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PERIODIC SOLUTIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

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INTRODUCTION

Since 1965 a number of authors has investigated the existence of the periodic solutions to abstract differential equations of the type

$$Lu \equiv \sum_{j=1}^{m} a_j \frac{d^j u}{dt^j}(t) + A\,u(t) = g(t) + F(t, u) \tag{0.1}$$

(see Taam [1], [2], Browder [3], [4], [5], Zend [6], Dezin [7], Simoněnko [8], Masuda [9], Dubinskij [10]). The papers [1]–[8] deal with such a first order equation ($m = 1$) which, roughly speaking, keeps the properties of the diffusion equation. In [9] a generalization of a telegraph equation ($m = 2$) is studied while in [10] m is an arbitrary natural number. In papers [1]–[9] the noncritical case is examined, i.e. there exists a sufficiently regular L^{-1} on the convenient space of the functions u periodic in t. The paper [10] is devoted also to some critical cases but only under the condition that L is an elliptic parabolic operator.

In this paper, some necessary and sufficient conditions for the existence of a periodie solution to (0.1) with $m = 1, 2$ are derived even under more general hypotheses on the operator L so that for $m = 1$ besides the diffusion equation also the Schrödinger equation is included and for $m = 2$ besides the telegraph equation also its generalization involving the term $Au_t(t)$ (representing the inner damping) and the wave and the beam equations (these special cases were studied earlier [11]–[15]) are included. We make use of the Poincaré method i.e. we investigate if in the problem $(\mathscr{M})$ given by (0.1) and

$$u^{(j)}(0) = \varphi_j\,, \quad j = 0, 1, \ldots, m - 1\,, \tag{0.2}$$

the initial data φ_j may be chosen in such a way that the corresponding solution to $(\mathscr{M})$ is periodic in t. We write the solution to $(\mathscr{M})$ by means of a semigroup generated by the operator $-A$ or with the aid of the spectral resolution of A which is equivalent in special cases to the eigenfunction expansion. This leads to some nonintrinsic requirements on g and F in (0.1) which have to belong to the domain of A^v for some v which means that g and F have to fulfil (to some extent) the boundary conditions imposed on u. The used procedure enables us to formulate the theorems on the stability or instability of the found solution. (The direct method of expanding of the sought periodic solution in a series of the form

$$\sum_{k=-\infty}^{\infty} u_k \exp\left(i\frac{2k\pi}{\omega}t\right)$$

has neither the advantage nor the disadvantage of our method mentioned above.)

In paragraph 1 we list some rules of the operational calculus and in the second section we prove a lemma on the normal solvability of the operator defined by (1.2.1). In paragraph 2 in Theorems 1.2.1 and 1.2.2 we derive necessary and sufficient conditions for the existence of a periodic solution to the first order linear equation while the solutions are looked for with values in a Banach or in a Hilbert space. In paragraph 3 an analogous problem for the second order linear equation of the type (3.1.1) and (3.2.1) respectively is investigated while we restrict ourselves to the solution with values in a Hilbert space only. In paragraph 4 one proves the existence of periodic solutions with small nonlinear perturbations of rather general type, while in the critical cases one indicates how to treat them. As an example of the resolution of a problem of the last type, in paragraph 5, the problem (5.1.1), (5.1.2) is studied rather thoroughly. Let us note that the restriction to the equations of first or second order is not inevitable for the used method but that the results would loose their lucidity.

1. NOTATIONS AND AUXILIARY RESULTS

1.1. Some properties of integrals in functional spaces. In this section we will introduce the notation and some properties of the spectral integrals and of the integrals of the abstract functions.

We will write $R = (-\infty, \infty)$, $R^+ = \langle 0, \infty)$. In the sequel H denotes a Hilbert space with the norm $\|\cdot\|$ induced by a scalar product $(\cdot, \cdot)$ and B denotes a Banach space with the norm $\|\cdot\|$ (both real or complex). If $J \subset R$ is a compact interval, $B_1 \subset B$ is a Banach space with the norm $\|\cdot\|_1$ and $k \geqq 0$ is an integer then $C^{(k)}(J; B_1)$ denotes a Banach space of all functions $u : J \to B_1$ having the continuous derivatives up to the order k with the norm $\|u\|_{k,B_1} = \max_{l=0,1,\ldots,k} \sup_{t\in J} \|u^{(l)}(t)\|_1$. In particular, for $x \in B_1 = \mathscr{D}(A^v)$, $(v \geqq 0)$, it is $\|x\|_1 = \|x\| + \|A^v x\|$ and we denote $\|\cdot\|_v = \|\cdot\|_{0,B_1}$. If $H_k \subset H$, $k = 1, 2, \ldots$ is a sequence of the subspaces of H orthogonal each other then we denote by $\sum_{k=1}^{\infty} H_k$ the direct sum of H_k. For $M \subset B$, M^c denotes the closure of M in B, $\mathscr{L}(M)$ the least linear subspace of B containing M and $\mathscr{B}(x_0; r; B) = \{x \in B; \|x - x_0\| \leqq r\}$. Further, introduce for $\nu \in R^+$, $\beta = 0$ or $\beta \neq 0$ and $M \subseteq R$ compact, the Banach spaces

$$
\begin{aligned}
(1.1.1) \qquad U_\nu(M) &= C^1(M; \mathscr{D}(A^\nu)) \cap C(M; \mathscr{D}(A^{\nu+1}))\,, \\
U_\nu(R^+) &= \{u \in U_\nu(\langle 0, \omega\rangle);\ u \text{ and } u' \text{ are } \omega\text{-periodic on } R^+\}\,, \\
U_\nu^\beta(M) &= C^2(M; \mathscr{D}(A^\nu)) \cap C^1(M; \mathscr{D}(A^{\nu+1/2}) \cap \mathscr{D}(\beta A^{\nu+1})) \cap \\
&\quad \cap C(M; \mathscr{D}(A^{\nu+1}) \cap \mathscr{D}(\beta A^{\nu+2}))\,, \\
U_\nu^\beta(R^+) &= \{u \in U_\nu^\beta(\langle 0, \omega\rangle);\ u, u' \text{ and } u'' \text{ are } \omega\text{-periodic on } R^+\}
\end{aligned}
$$

with the norms defined as the maximum of the norms in spaces included in the intersection on the right hand sides of (1.1.1). If A is an operator from the space B_1 into the space B_2 then we denote $\mathscr{D}(A)$, $\mathscr{R}(A) = \{Ax;\ x \in \mathscr{D}(A)\}$, $\mathscr{N}(A) = \{x \in \mathscr{D}(A);\ Ax = 0\}$, $\sigma(A)$ the domain, the range, the null-space and the spectrum of A respectively. Now let A be a selfadjoint operator in H with its corresponding resolution of the identity $E(\lambda)$. Then A may be written in the form of the abstract Stieltjes integral $Ax = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E(\lambda)\, x$ for

$$
x \in \mathscr{D}(A) = \left\{x \in H;\ \int_{-\infty}^{\infty} \lambda^2 \, \mathrm{d}\|E(\lambda)\, x\|^2 < \infty\right\}
$$

(see [17] p. 313). We shall write for $x \in \mathscr{D}(f(A))$, $f(A)\, x$ instead of $\int_{-\infty}^{\infty} f(\lambda)\, \mathrm{d}E(\lambda)\, x$. It is $\|f(A)\, x\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 \, \mathrm{d}\|E(\lambda)\, x\|^2$ for $x \in \mathscr{D}(f(A)) = \{x \in H, \int_{-\infty}^{\infty} |f(\lambda)|^2 \,.\, \mathrm{d}\|E(\lambda)\, x\|^2 < \infty\}$. If Λ is a subset of R and C_Λ is its characteristic function then we define

$$
\int_\Lambda f(\lambda)\, \mathrm{d}E(\lambda)\, x = \int_{-\infty}^{\infty} C_\Lambda(\lambda)\, f(\lambda)\, \mathrm{d}E(\lambda)\, x
$$

if the right hand side is defined. Clearly, the spectral measure of the set $\Lambda \subset R$ defined by $E(\Lambda) = \int_\Lambda 1 \, \mathrm{d}E(\lambda)$ is a projection as soon as the right hand side is defined.

Proposition 1.1.1. *Let $f(A)$ (where $f(\lambda)$, $\lambda \in R$, is any real or complex valued function) be a function of a selfadjoint operator* $A = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E(\lambda)$ (c.f. [17] p. 338). *Then $\mathscr{N}(f(A)) = E(\mathscr{N}(f))\, H$.*

Proof. If $x \in E(\mathscr{N}(f))\, H$ then

$$
\|f(A)\, x\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 \, \mathrm{d}\|E(\lambda)\, x\|^2 = \int_{\mathscr{N}(f)} 0 \, \mathrm{d}\|E(\lambda)\, x\|^2 = 0\,.
$$

On the contrary let $x \in \mathscr{N}(f(A))$. We can write x in the form $x = x_1 + x_2$, where

$x_1 \in E(\mathcal{N}(f))\, H$ and $x_2 \in (E(\mathcal{N}(f))\, H)^{\perp}$. Clearly

$$0 = \|f(A)\, x\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 \, \mathrm{d}\|E(\lambda)\,(x_1 + x_2)\|^2 =$$

$$= \int_{\mathcal{N}(f)} 0 \, \mathrm{d}\|E(\lambda)\, x_1\|^2 + \int_{R \setminus \mathcal{N}(f)} |f(\lambda)|^2 \, \mathrm{d}\|E(\lambda)\, x_2\|^2$$

hence $\|E(\lambda)\, x_2\|$ is constant on every component of $R \setminus \mathcal{N}(f)$. But as $x_2 = \int_{R \setminus \mathcal{N}(f)} 1 \, \mathrm{d}E(\lambda)\, x_2$, it is $x_2 = 0$.

Proposition 1.1.2. (cf. [16] p. 83). *Let $f(t, \lambda)$ be a function continuous in $t_0 \in R$ for any $\lambda \in R$ and let there exist a function $g(\lambda)$ and a number $\delta > 0$ such that $|f(t, \lambda)| \leqq g(\lambda)$ for $t \in \langle t_0 - \delta, t_0 + \delta \rangle$ and $\lambda \in R$. Let $g(A)$ be defined for a selfadjoint operator A. Then $f(t, A)\, x$ is a continuous function in t_0 for any $x \in \mathcal{D}(g(A))$.*

As a direct consequence of the preceding proposition we have

Proposition 1.1.3. *Let A be a selfadjoint operator and let $f(t, \lambda)$ be a function defined on $(t_0 - \delta, t_0 + \delta) \times R$, $(\delta > 0)$, having a derivative $\partial f/\partial t(t_0, \lambda)$, $t \in (t_0 - \delta, t_0 + \delta)$, $\lambda \in R$. Let there exist a function $g(\lambda)$ such that $g(A)$ is defined and $|\partial f/\partial t(t, \lambda)| \leqq g(\lambda)$ for $t \in (t_0 - \delta, t_0 + \delta)$, $\lambda \in R$ and let $f(t, A)\, x$ exist for at least one $t \in (t_0 - \delta, t_0 + \delta)$. Then $f(t, A)\, x$ exists for every $t \in (t_0 - \delta, t_0 + \delta)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(t, A)\, x) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(t, \lambda)\, \mathrm{d}E(\lambda)\, x\,, \quad t \in (t_0 - \delta, t_0 + \delta)\,, \quad x \in \mathcal{D}(g(A))\,.$$

At the end of paragraph we mention two propositions which are simple generalizations of the classical theorems on behaviour of integrals depending on the parameter.

Proposition 1.1.4. (cf. [18] p. 191). *Let $J \subset R$ be an interval and let $t_0 \in R$, $\delta > 0$. If $f : (t_0 - \delta, t_0 + \delta) \times J \to B$ is a function continuous in t_0 for any $\tau \in J$ and $g : J \to R^+$ is such that $\|f(t, \tau)\| \leqq g(\tau)$ for $[t, \tau] \in (t_0 - \delta, t_0 + \delta) \times J$ and $\int_J g(\tau)\, \mathrm{d}\tau$ exists then the function $F(t) = \int_J f(t, \tau)\, \mathrm{d}\tau$ is continuous in the point t_0.*

Proposition 1.1.5. (cf. [18] p. 191). *Let $a, c, t_0 \in R$, $\delta > 0$ and let the function $f : (t_0 - \delta, t_0 + \delta) \times \langle a, c \rangle \to B$ and $b : (t_0 - \delta, t_0 + \delta) \to \langle a, c \rangle$ fulfil the following assumptions:*

(a) *The integral $F(t) = \int_a^{b(t)} f(t, \tau)\, \mathrm{d}\tau$ exists for $t \in (t_0 - \delta, t_0 + \delta)$.*

(b) *The function $f(t, \tau)$ is continuous with respect to the set $\{[t, \tau];\ \tau \leqq b(t)\}$.*

(c) *There exist $K > 0$, a function $g : \langle a, c \rangle \to R^+$ integrable over $\langle a, c \rangle$ and a set $Q \subset R$ of measure zero such that*

$$\left|\frac{\partial f}{\partial t}(t, \tau)\right| \leqq g(\tau) \quad \text{for} \quad t \in (t_0 - \delta, t_0 + \delta)\,, \quad \tau \notin Q\,, \quad a < \tau < b(t_0) - K|t - t_0|\,.$$

(d) *There exists $b'(t_0)$.*

Then

$$F'(t_0) = \int_a^{b(t_0)} \frac{\partial f}{\partial t}(t_0, \tau)\,\mathrm{d}\tau + f(t_0, b(t_0))\, b'(t_0)\,.$$

1.2. A lemma on linear operator equations

If $\mathbf{H} = H \times H$ then for $[x_1, x_2], [y_1, y_2] \in \mathbf{H}$ we define the scalar product and the norm in $\mathbf{H}$ by

$$(([x_1, x_2], [y_1, y_2])) = (x_1, y_1) + (x_2, y_2)\,,$$
$$|||[x_1, x_2]||| = \sqrt{(([x_1, x_2], [x_1, x_2]))}\,.$$

Let A_{ij} $(i, j = 1, 2)$ be selfadjoint bounded operators in H and let $\mathbf{A} : \mathbf{H} \to \mathbf{H}$ be defined by

$$(1.2.1)\qquad \mathbf{A}[x_1, x_2] = [A_{11}x_1 + A_{12}x_2,\ A_{21}x_1 + A_{22}x_2],\ [x_1, x_2] \in \mathbf{H}\,.$$

Obviously the operator $\mathbf{A}^*$ is defined by

$$\mathbf{A}^*[x_1, x_2] = [A_{11}x_1 + A_{21}x_2,\ A_{12}x_1 + A_{22}x_2],\ [x_1, x_2] \in \mathbf{H}\,.$$

If $A_{12} = A_{21}$ then $\mathbf{A}$ is symmetric and hence selfadjoint.

Lemma 1.2.1. *Let A_{ij} $(i, j = 1, 2)$ be selfadjoint bounded operators in H. Let $A = \int_{-\infty}^{\infty} \lambda\,\mathrm{d}E(\lambda)$ and let the continuous functions $a_{ij}(\lambda)$ $(i, j = 1, 2, \lambda \in R)$ be such that*

$$(1.2.2)\qquad A_{ij} = \int_{-\infty}^{\infty} a_{ij}(\lambda)\,\mathrm{d}E(\lambda)\,, \quad i, j = 1, 2$$

and

$$\Lambda = \{\lambda \in R;\ a_{11}(\lambda)\, a_{22}(\lambda) = a_{12}(\lambda)\, a_{21}(\lambda)\} = \{\lambda_n\}_{n=1}^{\infty}\,.$$

Then, denoting

$$(1.2.3)\qquad D = \int_{-\infty}^{\infty} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda))\,\mathrm{d}E(\lambda)\,, \quad \textit{it is} \quad \mathcal{N}(D) = E(\Lambda)\, H$$

and

$$(1.2.4)\qquad \mathcal{N}(\mathbf{A}^*) = \Big\{[x_1, x_2] \in \mathbf{H};\ \int_\Lambda a_{11}(\lambda)\,\mathrm{d}E(\lambda)\, x_1 + \int_\Lambda a_{21}(\lambda)\,\mathrm{d}E(\lambda)\, x_2 = 0\,,$$
$$\int_\Lambda a_{12}(\lambda)\,\mathrm{d}E(\lambda)\, x_1 + \int_\Lambda a_{22}(\lambda)\,\mathrm{d}E(\lambda)\, x_2 = 0\,,\ x_1, x_2 \in \mathcal{N}(D)\Big\}\,.$$

If moreover there exists a $\delta_0 > 0$ such that

$$(1.2.5)\qquad |a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda)| \geqq \delta_0\,, \quad \lambda \in \sigma(A) \setminus \Lambda$$

then $\mathcal{R}(\mathbf{A})$ is closed.

Proof. First, let us prove (1.2.4).

a) Let $[x_1, x_2] \in \mathcal{N}(\mathbf{A}^*)$ i.e. let

(1.2.6) $$A_{11}x_1 + A_{21}x_2 = 0\,, \quad A_{12}x_1 + A_{22}x_2 = 0\,.$$

Multiplying the first equation (1.2.6) by A_{22} and subtracting the second one multiplied by A_{21} and making use of rules for calculating with spectral integrals we obtain $Dx_1 = 0$. Quite analogously we get $Dx_2 = 0$. Hence

$$x_1, x_2 \in \mathcal{N}(D) =$$
$$= \left\{ z \in H;\ \int_{-\infty}^{\infty} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda))^2\, \mathrm{d}(E(\lambda)\, z, z) = 0 \right\} = E(\Lambda)\, H\,.$$

For $x_1, x_2 \in E(\Lambda)\, H$ the system (1.2.6) reduces to

(1.2.6′) $$\int_\Lambda a_{11}(\lambda)\, \mathrm{d}E(\lambda)\, x_1 + \int_\Lambda a_{21}(\lambda)\, \mathrm{d}E(\lambda)\, x_2 = 0\,,$$
$$\int_\Lambda a_{12}(\lambda)\, \mathrm{d}E(\lambda)\, x_1 + \int_\Lambda a_{22}(\lambda)\, \mathrm{d}E(\lambda)\, x_2 = 0$$

and hence $[x_1, x_2]$ belongs to the set defined by the right hand side of (1.2.4).

b) If $[x_1, x_2]$ belongs to the set defined by the right hand side in (1.2.4) then $x_1, x_2 \in E(\Lambda)\, H$ and for these x_1, x_2 the systems (1.2.6) and (1.2.6′) are equivalent. Thus $[x_1, x_2] \in \mathcal{N}(\mathbf{A}^*)$. Let us prove the second part of Lemma i.e. that $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A})^c$. First we find easily that in consequence of the selfadjointness of the operator D and of (1.2.5) $\mathcal{R}(D) = \mathcal{R}(D)^c$ holds as if

$$f \perp \mathcal{N}(D) = E\,(\Lambda)\, H$$

then putting

(1.2.7) $$x = \int_{-\infty}^{\infty} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda))^{-1}\, \mathrm{d}E(\lambda) f =$$
$$= \int_{(-\infty,\infty)\setminus\Lambda} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda))^{-1}\, \mathrm{d}E(\lambda) f\,,$$

$Dx = f$ i.e. $f \in \mathcal{R}(D)$. Furthermore

(1.2.8) $$\|x\|^2 = \int_{(-\infty,\infty)\setminus\Lambda} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{21}(\lambda)\, a_{12}(\lambda))^{-2}\, \mathrm{d}(E(\lambda) f, f) \leqq \frac{1}{\delta_0^2} \|f\|^2\,.$$

Now, let

(1.2.9) $$[f_1, f_2] \perp \mathcal{N}(\mathbf{A}^*)\,.$$

Show that $[f_1, f_2] \in \mathscr{R}(\mathbf{A})$. In view of Proposition 1.1.1 and of $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ it is $\mathscr{N}(D) = \sum_{k=1}^{\infty} P_k H$ where $P_k = E(\lambda_k +) - E(\lambda_k)$ $(k = 1, 2, \ldots)$ and the condition $[x_1, x_2] \in \mathscr{N}(\mathbf{A}^*)$ is equivalent to

(1.2.10)
$$a_{11}(\lambda_k)\, x_{1,k} + a_{21}(\lambda_k)\, x_{2,k} = 0\,,$$
$$a_{12}(\lambda_k)\, x_{1,k} + a_{22}(\lambda_k)\, x_{2,k} = 0\,,$$
$$x_{1,k} = P_k x_1\,, \quad x_{2,k} = P_k x_2\,, \quad k = 1, 2, \ldots$$

But (1.2.9) is clearly equivalent to

(1.2.11)
$$(f_1, x_{1,k}) + (f_2, x_{2,k}) = 0 \quad \text{for} \quad x_{1,k}, x_{2,k} \in P_k H$$

fulfilling (1.2.10).

Choose a fixed k and write $a_{ij}(\lambda_k) = a_{ij}$ $(i, j = 1, 2)$. Suppose $a_{21} \neq 0$. Then by $(1.2.10_1)$ we have $x_{2,k} = -(a_{11}/a_{21})\, x_{1,k}$ and therefore (1.2.11) is equivalent to

$$\left(f_1 - \frac{a_{11}}{a_{21}} f_2, x_{1,k}\right) = 0\,, \quad x_{1,k} \in P_k H$$

i.e. to

(1.2.12)
$$P_k(A_{11} f_2 - A_{21} f_1) = 0\,.$$

Further, according to (1.2.12) and the equality $a_{11}a_{22} = a_{21}a_{12}$, it is

(1.2.13)
$$P_k(A_{22} f_1 - A_{12} f_2) = P_k\left(a_{22} f_1 - \frac{a_{11}a_{22}}{a_{21}} f_2\right) =$$
$$= \frac{a_{22}}{a_{21}} P_k(a_{21} f_1 - a_{11} f_2) = 0\,.$$

Hence writing $P = \sum_{k=1}^{\infty} P_k$ we have immediately from (1.2.12) and (1.2.13)

(1.2.14)
$$P(A_{11} f_2 - A_{21} f_1) = 0\,,$$
$$P(A_{22} f_1 - A_{12} f_2) = 0\,.$$

If $a_{21} = 0$ distinguishing several further cases we proceed quite analogously. Because of $P = E(\Lambda)$ and of (1.2.5) we can set

(1.2.15)
$$x_1 = \int_{-\infty}^{\infty} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda))^{-1}\, \mathrm{d}E(\lambda)\, (A_{22} f_1 - A_{12} f_2)\,,$$
$$x_2 = \int_{-\infty}^{\infty} (a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}(\lambda)\, a_{21}(\lambda))^{-1}\, \mathrm{d}E(\lambda)\, (A_{11} f_2 - A_{21} f_1)\,.$$

It may be easily verified that $[x_1, x_2]$ fulfils the equation $\mathbf{A}[x_1, x_2] = [f_1, f_2]$ and by (1.2.8) $|||[x_1, x_2]||| \leq c|||[f_1, f_2]|||$, c being a constant, which completes the proof.

2. PERIODIC SOLUTIONS TO THE FIRST ORDER LINEAR DIFFERENTIAL EQUATION

Let us investigate the equation

$$(2.1) \qquad u_t(t) + (A + \gamma I)\, u(t) = f(t), \quad t \in R^+$$

in a Banach space B, where A is a strongly positive, closed lineare operator, (i.e. $\|(\lambda I + A)^{-1}\| \leqq \text{const.}/(1 + |\lambda|)$, Re $\lambda \geqq 0$), $\mathscr{D}(A) \subseteq B \to B$, $-A$ generates a strongly continuous semigroup of linear bounded operators $T(t)$ $(t > 0$, $T(0) = I$, I being the identity operator) in B which have in consequence of the strong positiveness of A a holomorphic extension in the complex domain $\{z; \arg z < a, 0 < |z| < \infty\}$ $(0 < a \leqq \frac{1}{2}\pi)$ and $f : R^+ \to B$. A function $u : R^+ \to B$ is called a solution to (2.1) if

1° u is continuous on R^+
2° $u(t) \in \mathscr{D}(A)$ for $t > 0$
3° u_t is continuous on $(0, \infty)$
4° u fulfils the equation (2.1) on $(0, \infty)$.

Substituting

$$(2.2) \qquad u(t) = e^{-\gamma t}\, v(t), \quad t \geqq 0$$

in (2.1) we get

$$(2.3) \qquad v_t(t) + A\, v(t) = e^{\gamma t} f(t).$$

Proposition 2.1. (cf. [19] p. 170): *Let A be a strongly positive operator $(-A$ generating a semigroup $T(t))$ and let $A^\eta f(t)$ be continuous on R^+ for some $\eta > 0$. Then there exists a unique solution $v(t)$ of the equation (2.3) with the initial value*

$$(2.4) \qquad v(0) = \varphi \in B$$

and it is given by

$$(2.5) \qquad v(t) = T(t)\, \varphi + \int_0^t T(t - \tau)\, e^{\gamma\tau} f(\tau)\, d\tau, \quad t \in R^+.$$

Moreover if $\varphi \in \mathscr{D}(A^\nu)$ and $A^{\nu+\eta} f(t)$ is continuous on R^+ for some $\nu > 0$ then $u \in U_\nu(J)$ for any compact interval $J \subset R^+$.

The conditions ensuring the existence of a periodic solution in a Banach space and in a Hilbert space respectively are clarified by the following two theorems.

Theorem 2.1. *Let A be a strongly positive operator and let $f \in C(R^+; \mathscr{D}(A^{\nu+\eta}))$ $(\nu \geqq 0, \eta > 0$ being arbitrary$)$ be ω-periodic on R^+. Let $\mathscr{R}(e^{-\gamma\omega}\, T(\omega) - I)$ be closed.*

Then an ω-periodic solution u to (2.1) *exists if and only if*

$$\Pi \int_0^\omega e^{-\gamma\tau} T(\tau) f(\omega - \tau) \, d\tau = 0 , \tag{2.6}$$

where Π is the canonical transformation of B on $B/\mathscr{R}(e^{-\gamma\omega} T(\omega) - I)$. If the condition is satisfied then $u \in U_\nu(R^+)$.

Proof. Since all the solutions $u(t)$ of the equation (2.1) are given by (2.2), where $v(t)$ is a solution of (2.3), (2.4) with $\varphi \in B$ arbitrary, a necessary and sufficient condition for the existence of an ω-periodic solution (2.1) reads: There exists a $\varphi \in B$ such that

$$U(\varphi)(\omega) - U(\varphi)(0) = 0 , \tag{2.7}$$

where

$$U(\varphi)(t) = e^{-\gamma t} V(\varphi)(t) = e^{-\gamma t} T(t) \varphi + \int_0^t e^{-\gamma(t-\tau)} T(t-\tau) f(\tau) \, d\tau , \quad t \in R^+ .$$

The equation (2.7) is evidently equivalent to

$$(e^{-\gamma\omega} T(\omega) - I) \varphi = - \int_0^\omega e^{-\gamma\tau} T(\tau) f(\omega - \tau) \, d\tau. \tag{2.8}$$

As $\mathscr{R}(e^{-\gamma\omega} T(\omega) - I)$ is closed, the solution of (2.8) exists if and only if (2.6) holds. If (2.6) is fulfilled then (2.7) implies $\varphi \in \mathscr{D}(A^{\nu+\eta})$ and hence by Proposition 2.1 $u \in U_\nu(R^+)$.

Remark 2.1. If A^{-1} is compact then $T(t)$ is for $t > 0$ also compact and $\mathscr{R}(e^{-\gamma\omega} T(\omega) - I)$ is closed.

Theorem 2.2. *Let A be a selfadjoint operator $\mathscr{D}(A) \subseteq H \to H$ with* $\inf \sigma(A) = m > 0$ *and let $-\gamma$ be at most an isolated point of $\sigma(A)$. Let $f \in C(R^+; \mathscr{D}(A^{\nu+\eta}))$ ($\nu \geqq 0$, $\eta > 0$ arbitrary) be ω-periodic on R^+. Then an ω-periodic solution $u(t)$ to* (2.1) *exists iff*

$$\int_0^\omega e^{-\gamma\tau} (T(\tau) f(\omega - \tau), \zeta) \, d\tau = 0 \textit{ holds for } \quad \zeta \in \mathscr{N}(A + \gamma I) . \tag{2.9}$$

If the condition is satisfied then $u \in U_\nu(R^+)$.

Proof. According to the Theorem 2.1 it suffices to show that $\mathscr{R}(e^{-\gamma\omega} T(\omega) - I)$ is closed and that (2.9) is equivalent to (2.6). Since $e^{-\gamma\omega} T(\omega) - I = \int_m^\infty (e^{-\gamma\omega} e^{-\lambda\omega} - 1) \, dE(\lambda)$, where $E(\lambda)$ is a resolution of the identity corresponding to the operator A, is a selfadjoint operator, it suffices to prove that

$$\mathscr{N}(e^{-\gamma\omega} T(\omega) - I)^\perp \subset \mathscr{R}(e^{-\gamma\omega} T(\omega) - I) . \tag{2.10}$$

But if $\zeta \in \mathcal{N}(e^{-\gamma\omega} T(\omega) - I)^{\perp}$ and $\varepsilon > 0$ is sufficiently small then

$$\int_m^\infty (e^{-(\gamma+\lambda)\omega} - 1)^{-2} \, d(E(\lambda)\,\zeta, \zeta) = \left(\int_m^{-\gamma-\varepsilon} + \int_{-\gamma+\varepsilon}^\infty\right)$$

$$(e^{-(\gamma+\lambda)\omega} - 1)^{-2} \, d(E(\lambda)\,\zeta, \zeta) \leq \text{const} \, \|\zeta\|^2 < \infty \, .$$

Hence

$$(e^{-\gamma\omega} T(\omega) - I) \int_m^\infty (e^{-(\gamma+\lambda)\omega} - 1)^{-1} \, dE(\lambda)\,\zeta = \zeta \quad \text{and} \quad \zeta \in \mathcal{R}(e^{-\gamma\omega} T(\omega) - I) \, .$$

Condition (2.6) is obviously equivalent to

$$P \int_0^\omega e^{-\gamma\tau} \, T(\tau) f(\omega - \tau) \, d\tau = 0 \, , \tag{2.11}$$

where P is a projection on $\mathcal{N}(e^{-\gamma\omega} T(\omega) - I)$. However

$$\mathcal{N}(e^{-\gamma\omega} T(\omega) - I) = \left\{\zeta \in H \, ; \; \int_m^\infty (e^{-(\gamma+\lambda)\omega} - 1)^2 \, d(E(\lambda)\,\zeta, \zeta) = 0\right\} =$$

$$= (E(-\gamma +) - E(-\gamma))\, H = \mathcal{N}(A + \gamma I) \, ,$$

thus (2.11) is equivalent to (2.9). If (2.9) is satisfied then the smoothness of any ω-periodic solution $u(t)$ of (2.1) follows from $u(0) \in \mathcal{D}(A^{\nu+1})$ (it suffices to use Proposition 2.1).

Remark 2.2. Let the assumptions of Theorem 2.2 be fulfilled with $\nu = 0$. Define the operators Θ_0, Θ_1 by the equalities

$$\Theta_0 u = u_t + (A + \gamma I)\, u \, , \quad \Theta_1 u = -u_t + (A + \gamma I)\, u$$

for $u \in \mathcal{D}(\Theta_0) = \mathcal{D}(\Theta_1) = \{u \in U_0(\langle 0, \omega \rangle); \, u(0) = u(\omega)\}$. It may be shown easily that $\mathcal{D}(\Theta_0)$ is dense in $L_2(\langle 0, \omega \rangle; H)$. Since Θ_1^* is an extension of Θ_0 we may put $\Theta = \Theta_1^*$ for the closed extension of Θ_0. Further, we have $\mathcal{N}(\Theta_1) = \{u \in L_2(\langle 0, \omega \rangle; H); \, u(t) = \zeta \in \mathcal{N}(A + \gamma I)\}$ and $\mathcal{N}(\Theta^*) = \mathcal{N}(\Theta_1)^c$. In Theorem 2.2 we have proved that $\mathcal{N}(\Theta^*)^{\perp} \cap C(\langle 0, \omega \rangle; \mathcal{D}(A^n)) \subseteq \mathcal{R}(\Theta_0)$. Now prove that $\mathcal{N}(\Theta^*)^{\perp} \subseteq \mathcal{R}(\Theta)$. Indeed, if $f \in \mathcal{N}(\Theta^*)^{\perp}$ then there exist $f_n \in \mathcal{N}(\Theta^*)^{\perp} \cap C(\langle 0, \omega \rangle; \mathcal{D}(A^n))$ such that $f_n \to f$ in $L_2(\langle 0, \omega \rangle; H)$. As $f_n \in \mathcal{R}(\Theta_0)$, we can write the solutions $u_n \in \mathcal{D}(\Theta_0)$ of the equations

$$\Theta_0 u = f_n \, , \quad n = 1, 2, \ldots \, ,$$

in the form

$$u_n(t) = -e^{-\gamma t} \, T(t) \, [(I - P)(e^{-\gamma\omega} T(\omega) - I)]^{-1} \int_0^\omega e^{-\gamma\tau} \, T(\omega - \tau) f_n(\tau) \, d\tau +$$

$$+ \int_0^t e^{-\gamma(t-\tau)} \, T(t - \tau) f_n(\tau) \, d\tau \, ,$$

(cf. (2.2), (2.5), (2.8)), from which follows that the sequence u_n converges to some $u \in L_2(\langle 0, \omega \rangle; H)$. In virtue of closedness of Θ we have $u \in \mathscr{D}(\Theta)$ and $\Theta u = f$. So we have proved that Θ is normally solvable and that the necessary and sufficient condition for the existence of a solution $u \in \mathscr{D}(\Theta)$ of the equation $\Theta u = f$, $f \in L_2(\langle 0, \omega \rangle; H)$, reads:

(2.9′) $$\int_0^\omega (f(t), v(t))\, dt = 0 \text{ holds for every } v \in \mathscr{N}(\Theta^*) .$$

This solution may be considered as the generalized solution to (2.1).

The stability of the found ω-periodic solution is described by the following

Theorem 2.3. *Let the assumptions and the condition (2.9) of Theorem 2.2 be fulfilled. If $\gamma + m > 0$ and $\gamma + m \geqq 0$ and $\gamma + m < 0$ respectively then the found ω-periodic solution is exponentially stable and stable and unstable respectively.*

Proof. Consequently to the linearity of the problem the periodic solution has the same stability property as the trivial solution of the corresponding homogenous problem. Denote $T_1(t) = e^{-\gamma t}\, T(t)$ the semigroup generated by $-A - \gamma I$. If $\gamma + m > 0$, we have

$$\|T_1(t)\, \varphi\|^2 = \int_m^\infty e^{-2(\lambda+\gamma)t}\, d(E(\lambda)\, \varphi, \varphi) \leqq e^{-2(m+\gamma)t} \|\varphi\|^2 , \quad t \in R^+ ;$$

if $\gamma + m = 0$ we have for sufficiently small $\varepsilon > 0$ and $t \in R^+$

$$\|T_1(t)\, \varphi\|^2 = \left(\int_{\{-\gamma\}} + \int_{-\gamma+\varepsilon}^\infty\right) e^{-2(\lambda+\gamma)t}\, d(E(\lambda)\, \varphi, \varphi) \leqq \|E(-\gamma)\, \varphi\|^2 + e^{-2\varepsilon t}\|\varphi\|^2 ;$$

finally if $\gamma + m < 0$ we have for sufficiently small $\varepsilon > 0$ and $t \in R^+$

$$\|T_1(t)\, \varphi\|^2 = \left(\int_m^{-\gamma-\varepsilon} + \int_{-\gamma+\varepsilon}^\infty\right) e^{-2(\lambda+\gamma)t}\, d(E(\lambda)\, \varphi, \varphi) + \|(E(-\gamma+) - E(-\gamma))\, \varphi\|^2 \geqq$$
$$\geqq e^{2\varepsilon t}\|E(-\gamma - \varepsilon)\, \varphi\|^2 + \|(E(-\gamma+) - E(-\gamma))\, \varphi\|^2 ,$$

whence our assertion follows immediately.

Remark 2.3. In the case of a Banach space, provided that $-(A + \gamma I)$ generates a strongly continuous exponentially decreasing semigroup $T_1(t)$ (if $B = H$ and A is selfadjoint it is ensured by $m + \gamma > 0$) C. T. Taam in [1], [2] proved that

(2.12) $$\int_0^\infty T_1(s)\, f(t - s)\, ds$$

defines a unique ω-periodic solution which is exponentially stable. So far we have supposed that A in (2.1) si strongly positive what implies that $-A$ generates an analytic semigroup. To be able to investigate e.g. the ω-periodic solution of Schrödinger equation we shall now suppose that $-A$ generates a strongly continuous semigroup not necessarily analytic and that $\gamma = 0$. Of course we have to make more restrictive assumption on the right hand side of (2.1). The following Proposition may be easily proved.

Proposition 2.2 (cf. [19] p. 169): *Let $-A$ generate a strongly continuous semigroup $T(t)$ $(t \in R^+)$ and let $A f(t)$ be continuous on R^+. Then there exists a unique solution $u(t)$ of the equation* (2.1) *with the initial value*

$$u(0) = \varphi \in \mathscr{D}(A) \tag{2.13}$$

and it is given by

$$u(t) = T(t)\,\varphi + \int_0^t T(t-\tau)\,f(\tau)\,\mathrm{d}\tau\,, \quad t \in R^+ \tag{2.14}$$

moreover, $u_t(t)$ and $A\,u(t)$ are continuous functions on R^+.

Similarly to the Theorem 2.1 we can derive the following one.

Theorem 2.4. Let $-A$ *generates a strongly continuous semigroup $T(t)$ and let $f \in C(R^+; \mathscr{D}(A))$ be ω-periodic on R^+. Let $\mathscr{R}(T(\omega) - I)$ be closed. Then an ω-periodic solution to* (2.1) $(\gamma = 0)$ *exists iff* (2.6) *holds with* $\gamma = 0$.

The specialization to the Hilbert space case is rather different from that obtained in Theorem 2.2. We restrict ourselves to the case $A = iB$, B being selfadjoint.

Proposition 2.3. *Let $A = iB$, where B is a selfadjoint operator $\mathscr{D}(B) \subseteq H \to H$. If $\varphi \in \mathscr{D}(A^\nu)$, $A^\nu f(t)$ where $\nu > 0$, is continuous on R^+ and $u(t)$ is a function given by* (2.14) *then $u \in U_\nu(J)$ for any compact interval $J \subset R^+$.*

Theorem 2.5. *Let $A = iB$, where B is a selfadjoint operator $\mathscr{D}(B) \subseteq H \to H$ such, that there exist constants $c > 0$, $\varrho \geqq 0$ that*

$$\min_{l=0,\pm1,\ldots} \left|\lambda - \frac{2\pi l}{\omega}\right| \geqq \frac{c}{\lambda^\varrho}\,, \quad \lambda \in M = \sigma(B) \setminus \left\{\frac{2k\pi}{\omega}\right\}_{k=0,\pm1,\ldots} \tag{2.15}$$

holds and that there exists some real β in the resolvent set of B. Let $f \in C(R^+; \mathscr{D}(B^{1+\varrho+\nu}))$ be ω-periodic on R^+. Then an ω-periodic solution $u(t)$ to (2.1) *(with $\gamma = 0$) exists iff*

$$\int_0^\omega (T(\tau)\,f(\omega - \tau), \zeta)\,\mathrm{d}\tau = 0\,, \quad \zeta \in \mathscr{N} = \sum_{k=-\infty}^{\infty} \mathscr{N}\left(B + \frac{2k\pi}{\omega} I\right), \tag{2.16}$$

where $T(t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\,\mathrm{d}E(\lambda)$ $(t \in R^+$, $E(\lambda)$ is the resolution of the identity cor-

responding to B) is a semigroup generated by $-A$. If the condition is satisfied then $u \in U_\nu(R^+)$.

Proof. It is clear that

$$\mathcal{N}((B-\beta I)^\varrho (T(\omega)-I)) = \mathcal{N}(T(\omega)-I) =$$

$$= \sum_{k=-\infty}^{\infty} \left(E\left(\frac{2k\pi}{\omega}+\right) - E\left(\frac{2k\pi}{\omega}\right)\right) H = \sum_{k=-\infty}^{\infty} \mathcal{N}\left(B + \frac{2k\pi}{\omega} I\right) = \mathcal{N} .$$

Since $T(\omega) = \int_{-\infty}^{\infty} e^{-i\lambda\omega} \, dE(\lambda)$, we have for $\zeta \in \mathcal{N}^\perp$

$$\|(B-\beta I)^\varrho (T(\omega)-I)\zeta\|^2 = \int_{-\infty}^{\infty} |\lambda-\beta|^{2\varrho} |e^{i\lambda\omega}-1|^2 \, d(E(\lambda)\zeta,\zeta) =$$

$$= 4 \int_M |\lambda-\beta|^{2\varrho} \sin^2 \frac{\omega}{2}\lambda \, d(E(\lambda)\zeta,\zeta) .$$

But if $\lambda \in M$ and $k(\lambda)$ is such an integer that $|\lambda - 2k(\lambda)\pi/\omega| = \min_{l=0,\pm1,\ldots} |\lambda - 2l\pi/\omega|$ then using (2.15) we obtain

$$|\lambda-\beta|^{2\varrho} \sin^2 \frac{\omega}{2}\lambda = \frac{\omega}{2} |\lambda-\beta|^{2\varrho} \left|\lambda - \frac{2k(\lambda)\pi}{\omega}\right| \frac{\sin^2\left(\frac{\omega}{2}\lambda - k(\lambda)\pi\right)}{\left|\frac{\omega}{2}\lambda - k(\lambda)\pi\right|^2} \geqq$$

$$\geqq \frac{\omega}{2} c^2 \min_{\lambda \in M} \left(\frac{|\lambda-\beta|}{\lambda}\right)^{2\varrho} = c_1^2 , \quad c_1 > 0 ,$$

and accordingly $\|(B-\beta I)^\varrho (T(\omega)-I)\zeta\| \geqq 2c_1\|\zeta\|$. So we have proved that $\mathcal{R}((B-\beta I)^\varrho (T(\omega)-I))$ is closed. As the β belongs to the resolvent set of B, we can write every $\varphi \in H$ in the form $\varphi = (B-\beta I)^\varrho \tilde{\varphi}$, where $\tilde{\varphi} = (B-\beta I)^{-\varrho} \varphi$. Writing the solution $u(t)$ of the equation (2.1) with the initial condition $u(0) = \varphi$ in the form (2.14) we see that the necessary and sufficient condition for $u(t)$ to be ω-periodic is that the equation

$$(B-\beta I)^\varrho (T(\omega)-I)\tilde{\varphi} = -\int_0^\omega T(\omega-\tau) f(\tau) \, d\tau \tag{2.17}$$

has a solution $\tilde{\varphi} \in \mathcal{D}(B^\varrho)$. It is easy to see in virtue of $f(t) \in \mathcal{D}(B^{1+\varrho+\nu})$ $(t \in R^+)$ that if (2.17) has a solution $\tilde{\varphi} \in H$ then $\tilde{\varphi} \in \mathcal{D}(B^{1+\varrho+\nu})$. The solution $\tilde{\varphi} \in H$ of (2.17) exists iff

$$\int_0^\omega T(\omega-\tau) f(\tau) \, d\tau \in \mathcal{N}((B-\beta I)^\varrho (T(\omega)-I))^\perp . \tag{2.18}$$

But as $\mathcal{N}((B-\beta I)^\varrho (T(\omega)-I))^\perp = \mathcal{N}^\perp$, it is obvious after simple calculations that (2.18) is equivalent to (2.16). Finally if (2.16) is satisfied then (2.17) holds with

$\tilde{\varphi} \in \mathscr{D}(B^{1+\varrho+\nu})$ which implies $u(0) \in \mathscr{D}(B^{1+\nu})$ for the ω-periodic solution $u(t)$ and hence $u \in U_\nu(R^+)$.

Remark 2.4. Let the assumption of Theorem 2.3 be fulfilled with $\nu = 1$. Define the operators Θ_0, Θ_1 by the equalities

$$\Theta_0 u = B^\varrho u_t + iB^{\varrho+1} u ,$$

$$\Theta_1 u = -B^\varrho u_t - iB^{\varrho+1} u , \quad \text{for} \quad u \in \mathscr{D}(\Theta_0) = \mathscr{D}(\Theta_1) =$$
$$= \{u \in C^1(\langle 0, \omega\rangle; \mathscr{D}(B^\varrho)) \cap C(\langle 0, \omega\rangle; \mathscr{D}(B^{\varrho+1})); \ u(0) = u(\omega)\} .$$

We can similarly as in Remark 2.2 prove that any closed extension Θ of Θ_0 is normally solvable and that the necessary and sufficient condition for the existence of the solution u^* to the equation

$$\Theta u = f$$

is the same but now it is

$$\mathscr{N}(\Theta_1) = \left\{u \in \mathscr{D}(\Theta_1); \ u(t) = T(t)\,\zeta , \quad \text{where} \quad \zeta \in \sum_{k=-\infty}^{\infty} \mathscr{N}\left(B + \frac{2k\pi}{\omega} I\right)\right\}$$

and of course $\mathscr{N}(\Theta^*) = \mathscr{N}(\Theta_1)^c$.

Theorem 2.6. *Let the assumptions and the condition* (2.16) *of Theorem* 2.5 *be fulfilled. Then the found ω-periodic solution of* (2.1) (*with $\gamma = 0$*) *is stable.*

The proof follows immediately from the equality $\|\int_{-\infty}^{\infty} e^{-i\lambda t}\, dE(\lambda)\, \varphi\|^2 = \|\varphi\|^2$ holding for $\varphi \in H$, $t \in R^+$.

3. PERIODIC SOLUTIONS TO THE SECOND ORDER LINEAR DIFFERENTIAL EQUATION

For the sake of uniformity of the treatment we work here only in a Hilbert space since only some special cases we are able to investigate in a Banach space.

3.1. Equation with a dissipative term. Let the equation

(3.1.1) $$u_{tt}(t) + 2(\alpha + \beta A)\, u_t(t) + (A + \gamma I)\, u(t) = f(t) , \quad t \in R^+$$

with the initial data

(3.1.2) $$u(0) = \varphi \in \mathscr{D}(A) , \quad (\alpha + \beta A)\, u(0) + u_t(0) = \psi \in H ,$$

where A is a selfadjoint operator $\mathscr{D}(A) \subset H \to H$, $A = \int_m^\infty \lambda\, dE(\lambda)$, $m > 0$, $\alpha, \beta \geqq 0$, γ are real numbers and $f : R^+ \to H$ be given. A function $u(t)$ is called the solution of (3.1.1), (3.1.2) if $u \in U_2(J)$, for any compact interval $J \subset R^+$, $u(t)$ satisfies (3.1.1) in R^+ and $u(0)$, $u_t(0)$ satisfy (3.1.2).

Proposition 3.1.1. *If $u(t)$ is a solution of* (3.1.1)–(3.1.2) *then*

$$u(t) = J(t)\,\varphi + K(t)\,\psi + \int_0^t K(t-\tau) f(\tau)\,\mathrm{d}\tau\,, \quad t \in R^+\,, \tag{3.1.3}$$

where

$$J(t) = \int_m^\infty e^{-(\alpha+\beta\lambda)t} \cos t[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}]\,\mathrm{d}E(\lambda)\,,$$

$$K(t) = \int_m^\infty e^{-(\alpha+\beta\lambda)t} \frac{\sin t[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}]}{[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}]}\,\mathrm{d}E(\lambda)\,.$$

On the other hand if $u(t)$ is given by (3.1.3) *with $\varphi \in \mathscr{D}(A) \cap \mathscr{D}(\beta A^2)$, $\psi \in \mathscr{D}(A^{1/2}) \cap \cap \mathscr{D}(\beta A)$, $f \in C(R^+, \mathscr{D}(A^{1/2}) \cap \mathscr{D}(\beta A))$ then $u(t)$ is a solution to* (3.1.1)–(3.1.2).

Proof. Let $u(t)$ be a solution to (3.1.1)–(3.1.2). Writing in (3.1.1) τ instead of t aplying the operator $K(t-\tau)$ to it integrating by parts and making use of (3.1.2) we verify easily that $u(t)$ satisfies (3.1.3).

On the contrary if $u(t)$ is given by (3.1.3) with $\varphi \in \mathscr{D}(A) \cap \mathscr{D}(\beta A^2)$, $\psi \in \mathscr{D}(A^{1/2}) \cap \cap \mathscr{D}(\beta A)$, $f \in C(R^+; \mathscr{D}(A^{1/2}) \cap \mathscr{D}(\beta A))$ then using Proposition 1.1.2–1.1.5 we find easily that $u(t)$ is a solution to (3.1.1)–(3.1.2).

If a smoother solution is required then the following proposition the proof of which follows from (3.1.3) and from. Propositions 1.1.2–1.1.5 may be useful.

Proposition 3.1.2. *If $\varphi \in \mathscr{D}(A^{\nu+1}) \cap \mathscr{D}(\beta A^{\nu+2})$, $\psi \in \mathscr{D}(A^{\nu+1/2}) \cap \mathscr{D}(\beta A^{\nu+1})$, $f \in \in C(R^+; \mathscr{D}(A^{\nu+1/2}) \cap \mathscr{D}(\beta A^{\nu+1}))$, $(\nu \geqq 0)$ and $u(t)$ is a function defined by* (3.1.3) *then $u \in U^\beta_\nu(\langle 0, \omega \rangle)$, $\forall \omega > 0$.*

Since a solution $u(t) = U(\varphi, \psi)(t)$ of (3.1.1)–(3.1.2) is uniquely determined by φ and ψ, there exists an ω-periodic solution to (3.1.1) iff there exist $\varphi \in \mathscr{D}(A)$, $\psi \in \in \mathscr{D}(A^{1/2}) \cap \mathscr{D}(\beta A)$ such that

$$\begin{aligned} U(\varphi, \psi)(\omega) - U(\varphi, \psi)(0) &= 0\,, \\ U_t(\varphi, \psi)(\omega) - U_t(\varphi, \psi)(0) &= 0\,. \end{aligned} \tag{3.1.4}$$

Inserting (3.1.3) into (3.1.4), denoting $\varkappa = \frac{1}{2}$ if $\beta = 0$ and $\varkappa = 1$ if $\beta \neq 0$ and putting

$$\tilde{\varphi} = A^\varkappa \varphi\,, \tag{3.1.5}$$

we have

$$\begin{aligned} -A^\varkappa K(\omega)\,\psi + (I - J(\omega))\,\tilde{\varphi} &= A^\varkappa \int_0^\omega K(\omega - \tau) f(\tau)\,\mathrm{d}\tau \\ (I - J(\omega))\,\psi + A^{-\varkappa}(A + \gamma I - (\alpha I + \beta A)^2)\,K(\omega)\,\tilde{\varphi} &= \int_0^\omega J(\omega - \tau) f(\tau)\,\mathrm{d}\tau\,. \end{aligned} \tag{3.1.6}$$

Applying Lemma 1.2.1 to (3.1.6) we get the following

Theorem 3.1.1. *Let $\alpha, \beta \in R^+$, $\alpha + \beta > 0$, $\gamma, \nu \geqq 1$ be real numbers and let $-\gamma$ be at most an isolated point of $\sigma(A)$. Let $f \in C(R^+; \mathscr{D}(A^{\nu+1/2}) \cap \mathscr{D}(\beta A^{\nu+1}))$ be ω-periodic on R^+. Then there exists an ω-periodic solution $u(t)$ to (3.1.1) iff*

$$\int_0^\omega (f(\tau), \zeta)\,\mathrm{d}\tau = 0\,, \quad \zeta \in \mathscr{N}(A + \gamma I)\,. \tag{3.1.7}$$

If the condition is satisfied then $u \in U_\nu^\beta(R^+)$.

Proof. Denote

$$A_{ij} = \int_m^\infty a_{ij}(\lambda)\,\mathrm{d}E(\lambda)\,, \quad i, j = 1, 2, \ldots\,, \tag{3.1.8}$$

$$f_1 = \int_0^\omega e^{-(\alpha+\beta A)(\omega-\tau)} A^\varkappa \frac{\sin \tau[\sqrt{(A + \gamma - (\alpha + \beta A)^2)}]}{\sqrt{(A + \gamma - (\alpha + \beta A)^2)}} f(\tau)\,\mathrm{d}\tau\,,$$

$$f_2 = \int_0^\omega e^{-(\alpha+\beta A)(\omega-\tau)} \cos \tau[\sqrt{(A + \gamma - (\alpha + \beta A)^2)}] f(\tau)\,\mathrm{d}\tau\,,$$

$$D = A_{11}A_{22} - A_{12}A_{21}\,,$$

where

$$a_{11}(\lambda) = -e^{-(\alpha+\beta\lambda)\omega}\lambda^\varkappa \frac{\sin \omega[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2}]}{\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}}\,,$$

$$a_{12}(\lambda) = a_{21}(\lambda) = 1 - e^{-(\alpha+\beta\lambda)\omega} \cos \omega[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}]\,,$$

$$a_{22}(\lambda) = e^{-(\alpha+\beta\lambda)\omega} \frac{\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}}{\lambda^\varkappa} \sin \omega[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}]\,,$$

$$\lambda \in \langle\, m, \infty)\,.$$

Clearly,

$$d(\lambda) \equiv a_{11}(\lambda)\, a_{22}(\lambda) - a_{12}^2(\lambda) =$$
$$= -e^{-2(\alpha+\beta\lambda)\omega} + 2e^{-(\alpha+\beta\lambda)\omega} \cos \omega[\sqrt{\lambda + \gamma - (\alpha + \beta\lambda)^2}] - 1$$

and

$$\Lambda = \{\lambda \in \langle m, \infty);\ d(\lambda) = 0\} \subseteq \{-\gamma\}\,.$$

Further

$$d(\lambda) \leqq -e^{-2(\alpha+\beta\lambda)\omega} + 2e^{-(\alpha+\beta\lambda)\omega} - 1 = -(1 - e^{-(\alpha+\beta\lambda)\omega})^2 \leqq$$
$$\leqq -(1 - e^{-(\alpha+\beta m)})^2 < 0$$

for $\lambda \in \langle m, \infty)$ such that $\lambda + \gamma - (\alpha + \beta\lambda)^2 \geqq 0$ and

$$|d(\lambda)| = |e^{-2\alpha\omega} - 2e^{-\alpha\omega} \operatorname{ch} \omega[\sqrt{((\alpha + \beta\lambda)^2 - \lambda - \gamma)}]| > 0$$

for $\lambda \in \langle m, \infty) \setminus \Lambda$ such that $\lambda + \gamma - (\alpha + \beta\lambda)^2 < 0$. As $-\gamma$ is an isolated point of $\sigma(A)$, there exists $\delta_0 > 0$ such that $|d(\lambda)| \geqq \delta_0$ for $\lambda \in \sigma(A) \setminus \Lambda$ (it suffices to set

$$\delta_0 = \min\left[(1 - e^{-(\alpha+\beta m)})^2,\ \inf\left\{\left|\operatorname{ch}\alpha\omega - \operatorname{ch}\omega[\sqrt{(\lambda + \gamma - (\alpha + \beta\lambda)^2)}]\right|;\right.\right.$$
$$\left.\left.\lambda \in \sigma(A) \cap \{\lambda \in R;\ \lambda + \gamma - (\alpha + \beta\lambda)^2 < 0\} \setminus \Lambda\right\}\right].$$

Hence all assumptions of Lemma 1.2.1 are satisfied and a solution to (3.1.6) exists iff

$$(3.1.9) \qquad [f_1, f_2] \perp \mathcal{N}(\mathbf{A}),$$

where $\boldsymbol{A}$ is defined by (1.2.1). But

$$(3.1.10) \qquad \mathcal{N}(\mathbf{A}) = \left\{[\psi, \tilde{\varphi}] \in \mathbf{H};\ \psi, \tilde{\varphi} \in \mathcal{N}(A + \gamma I),\ \psi = \frac{\alpha - \beta\gamma}{|\gamma|^\varkappa}\tilde{\varphi}\right\},$$

since $\mathcal{N}(D) = (E(-\gamma+) - E(-\gamma))H = \mathcal{N}(A + \gamma I)$ and the system (3.1.6) with $\boldsymbol{f} = 0$ is on $\mathcal{N}(A + \gamma I)$ equivalent to the equation

$$\psi = \frac{\alpha - \beta\gamma}{|\gamma|^\varkappa}\tilde{\varphi} \quad (\text{for } \gamma \neq 0;\ \text{otherwise } \alpha\tilde{\varphi} = 0).$$

It remains to prove that (3.1.9) is equivalent to (3.1.7). But (3.1.9) may be rewritten in the form

$$(f_1, \psi) + (f_2, \tilde{\varphi}) = 0, \quad [\psi, \tilde{\varphi}] \in \mathcal{N}(\mathbf{A})$$

i.e.

$$(3.1.10) \qquad \left(f_2 + \frac{\alpha - \beta\gamma}{|\gamma|^\varkappa} f_1, \tilde{\varphi}\right) = 0, \quad \tilde{\varphi} \in \mathcal{N}(A + \gamma I).$$

Denoting $P = E(\gamma+) - E(\gamma)$ and using (3.1.8) we have for $\psi \in \mathcal{N}(A + \gamma I)$

$$0 = \left(\int_0^\omega e^{-(\alpha+\beta A)\tau}\left[A^\varkappa \frac{\sin\tau(\sqrt{(A + \gamma - (\alpha + \beta A)^2)})}{\sqrt{(A + \gamma - (\alpha + \beta A)^2)}} + \right.\right.$$
$$\left.\left. + \frac{|\gamma|^\varkappa}{\alpha - \beta\gamma}\cos\tau(\sqrt{(A + \gamma - (\alpha + \beta A)^2)})\right] f(\tau)\,\mathrm{d}\tau, P\psi\right) =$$
$$= \left(\int_0^\omega e^{-(\alpha+\beta A)\tau}\left[A^\varkappa \frac{\operatorname{sh}\tau(\sqrt{((\alpha + \beta A)^2 - A - \gamma)})}{\sqrt{((\alpha + \beta A)^2 - A - \gamma)}} + \right.\right.$$
$$\left.\left. + \frac{|\gamma|^\varkappa}{\alpha - \beta\gamma}\operatorname{ch}\tau(\sqrt{((\alpha + \beta A)^2 - A - \gamma)})\right] P f(\tau)\,\mathrm{d}\tau, \psi\right) =$$
$$= \left(\int_0^\omega e^{-(\alpha-\beta\gamma)\tau}\left[\frac{|\gamma|^\varkappa}{|\alpha - \beta\gamma|}\operatorname{sh}\tau|\alpha - \beta\gamma| + \frac{|\gamma|^\varkappa}{\alpha - \beta\gamma}\operatorname{ch}\tau|\alpha - \beta\gamma|\right] P f(\tau)\,\mathrm{d}\tau, \psi\right) =$$
$$= \frac{|\gamma|^\varkappa}{|\alpha - \beta\gamma|}\int_0^\omega (f(\tau), \psi)\,\mathrm{d}\tau$$

for $\alpha \neq \beta\gamma$; otherwise we proceed anologously. Expressing the solution $[\psi, \tilde{\varphi}]$ of (3.1.6) similarly as in (1.2.16) and making use of Proposition 3.1.2 we find easily that every ω-periodic solution of (3.1.1) belongs to $U_\nu^\beta(R^+)$.

Remark 3.1.1. Let the assumptions of Theorem 3.1.1 be fulfilled with $\nu = 1$. Define the operators Θ_0, Θ_1 by the equalities

$$\Theta_0 u = u_{tt} + (\alpha + \beta A) u_t + (A + \gamma) u$$

$$\Theta_1 u = u_{tt} - (\alpha + \beta A) u_t + (A + \gamma) u$$

for $u \in \mathscr{D}(\Theta_0) = \mathscr{D}(\Theta_1) = \{u \in U_0^\beta(\langle 0, \omega \rangle), u(0) = u(\omega), u_t(0) = u_t(\omega)\}$. If we define the operator Θ in the same way as in Remark 2.2 we can prove similarly the normal solvability of Θ. A solution to the equation $\Theta u = f$ exists iff (2.9′) is fulfilled. Here $\mathscr{N}(\Theta_1) = \{u \in \mathscr{D}(\Theta_1); u(t) = \zeta \in \mathscr{N}(A + \gamma I)\}$ and, of course, $\mathscr{N}(\Theta^*) = \mathscr{N}(\Theta_1)^c$.

Corollary 3.1.1. *If $\gamma + m > 0$ then $-\gamma \notin \sigma(A)$, (3.1.7) is fulfilled for every $f \in C(R^+; \mathscr{D}(A^{1/2}) \cap \mathscr{D}(\beta A))$ and there exists a unique ω-periodic solution to (3.1.1). It may be easily shown that this solution is given by*

$$u(t) = \int_0^\infty K(s) f(t - s) \, ds \,, \quad t \in R^+ \,.$$

This procedure may be generalized for the case of the Banach space. That will be shown in the paper of M. Sova which is in preparation.

Theorem 3.1.2. *Let the assumptions and the condition (3.1.7) of Theorem 3.1.1 be fulfilled. If $\gamma + m > 0$ and $\gamma + m \geqq 0$ and $\gamma + m < 0$ respectively then every ω-periodic solution is exponentially stable and stable and unstable respectively.*

The proof is analogous to that of Theorem 2.3.

3.2. Equation without a dissipative term. In this section the equation

$$u_{tt}(t) + (A + \gamma I) u(t) = f(t) \,, \quad t \in R^+ \tag{3.2.1}$$

is dealt with.

Theorem 3.2.1. *Let $\gamma, \nu \geqq 0$ be real numbers. Let A be a selfadjoint operator $\mathscr{D}(A) \subset H \to H$, $\inf \sigma(A) = m > 0$ and let there exist numbers $c > 0$, $\varrho \geqq 0$ such that*

$$\inf_{k=0,\pm 1,\dots} \left| \sqrt{(\lambda + \gamma)} - \frac{2k\pi}{\omega} \right| \geqq \frac{c}{\lambda^\varrho}, \quad \lambda \in \sigma(A) \setminus \left\{ \frac{4k^2\pi^2}{\omega^2} - \gamma \right\}_{k=0,\pm 1,\dots} \tag{3.2.2}$$

holds. Let $f \in C(R^+; \mathscr{D}(A^{\varrho+\nu+1/2}))$ *be* ω*-periodic on* R^+. *Then there exists an* ω*-periodic solution* $u(t)$ *to* (3.2.1) *iff*

$$\text{(3.2.3)} \qquad \int_0^\omega \left(\frac{A^{1/2} \sin \tau(\sqrt{(A + \gamma I)})}{\sqrt{(A + \gamma I)}} f(\tau), \psi \right) d\tau = 0 \quad \textit{for}$$

$$\psi \in \mathscr{N}_1 = \mathscr{N}\left(\frac{\sin \frac{1}{2}\omega(\sqrt{(A + \gamma I)})}{\sqrt{(A + \gamma I)}} \right), \int_0^\omega (\cos \tau(\sqrt{(A + \gamma I)}) f(\tau), \varphi) \, d\tau = 0 \quad \textit{for}$$

$$\varphi \in \mathscr{N}_2 = \mathscr{N}\left(\sin \frac{\omega}{2} (\sqrt{(A + \gamma I)}) \right).$$

If the condition is satisfied then $u \in U_\nu^0(R^+)$.

Proof. Now an ω-periodic solution to (3.2.1) exists iff the system

$$\text{(3.2.4)} \qquad - \frac{\sin \omega(\sqrt{A + \gamma I})}{\sqrt{(A + \gamma I)}} \psi + (1 - \cos \omega(\sqrt{(A + \gamma I)})) \varphi = = \int_0^\omega \frac{\sin \tau(\sqrt{A + \gamma I})}{\sqrt{(A + \gamma I)}} f(\tau) \, d\tau$$

$$(1 - \cos \omega(\sqrt{(A + \gamma I)})) \psi + \sqrt{(A + \gamma I)} \sin \omega (\sqrt{(A + \gamma I)}) \varphi = = \int_0^\omega \cos \tau(\sqrt{(A + \gamma I)}) f(\tau) \, d\tau$$

for the initial values $u(0) = \varphi$, $u_t(0) = \psi$ has a solution. Retaining the notation (3.1.8), (3.1.5) with $\alpha = \beta = 0$, (3.2.4) is equivalent to

$$\mathbf{A}[\psi, \tilde{\varphi}] = [f_1, f_2],$$

where $\mathbf{A}$ is given by (1.2.1). In the sequel we make use the following

Lemma 3.2.1. *If* $g_1, g_2 \in \mathscr{D}(A^\varrho)$ *then* $[g_1, g_2] \in \mathscr{R}(\mathbf{A})$ *iff* $[g_1, g_2] \in \mathscr{N}(\mathbf{A})^\perp$.

Proof of Lemma. Obviously

$$d(\lambda) = a_{11}(\lambda) \, a_{22}(\lambda) - a_{12}^2(\lambda) = -4 \sin^2 \frac{\omega}{2} (\sqrt{(\lambda + \gamma)})$$

and

$$\Lambda = \{\lambda \in R; d(\lambda) = 0\} = \left\{ \frac{4k^2\pi^2}{\omega^2} - \gamma \right\}_{k=0,\pm1,\dots}$$

Set $[g_1, g_2] \in \mathscr{N}(\mathbf{A})^\perp$. Let $P_k = E(\lambda_k +) - E(\lambda_k)$ where, $\lambda_k = 4k^2\pi^2/\omega^2 - \gamma$, $k =$

$= 0, \pm 1, \ldots$. Similarly as in the proof of Lemma 1.2.1 we find easily

$$P_k(A_{21}g_1 - A_{11}g_2) = 0\,,$$
$$P_k(A_{22}g_1 - A_{12}g_2) = 0\,, \quad k = 0, \pm 1, \ldots$$

Hence $A_{21}g_1 - A_{11}g_2$, $A_{22}g_1 - A_{12}g_2$ are ortogonal to

$$PH = \sum_{k=-\infty}^{\infty} P_k H = \mathcal{N}(D)\,.$$

Since

$$\int_m^\infty d^{-2}(\lambda)\, \mathrm{d}(E(\lambda)\,(A_{21}g_1 - A_{11}g_2),\, A_{21}g_1 - A_{11}g_2) =$$
$$= \int_m^\infty \frac{\sin^2 \frac{1}{2}\omega(\sqrt{(\lambda + \gamma)})}{16 \sin^4 \frac{1}{2}\omega(\sqrt{(\lambda + \gamma)})} \Bigg(\mathrm{d}E(\lambda) \Bigg(2 \sin \tfrac{1}{2}\omega(\sqrt{(A + \gamma)})\, g_1 +$$
$$+ \left(\frac{A}{A + \gamma I}\right)^{1/2} \cos \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)})\, g_2\Bigg),$$
$$\sin \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)})\, g_1 + \left(\frac{A}{A + \gamma I}\right)^{1/2} \cos \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)})\, g_2\Bigg) \leqq$$
$$\leqq \text{const.} \int_m^\infty \lambda^{2\varrho}(\mathrm{d}E(\lambda)\,\zeta, \zeta) = \text{const.}\ \|A^\varrho \zeta\|^2 < \infty\,,$$

where

$$\zeta = 2 \sin \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)})\, g_1 + \left(\frac{A}{A + \gamma I}\right)^{1/2} \cos \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)})\, g_2 \in \mathcal{D}(A^\varrho)\,,$$

we may put

$$\psi = \int_m^\infty d^{-1}(\lambda)\, \mathrm{d}E(\lambda)\,(A_{21}g_1 - A_{11}g_2)\,.$$

According to

$$\int_m^\infty d^{-2}(\lambda)\, \mathrm{d}(E(\lambda)\,(A_{22}g_1 - A_{12}g_2),\, A_{22}g_1 - A_{12}g_2) < \infty$$

we may put analogously

$$\tilde{\varphi} = \int_m^\infty d^{-1}(\lambda)\, \mathrm{d}E(\lambda)\,(A_{22}g_1 - A_{12}g_2)\,.$$

It may be verified easily that $[\psi, \tilde{\varphi}]$ satisfies the system $\boldsymbol{A}[\psi, \tilde{\varphi}] = [g_1, g_2]$ and hence $[g_1, g_2] \in \mathcal{R}(\boldsymbol{A})$. The necessity of the condition follows from the selfadjointness of the operator $\boldsymbol{A}$.

To complete the proof of Theorem 3.2.1 it remains to show that

$$[f_1, f_2] \in \mathcal{N}(\mathbf{A})^{\perp} \tag{3.2.5}$$

is equivalent to (3.2.3). But

$$\mathcal{N}(\mathbf{A}) = \{[\psi, \tilde{\varphi}] \in \mathbf{H};\ \psi \in \mathcal{N}_1,\ \tilde{\varphi} \in \mathcal{N}_2\} \tag{3.2.6}$$

because $\mathcal{N}(\mathbf{A}) \subset \{[\psi, \tilde{\varphi}] \in \mathbf{H};\ \psi, \tilde{\varphi} \in \mathcal{N}_2\}$, the set on the right hand side of (3.2.6) is a subset of $\mathcal{N}(\mathbf{A})$ and $[\psi, \tilde{\varphi}] \in \mathcal{N}(\mathbf{A})$ implies $P_0\psi = 0$. With respect to $B^{\varkappa}\mathcal{N}_1 = \mathcal{N}_1$, $B^{\varkappa}\mathcal{N}_2 = \mathcal{N}_2$ for an arbitrary $\varkappa$ we may rewrite (3.2.5) in the form

$$(\psi, f_1) = 0\,, \quad \psi \in \mathcal{N}_1\,,$$
$$(\varphi, f_2) = 0\,, \quad \varphi \in \mathcal{N}_2\,,$$

which is evidently equivalent to (3.2.3). The smoothness of the found solution may be proved analogously as in the preceding theorems.

Remark 3.2.1. Let the assumptions of Theorem 3.2.1 be fulfilled with $\nu = 1$. Define the operators Θ_0, Θ_1 by the equalities

$$\Theta_0 u = A^{\varrho} u_{tt} + A^{\varrho}(A + \gamma I)\, u$$

$$\Theta_1 u = \Theta_0 u \quad \text{for} \quad u \in \mathcal{D}(\Theta_0) =$$
$$= \{u \in U^0_{\varrho}(\langle 0, \omega \rangle;\ u(0) = u(\omega),\ u_t(0) = u_t(\omega)\}\,.$$

The statement of Remark 3.1.1 remains valid with the only change that now it is

$$\mathcal{N}(\Theta_1) = \mathcal{N}(\Theta_0) = \Big\{u \in L_2(\langle 0, \omega\rangle; H),\ u(t) = J(t)\, A^{-1/2}\tilde{\varphi} + K(t)\,\psi\,,$$
$$\text{where} \quad \tilde{\varphi} \in \mathcal{N}(\sin \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)})) \cap \mathcal{D}(A^{1/2})\,,$$
$$\psi \in \mathcal{N}\left(\frac{\sin \frac{1}{2}\omega(\sqrt{(A + \gamma I)})}{\sqrt{(A + \gamma I)}}\right) \cap \mathcal{D}(A^{1/2})\Big\}\,.$$

The proof is analogous to that in Remark 2.2.

Theorem 3.2.2. *Let the assumptions and the condition* (3.2.3) *of Theorem* 3.2.1 *be fulfilled. If* $m + \gamma > 0$ *and* $m + \gamma \leqq 0$ *respectively then every* ω*-periodic solution is stable and unstable respectively.*

Proof. If $\gamma + m > 0$ then $\|K(t)\| \leqq 1$, $\|J(t)\| \leqq 1/\sqrt{(m + \gamma)}$, $t \in R^+$. If $\gamma + m \leqq 0$ then

$$\|K(t)\,\varphi\| \geqq \|E(\gamma+)\,\varphi\|\,,$$
$$\|J(t)\,\psi\| \geqq t \inf_{\tau \in R} \frac{\operatorname{sh} \tau}{\tau}\, \|E(-\gamma+)\,\psi\|\,, \quad \varphi, \psi \in H\,, \quad t \in R^+\,.$$

From these inequalities our assertion follows immediately.

4.1. First order equation. Let us consider the equation

$$(4.1.1) \qquad u_t(t) + (A + \gamma I)\, u(t) = F(t, u)\,, \quad t > 0\,,$$

where A is a strongly positive operator $\mathscr{D}(A) \subseteq B \to B$, $-A$ generates a strongly continuous semigroup $T(t)$ $(t \in R^+)$ and $F : R^+ \times \mathscr{D}_F \to B$ is a nonlinear operator in a Banach space B ω-periodic in t. It is known that $T(t)$ has a holomorphic extension in some complex domain $\{z;\ \arg z < a,\ 0 < |z| < \infty\}$, $0 < a \leqq \frac{1}{2}\pi$ (see [17] p. 254). Let us note that the operator F is not necessarily a Němyckii operator and that we write $F(t, u)$ instead of $F(t, u)(t)$ and e.g. $\int F(\tau, e^{\gamma\tau} u(\tau))\, d\tau$ instead of $\int F(\tau, e^{\gamma(\cdot)} u(\cdot))(\tau)\, d\tau$. In the following keep the definition of the solution given in Section 1.3. Since the investigation of the existence of periodic solutions to (4.1.1) is rather complicated in critical cases, we restrict ourselves to the case $B = H$. To the study of this problem in a Banach space theory of M. Sova [20] may be used.

Theorem 4.1.1. *Let A be a selfadjoint operator $\mathscr{D}(A) \subseteq H \to H$ with* $\inf \sigma(A) = m > 0$ *and let $-\gamma$ be at most an isolated point of $\sigma(A)$. Then an ω-periodic solution to* (4.1.1) *exists iff the system*

$$(4.1.2) \qquad u(t) = T(t)(\varphi_1 + \varphi_2) + \int_0^t e^{\gamma\tau}\, T(t - \tau)\, F(\tau, e^{-\gamma\tau} u(\tau))\, d\tau\,,$$

$$P \int_0^\omega e^{\gamma\tau}\, T(\omega - \tau)\, F(\tau, e^{-\gamma\tau} u(\tau))\, d\tau = 0\,,$$

$$\varphi_2 = -[(I - P)(e^{-\gamma\omega}\, T(\omega) - I)]^{-1} \int_0^\omega e^{-\gamma(\omega - \tau)}\, T(\omega - \tau)\, F(\tau, e^{-\gamma\tau} u(\tau))\, d\tau\,,$$

where P is a projection on $\mathscr{N}(A + \gamma I)$, $\varphi_1 = P\varphi$, $\varphi_2 = (I - P)\,\varphi$ has a solution

$$(4.1.3) \qquad \varphi_1 = \varphi_1^* \in H\,, \quad \varphi_2 = \varphi_2^* \in \mathscr{D}(A)\,, \quad u = u^* \in U_0(R^+)\,.$$

If the condition is satisfied then the sought ω-periodic solution is $u^*(t)$.

Proof. The existence of $[(I - P)(e^{-\gamma\omega}\, T(\omega) - I)]^{-1}$ was shown in the proof of Theorem 2.3. Let the system (4.1.2) have a solution (4.1.3). It may be verified easily that $u^*(t)$ is a solution of (4.1.1). Further by $(4.1.2_2)$ and $(4.1.2_3)$

$$u^*(\omega) - u^*(0) = (e^{-\gamma\omega}\, T(\omega) - I)(\varphi_1^* + \varphi_2^*) + \int_0^\omega e^{-\gamma(\omega - \tau)}\, T(\omega - \tau)$$

$$F(\tau, u^*(\tau))\, d\tau = (I - P)(e^{-\gamma\omega}\, T(\omega) - I)\,\varphi_2^* +$$

$$+ (I - P) \int_0^\omega e^{-\gamma(\omega - \tau)}\, T(\omega - \tau)\, F(\tau, u^*(\tau))\, d\tau = 0\,,$$

which proves the ω-periodicity of $u^*(t)$ in consequence of a unique determination of a solution by its initial data. On the other hand if (4.1.1) has an ω-periodic solution u^* then we find easily that $\varphi = \varphi^* = u^*(0)$ and $u = u^*(t)$ satisfy the system (4.1.2).

Remark 4.1.1. Evidently, if $\gamma + m > 0$ then $P = 0$, $\varphi_1 = 0$, $\varphi_2 = \varphi$ and $(4.1.2_2)$ may be cancelled.

Corollary 4.1.1. *Let the assumptions of Theorem* 4.1.1 *be fulfilled with* $\gamma + m > 0$, $F(t, u) = \varepsilon \tilde{F}(t, u)$, *where* $\tilde{F}(\cdot, \cdot)$ *maps* $R^+ \times C(\langle 0, \omega \rangle; \mathscr{D}(A^\nu))$ *into* $C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu-\eta}))$ *with some* $\nu \geqq 1$, $0 \leqq \eta < 1$ *and let it be*

$$\|\tilde{F}(t, u_1) - \tilde{F}(t, u_2)\|_{\nu-\eta} \leqq L\|u_1 - u_2\|_\nu$$
$$t \in R^+ \,, \quad u_1, u_2 \in C(\langle 0, \omega \rangle; \mathscr{D}(A^\nu)) \,.$$

Then there exists for sufficiently small $\varepsilon > 0$ *a unique* ω-*periodic solution* $u^*(\varepsilon)(\cdot) \in U_{\nu-1}(R^+)$ *to* (4.1.1) *and it is continuous in* ε *in the norm of* $U_{\nu-1}(R^+)$.

Proof. Applying the Banach fixed point theorem to the system (4.1.2) for $[u, \varphi] \in C(\langle 0, \omega \rangle; \mathscr{D}(A^\nu)) \times \mathscr{D}(A^\nu)$, we find an ω-periodic function $u = u^*(\cdot) \in C(R^+; \mathscr{D}(A^\nu))$. But as, in virtue of the assumption, $F(\cdot, u^*(\cdot)) \in C(R^+; \mathscr{D}(A^{\nu-\eta}))$ and $u^*(t)$ satisfies the equation $(4.1.2_1)$, it is by the proposition 3.1.2 $u^* \in U_{\nu-1}(R^+)$. (Even a simpler proof follows from the existence of the fixed point to the operator

$$\varepsilon \int_0^\infty T(\tau) \tilde{F}(t - \tau, u(t - \tau)) \, d\tau, \text{ cf. Remark 2.3.)}$$

Now let us investigate the nonlinear counterpart of Theorem 2.5.

Theorem 4.2.1. *Let* $A = iB$, *where* B *is a selfadjoint operator* $\mathscr{D}(B) \to H$ *such that there exist constants* $c > 0$, $\varrho \geqq 0$, *that* (2.15) *holds and that there exists a real* β *in the resolvent set of* B *and let* $F(\cdot, u) \in C(R^+; \mathscr{D}(B^\varrho))$ *for* $u \in U_0(R^+)$. *Then there exists an* ω-*periodic solution to* (4.1.1) *with* $\gamma = 0$ *iff the system*

$$u(t) = T(t)(\varphi_1 + \varphi_2) + \int_0^t T(t - \tau) F(\tau, u(\tau)) \, d\tau \,, \tag{4.1.4}$$
$$P \int_0^\omega T(\omega - \tau)(B - \beta I)^\varrho F(\tau, u(\tau)) \, d\tau = 0 \,,$$
$$\varphi_2 = [(I - P)(B - \beta I)^\varrho (T(\omega) - I)]^{-1} \int_0^\omega T(\omega - \tau)(B - \beta I)^\varrho F(\tau, u(\tau)) \, d\tau \,,$$

where $T(t)$ *is a semigroup generated by* $-A$, P *is a projection on to*

$$\sum_{k=-\infty}^{\infty} \mathscr{N}\left(B + \frac{2k\pi}{\omega} I\right) H \,, \quad \varphi_1 = P\varphi \,, \quad \varphi_2 = (I - P)\varphi \,,$$

has a solution $\varphi_1 = \varphi_1^* \in \mathscr{D}(A)$, $\varphi_2 = \varphi_2^* \in \mathscr{D}(A)$, $u = u^*(\cdot) \in U_0(R^+)$. *If the condition is satisfied then the sought solution is* $u^*(t)$.

The proof is quite analogous to that of the preceding theorem. The existence of $[(I - P)(B - \beta I)^{\varrho}(T(\omega) - I)]^{-1}$ was shown in the proof of Theorem 2.5.

4.2. Second order equation with a dissipative term. Let us consider the equation

$$u_{tt}(t) + (\alpha I + \beta A) u_t(t) + (A + \gamma I) u(t) = F(t, u), \quad t \in R^+, \tag{4.2.1}$$

where A is a selfadjoint operator in H, $\inf \sigma(A) = m > 0$ and $F : R^+ \times \mathscr{D}_F \to H$ is a nonlinear operator in H ω-periodic in t and $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$, γ are real numbers. Let us keep the notation of paragraph 3.1 and the definition of the solution as given there.

Theorem 4.2.1. *Let* $-\gamma$ *be at most an isolated point of* $\sigma(A)$. *Then an* ω*-periodic solution to* (4.2.1) *exists iff the system*

$$u(t) = J(t) A^{-\varkappa}(\tilde{\varphi}_1 + \tilde{\varphi}_2) + K(t)(\psi_1 + \psi_2) + \int_0^t K(t - \tau) F(\tau, u(\tau))\, d\tau, \tag{4.2.2}$$

$$P \int_0^\omega F(\tau, u(\tau))\, d\tau = 0,$$

$$\psi_2 = 2[(I - P) D]^{-1} \int_0^\omega e^{-(\alpha + \beta A)(\omega - \tau)} \sin(\tau - \tfrac{1}{2}\omega) [\sqrt{(A + \gamma I - (\alpha + \beta A)^2)}] *$$

$$* \sin \tfrac{1}{2}\omega[\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}] F(\tau, u(\tau))\, d\tau,$$

$$\tilde{\varphi}_2 = -2[(I - P) D]^{-1} \int_0^\omega \frac{A^{\varkappa} e^{-(\alpha + \beta A)(\omega - \tau)}}{\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}} \cos(\tau - \tfrac{1}{2}\omega) .$$

$$. [\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}] * \sin \tfrac{1}{2}\omega[\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}] F(\tau, u(\tau))\, d\tau,$$

where P *is a projection on* $\mathscr{N}(A + \gamma I)$, $\tilde{\varphi}_1 = P\tilde{\varphi}$, $\tilde{\varphi}_2 = (I - P)\tilde{\varphi}$, $\psi_2 = (I - P)\psi$, $\psi_1 = P\psi$, $\varkappa = \frac{1}{2}$ *if* $\beta = 0$ *and* $\varkappa = 1$ *if* $\beta \neq 0$, *has a solution*

$$\varphi_1 = \varphi_1^* \in H, \quad \varphi_2 = \varphi_2^* \in \mathscr{D}(A^{1/2}) \cap \mathscr{D}(\beta A), \quad \psi_1 = \psi_1^* \in H, \tag{4.2.3}$$

$$\psi_2 = \psi_2^* \in \mathscr{D}(A^{1/2}) \cap \mathscr{D}(\beta A), \quad u = u^*(\cdot) \in U_0^\beta(R^+).$$

If the condition is satisfied then the sought ω*-periodic solution is* $u^*(t)$.

Proof. The existence of $[(I - P) D]^{-1}$ was shown in the proof of Theorem 3.1.1. Let the system (4.2.2) have a solution (4.2.3). It may be verified easily that $u^*(t)$ is a solution of (4.2.1). Further, the ω-periodicity of $u^*(t)$ is equivalent to

$$u^*(\omega) - u^*(0) = 0, \quad u_t^*(\omega) - u_t^*(0) = 0, \tag{4.2.4}$$

since every solution of (4.2.1) is uniquely determined by its initial data. But (4.2.4) is equivalent to

(4.2.5) $$\mathbf{A}[\psi, \tilde{\varphi}] = [f_1, f_2],$$

where $\mathbf{A}$ is given by (1.2.1), (3.1.8) and

$$f_1 = \int_0^\omega A^\varkappa K(\omega - \tau) F(\tau, u(\tau)) \, d\tau,$$

$$f_2 = \int_0^\omega J(\omega - \tau) F(\tau, u(\tau)) \, d\tau.$$

It is shown in the proof of Theorem 3.1.1 that $\mathscr{R}(\mathbf{A})$ is closed; so (4.2.5) is equivalent to

(4.2.6) $$\mathbf{P}[f_1, f_2] = 0,$$
$$[\psi_2, \tilde{\varphi}_2] = [(\mathbf{I} - \mathbf{P}) \mathbf{A}]^{-1} [f_1, f_2],$$

where $\mathbf{P}$ is a projection on $\mathscr{N}(\mathbf{A})$, $[\psi_1, \tilde{\varphi}_1] = \mathbf{P}[\psi, \tilde{\varphi}]$, $[\psi_2, \tilde{\varphi}_2] = (\mathbf{I} - \mathbf{P})[\psi, \tilde{\varphi}]$. Using (3.1.10) we obtain after simple calculations that (4.2.6) is equivalent to the system ($4.2.2_2$), ($4.2.2_3$), ($4.2.2_4$). Hence $u^*(t)$ is ω-periodic.

On the other hand, if (4.2.1) has an ω-periodic solution $u^*(t)$ then we find easily that $\psi^* = u_t^*(0)$, $\tilde{\varphi}^* = A^\varkappa u^*(0)$ and $u^*(t)$ satisfy the system (4.2.2).

Theorem 4.2.2. *Let $\gamma + m > 0$ and $F(t, u) = h(t) + \varepsilon \tilde{F}(t, u)$, where $h \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu + 1/2}))$, $\tilde{F}(\cdot, u(\cdot)) \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu + 1/2}))$ for $u \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1})$ and*

(4.2.7) $$\|\tilde{F}(\cdot, u_1(\cdot)) - \tilde{F}(\cdot, u_2(\cdot))\|_{\nu + 1/2} \leqq L \|u_1 - u_2\|_{\nu + 1} \quad \textit{for}$$
$$u_1, u_2 \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1}))$$

if $\beta = 0$ and where

$$h \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1})), \quad \tilde{F}(\cdot, u(\cdot)) \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1}))$$

for $u \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+2}))$ and

$$\|\tilde{F}(\cdot, u_1(\cdot)) - \tilde{F}(\cdot, u_2(\cdot))\|_{\nu+1} \leqq L \|u_1 - u_2\|_{\nu+2}$$

for $u_1, u_2 \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+2}))$ if $\beta \neq 0$, and h and $\tilde{F}$ are ω-periodic in t on R^+. Then there exists for sufficiently small $\varepsilon > 0$ a unique ω-periodic solution $u^*(\varepsilon)(t)$ and $u^*(\varepsilon)(\cdot) \in U_\nu^\beta(R^+)$, $u^*(\varepsilon)(\cdot)$ is continuous in the norm of $U_\nu^\beta(R^+)$.

Proof. It is clear that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

(4.2.8) $$\|J(t)\| \leqq c_1, \quad \|A^\varkappa K(t)\| \leqq c_2, \quad t \in \langle 0, \omega \rangle.$$

Suppose that $\beta = 0$ (for the case of $\beta \neq 0$ the procedure is quite analogous). Let $u \in U_\nu^0(\langle 0, \omega \rangle)$. Then $u \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1}))$ and accordingly $\tilde{F}(\cdot, u(\cdot)) \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1/2}))$. From this we find easily that $\int_0^t K(t - \tau) \tilde{F}(\tau, u(\tau)) \, d\tau$ belongs to $U_\nu^0(\langle 0, \omega \rangle)$. We see that the operator $G_1(\varepsilon)(\psi, \tilde{\varphi}, u)$ defined by the right hand side of $(4.2.2_1)$ (where in virtue of $P = 0$ it is $\psi_1 = \tilde{\varphi}_1 = 0$, $\psi_2 = \psi$, $\tilde{\varphi}_2 = \tilde{\varphi}$) maps $R^+ \times \mathscr{D}(A^{\nu+1/2}) \times \mathscr{D}(A^{\nu+1/2}) \times U_\nu^0(\langle 0, \omega \rangle)$ into $U_\nu^0(\langle 0, \omega \rangle)$. Further, according to the (4.2.7) and (4.2.8) G_1 fulfils the Lipschitz condition

$$\|G_1(\varepsilon)(\psi, \tilde{\varphi}, u_1) - G_1(\varepsilon)(\psi, \tilde{\varphi}, u_2)\|_{\nu+1} \leqq \varepsilon \omega c_2 L \|u_1 - u_2\|_{\nu+1}$$

for every $\psi, \tilde{\varphi} \in H$, $u_1, u_2 \in U_\nu^0(\langle 0, \omega \rangle)$. According to the Banach fixed point theorem there exists an $\varepsilon_1 > 0$ such that for $\varepsilon \in \langle 0, \varepsilon_1 \rangle$ there exists a unique solution $u^*(\varepsilon)(\psi, \tilde{\varphi}) \in U_\nu^0(\langle 0, \omega \rangle)$ for every fixed $\psi, \tilde{\varphi} \in \mathscr{D}(A^{\nu+1/2})$ and this solution clearly satisfies the condition

$$(4.2.9) \qquad \|u(\varepsilon)(\psi_1, \tilde{\varphi}_1) - u(\varepsilon)(\psi_2, \tilde{\varphi}_2)\|_{\nu+1} \leqq (1 + \varepsilon c_2 L)^{-1} m^{-1} \max(c_1, c_2) \times$$
$$+ (\|\psi_1 - \psi_2\|_{\nu+1/2} + \|\tilde{\varphi}_1 - \tilde{\varphi}_2\|_{\nu+1/2}), \quad \psi_i, \tilde{\varphi}_i \in \mathscr{D}(A^{\nu+1/2}) \quad (i = 1, 2,).$$

It is clear that the operator $G_2(\varepsilon)(\psi, \tilde{\varphi})$ from $H \times H$ into itself defined by the right hand sides of $(4.2.2_3)$, $(4.2.2_4)$ maps $\mathscr{D}(A^{\nu+1/2}) \times \mathscr{D}(A^{\nu+1/2})$ into itself. We obtain easily from (4.2.7), (4.2.9) that the assumptions of Banach fixed point theorem are satisfied in $\mathscr{B}(\psi_0, \tilde{\varphi}_0, \varepsilon_2; \mathscr{D}(A^{\nu+1/2}) \times \mathscr{D}(A^{\nu+1/2}))$, where

$$\psi_0 = 2[(I - P) D]^{-1} \int_0^\omega \frac{A^\varkappa e^{-(\alpha I + \beta A)(\omega - \tau)}}{\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}} \sin(\tau - \tfrac{1}{2}\omega) \, .$$
$$. \sqrt{[(A + \gamma I - (\alpha I + \beta A)^2)]} * \sin \tfrac{1}{2}\omega[\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}] \, h(\tau) \, d\tau \, ,$$

$$\tilde{\varphi}_0 = -2[(I - P) D]^{-1} \int_0^\omega \frac{A^\varkappa e^{-(\alpha I + \beta A)(\omega - \tau)}}{\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}} \cos(\tau - \tfrac{1}{2}\omega) \, .$$
$$. [\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}] * \sin \tfrac{1}{2}\omega[\sqrt{(A + \gamma I - (\alpha I + \beta A)^2)}] \, h(\tau) \, d\tau$$

ε_2 is sufficiently small and so there exists a unique pair $[\psi^*(\varepsilon), \tilde{\varphi}^*(\varepsilon)] \in \mathscr{D}(A^{\nu+1/2}) \times \mathscr{D}(A^{\nu+1/2})$ $(\varepsilon \in \langle 0, \varepsilon_2 \rangle)$ satisfying the system $(4.2.2_3)$, $(4.2.2_4)$. Because of the continuity of $\psi^*(\varepsilon)$, $\tilde{\varphi}^*(\varepsilon)$ in $\varepsilon \in \langle 0, \varepsilon_2 \rangle$ in the norm of $\mathscr{D}(A^{\nu+1/2})$ and of the continuity of $u^*(\varepsilon)(\psi, \tilde{\varphi})$ in ε, ψ and $\tilde{\varphi}$, u^* is continuous in $\varepsilon \in \langle 0, \varepsilon^* \rangle$ where $\varepsilon^* = \min(\varepsilon_1, \varepsilon_2)$, in the norm of $U_\nu^0(\langle 0, \omega \rangle)$. The assertion of Theorem in the full generality follows from the ω-periodicity of $u^*(\varepsilon)(\psi^*(\varepsilon), \tilde{\varphi}^*(\varepsilon))(t)$. (Even a simpler proof follows from the existence of the fixed point to the operator $\int_0^\infty K(s) F(t - s, u(t - s)) \, ds$ (cf. Corollary 3.1.1).)

4.3. Second order equation without a dissipative term. Now let us consider the equation

$$(4.3.1) \qquad u_{tt}(t) + (A + \gamma I) u(t) = F(t, u), \quad t \in R^+,$$

with A, γ and F as in Section 4.2. We still keep the notation of Section 3.1. Besides, let us introduce the projections P and Q respectively on $\mathcal{N}(\sin \omega(\sqrt{(A + \gamma I)})/\sqrt{(A + \gamma I)})$ and on $\mathcal{N}(\sin \omega(\sqrt{(A + \gamma I)}))$ respectively.

Theorem 4.3.1. *Let there exist $c > 0$ and $\varrho \geqq 0$ such that (3.2.2) holds and let $F(\cdot, u) \in C(R^+; \mathcal{D}(A^\varrho))$ for $u \in U_0^0(R^+)$. Then an ω-periodic solution to (4.3.1) exists iff the system*

(4.3.2)
$$u(t) = J(t)\, A^{-1/2}(\tilde\varphi_1 + \tilde\varphi_2) + K(t)\,(\psi_1 + \psi_2) + \int_0^t K(t - \tau)\, F(\tau, u(\tau))\, d\tau\,,$$

$$P \int_0^\omega \frac{A^{1/2} \sin \tau(\sqrt{(A + \gamma I)})}{\sqrt{(A + \gamma I)}} * A^\varrho F(\tau, u(\tau))\, d\tau = 0\,,$$

$$Q \int_0^\omega \cos \tau(\sqrt{(A + \gamma I)}) * A^\varrho F(\tau, u(\tau))\, d\tau = 0\,,$$

$$\psi_2 = 2[(I - P)\, A^\varrho D]^{-1} \int_0^\omega \sin (\tau - \tfrac{1}{2}\omega)\,(\sqrt{A + \gamma I}) * $$
$$* \sin \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)}) * A^\varrho F(\tau, u(\tau))\, d\tau\,,$$

$$\tilde\varphi_2 = -\, 2[(I - Q)\, A^\varrho D]^{-1} \int_0^\omega \left(\frac{A}{A + \gamma I}\right)^{1/2} \cos (\tau - \tfrac{1}{2}\omega)\,(\sqrt{A + \gamma I}) * $$
$$* \sin \tfrac{1}{2}\omega(\sqrt{(A + \gamma I)}) * A^\varrho F(\tau, u(\tau))\, d\tau\,,$$

where $\psi_1 = P\psi$, $\psi_2 = (I - P)\,\psi$, $\tilde\varphi_1 = Q\tilde\varphi$, $\tilde\varphi_2 = (I - Q)\,\tilde\varphi$, $(\psi, \tilde\varphi \in H)$, has a solution

(4.3.3)
$$\psi_i^*, \tilde\varphi_i^* \in \mathcal{D}(A^{1/2})\,, \quad (i = 1, 2)\,, \quad u^* \in U_0^0(R^+)\,.$$

If the condition is satisfied then the sought ω-periodic solution is $u^*(t)$.

Proof. The existence of $[(I - P)\, A^\varrho D]^{-1}$ and $[(I - Q)\, A^\varrho D]^{-1}$ was shown in the proof of Theorem 3.2.1. Let the system (4.3.2) have the solution (4.3.3). It is clear that $u^*(t)$ is a solution of (4.3.1) and that it is ω-periodic iff the system

$$\mathbf{A}[\psi, \tilde\varphi] = [f_1, f_2]\,,$$

where $\mathbf{A}$ is given by (1.2.1), (3.1.8) (with $\alpha = \beta = 0$) and

$$f_1 = \int_0^\omega A^{1/2} K(\omega - \tau)\, F(\tau, u^*(\tau))\, d\tau\,,$$

$$f_2 = \int_0^\omega J(\omega - \tau)\, F(\tau, u^*(\tau))\, d\tau\,,$$

has a solution $[\psi^*, \tilde\varphi^*] \in \mathbf{H}$.

It follows from (3.2.6) that $\mathbf{A}[\psi_1^*, \tilde{\varphi}_1^*] = 0$. As $\psi_i^*, \tilde{\varphi}_i^*$ $(i = 1, 2)$ satisfy $(4.3.2_{2-5})$ we have $\mathbf{A}[A^\varrho \psi_2^*, A^\varrho \tilde{\varphi}_2^*] = [A^\varrho f_1, A^\varrho f_2]$ and hence

$$\mathbf{A}[\psi^*, \tilde{\varphi}^*] = [f_1, f_2] \,.$$

On the contrary if there exists an ω-periodic solution $u^*(t)$ then it is easy that $\psi_i^*, \tilde{\varphi}_i^*$ $(i = 1, 2)$, $u^*(t)$ given by $\psi^* = u_t^*(0)$, $\tilde{\varphi}^* = A^{1/2} u^*(0)$, $\psi_1^* = P\psi^*$, $\psi_2^* = (I - P)\psi^*$, $\tilde{\varphi}_1^* = P\tilde{\varphi}^*$, $\tilde{\varphi}_2^* = (I - P)\tilde{\varphi}^*$ satisfy the system (4.3.2).

At the end of the paragraph we introduce one special procedure of solving of the system (4.3.2). This procedure is used in paragraph 5, where an example is given.

Theorem 4.3.2. *Let the assumption of Theorem* 4.3.1 *be fulfilled and let* $F(t, u) = \varepsilon \tilde{F}(t, u)$. *Suppose that the operator* $\tilde{F}$ *has the continuous Gâteaux derivative* $\tilde{F}_u'(\cdot, u) : C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1})) \to C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+\varrho+1/2}))$ $(\nu \geqq 1)$ *and that the equation*

$$(4.3.4) \qquad \hat{G}(\psi_1, \tilde{\varphi}_1) \equiv \left[P \int_0^\omega \frac{A^{1/2} \sin \tau(\sqrt{(A + \gamma)})}{\sqrt{(A + \gamma I)}} A^\varrho \tilde{F}(\tau, J(\tau) A^{-1/2} \tilde{\varphi}_1 + K(\tau) \psi_1) \, d\tau \,, \right.$$
$$\left. Q \int_0^\omega \cos \tau(\sqrt{(A + \gamma I)}) * A^\varrho \tilde{F}(\tau, J(\tau) A^{-1/2} \tilde{\varphi}_1 + K(\tau) \psi_1) \, d\tau \right] = [0, 0] \,,$$

have a solution $\psi_1^0 \in P\,\mathscr{D}(A^{\nu+1/2})$, $\tilde{\varphi}_1^0 \in Q\,\mathscr{D}(A^{\nu+1/2})$ *such that there exists* $[\hat{G}'_{[\psi_1, \tilde{\varphi}_1]}(\psi_1^0, \varphi_1^0)]^{-1}$ *continuous as the mapping* $P\,\mathscr{D}(A^{\nu+1/2}) \times Q\,\mathscr{D}(A^{\nu+1/2})$ *into itself. Then there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ *there exists a unique* ω*-periodic solution* $u^*(\varepsilon)(t)$ *of* (4.1.1) *such that* $u^*(0)(t) = J(t) A^{-1/2} \tilde{\varphi}_1^0 + K(t) \psi_1^0$ *and moreover*

$$u^*(\cdot)(\cdot) \in C(\langle 0, \varepsilon_0 \rangle; U_\nu^0(R^+)) \,.$$

Proof. It is clear that for the existence of an ω-periodic solution $u^* \in U_\nu^0(R^+)$ it is sufficient to show that the system (4.3.2) have the solution $u_1 = u^* \in C(\langle 0, \omega \rangle; \mathscr{D}(A^{\nu+1})) = B_1$,

$$u_2 = \psi_1 \in P\,\mathscr{D}(A^{\nu+1/2}) = B_2 \,, \quad u_3 = \tilde{\varphi}_1 \in Q\,\mathscr{D}(A^{\nu+1/2}) = B_3 \,,$$
$$u_4 = \psi_2 \in (I - P)\,\mathscr{D}(A^{\nu+1/2}) = B_4 \,, \quad u_5 = \tilde{\varphi}_2 \in (I - Q)\,\mathscr{D}(A^{\nu+1/2}) = B_5 \,,$$

where B_1, B_2, B_3, B_4, B_5 respectively are the Banach spaces with the norms defined by $\|u_1\|_{(1)} = \|u_1\|_{\nu+1}$, $\|u_2\|_{(2)} = \|u_2\|_{\nu+1/2}$, $\|u_3\|_{(3)} = \|u_3\|_{(2)}$, $\|u_4\|_{(4)} = \|u_4\|_{(2)}$, $\|u_5\|_{(5)} = \|u_5\|_{(2)}$ respectively for $u_i \in B_i$, $i = 1, \ldots, 5$ respectively. Let us investigate the system (4.3.2) in the Banach space $\mathbf{B} = B_1 \times B_2 \times B_3 \times B_4 \times B_5$ $(\|\mathbf{u}\|_B = \sum_{i=1}^5 \|u_i\|_{(i)}$, where $\mathbf{u} = [u_1, u_2, u_3, u_4, u_5])$. Define the operator

$$\mathbf{G}(\varepsilon)(\mathbf{u}) = [G_1(\varepsilon)(\mathbf{u}), G_2(\varepsilon)(\mathbf{u}), \ldots, G_5(\varepsilon)(\mathbf{u})] \,,$$

where $\boldsymbol{u} = [u_1, u_2, \ldots, u_5] \in \mathbf{B}$,

$$G_1(\varepsilon)(\boldsymbol{u})(t) = u_1(t) - J(t)\,A^{-1/2}(u_3 + u_5) - K(t)\,(u_2 + u_4) - \\ -\varepsilon \int_0^t K(t-\tau)\,\tilde{F}(\tau, u_1(\tau))\,\mathrm{d}\tau$$

$$G_2(\varepsilon)(\boldsymbol{u}) = P \int_0^{\omega} \frac{A^{1/2} \sin \tau(\sqrt{A + \gamma I})}{\sqrt{(A + \gamma I)}}\,\tilde{F}(\tau, u_1(\tau))\,\mathrm{d}\tau\,,$$

$$G_3(\varepsilon)(\boldsymbol{u}) = Q \int_0^{\omega} \cos \tau(\sqrt{A + \gamma I})\,\tilde{F}(\tau, u_1(\tau))\,\mathrm{d}\tau\,,$$

$$G_4(\varepsilon)(\boldsymbol{u}) = u_4 - 2\varepsilon[(I - P)\,A^{\varrho}D]^{-1} \int_0^{\omega} \sin(\tau - \tfrac{1}{2}\omega)(\sqrt{A + \gamma I}) * \\ * \sin \tfrac{1}{2}\omega(\sqrt{A + \gamma I}) * A^{\varrho}\tilde{F}(\tau, u_1(\tau))\,\mathrm{d}\tau\,,$$

$$G_5(\varepsilon)(\boldsymbol{u}) = u_5 + 2\varepsilon[(I - Q)\,A^{\varrho}D]^{-1} \int_0^{\omega} \left(\frac{A}{A + \gamma I}\right)^{1/2} * \\ * \cos(\tau - \tfrac{1}{2}\omega)(\sqrt{A + \gamma I}) * \sin \tfrac{1}{2}\omega(\sqrt{A + \gamma I}) * A^{\varrho}\tilde{F}(\tau, u_1(\tau))\,\mathrm{d}\tau\,.$$

Clearly we have $\boldsymbol{G}(\varepsilon)(\boldsymbol{u}) \in \mathbf{B}$ for $\boldsymbol{u} \in \mathbf{B}$ and for any $\varepsilon \geqq 0$. Further it is obvious that for

$$\boldsymbol{u} = \boldsymbol{u}^0 = [J(\cdot)\,A^{-1/2}\psi_1^0 + K(\cdot)\,\tilde{\varphi}_1^0, \psi_1^0, \tilde{\varphi}_1^0, 0, 0]$$

it is

$$\boldsymbol{G}(0)(\boldsymbol{u}^0) = 0$$

and that the operator $\boldsymbol{G}(\varepsilon)(\boldsymbol{u})$ have the continuous Frèchet derivative $\boldsymbol{G}'_u(\varepsilon)(\boldsymbol{u})$ in $\mathbf{B}$ for any $\varepsilon \geqq 0$. Prove that there exists $[\boldsymbol{G}'_u(0)(\boldsymbol{u}^0)]^{-1} : \mathbf{B} \to \mathbf{B}$ continuous. Let $\boldsymbol{h} = [h_1, h_2, \ldots, h_5] \in \mathbf{B}$ be arbitrary. Solve the equation

$$\boldsymbol{G}'_u(0)(\boldsymbol{u}^0)\,\bar{\boldsymbol{u}} = \boldsymbol{h}\,.$$

As $\boldsymbol{G}'_u(0)(\boldsymbol{u}^0)\,\bar{\boldsymbol{u}} = [\bar{u}_1 - J(\cdot)\,A^{-1/2}(\bar{u}_3 + \bar{u}_5) - K(\cdot)(\bar{u}_2 + \bar{u}_4),$

$$P \int_0^{\omega} \frac{A^{1/2} \sin \tau(\sqrt{(A + \gamma I)})}{\sqrt{(A + \gamma I)}}\,A^{\varrho}F'_u(\tau, J(\tau)\,A^{-1/2}(\tilde{\varphi}_1^0 + \tilde{\varphi}_2^0) + K(\tau)(\psi_1^0 + \psi_2^0))\,\bar{u}_1(\tau)\,\mathrm{d}\tau\,,$$

$$Q \int_0^{\omega} \cos \tau(\sqrt{(A + \gamma I)}) * A^{\varrho}F'_u(\tau, J(\tau)\,A^{-1/2}(\tilde{\varphi}_1^0 + \tilde{\varphi}_2^0) + K(\tau)(\psi_1^0 + \psi_2^0))\,\bar{u}_1(\tau)\,\mathrm{d}\tau\,,$$

$$\bar{u}_4, \bar{u}_5] \quad \text{and} \quad [\hat{G}'_{[\psi_1,\varphi_1]}(\psi_1^0, \tilde{\varphi}_1^0)]^{-1}$$

is a continuous mapping $B_2 \times B_3$ into itself we may express easily $\bar{\boldsymbol{u}}$ by means of $\boldsymbol{h}$. Besides, we find that

$$c\,\|\boldsymbol{h}\|_{\mathbf{B}} \geqq \|\bar{\boldsymbol{u}}\|_{\mathbf{B}}$$

with a constant $c > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{u}$. We see that all the assumptions of the implicit function theorem for the equation

$$G(\varepsilon)(u) = 0 \tag{4.3.5}$$

are fulfilled (see e.g. [21] p. 305) and so there exists for sufficiently small $\varepsilon > 0$ a unique solution $\boldsymbol{u}^*(\varepsilon) \in \mathbf{B}$ to (4.3.5) such that $\boldsymbol{u}^*(0) = \boldsymbol{u}^0$ and $\boldsymbol{u}^*(\varepsilon)$ is continuous in ε.

5.1. An example. Let Ω be a parallelepiped, $\Omega = (0, \pi a_1) \times (0, \pi a_2) \times \dots \dots \times (0, \pi a_n)$, $a_i > 0$ $(i = 1, \dots, n)$ and $H = L_2(\Omega)$. Let the operator A be defined by

$$Av = -\sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} \quad \text{for} \quad v \in \mathscr{D}(A) = W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega) \tag{5.1.0}$$

(in the sense of distributions).

Lemma 5.1.1. *The operator A is selfadjoint.*

Proof. By the Neumann theorem (see [22] p. 121) it suffices to show that $\mathscr{R}(A) = H$. Let $g \in H$ and write it in the form

$$g(\boldsymbol{x}) = \sum_{\substack{1 \le k_j < \infty \\ j = 1, \dots, n}} g_{\boldsymbol{k}} \sin \frac{k_1 x_1}{a_1} \sin \frac{k_2 x_2}{a_2} \dots \sin \frac{k_n x_n}{a_n},$$

$$\boldsymbol{k} = [k_1, \dots, k_n], \quad \boldsymbol{x} = [x_1, x_2, \dots, x_n] \in \Omega .$$

It is clear that the relation

$$v(\boldsymbol{x}) = \sum_{\substack{1 \le k_j < \infty \\ j = 1, \dots, n}} \left(\sum_{i=1}^{n} \frac{k_i^2}{a_i^2} \right)^{-1} g_{\boldsymbol{k}} \sin \frac{k_1}{a_1} x_1 \dots \sin \frac{k_n}{a_n} x_n, \quad \boldsymbol{x} \in \Omega \tag{5.1.1}$$

defines such an element of $W_2^2(\Omega)$ that $Av = g$. Since every finite sum in (5.1.1) vanishes on $\partial\Omega$, $v = 0$ in $L_q(\partial\Omega)$ with q arbitrary and we have $v \in \mathring{W}_2^1(\Omega)$ (see [23], pp. 86–87, Th. 4.6, 4.7, 4.10).

Lemma 5.1.2. *The spectrum $\sigma(A)$ of the operator A consists of the point spectrum $\sigma_p(A) = \{\lambda_{\boldsymbol{k}} = \sum_{i=1}^{n} k_i^2/a_i^2;\ \boldsymbol{k} = [k_1, k_2, \dots, k_n],\ 1 \le k_j < \infty\ (j = 1, \dots, n)$ integers$\}$ and to the eigenvalue $\lambda_{\boldsymbol{k}}$ there corresponds the eigenfunction $v_{\boldsymbol{k}}(x) = \sin (k_1 x_1/a_1) \dots \sin (k_n x_n/a_n)$.*

Proof. The discreteness of $\sigma(A)$ follows from [24] (p. 250, Th. 14.6) and the remaining facts may be verified easily by a straitforward calculation.

Lemma 5.1.3. *Let* $\omega = 2\pi p_0/q_0$, $a_j = p_j/q_j$ $(j = 1, \ldots, n)$, *where* p_j, q_j $(j = 0, \ldots, n)$ *are relatively prime positive integers. Then for* $\lambda \in \sigma(A) \setminus \{4k^2\pi^2/\omega^2\}_{k=1}^{\infty}$ *the relation*

$$\min_{k=0,1,\ldots} \left| \sqrt{\lambda} - \frac{2k\pi}{\omega} \right| \geqq \frac{1}{3p^2 \sqrt{\lambda}}, \quad \textit{where} \quad p = \prod_{j=0}^{n} p_j \tag{5.1.2}$$

holds.

Proof. Indeed, if $\boldsymbol{k} = [k_1, \ldots, k_n]$, and l_0 is such an integer that

$$\min_{l=0,1,\ldots} \left| \left(\sqrt{\sum_{j=1}^{n} \frac{k_j^2}{a_j^2}} \right) - \frac{2l\pi}{\omega} \right| = \left| \sqrt{\sum_{j=1}^{n} \frac{k_j^2}{a_j^2}} - \frac{2l_0\pi}{\omega} \right|$$

then

$$\left| \left(\sqrt{\sum_{j=1}^{n} \frac{k_j^2}{a_j^2}} \right) - \frac{2l_0\pi}{\omega} \right| = \frac{\left| \sum_{j=1}^{n} \frac{k_j^2 q_j^2}{p_j^2} - \frac{l_0^2 q_0^2}{p_0^2} \right|}{\left(\sqrt{\sum_{j=1}^{n} \frac{k_j^2 q_j^2}{p_j^2}} \right) + \frac{l_0 q_0}{p_0}} \geqq \frac{1}{3 \left(\sqrt{\sum_{j=1}^{n} \frac{k_j^2 q_j^2}{p_j^2}} \right) \prod_{j=0}^{n} p_j^2}.$$

Lemma 5.1.4. *It is* $\mathscr{N} = \mathscr{N}_1 = \mathscr{N}\left(\frac{\sin \frac{1}{2}\omega \sqrt{A}}{\sqrt{A}} \right) = \mathscr{N}_2 = \mathscr{N}(\sin \frac{1}{2}\omega \sqrt{A}) = \mathscr{L}\left(\left\{ v_{\boldsymbol{k}}(\boldsymbol{x}) = \sin \frac{k_1 x_1}{a_1} \ldots \sin \frac{k_n x_n}{a_n} ;\ \boldsymbol{x} \in \Omega \right. \right.$ *there exists a positive integer* l *such that* $\left. \left. \sum_{j=1}^{n} \frac{k_j^2}{a_j^2} = \frac{l^2 q_0^2}{p_0^2} \right\} \right)^c$.

Proof. The assertion is clear by Proposition 1.1.1.

Theorem 5.1.1. *Let the equation* (3.2.1) *be given with* A *defined by* (5.1.0), $(\gamma = 0)$ *and let* $f \in C(R^+; \mathscr{D}(A^{\nu+1}))$, $(\nu \geqq 0)$, *be* ω*-periodic on* R^+. *Let* ω *and* a_j $(j = 1, \ldots, n)$ *fulfil the assumption of Lemma* 5.1.3. *Then an* ω*-periodic solution* $u(t)$ *of the equation* (3.2.1) *exists iff*

$$\int_0^{\omega} \int_{\Omega} f(t, \boldsymbol{x})\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = 0 \quad \textit{holds for every} \quad v \in \mathscr{N}. \tag{5.1.3}$$

If the condition is satisfied then $u \in U_{\nu}^{0}(R^+)$.

Proof. Let us verify that all the assumptions of Theorem 3.2.1 are satisfied. The selfadjointness of A is guaranteed by Lemma 5.1.1. The assumption (3.2.2) follows from (5.1.2) with $\varrho = \frac{1}{2}$. The condition (5.1.3) is evidently equivalent do (3.2.3) in virtue of Lemma 5.1.4.

Remark 5.1.1. Since $\mathscr{D}(A^\nu) \subset W_2^{2\nu}(\Omega)$, the found solution is a classical one as soon as $\nu > 1 + \frac{1}{4}n$ by the Sobolev imbedding theorem.

Lemma 5.1.5. *Let $n = 2, 3$ and let $f = f(t, \mathbf{x}, u)$ be a function which is continuous on $\langle 0, T\rangle \times \Omega^c \times (-\infty, \infty)$, $(T > 0)$, together with its derivatives*

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial f}{\partial u}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \frac{\partial^2 f}{\partial x_i \partial u}, \quad \frac{\partial^2 f}{\partial u^2},$$

$$1 \leqq i, j \leqq n \quad \text{and let} \quad f(t, \mathbf{x}, 0) = 0 \quad \text{for} \quad t \in \langle 0, T\rangle, \ \mathbf{x} \in \partial\Omega .$$

Then for any $v(\cdot, \cdot) \in C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$, $f(\cdot, \cdot, v(\cdot, \cdot))$ belongs to $C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$. Moreover, if

$$f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial u}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_i \partial u}, \frac{\partial^2 f}{\partial u^2}, \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}, \frac{\partial^3 f}{\partial x_i \partial x_j \partial u}, \frac{\partial^3 f}{\partial x_i \partial u^2}, \frac{\partial^3 f}{\partial u^3}$$

$$(1 \leqq i, j, k \leqq n)$$

are continuous in all the variables then $f(\cdot, \cdot, u(\cdot, \cdot))$ fulfils the Lipschitz condition

$$\|f(\cdot, \cdot, u_1(\cdot, \cdot)) - f(\cdot, \cdot, u_2(\cdot, \cdot)\|_1 \leqq L\|u_1(\cdot, \cdot) - u_2(\cdot, \cdot)\|_1 , \tag{5.1.4}$$

where $u_1(\cdot, \cdot), u_2(\cdot, \cdot) \in C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$ and it has continuous Gâteaux derivative $f'_u(\cdot, \cdot, u(\cdot, \cdot))$ as the mapping from $C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$ into itself for any $u \in C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$.

The proof follows readily from [25] Lemma 1.2, from the Sobolev imbedding theorem and from the lemma on traces (see [23] pp. 86–87).

Lemma 5.1.6. Let $n = 4$, and let $f = f(t, \mathbf{x}, u)$ be a function defined on $\langle 0, T\rangle \times \Omega^c \times (-\infty, \infty)$, $(T > 0)$, such that $f(t, \mathbf{x}, 0) = 0$, $t \in \langle 0, T\rangle$, $\mathbf{x} \in \partial\Omega$. *Suppose that for any $u = u(\cdot, \cdot) \in C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$ the following inclusions are valid:*

$$\frac{\partial f}{\partial x_i}(\cdot, \cdot, u), \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(\cdot, \cdot, u) \in C(\langle 0, T\rangle; L_2(\Omega)),$$

$$\frac{\partial f}{\partial u}(\cdot, \cdot, u), \quad \frac{\partial^2 f}{\partial u^2}(\cdot, \cdot, u) \in C(\langle 0, T\rangle; L_\infty(\Omega)),$$

$$\frac{\partial^2 f}{\partial x_i \partial u}(\cdot, \cdot, u) \in C(\langle 0, T\rangle; L_4(\Omega)), \quad i, j = 1, 2, 3, 4 .$$

Then $f(\cdot, \cdot, u) \in C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$.

Moreover, if all the derivatives mentioned above fulfil the Lipschitz condition in the corresponding spaces and if they have as the operators from $C(\langle 0, T\rangle; W_2^2(\Omega) \cap \cap W_2^1(\Omega))$ into corresponding spaces the continuous Gâteaux derivatives then $f(\cdot, \cdot, u)$ fulfils the Lipschitz condition (5.1.4) and there exists a continuous Gâteaux derivative $f'_u(\cdot, \cdot, u)$ as the mapping from $C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$ into itself for any $u \in C(\langle 0, T\rangle; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$.

The proof follows from the papers mentioned in Lemma 5.1.5.

Theorem 5.1.2. *Let the equation*

$$(5.1.5) \qquad u_{tt}(t, \mathbf{x}) + A\, u(t, \mathbf{x}) = \varepsilon[u - h(t, \mathbf{x}) + \varepsilon f(t, \mathbf{x}, u)]\,, \quad t \in R^+\,, \quad \mathbf{x} \in \Omega$$

be given with A defined by (5.1.1) and with a_j fulfilling the assumption of Lemma 5.1.3. Let $n = 2, 3$ and $n = 4$ respectively. Suppose that $h(\cdot, \cdot) \in C(\langle 0, \omega\rangle; W_2^2(\Omega) \cap \cap \mathring{W}_2^1(\Omega))$ and that $f(t, \mathbf{x}, u)$ fulfils the assumptions of Lemma 5.1.5 and Lemma 5.1.6 respectively with $T = \omega$. Let h and f be ω-periodic in t with $\omega = 2\pi p_0/q_0$, where p_0, q_0 are relatively prime positive integers. Then for sufficiently small $\varepsilon > 0$ there exists a unique ω-periodic solution to (5.1.5) which is continuous in ε in the norm of $U_0^0(\langle 0, \omega\rangle)$.

Proof. Denote

$$\mathscr{S} = \left\{ \mathbf{k} = [k_1, k_2, \ldots, k_n];\ \frac{l^2 q_0^2}{p_0^2} = \sum_{j=1}^{n} \frac{k_j^2}{a_j^2} \quad \textit{for some integer } l \right\}.$$

Obviously it is sufficient to verify the assumptions of Theorem 4.3.2. Here we set $\tilde{F}(t, u) = u - h(t, \cdot) + \varepsilon f(t, \cdot, u)$. The existence of the continuous Gâteaux derivative of $\tilde{F}$ is guaranteed by Lemma 5.1.5 and Lemma 5.1.6 respectively (for $n = 2, 3$ and for $n = 4$ respectively). After some arrangements the equation (4.3.4) may be now written in the form

$$(5.1.6) \qquad \hat{G}_1(0)(\psi_1, \tilde{\varphi}_1) = \sum_{\mathbf{k} \in \mathscr{S}} \left[\psi_{\mathbf{k}} - \sqrt{\lambda_{\mathbf{k}}} \left(\int_0^\omega \sin^2 \tau \sqrt{\lambda_{\mathbf{k}}}\, \mathrm{d}\tau \right)^{-1} . \right.$$
$$\left. . \left(\int_0^\omega h_{\mathbf{k}}(\tau) \sin \tau \sqrt{\lambda_{\mathbf{k}}}\, \mathrm{d}\tau \right) \right] v_{\mathbf{k}} = 0,$$

$$\hat{G}_2(0)(\psi_1, \tilde{\varphi}_1) = \sum_{\mathbf{k} \in \mathscr{S}} \left[\tilde{\varphi}_{\mathbf{k}} - \sqrt{\lambda_{\mathbf{k}}} \left(\int_0^\omega \cos^2 \tau \sqrt{\lambda_{\mathbf{k}}}\, \mathrm{d}\tau \right)^{-1} . \right.$$
$$\left. . \int_0^\omega h\,(\tau) \cos \tau \sqrt{\lambda_{\mathbf{k}}}\, \mathrm{d}\tau \right] v_{\mathbf{k}} = 0\,,$$

which yields immediately the solution

$$\psi_1^0 = \frac{2}{\omega} \sum_{k \in \mathscr{S}} \sqrt{\lambda_k} \int_0^\omega h_k(\tau) \sin \tau \sqrt{\lambda_k}\, d\tau\, v_k ,$$

$$\tilde{\varphi}_1^0 = \frac{2}{\omega} \sum_{k \in \mathscr{S}} \sqrt{\lambda_k} \int_0^\omega h_k(\tau) \cos \tau \sqrt{\lambda_k}\, d\tau\, v_k .$$

(Here ψ_k, φ_k and $h_k(t)$ respectively are Fourier coefficients of the functions ψ, φ and $h(t, \cdot)$ respectively.) Finally, because of (5.1.6), $[\hat{G}'_{[\psi_1,\varphi_1]}(\psi_1^0, \tilde{\varphi}_1^0)]^{-1}$ is the identity operator, which completes the proof.

Remark 5.1.2. Evidently, we may require that the function $f(t, \cdot, u)$ fulfils the assumptions of Theorem 5.1.2 only in the neighbourhood of $u = u_0(t, \cdot) = J(t) A^{-1/2} \tilde{\varphi}_1^0(\cdot) + K(t) \psi_1^0(\cdot)$ (cf. Theorem 4.3.2).

Remark 5.1.3. We are not able to surpass $n = 4$ for the number of space dimensions according to the term $(\partial^2 f/\partial u^2)(\partial^2 f/\partial x_j\, \partial x_k)$ in the total derivative $\partial^2 f/\partial x^2$. We could overcome this limitation looking for less smooth solutions putting $\mathscr{D}(A) = \mathring{W}_2^1(\Omega)$. On the other hand the restriction on f to depend only on u and not on its derivatives is unavoidable in the frames of the present theory.

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