) $2<r<n$, $n/2<q<n$;

(Case II) $2n/(n-1)\leq r<n$, $n/2<q<n$.

For such r and q, we assume that f belongs to the class

$$(1.1)\qquad f\in BC(\boldsymbol{R};\ L^p\cap L^l)$$

for $1<p, l<\infty$ with $1/r+2/n<1/p$, $1/q<1/l<1/q+1/n$ provided $n\geq 4$ in both (Case I) and (Case II).

If $n=3$ in (Case I), assume moreover that

(1.2) $P_p f(s) = A_p^\delta g(s) \, (s \in \boldsymbol{R})$ with some $g \in BC(\boldsymbol{R}; D(A_p^\delta))$ and $f \in BC(\boldsymbol{R}; L^l)$

for $1 < p < \min\{r, q\}$ and $\delta > 0$ satisfying $3/2p + \delta > 1 + \max\{1 + 3/2r, 1/2 + 3/2q\}$ and for $1/q < 1/l < 1/q + 1/3$.

Our result now reads:

THEOREM 1. *Let Ω and f satisfy Assumption* 1 *and Assumption* 2, *respectively. Suppose that $f(t) = f(t + \omega)$ for all $t \in \boldsymbol{R}$ with some $\omega > 0$. Then there is a constant $\eta = \eta(n, r, q, p, l, \delta) > 0$ such that if*

$$\sup_{s \in \boldsymbol{R}} \|P_p f(s)\|_p + \sup_{s \in \boldsymbol{R}} \|P_l f(s)\|_l \leq \eta \quad \text{for } n \geq 4 \text{ in (Case I) and (Case II)},$$

$$\sup_{s \in \boldsymbol{R}} \|g(s)\|_p + \sup_{s \in \boldsymbol{R}} \|P_l f(s)\|_l \leq \eta \quad \text{for } n = 3 \text{ in (Case I)},$$

we have a periodic solution u of (I.E.) *with the same period ω as f in the class $BC(\boldsymbol{R}; L_\sigma^r)$ with $\nabla u \in BC(\boldsymbol{R}; L^q)$.*

Such a solution u is unique within this class provided $\sup_{s \in \boldsymbol{R}} \|u(s)\|_r + \sup_{s \in \boldsymbol{R}} \|\nabla u(s)\|_q$ *is sufficiently small.*

Concerning the existence of solutions to (E), we have:

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *let us assume furthermore that f is a Hölder continuous function on $\boldsymbol{R}$ with values in L^n. Then the periodic solution u given by Theorem* 1 *has the following additional properties*:

(i) $u \in BC(\boldsymbol{R}; L_\sigma^n) \cap C^1(\boldsymbol{R}; L_\sigma^n)$;

(ii) *$u(t) \in D(A_n)$ for all $t \in \boldsymbol{R}$ and $A_n u \in C(\boldsymbol{R}; L_\sigma^n)$*;

(iii) *u satisfies* (E) *in L_σ^n for all $t \in \boldsymbol{R}$.*

REMARKS. (1) Taking $n = 3$, $2 < r < 3$ and $q = 2$ in (Case I), our theorems include Maremonti [15, Theorem 1] and [16, Theorem 2].

(2) The first condition of (1.2) seems to be artificial, but it may be replaced by $f(s) = \operatorname{div} F(s)$ with some $F = \{F_{i,j}\}_{i,j=1,2,3} \in BC(\boldsymbol{R}; H^{1,p}(\Omega))$ for $1 < p < \infty$ satisfying $1/r + 1/3 < 1/p$.

(3) When Ω is a bounded domain in $\boldsymbol{R}^n$ $(n \geq 2)$, the above results also hold and we can relax the assumption on the external force. Indeed, it suffices to assume that $f \in BC(\boldsymbol{R}; L^r)$ with $\sup_{s \in \boldsymbol{R}} \|Pf(s)\|_r$ small for $r > n/2$. Under such a hypothesis, there is a periodic solution u of (I.E.) in the class $u \in BC(\boldsymbol{R}; D(A_r^{1/2}))$.

2. Preliminaries. Throughout this paper, we shall denote by C various constants. In particular, $C = C(*, \cdots, *)$ will denote the constants which depend only on the quantities appearing in parentheses. Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u$, $P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u$, $A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively. Let us first recall the following

L^p-L^r estimates for the semigroup $\{e^{-tA}\}_{t\geq 0}$:

LEMMA 2.1 (Kato [12], Ukai [26], Giga-Sohr [9], Iwashita [10], Borchers-Miyakawa [1], [2]).

(1) *Let Ω be as in* (Case I) *of Assumption* 1. *Then there holds*

$$\|e^{-tA}a\|_r \leq Ct^{-n(1/p-1/r)/2}\|a\|_p, \qquad 1<p\leq r<\infty$$

$$\|\nabla e^{-tA}a\|_r \leq Ct^{-n(1/p-1/r)/2-1/2}\|a\|_p, \qquad 1<p\leq r<\infty$$

for all $a\in L^p_\sigma$ and all $t>0$, where $C=C(n, p, r)$.

(2) *Let Ω be as in* (Case II) *of Assumption* 1. *Then there holds*

$$\|e^{-tA}a\|_r \leq Ct^{-n(1/p-1/r)/2}\|a\|_p, \qquad 1<p\leq r<\infty$$

$$\|\nabla e^{-tA}a\|_r \leq Ct^{-n(1/p-1/r)/2-1/2}\|a\|_p, \qquad 1<p\leq r\leq n$$

for all $a\in L^p_\sigma$ and all $t>0$, where $C=C(n, p, r)$.

Using this lemma, we shall estimate the nonlinear term of (I.E.).

LEMMA 2.2. *Let r and q be as in Theorem* 1 *according to the* (Case I) *and* (Case II) *of Assumption* 1. *Define a function space Y and a bilinear operator $G(\cdot,\cdot)$ on Y by*

$$Y\equiv\{u\in BC(\boldsymbol{R}; L^r_\sigma); \nabla u\in BC(\boldsymbol{R}; L^q)\},$$

$$G(u, v)(t)\equiv -\int_{-\infty}^{t} e^{-(t-s)A}P(u\cdot\nabla v)(s)ds \qquad u, v\in \boldsymbol{Y},$$

respectively. Then we have $G(u, v)\in \boldsymbol{Y}$ with

$$\sup_{s\in\boldsymbol{R}}\|G(u, v)(s)\|_r \leq C_1\left(\sup_{s\in\boldsymbol{R}}\|u(s)\|_r \sup_{s\in\boldsymbol{R}}\|v(s)\|_r + \sup_{s\in\boldsymbol{R}}\|u(s)\|_r \sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q\right) \tag{2.1}$$

$$\sup_{s\in\boldsymbol{R}}\|\nabla G(u, v)(s)\|_q \leq C_1\left(\sup_{s\in\boldsymbol{R}}\|u(s)\|_r \sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q + \sup_{s\in\boldsymbol{R}}\|\nabla u(s)\|_q \sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q\right) \tag{2.2}$$

for all $u, v\in\boldsymbol{Y}$, where $C_1=C_1(n, r, q)$.

PROOF. Set

$$\begin{aligned} G(u, v)(t) &= -\int_{-\infty}^{t-1} e^{-(t-s)A}P(u\cdot\nabla v)(s)ds - \int_{t-1}^{t} e^{-(t-s)A}P(u\cdot\nabla v)(s)ds \\ &\equiv -I_1(t)-I_2(t). \end{aligned} \tag{2.3}$$

By integration by parts and Lemma 2.1 we have

$$|(I_1(t), \phi)| = \left| -\int_{-\infty}^{t-1} (u(s)\cdot\nabla e^{-(t-s)A}\phi, v(s))ds \right|$$
$$\leq \int_{-\infty}^{t-1} \|\nabla e^{-(t-s)A}\phi\|_{(r/2)'}\|u(s)\cdot v(s)\|_{r/2}ds$$
$$\leq C\int_{-\infty}^{t-1} (t-s)^{-n(1/r'-1/(r/2)')/2-1/2}\|\phi\|_{r'}\|u(s)\|_r\|v(s)\|_r ds$$
$$\leq C\sup_{s\in\boldsymbol{R}}\|u(s)\|_r \sup_{s\in\boldsymbol{R}}\|v(s)\|_r \int_{-\infty}^{t-1}(t-s)^{-n/2r-1/2}ds\|\phi\|_{r'}$$
$$\leq C\sup_{s\in\boldsymbol{R}}\|u(s)\|_r \sup_{s\in\boldsymbol{R}}\|v(s)\|_r\|\phi\|_{r'}$$

for all $\phi\in C_{0,\sigma}^{\infty}$ and all $t\in\boldsymbol{R}$, where $C=C(n, r)$. Note that, in (Case II), on account of such restriction as Lemma 2.2 (2), we need $(r/2)'\leq n$, i.e., $2n'\leq r$. By duality there holds

$$\|I_1(t)\|_r\leq C\sup_{s\in\boldsymbol{R}}\|u(s)\|_r\sup_{s\in\boldsymbol{R}}\|v(s)\|_r \qquad \text{for all} \quad t\in\boldsymbol{R}, \tag{2.4}$$

where $C=C(n, r)$. Similarly, Lemma 2.1 and the Hölder inequality yield

$$\|I_2(t)\|_r\leq\int_{t-1}^{t}\|e^{-(t-s)A}P(u\cdot\nabla v)(s)\|_r ds$$
$$\leq\int_{t-1}^{t}(t-s)^{-n(1/r+1/q-1/r)/2}\|u(s)\|_r\|\nabla v(s)\|_q ds \tag{2.5}$$
$$\leq\sup_{s\in\boldsymbol{R}}\|u(s)\|_r\sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q$$

for all $t\in\boldsymbol{R}$ with $C=C(n, r, q)$. Now (2.1) follows from (2.3)–(2.5).

To show (2.2), we make use of the following inequality of the Sobolev type due to Giga-Sohr [9, Corollary 2.2 (ii)]:

$$\|\phi\|_{q^*}\leq C\|\nabla\phi\|_q \quad \text{with } q^*=nq/(n-q) \text{ for all } \phi\in L^r \text{ with } \nabla\phi\in L^q, \tag{2.6}$$

where $C=C(n, q)$. It should be noted that (2.6) holds even though ϕ does not vanish on the boundary $\partial\Omega$. Now it follows from Lemma 2.1 and (2.6) that

$$\|\nabla G(u, v)(t)\|_q\leq\int_{-\infty}^{t-1}\|\nabla e^{-(t-s)A}P(u\cdot\nabla v)(s)\|_q ds+\int_{t-1}^{t}\|\nabla e^{-(t-s)A}P(u\cdot\nabla v)(s)\|_q ds$$
$$\leq C\int_{-\infty}^{t-1}(t-s)^{-n(1/r+1/q-1/q)/2-1/2}\|u(s)\|_r\|\nabla v(s)\|_q ds$$

$$+C\int_{t-1}^{t}(t-s)^{-n(1/q^*+1/q-1/q)/2-1/2}\|u(s)\|_{q^*}\|\nabla v(s)\|_q ds$$

$$\leq C\sup_{s\in\boldsymbol{R}}\|u(s)\|_r\sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q\int_{-\infty}^{t-1}(t-s)^{-n/2r-1/2}ds$$

$$+C\sup_{s\in\boldsymbol{R}}\|u(s)\|_{q^*}\sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q\int_{t-1}^{t}(t-s)^{-n/2q}ds$$

$$\leq C\Big(\sup_{s\in\boldsymbol{R}}\|u(s)\|_r\sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q+\sup_{s\in\boldsymbol{R}}\|\nabla u(s)\|_q\sup_{s\in\boldsymbol{R}}\|\nabla v(s)\|_q\Big)$$

for all $t\in\boldsymbol{R}$ with $C=C(n,r,q)$, which yields (2.2). This completes the proof of Lemma 2.2.

We shall next show bounds for the external force.

LEMMA 2.3. *Let f be as in* (1.1) *and* (1.2) *of Assumption* 2. *Let*

$$F(t)\equiv\int_{-\infty}^{t}e^{-(t-s)A}Pf(s)ds\,,\qquad t\in\boldsymbol{R}\,.$$

Then we have $F\in Y$ and the following estimates hold:

$$\left.\begin{array}{l}\sup_{s\in\boldsymbol{R}}\|F(s)\|_r\\ \sup_{s\in\boldsymbol{R}}\|\nabla F(s)\|_q\end{array}\right\}\leq C\Big(\sup_{s\in\boldsymbol{R}}\|Pf(s)\|_p+\sup_{s\in\boldsymbol{R}}\|Pf(s)\|_l\Big)\tag{2.7}$$

with $C=C(n,r,q,p,l)$ provided f satisfies (1.1);

$$\left.\begin{array}{l}\sup_{s\in\boldsymbol{R}}\|F(s)\|_r\\ \sup_{s\in\boldsymbol{R}}\|\nabla F(s)\|_q\end{array}\right\}\leq C\Big(\sup_{s\in\boldsymbol{R}}\|g(s)\|_p+\sup_{s\in\boldsymbol{R}}\|Pf(s)\|_l\Big)\tag{2.8}$$

with $C=C(n,r,q,p,l,\delta)$ provided f satisfies (1.2).

PROOF. If f satisfies (1.1), we have by Lemma 2.1

$$\|F(t)\|_r\leq\int_{-\infty}^{t-1}\|e^{-(t-s)A}Pf(s)\|_r ds+\int_{t-1}^{t}\|e^{-(t-s)A}Pf(s)\|_r ds$$

$$\leq C\sup_{s\in\boldsymbol{R}}\|Pf(s)\|_p\int_{-\infty}^{t-1}(t-s)^{-n(1/p-1/r)/2}ds$$

$$+C\sup_{s\in\boldsymbol{R}}\|Pf(s)\|_l\int_{t-1}^{t}(t-s)^{-n(1/l-1/r)/2}ds$$

for all $t\in\boldsymbol{R}$, where $C=C(n,r,q,p,l)$. Since $n/2<q$ and $r<n$, we have by hypothesis on

l that $1/l<2/n+1/r$ and hence the second integral above converges. So does the first integral by hypothesis on p and we obtain (2.7) for $\sup_{s\in \mathbf{R}}\|F(s)\|_r$. Similarly, we have by Lemma 2.1

$$\|\nabla F(t)\|_q \le C\sup_{s\in \mathbf{R}}\|Pf(s)\|_p\int_{-\infty}^{t-1}(t-s)^{-n(1/p-1/q)/2-1/2}ds + C\sup_{s\in \mathbf{R}}\|Pf(s)\|_l\int_{t-1}^{t}(t-s)^{-n(1/l-1/q)/2-1/2}ds$$

for all $t\in \mathbf{R}$ with $C=C(n, r, q, p, l)$. Since $1/p>1/r+2/n>1/n+1/q$, the first integral above converges and we obtain (2.7) for $\sup_{s\in \mathbf{R}}\|\nabla F(s)\|_q$.

To show (2.8), we shall make use of the estimate

$$\|A^\delta e^{-tA}a\|_p \le Ct^{-\delta}\|a\|_p \quad \text{for all} \quad a\in L^p_\sigma, \quad t>0 \tag{2.9}$$

with $C=C(p, \delta)$. This is an immediate consequence of Proposition 1. Therefore, if f satisfies (1.2), it follows from Lemma 2.1 and (2.9) that

$$\begin{aligned}\|F(t)\|_r &\le \int_{-\infty}^{t-1}\|e^{-(t-s)A}A^\delta g(s)\|_r ds + \int_{t-1}^{t}\|e^{-(t-s)A}Pf(s)\|_r ds\\ &\le C\int_{-\infty}^{t-1}(t-s)^{-3(1/p-1/r)/2}\|A^\delta e^{-(t-s)A/2}g(s)\|_p ds\\ &\quad + C\int_{t-1}^{t}(t-s)^{-3(1/l-1/r)/2}\|Pf(s)\|_l ds\\ &\le C\int_{-\infty}^{t-1}(t-s)^{-3(1/p-1/r)/2-\delta}\|g(s)\|_p ds\\ &\quad + C\sup_{s\in \mathbf{R}}\|Pf(s)\|_l\int_{t-1}^{t}(t-s)^{-3(1/l-1/r)/2}ds\\ &\le C\left(\sup_{s\in \mathbf{R}}\|g(s)\|_p + \sup_{s\in \mathbf{R}}\|Pf(s)\|_l\right)\end{aligned}$$

for all $t\in \mathbf{R}$ with $C=C(n, r, q, p, l, \delta)$. This yields (2.8) for $\sup_{s\in \mathbf{R}}\|F(s)\|_r$. Similarly, we can deal with ∇F to show (2.8) for $\sup_{s\in \mathbf{R}}\|\nabla F(s)\|_q$ and the proof of Lemma 2.3 is complete.

3. Existence of periodic solution; Proof of Theorem 1. Using Lemmas 2.2 and 2.3, we shall prove the existence and uniqueness of solution to the integral equation (I.E.) by successive approximation. Let us recall the function space Y and the bilinear operator $G(\cdot, \cdot)$ on Y introduced in Lemma 2.2. Equipped with the norm $\|\cdot\|_Y$ defined by

$$\|u\|_Y \equiv \sup_{s\in\mathbf{R}} \|u(s)\|_r + \sup_{s\in\mathbf{R}} \|\nabla u(s)\|_q ,$$

Y is a Banach space. We construct a periodic solution of (I.E.) according to the scheme

$$u_0(t) = \int_{-\infty}^{t} e^{-(t-s)A} Pf(s)ds , \tag{3.1}$$

$$u_{m+1}(t) = u_0(t) + G(u_m, u_m)(t) , \qquad m = 0, 1, \dots . \tag{3.2}$$

By Lemma 2.3, we have $u_0 \in Y$ with

$$\|u_0\|_Y \leq C\left(\sup_{s\in\mathbf{R}} \|Pf(s)\|_p + \sup_{s\in\mathbf{R}} \|Pf(s)\|_l \right) \tag{3.3}$$

provided f satisfies (1.1);

$$\|u_0\|_Y \leq C\left(\sup_{s\in\mathbf{R}} \|g(s)\|_p + \sup_{s\in\mathbf{R}} \|Pf(s)\|_l \right) \tag{3.4}$$

provided f satisfies (1.2). Since f is a periodic function with period ω, we can easily verify that u_0 is also periodic with the same period ω. By induction and Lemma 2.2, so is u_m for all $m = 0, 1, \dots$. Moreover, it follows from (2.1) and (2.2) that

$$\|u_{m+1}\|_Y \leq \|u_0\|_Y + \|G(u_m, u_m)\|_Y \leq \|u_0\|_Y + C_* \|u_m\|_Y^2 , \qquad m = 0, 1, \dots,$$

where $C_* = 2C_1$. Hence if

$$\|u_0\|_Y < \frac{1}{4C_*} , \tag{3.5}$$

then there holds

$$\|u_m\|_Y \leq \frac{1 - \sqrt{1 - 4C_* \|u_0\|_Y}}{2C_*} \equiv K < \frac{1}{2C_*} \qquad \text{for all} \quad m = 0, 1, \dots . \tag{3.6}$$

By (3.3) and (3.4), we can take the constant η in Theorem 1 so that the condition (3.5) is satisfied.

Now assume (3.5). Setting $w_m \equiv u_m - u_{m-1}$ ($u_{-1} \equiv 0$), we have

$$w_{m+1}(t) = G(u_m, u_m)(t) - G(u_{m-1}, u_{m-1})(t) = G(w_m, u_m)(t) - G(u_{m-1}, w_m)(t)$$

and Lemma 2.2 and (3.6) yield

$$\begin{aligned} \|w_{m+1}\|_Y &\leq \|G(w_m, u_m)\|_Y + \|G(u_{m-1}, w_m)\|_Y \\ &\leq C_*(\|w_m\|_Y \|u_m\|_Y + \|u_{m-1}\|_Y \|w_m\|_Y) \leq 2C_* K \|w_m\|_Y \\ &\leq \cdots \leq (2C_* K)^{m+1} \|u_0\|_Y \end{aligned} \tag{3.7}$$

for all $m=0, 1, \dots$. Since $u_m(t)=\sum_{j=0}^{m} w_j(t)$, we see by (3.6) and (3.7) that there exists a function $u\in Y$ such that

$$u_m \to u \quad \text{in } Y \text{ as } m\to\infty . \tag{3.8}$$

Clearly, such a limit $u(t)$ is also periodic in t with the same period ω as f. As in (3.7), we have by (3.6) that

$$\begin{aligned} \|G(u_m, u_m)-G(u, u)\|_Y &\le \|G(u_m-u, u_m)\|_Y+\|G(u, u_m-u)\|_Y \\ &\le C_*\|u_m-u\|_Y\|u_m\|_Y+C_*\|u\|_Y\|u_m-u\|_Y \\ &< \|u_m-u\|_Y \end{aligned} \tag{3.9}$$

for all m, from which follows that

$$G(u_m, u_m) \to G(u, u) \quad \text{in } Y \text{ as } m\to\infty . \tag{3.10}$$

Now letting $m\to\infty$ in (3.2), we see by (3.8) and (3.10) that u is a desired periodic solution of the integral equation (I.E.).

It remains to show the uniqueness. Suppose that $v\in Y$ is another solution of (I.E.) with $\|v\|_Y\le K$, where K is the same constant as in (3.6). Then we have as in (3.9) that

$$\|u-v\|_Y\le C_*(\|u-v\|_Y\|u\|_Y+\|v\|_Y\|u-v\|_Y)\le 2C_*K\|u-v\|_Y .$$

Since $2C_*K<1$, there holds $u=v$ and the assertion on uniqueness follows. This proves Theorem 1.

4. Regularity of solutions to (I.E.); Proof of Theorem 2. In this section, we shall show that the periodic solution u constructed in the preceding section is actually a solution of the differential equation (E). To this end, we need the local existence of strong solutions to the initial-boundary value problem for (N-S). In particular, it is important to give the time-interval of existence in terms of the prescribed data. Here we follow the argument of Kato [12] and Giga [7].

Let us first define the strong solution of the initial value problem for (N-S).

DEFINITION. Let $a\in L^n_\sigma$ and let $Pf\in C((t_0, t_1); L^n_\sigma)$, where $t_0<t_1$. Then a measurable function v on $\Omega\times(t_0, t_1)$ is called a *strong* solution of (N-S) on (t_0, t_1) with the initial data a at t_0 if

(i) $v\in BC([t_0, t_1); L^n_\sigma)\cap C^1((t_0, t_1); L^n_\sigma)$;

(ii) $v(t)\in D(A_n)$ for $t_0<t<t_1$ and $A_n u\in C((t_0, t_1); L^n_\sigma)$;

(iii)

$$\begin{cases} \dfrac{dv}{dt}+Av+P(v\cdot\nabla v)=Pf \quad \text{in} \quad L^n_\sigma \text{ for } t_0<t<t_1 , \\ v(t_0)=a . \end{cases}$$

Our result on the local existence of strong solutions now reads:

LEMMA 4.1. *Let $n/2<q<n$ and let $1<l<\infty$ satisfy $1/q<1/l<1/q+1/n$. Assume that $a\in L^n_\sigma\cap L^{q^*}_\sigma$ with $q^*=nq/(n-q)$, $f\in BC(\boldsymbol{R}; L^l)$ and that $Pf(\cdot)$ is a Hölder continuous function on $\boldsymbol{R}$ with values in L^n_σ. Then there exists $T>0$ such that for every $t_0\in\boldsymbol{R}$, we have a unique strong solution v of* (N-S) *on (t_0, t_0+T) with the initial data a at t_0. Moreover, v has the additional property $v\in BC([t_0, t_0+T); L^{q^*}_\sigma)$ with*

$$\sup_{t_0\le t<t_0+T}\|v(t)\|_{q^*}\le C_2\,, \tag{4.1}$$

where $C_2=C_2(\|a\|_{q^*}, \|Pf\|_{BC(\boldsymbol{R}; L^l)})$ is independent of t_0. Here T is estimated as

$$T=C_3(\|a\|_{q^*}+\|Pf\|_{BC(\boldsymbol{R}; L^l_\sigma)})^{-2q^*/(q^*-n)} \tag{4.2}$$

with $C_3=C_3(n, q, l)$ independent of a, f and t_0.

REMARK. When $f\equiv 0$, Giga [7, Theorem 4] obtained (4.2) by making use of the fact that $A_r^{-1/2}P_r(\partial/\partial x_j)\,(j=1, \dots, n)$ is a bounded operator from L^r into L^r_σ for $1<r<\infty$. Our proof below seems to be rather elementary; we use Lemma 2.1 and integration by parts.

PROOF OF LEMMA 4.1. The proof is similar to that of Kato [12] and Giga [7]. However, we give it for completeness. It suffices only to construct the solution v of the integral equation:

$$v(t)=e^{-(t-t_0)A}a+\int_{t_0}^{t}e^{-(t-s)A}Pf(s)ds-\int_{t_0}^{t}e^{-(t-s)A}P(v\cdot\nabla v)(s)ds\,,\quad t_0<t<t_0+T \tag{4.3}$$

in the class

$$v\in BC([t_0, t_0+T); L^n_\sigma\cap L^{q^*}_\sigma)\quad\text{with}\quad (t-t_0)^{1/2}\nabla v\in BC([t_0, t_0+T); L^n)\,. \tag{4.4}$$

Indeed, with the aid of Kozono-Ogawa [13, Lemma A.4], the assumption on Pf and a general theory of holomorphic semigroup guarantee that the solution v of (4.3) in the class (4.4) satisfies the properties (i), (ii) and (iii) of Lemma 4.1 (see, e.g., Tanabe [24, Theorem 3.3.4]).

Since this lemma deals with only local existence of solutions, we may assume that $0<T\le 1$. Let us solve (4.3) by successive approximation:

$$v_0(t)=e^{-(t-t_0)A}a+\int_{t_0}^{t}e^{-(t-s)A}Pf(s)ds\,, \tag{4.5}$$

$$v_{m+1}(t)=v_0(t)-\int_{t_0}^{t}e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds\,. \tag{4.6}$$

Taking $\alpha=n/q-1$, we have by assumption $0<\alpha<1$ and $q^*=n/\alpha$. Let us first show

$$\sup_{t_0<t<t_0+T}(t-t_0)^{(1-\alpha)/2}\|v_m(t)\|_{n/\alpha}\le K_m\,,\qquad m=0, 1, \dots \tag{4.7}$$

with some constant K_m. In fact, by Lemma 2.1 and the Sobolev inequality (2.6), we have

$$\begin{aligned}\|v_0(t)\|_{n/\alpha} &\le \|e^{-(t-t_0)A}a\|_{n/\alpha}+\int_{t_0}^{t}\|e^{-(t-s)A}Pf(s)\|_{n/\alpha}ds\\ &\le C\|a\|_{n/\alpha}+C\int_{t_0}^{t}\|\nabla e^{-(t-s)A}Pf(s)\|_q ds\\ &\le C\|a\|_{n/\alpha}+C\int_{t_0}^{t}(t-s)^{-n(1/l-1/q)/2-1/2}\|Pf(s)\|_l ds\\ &\le C\|a\|_{n/\alpha}+C\|Pf\|_{BC(\boldsymbol{R};L^l_\sigma)}(t-t_0)^{-(1-\alpha)/2+3/2-n/2l}\end{aligned}$$

for all $t_0<t<t_0+T$ with $C=C(n, q, l)$ independent of t_0. Since $1/l<1/q+1/n$, we have $(1-\alpha)/2<(3-n/l)/2$ and hence the above estimate yields

$$\sup_{t_0<t<t_0+T}(t-t_0)^{(1-\alpha)/2}\|v_0(t)\|_{n/\alpha}\le C_4T^{(1-\alpha)/2}(\|a\|_{n/\alpha}+\|Pf\|_{BC(\boldsymbol{R};L^l_\sigma)})\,.$$

Then K_0 may be chosen as

$$K_0=C_4T^{(1-\alpha)/2}(\|a\|_{n/\alpha}+\|Pf\|_{BC(\boldsymbol{R};L^l_\sigma)})\,, \tag{4.8}$$

where $C_4=C_4(n, q, l)$ is independent of t_0. Suppose that (4.7) is true. By Lemma 2.1 and integration by parts there holds

$$\begin{aligned}\left|\left(-\int_{t_0}^{t}e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds,\ \phi\right)\right| &= \left|\int_{t_0}^{t}(v_m(s)\cdot\nabla e^{-(t-s)A}\phi,\ v_m(s))ds\right|\\ &\le\int_{t_0}^{t}\|v_m\otimes v_m(s)\|_{n/2\alpha}\|\nabla e^{-(t-s)A}\phi\|_{n/(n-2\alpha)}ds\\ &\le C\int_{t_0}^{t}(t-s)^{-\alpha/2-1/2}\|v_m(s)\|^2_{n/\alpha}ds\cdot\|\phi\|_{n/(n-\alpha)}\\ &\le CB((1-\alpha)/2,\ \alpha)K_m^2(t-t_0)^{-(1-\alpha)/2}\|\phi\|_{n/(n-\alpha)}\end{aligned}$$

for all $\phi\in C^\infty_{0,\sigma}$ and all $t\in(t_0, t_0+T)$, where $B(\cdot,\cdot)$ denotes the beta function and $C=C(n, q)$. By duality we have

$$\left\|\int_{t_0}^{t}e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds\right\|_{n/\alpha}\le C_5K_m^2(t-t_0)^{-(1-\alpha)/2}\,,\qquad t_0<t<t_0+T$$

and hence we may define K_{m+1} as

$$K_{m+1}\equiv K_0+C_5K_m^2\,, \tag{4.9}$$

where $C_5=C_5(n, q)$ is independent of t_0. An elementary consideration shows that if

(4.10) $$K_0 < \frac{1}{4C_5},$$

then there holds

(4.11) $$K_m \le \frac{1-\sqrt{1-4C_5K_0}}{2C_5} \equiv k < \frac{1}{2C_5} \quad \text{for all} \quad m=0, 1, \ldots .$$

Assume (4.10) for a moment. Then in the same way as in (3.8), the uniform estimate (4.11) with respect to m yields a function v with $(t-t_0)^{(1-\alpha)/2}v(\cdot) \in BC([t_0, t_0+T); L_\sigma^{n/\alpha})$ such that

(4.12) $$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|v_m(t)-v(t)\|_{n/\alpha} = 0 .$$

We shall next show that if K_0 is sufficiently small, then the limit v also satisfies $(t-t_0)^{1/2}\nabla v(\cdot) \in BC([t_0, t_0+T); L^n)$ with

(4.13) $$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} (t-t_0)^{1/2} \|\nabla v_m(t)-\nabla v(t)\|_n = 0 .$$

To this end, let us prove that

(4.14) $$\sup_{t_0<t<t_0+T} (t-t_0)^{1/2} \|\nabla v_m(t)\|_n \le L_m \quad \text{for} \quad m=0, 1, \ldots .$$

By Lemma 2.1, there holds

$$\|\nabla v_0\|_n \le C(t-t_0)^{-1/2}\|a\|_n + C(t-t_0)^{1/2}\|Pf\|_{BC((t_0,t_0+T);L_\sigma^n)} \quad \text{for all} \quad t\in(t_0, t_0+T)$$

and hence we may take L_0 as

$$L_0 \equiv C(\|a\|_n + \|Pf\|_{BC((t_0,t_0+T);L_\sigma^n)}),$$

where $C=C(n)$ is independent of t_0. Moreover, it follows from (4.7), (4.11) and (4.14) that

$$\begin{aligned} \left\|\nabla\int_{t_0}^{t} e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds\right\|_n &\le \int_{t_0}^{t} (t-s)^{-n(1/n+\alpha/n-1/n)/2-1/2}\|v_m(s)\|_{n/\alpha}\|\nabla v_m(s)\|_n ds \\ &\le CK_mL_m\int_{t_0}^{t} (t-s)^{-\alpha/2-1/2}(s-t_0)^{\alpha/2-1}ds \\ &\le C_6kL_m(t-t_0)^{-1/2} \end{aligned}$$

for all $t_0<t<t_0+T$, where $C_6=C_6(n, q)$ is independent of t_0. Hence we may take L_{m+1} as

$$L_{m+1} \equiv L_0 + C_6kL_m ,$$

which shows that $\{L_m\}_{m=0}^\infty$ is a *linear recurrence*. If

(4.15) $$k<1/C_6\,,$$

then we have a uniform bound of $\{L_m\}_{m=0}^{\infty}$ as

$$L_m \le \frac{L_0}{1-C_6 k} \equiv L \qquad \text{for all} \quad m=0, 1, \dots .$$

Assume (4.15) for a moment. Then it is easy to see that the limit v satisfies (4.13).

To prove $v \in BC([t_0, t_0+T); L^n_\sigma \cap L^{q^*}_\sigma)$, we need to show

(4.16) $$\sup_{t_0<t<t_0+T} \|v_m(t)\|_{n/\mu} \le M_{\mu,m} \quad (m=0, 1, \dots) \qquad \text{for } \mu=\alpha \text{ and } \mu=1\,.$$

Calculation similar to (4.7) and (4.14) yields $M_{\mu,0}$ as

$$M_{\alpha,0}=C(\|a\|_{n/\alpha}+\|Pf\|_{BC(\mathbf{R};L^l_\sigma)})\,, \qquad M_{1,0}=C(\|a\|_n+\|Pf\|_{BC([t_0,t_0+T);L^n_\sigma)})\,,$$

where $C=C(n, l, \mu)$ is independent of t_0. Notice that $0<T\le 1$. Suppose that (4.16) is true. Then by Lemma 2.1, (4.11) and integration by parts, we have

$$\begin{aligned}
&\left|\left(-\int_{t_0}^{t} e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds,\ \phi\right)\right| \\
&\le \left|\int_{t_0}^{t} (v_m(s)\cdot\nabla e^{-(t-s)A}\phi,\ v_m(s))ds\right| \\
&\le \int_{t_0}^{t} \|v_m(s)\|_{n/\alpha}\|v_m(s)\|_{n/\mu}\|\nabla e^{-(t-s)A}\phi\|_{n/(n-\alpha-\mu)}ds \\
&\le CK_mM_{\mu,m}\int_{t_0}^{t}(t-s)^{-\alpha/2-1/2}(s-t_0)^{-(1-\alpha)/2}ds\cdot\|\phi\|_{n/(n-\mu)} \\
&\le CkM_{\mu,m}B((1-\alpha)/2, (1+\alpha)/2)\|\phi\|_{n/(n-\mu)}
\end{aligned}$$

for all $\phi\in C^\infty_{0,\sigma}$ and all $t_0<t<t_0+T$, where $C=C(n, q, \mu)$. By duality we may take $M_{\mu,m+1}$ as

$$M_{\mu,m+1}=M_{\mu,0}+C_7kM_{\mu,m} \qquad \text{for} \quad \mu=\alpha, 1\,,$$

where $C_7=C_7(n, q, \mu)$ is independent of t_0. If

(4.17) $$k<1/C_7\,,$$

then there holds

$$M_{\mu,m}\le\frac{M_{\mu,0}}{1-C_7k} \quad (\mu=\alpha, 1) \qquad \text{for all } m=0, 1, \dots,$$

which yields $v\in BC([t_0, t_0+T); L^n_\sigma\cap L^{n/\alpha}_\sigma)$ with

$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} \|v_m(t)-v(t)\|_{n/\mu}=0 \qquad \text{for} \quad \mu=\alpha, 1\,. \tag{4.18}$$

In particular, the constant C_2 in (4.1) can be given as $C_2=M_{\alpha,0}/(1-C_7k)$. Now we see that under the conditions (4.10), (4.15) and (4.17), the limit v belongs to the class in (4.4). Moreover, there holds

$$\int_{t_0}^{t} e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds \to \int_{t_0}^{t} e^{-(t-s)A}P(v\cdot\nabla v)(s)ds \quad \text{in } L^n_\sigma \tag{4.19}$$

uniformly in $t\in[t_0, t_0+T)$ as $m\to\infty$. Indeed, by Lemma 2.1, (4.11) and (4.14) we have

$$\begin{aligned}
&\left\|\int_{t_0}^{t} e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds-\int_{t_0}^{t} e^{-(t-s)A}P(v\cdot\nabla v)(s)ds\right\|_n\\
&\le \int_{t_0}^{t} \|e^{-(t-s)A}P((v_m-v)\cdot\nabla v_m)(s)\|_n ds+\int_{t_0}^{t} \|e^{-(t-s)A}P(v\cdot\nabla(v_m-v))(s)\|_n ds\\
&\le \int_{t_0}^{t} (t-s)^{-n(\alpha/n+1/n-1/n)/2}\|v_m(s)-v(s)\|_{n/\alpha}\|\nabla v_m(s)\|_n ds\\
&\quad+\int_{t_0}^{t} (t-s)^{-n(\alpha/n+1/n-1/n)/2}\|v(s)\|_{n/\alpha}\|\nabla v_m(s)-\nabla v(s)\|_n ds\\
&\le CB(1-\alpha/2, \alpha/2)\Bigg(L \sup_{t_0<s<t_0+T} (s-t_0)^{(1-\alpha)/2}\|v_m(s)-v(s)\|_{n/\alpha}\\
&\quad+k \sup_{t_0<s<t_0+T} (s-t_0)^{1/2}\|\nabla v_m(s)-\nabla v(s)\|_n\Bigg)
\end{aligned}$$

for all $t_0<t<t_0+T$, from which and (4.12–13) we obtain (4.19). Now, letting $m\to\infty$ in (4.6), we see by (4.18) and (4.19) that v is a solution of (4.3). The proof for uniqueness is standard, so we may omit it (see [3], [8]).

It remains to estimate the time-interval T of existence in terms of the prescribed data. Since k is determined by (4.11), there exists a constant $\kappa=\kappa(n, q, l)$ independent of t_0 such that if $K_0\le\kappa$, then all conditions (4.10), (4.15) and (4.17) are satisfied. Now from (4.8) we see that T may be chosen as

$$T\equiv\left(\frac{\kappa}{C_4(\|a\|_{n/\alpha}+\|Pf\|_{BC(\boldsymbol{R};L^l_\sigma)})}\right)^{2/(1-\alpha)},$$

which shows (4.2) and proves Lemma 4.1.

PROOF OF THEOREM 2. Let u be the periodic solution of the integral equation (I.E.) given by Theorem 1. Since $u\in \boldsymbol{Y}$, we have by (2.6) that $u\in BC(\boldsymbol{R}; L^n_\sigma\cap L^{q^*}_\sigma)$, where $q^*=nq/(n-q)$. Let

$$T=C_3(\|u\|_{BC(\boldsymbol{R};L^{q^*}_\sigma)}+\|Pf\|_{BC(\boldsymbol{R};L^1_\sigma)})^{-2q^*/(q^*-n)},$$

where C_3 is the same constant as in (4.2). Then by Lemma 4.1, for every $t_0\in\boldsymbol{R}$ there exists a unique strong solution v of (N-S) on (t_0, t_0+T) with the initial data $u(t_0)$. By (3.6) and (4.1), we have

$$\sup_{t_0<s<t_0+T}\|v(s)\|_{q^*}+\sup_{t_0<s<t_0+T}\|\nabla u(s)\|_q\le C_2+K\equiv C_8, \tag{4.20}$$

where C_8 is independent of t_0. By (4.3) with a replaced by $u(t_0)$ and by (I.E.), it is easy to see

$$\begin{aligned} u(t)-v(t)&=-\int_{t_0}^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds+\int_{t_0}^t e^{-(t-s)A}P(v\cdot\nabla v)(s)ds\\ &=-\int_{t_0}^t e^{-(t-s)A}P((u-v)\cdot\nabla u)(s)ds-\int_{t_0}^t e^{-(t-s)A}P(v\cdot\nabla(u-v))(s)ds\\ &\equiv J_1(t)+J_2(t), \qquad t_0<t<t_0+T. \end{aligned}\tag{4.21}$$

By Lemma 2.1 there holds

$$\begin{aligned} \|J_1(t)\|_n&\le C\int_{t_0}^t (t-s)^{-n(1/n+1/q-1/n)/2}\|u(s)-v(s)\|_n\|\nabla u(s)\|_q ds\\ &\le C\sup_{s\in\boldsymbol{R}}\|\nabla u(s)\|_q\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_n(t-t_0)^{1-n/2q}, \end{aligned}\tag{4.22}$$

for all $t_0<t<t_0+T$, where $C=C(n,q)$ is independent of t_0. By integration by parts we have

$$\begin{aligned} |(J_2(t),\phi)|&=\left|\int_{t_0}^t (v(s)\cdot\nabla e^{-(t-s)A}\phi, u(s)-v(s))ds\right|\\ &\le C\int_{t_0}^t \|v(s)\|_{q^*}\|\nabla e^{-(t-s)A}\phi\|_{q'}\|u(s)-v(s)\|_n ds\\ &\le C\sup_{t_0<s<t_0+T}\|v(s)\|_{q^*}\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_n\int_{t_0}^t (t-s)^{-n(1/n'-1/q')/2-1/2}ds\|\phi\|_{n'}\\ &\le C\sup_{t_0<s<t_0+T}\|v(s)\|_{q^*}\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_n(t-t_0)^{1-n/2q}\|\phi\|_{n'} \end{aligned}$$

for all $\phi\in C^\infty_{0,\sigma}$ and all $t_0<t<t_0+T$, where $C=C(n,q)$. By duality,

$$\|J_2(t)\|_n\le C\sup_{t_0<s<t_0+T}\|v(s)\|_{q^*}\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_n(t-t_0)^{1-n/2q} \tag{4.23}$$

for all $t_0<t<t_0+T$. Now it follows from (4.20–23) that

$$\|u(t)-v(t)\|_n \le C_9 \sup_{t_0<s<t_0+t} \|u(s)-v(s)\|_n (t-t_0)^{1-n/2q}, \qquad t_0<t<t_0+T$$

with C_9 independent of t_0. Defining $\tau \equiv \min\{(1/2C_9)^{2q/(2q-n)}, T\}$, we obtain from the above estimate that

$$\|u(t)-v(t)\|_n \le C_9 \tau^{1-n/2q} \sup_{t_0<s<t_0+t} \|u(s)-v(s)\|_n$$

$$\le \frac{1}{2} \sup_{t_0<s<t_0+t} \|u(s)-v(s)\|_n$$

for all $t_0 \le t \le t_0+\tau$, which yields

$$u \equiv v \qquad \text{on} \quad [t_0, t_0+\tau]\,.$$

Since τ can be taken independently of t_0, we have

$$u \equiv v \qquad \text{on} \quad [t_0, t_0+T)\,.$$

Now, since t_0 is arbitrary, it follows from Lemma 4.1 that u has the desired properties (i), (ii) and (iii) in Theorem 2. □

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