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# PERIODIC SOLUTIONS TO WEAKLY NONLINEAR AUTONOMOUS WAVE EQUATIONS

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## 0. INTRODUCTION.

In [13], one of the authors has derived a necessary and sufficient condition for the existence of a $2\pi$-periodic or $2\pi p/q$-periodic solution to the system

$$\square u \equiv u_{tt} - u_{xx} = g(t, x)\,, \quad t \in R\,, \quad x \in [0, \pi]\,, \tag{0.1}$$

$$u(t, 0) = u(t, \pi) = 0\,, \quad t \in R\,, \tag{0.2}$$

provided $g$ is sufficiently smooth and, respectively, $2\pi$-periodic or $2\pi p/q$-periodic in $t$, as well as necessary or sufficient conditions for the corresponding weakly non-linear problem given by (0.2) and

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x)\,. \tag{0.3}$$

(The analogous problem for some other boundary conditions was dealt with in [14].)

Here we want to study the existence of $\omega$-periodic solutions to the corresponding autonomous problem $(\mathscr{A}^{\varepsilon}_{\omega})$ given by

$$u_{tt} - u_{xx} = \varepsilon f(x, u, u_t, u_x, \varepsilon)\,, \quad t \in R\,, \quad x \in [0, \pi]\,, \tag{0.4}$$

$$u(t, 0) = u(t, \pi) = 0\,, \quad t \in R\,, \tag{0.5}$$

$$u(t + \omega, x) = u(t, x)\,, \quad t \in R\,, \quad x \in [0, \pi]\,. \tag{0.6}$$

Let us recall two characteristic properties of this problem:

Firstly, the period $\omega$ of the sought solution is here an unknown which in general depends on $\varepsilon$ and will be looked for in the neighbourhood of the periods $\omega_n = 2\pi n^{-1}$,

$n = 1, 2, \ldots$ of the free oscillations. This fact is revealed even by the linear problem $(\mathscr{A}_\omega^\varepsilon)$ given by (0.4)–(0.6) with $f(x, u, u_t, u_x, \varepsilon) = u$ which has the sequence of solutions

$$u_n(t, x) = a_n \cos\left(\sqrt{(n^2 - \varepsilon)}\, t - \alpha_n\right) \sin nx$$

with periods $\omega_n = 2\pi/\sqrt{(n^2 - \varepsilon)}$.

Secondly, if $u$ is a solution of (0.4)–(0.6) then $\bar{u}(t, x) \equiv u(t + h, x)$ ($h \in R$, fixed) is a solution of (0.4)–(0.6) as well.

The existence of $\omega$-periodic solutions to the equation

$$\Box u = f(x, Du, \varepsilon)\,, \quad x \in \Omega\,, \tag{0.7}$$

is dealt with in several papers. V. A. VITT [15] investigates the existence of an $\omega$-periodic solution to (0.7) with $\Omega = (0, 1)$ and with some nonlinear boundary conditions. The procedure of his is formal. The same procedure is used by M. E. ŽABOTINSKIJ [16] (again in a purely formal way) to investigate an $\omega$-periodic solution of more general problem described by the equations ($\Omega \subset R^3$)

$$\varrho(x)\, u_{tt} - Lu = \varepsilon f(x, u, \varepsilon)\,, \quad t \in R\,, \quad x \in \Omega\,,$$

$$u_{tt} - \Lambda u = \varepsilon F(x, u, \varepsilon)\,, \quad t \in R\,, \quad x \in \partial\Omega\,,$$

where $L$ is an operator of the form $Lu = -q(x)\, u + \sum_{j=1}^{3} (pu_{x_j})_{x_j}$, $\varrho, q, p$ are analytic functions of $x$ and $\Lambda$ is a linear operator. J. P. FINK, W. S. HALL and S. KHALILI [5] look for $2\pi/\omega$-periodic solutions of (0.7) with $f(x, Du, \varepsilon) = u^3$, $f(x, Du, \varepsilon) = \alpha(u + \beta u^3)$, $f(x, Du, \varepsilon) = -M^2 \sin u$ and with boundary conditions (0.2). They obtain a periodic solution (as well as its period) as a formal power series in a small parameter. J. B. KELLER and L. TING [7] look for $2\pi/\omega$-periodic solutions to $u_{tt} - u_{xx} = f(u)$, $x \in [0, \pi]$ with (0.2). Here even the formal procedure is not quite correct since the authors do not take into account that the right hand side of (3.12) has to be orthogonal to each $2\pi$-periodic function which satisfies the homogeneous equation. (Similarly in [10].) S. I. POCHOŽAJEV [12] investigates the existence of a periodic solution under general assumptions. His theory includes the equation (0.7) with $\Omega = (0, \pi)$ only in the case $f(x, Du, \varepsilon) = g_1(x)\, u + g_2(x)\, u_x$. G. PETIAU [11] and B. A. FLEISHMAN [6] find periodic solutions to (0.7) with $f(x, Du, \varepsilon) = \alpha u + \beta u^3$ and $x \in R$, $x \in R^3$. The periodic solution is of the form $\psi(kx - c(t - t_0))$, where $\psi$ is expressed by Jacobi elliptic functions. In the work [1] M. S. BERGER investigates the existence of infinite countable number of distinct periodic solutions to $p(x)\, u_{tt} - \Delta u + f(x, u) = 0$ in $R \times G$, $u|_{\partial G} = 0$, ($G$ is a bounded domain in $R^n$) under certain assumptions on $f$. However, some crucial points in his proof are obscure. J. KURZWEIL [8] (§5) proves the existence of $\omega(\varepsilon)$-periodic solution to the equation

$$u_{tt} - u_{xx} = -\varepsilon[(h(2^{-1}(u_x + u_t)) - h(2^{-1}(u_x - u_t))) \cos 2x + u_t]$$

and (0.2) with the aid of the averaging principle assuming that $\varepsilon$ positive is sufficiently small and $h$ fulfils some assumptions. Thus his paper represents the only strictly proved result on the existence of a solution to a nonlinear problem of the type $(\mathscr{A}^{\varepsilon}_{\omega(\varepsilon)})$. The following papers are closely related to the preceding ones. M. H. MILMAN and J. B. KELLER [10] study $2\pi/\omega$-periodic solutions to the equation $u_{tt} - u_{xx} + u = \varepsilon f(u_t)$, $x \in [0, \pi]$ with boundary conditions (0.2) using the expansion with respect to the small parameter. M. S. Berger [2], [3] looks for a periodic solution to $u_{tt} = \Delta u - m^2 u + k(|x|)\,|u|^{\sigma} u$, $(t, x) \in R \times R^3$, $m > 0$, in the form $u(t, x) = e^{i\lambda t} v(x)$ $(0 \not\equiv v(x) \to 0$ exponentially as $|x| \to \infty)$ provided $0 \leqq k_1 \leqq k(|x|) \leqq k_2 < +\infty$ and $0 < \sigma < 4$.

Short before completing this paper we received a preprint of a paper by J. P. FINK, W. S. HALL and A. R. HAUSRATH: "Discontinuous periodic solutions for an autonomous wave equation" in which the authors investigate the existence of $2\pi$-periodic solutions to the system $y_{1t} = y_{2x} + \varepsilon(y_1 - y_1^3)$, $y_{2t} = y_{1x}$. We had been informed about its preparation and results sooner from the correspondence with W. S. Hall which encouraged our investigation whose results are found in §2.

This paper consists of two paragraphs. In paragraph 1 we introduce Banach spaces of piecewise regular functions and define the notion of a generalized solution of a boundary value problem for the wave equation. This part further includes the assertion that a generalized solution coincides with a solution of a certain integro-differential equation. In Theorem 1.1 necessary and sufficient conditions are stated for the existence of a solution to the problem $(\mathscr{A}^{\varepsilon}_{\omega})$ in the class of piecewise regular functions. In Remark 1.5 we show the reasons why we consider this type of solutions. (Regarding nonregular periodic solutions of the wave equations let us refer to interesting examples due to J. Kurzweil [8], [9] who constructs regular solutions of certain types of the wave equations that "converge" with increasing time to a piecewise regular $2\pi$-periodic functions. In paragraph 2 we prove the existence of $2\pi$-periodic piecewise continuous solutions to the problem (0.4), (0.5) with $f = -\alpha u + \beta u^3$, $\alpha/\beta > 0$, $f = (\gamma + u^2)\, u_t$ and $2\pi$-periodic continuous and piecewise regular solutions to the problem (0.4), (0.5) with $f = -\alpha u_t + \beta u_t^3$, $\alpha/\beta > 0$.

The solution $u$ of the problem (0.4), (0.5) defined on $(T_1, T_2) \times [0, \pi]$ is extended to the set $(T_1, T_2) \times R$ as a function fulfilling

$$u(t, x) = -u(t, -x) = u(t, x + 2\pi)\,.$$

In the sequel we consider only the functions $u$ extended in the variable $x$ to $R$ in this way.

## 1. GENERAL CONSIDERATIONS.

Let be given a set $S$ with the following three properties:

(1.1) (i) $S \subset R$, $x \in S$ implies $-x \in S$,
(ii) $x \in S$ implies $x + 2\pi \in S$, $x - 2\pi \in S$,
(iii) $S \cap (0, 2\pi)$ is a finite set.

Given $S$ with the property (1.1), we denote

$$\hat{S} = \{(t, x);\ t \in R,\ x \in R,\ x + t \in S \text{ or } x - t \in S\}\,.$$

For such $S$, $\hat{S}$ and $\omega \geqq 0$ we define the function spaces $\mathscr{D}^k_{2\pi}(S)$, $\mathscr{D}^k_{\omega,2\pi}(\mathscr{T}; \hat{S})$ (where $\mathscr{T} = (T_1, T_2)$, $-\infty \leqq T_1 < T_2 \leqq +\infty$) as follows:

**Definition 1.1.** A function is an element of $\mathscr{D}^k_{2\pi}(S)$ if it is defined on $R \setminus S$, $2\pi$-periodic and uniformly continuous together with its derivatives up to order $k$ on each open component of $R \setminus S$.

We denote

$$|s|_0 = \sup\ \{|s(x)|;\ x \in R \setminus S\}\,,$$

$$|s|_k = \max\ \{|s^{(p)}|_0;\ p = 0, 1, \ldots, k\}\,.$$

The space $\mathscr{D}^k_{2\pi}(S)$ equipped with the norm $|\cdot|_k$ is a Banach space.

**Definition 1.2.** A function $u = u(t, x)$ is an element of $\mathscr{D}^k_{0,2\pi}(\mathscr{T}; \hat{S})$ if it is defined on $\mathscr{T} \times R \setminus \hat{S}$, $2\pi$-periodic and odd in $x$ and uniformly continuous together with its derivatives up to order $k$ on each open component of $\mathscr{T} \times R \setminus \hat{S}$. We denote

$$\|u\|_0 = \sup\ \{|u(t, x)|;\ (t, x) \in \mathscr{T} \times R \setminus \hat{S}\}\,,$$

$$\|u\|_k = \max\ \{\|\partial^{i+j}u/\partial t^i\, \partial x^j\|_0;\ i, j \text{ nonnegative integers},\ i + j \leqq k\}\,.$$

The space $\mathscr{D}^k_{0,2\pi}(\mathscr{T}; \hat{S})$ equipped with the norm $\|\cdot\|_k$ is a Banach space (for $-\infty < T_1 < T_2 < +\infty$).

**Definition 1.3.** Let $\omega > 0$ be given. We denote by $\mathscr{D}^k_{\omega,2\pi}(R; \hat{S})$ the subspace of $\mathscr{D}^k_{0,2\pi}(R; \hat{S})$ containing those functions that are $\omega$-periodic in $t$, i.e., $u \in \mathscr{D}^k_{0,2\pi}(R; \hat{S})$ is an element of $\mathscr{D}^k_{\omega,2\pi}(R; \hat{S})$ if $u(t + \omega, x) = u(t, x)$ is fulfilled for every $(t, x) \in R^2 \setminus \hat{S}$ such that $(t + \omega, x) \in R^2 \setminus \hat{S}$. This space is equipped with the norm of the space $\mathscr{D}^k_{0,2\pi}((0, \omega); \hat{S})$.

Remark 1.1. For $S = \emptyset$ the spaces $\mathscr{D}^k_{\omega,2\pi}(\mathscr{T}; \emptyset)$, $\mathscr{D}^k_{2\pi}(\emptyset)$ are simply function spaces of class $\mathscr{C}^k$.

Further let be given a function $f$, which describes the nonlinearity in our equation, fulfilling:

(1.2) $f = f(x, y_0, y_1, y_2, \varepsilon)$ is continuous on $R^4 \times [-\varepsilon_0, \varepsilon_0]$,

$f$ has continuous first derivatives with respect to $x, y_0, y_1, y_2$ and $f$ fulfils on $R^4 \times [-\varepsilon_0, \varepsilon_0]$:

$$f(x + 2\pi, y_0, y_1, y_2, \varepsilon) = f(x, y_0, y_1, y_2, \varepsilon),$$

$$f(x, y_0, y_1, y_2, \varepsilon) = -f(-x, -y_0, -y_1, y_2, \varepsilon).$$

In the sequel we often use

**Definition 1.4.** Let $S$ with the properties (1.1), a function $f$ fulfilling (1.2) and $T_1, T_2 \in R \cup \{-\infty, +\infty\}$, $-\infty \leqq T_1 \leqq 0 < T_2 \leqq +\infty$, be given.

A function $u \in \mathscr{D}^2_{0,2\pi}((T_1, T_2); \hat{S})$ is called a generalized solution to

$$\text{(1.3.1)} \qquad \Box u \equiv u_{tt} - u_{xx} = \varepsilon f(x, u, u_t, u_x, \varepsilon),$$

$$\text{(1.3.2)} \qquad u(t, 0) = u(t, \pi) = 0$$

if $u$ fulfils

$$\text{(1.4)} \qquad \iint_{(T_1,T_2)\times R} \{u \Box \varphi - \varepsilon f(x, u, u_t, u_x, \varepsilon)\,\varphi\} = 0$$

for every $\varphi \in \mathscr{C}^\infty_0((T_1, T_2) \times R)$.

Using this definition we can formulate

**Lemma 1.1.** *The function $u \in \mathscr{D}^2_{0,2\pi}((T_1, T_2); \hat{S})$ is a generalized solution to (1.3) if and only if there exists a function $s \in \mathscr{D}^2_{2\pi}(S)$ such that the relation*

$$\text{(1.5)} \qquad u(t, x) = s(x + t) - s(-x + t) + \frac{\varepsilon}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} F(u)(\varepsilon)(\tau, \xi)\,d\xi\,d\tau$$

*holds for all $(t, x) \in (T_1, T_2) \times R \setminus \hat{S}$ where*

$$\text{(1.6)} \qquad F(u)(\varepsilon)(t, x) = f(x, u(t, x), u_t(t, x), u_x(t, x), \varepsilon).$$

Remark 1.2. The function $u$ in Lemma 1.1 determines $s$ uniquely if we impose on $s$ an additional condition

$$\text{(1.7)} \qquad \int_0^{2\pi} s(\xi)\,d\xi = 0.$$

Proof of Lemma 1.1. Let two functions $u \in \mathscr{D}^2_{0,2\pi}((T_1, T_2); \hat{S})$, $s \in \mathscr{D}^2_{2\pi}(S)$ fulfilling (1.5) for every $(t, x) \in (T_1, T_2) \times R \setminus \hat{S}$ and a function $\varphi \in \mathscr{C}^\infty_0((T_1, T_2) \times R)$ be given. We prove in several steps that (1.4) holds.

A 1) For $v(t, x) = s(x + t) - s(-x + t)$ we have

$$\iint_{(T_1,T_2)\times R} v \Box \varphi = 0$$

as a consequence of the following argument: Denoting $\hat{v}(t, x) = s(x + t)$, $\psi(t, x) = \varphi_t(t, x) + \varphi_x(t, x)$, we obtain

$$\iint_{(T_1,T_2)\times R} \hat{v} \,\Box\, \varphi = \iint_{(T_1,T_2)\times R} s(x + t) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \psi(x, t) = 0$$

because $s(x + t)$ is a constant on every straight line parallel to the direction $(-1, 1)$ (as far as it is defined). Similarly for $s(-x + t)$.

A 2) Having in mind (1.6) we denote further $g(t, x) = \varepsilon F(u)(\varepsilon)(t, x)$ and

$$\hat{u}(t, x) = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} g(\tau, \xi)\, d\xi\, d\tau\,. \tag{1.8}$$

Now we must show that $\hat{u}$ fulfils

$$\iint_{(T_1,T_2)\times R} \{\hat{u} \,\Box\, \varphi - g\varphi\} = 0 \tag{1.9}$$

to prove the first part of our Lemma.

A 3) A straightforward computation shows that $g \in \mathscr{D}^1_{0,2\pi}((T_1, T_2); \hat{S})$ and that $\hat{u}$ given by (1.8) is a continuous function on $(T_1, T_2) \times R$ fulfilling $\hat{u}(t, x) = -\hat{u}(t, -x) = \hat{u}(t, x + 2\pi)$ for every $(t, x) \in (T_1, T_2) \times R$. Further, one can immediately verify that for $(t, x) \in (T_1, T_2) \times R \setminus \hat{S}$ the following formulae hold:

$$\hat{u}_t(t, x) = 2^{-1} \int_0^t [g(\tau, x + t - \tau) + g(\tau, x - t + \tau)]\, d\tau\,,$$

$$\hat{u}_x(t, x) = 2^{-1} \int_0^t [g(\tau, x + t - \tau) - g(\tau, x - t + \tau)]\, d\tau\,.$$

We describe the second derivative of $\hat{u}$, too. For every $(t, x) \in R^2 \setminus \hat{S}$ we denote by $M(t, x)$ the intersection of $\hat{S}$ and the two segments that join the point $(t, x)$ with the axis $t = 0$ and have the directions $(1, -1)$ and $(1, 1)$. To every point $P \in M(t, x)$ there exist just two components of $R^2 \setminus \hat{S}$ such that $P$ is an element of their closures. We denote these components by $Q_P^-, Q_P^+$ in such a way that there exists $(t^-, x^-) \in Q_P^-$ fulfilling $t^- < t^+$ for every $(t^+, x^+) \in Q_P^+$. Further we denote by $g^+(P)$ or $g^-(P)$ the limit of the function $g \in \mathscr{D}^1_{0,2\pi}((T_1, T_2; \hat{S})$ at the point $P$ with respect to $Q_P^-$ or $Q_P^+$, respectively. Then we have ($(t, x) \in R^2 \setminus \hat{S}$)

$$\hat{u}_{tt}(t, x) - g(t, x) = \hat{u}_{xx}(t, x) =$$

$$= 2^{-1} \int_0^t (g_x(\tau, x + t - \tau) - g_x(\tau, x - t + \tau))\, d\tau +$$

$$+ 2^{-1} (\operatorname{sign} t) \sum_{P \in M(t,x)} (g^-(P) - g^+(P))\,.$$

These formulae show that $\hat{u}_t$, $\hat{u}_x$, $\hat{u}_{tt}$, $\hat{u}_{xx}$ (and similarly $\hat{u}_{tx}$) are uniformly continuous functions on each open component of $(T_1, T_2) \times R \setminus \hat{S}$. Thus $\hat{u} \in \in \mathscr{D}^2_{0,2\pi}((T_1, T_2), \hat{S})$ and $\Box \hat{u} = g$ holds on each component of $(T_1, T_2) \times R \setminus \hat{S}$.

A 4) Let us denote by $\Omega$ the interior of the set supp $\varphi$ and by $\Omega_k$ the open components of the set $\Omega \setminus \hat{S}$. Then (1.9) may be written in the form

$$(1.10) \qquad \sum_k \iint_{\Omega_k} \{\hat{u} \Box\varphi - g\varphi\} = \sum_k \iint_{\Omega_k} \{\hat{u} \Box\varphi - \Box\hat{u}\, \varphi\} =$$

$$= \sum_k \int_{\partial\Omega_k} \hat{u}\, (\varphi_t, \, -\varphi_x) \circ n_k \,\mathrm{d}s - \int_{\partial\Omega_k} \varphi\, (\hat{u}_t, \, -\hat{u}_x) \circ n_k \,\mathrm{d}s$$

where $\partial\Omega_k$ denotes the boundary of $\Omega_k$, $n_k$ is the vector of the outer normal to $\Omega_k$ and "$\circ$" denotes the scalar product in $R^2$. The expression $\varphi(\hat{u}_t, -\hat{u}_x) \circ n_k$ may have non zero values only on such parts of $\partial\Omega_k$ that are subsets of $\hat{S}$. Let $d$ be a segment with the endpoints $P, Q$, $d \subset \hat{S}$ such that $d \subset \partial\Omega_k \cap \partial\Omega_l$. Then

$$(1.11) \qquad n_k = -n_l \quad \text{on} \quad d$$

and

$$\int \varphi(\hat{u}_t, \, -\hat{u}_x) \circ n_k \,\mathrm{d}s = - \int_d \hat{u}(\varphi_t, \, -\varphi_x) \circ n_k \,\mathrm{d}s \pm (\hat{u}(P) \,.\, \varphi(P) - \hat{u}(Q) \,.\, \varphi(Q)) \,,$$

$$\int_d \varphi(\hat{u}_t, \, -\hat{u}_x) \circ n_l \,\mathrm{d}s = - \int_d \hat{u}(\varphi_t, \, -\varphi_x) \circ n_l \,\mathrm{d}s \mp (\hat{u}(P) \,.\, \varphi(P) - \hat{u}(Q) \,.\, \varphi(Q)) \,.$$

Hence we see that (1.10) equals

$$2 \sum_k \int_{\partial\Omega_k} \hat{u}(\varphi_t, \, -\varphi_x) \circ n_k \,\mathrm{d}s \,.$$

This expression is equal to zero as a consequence of (1.11) and of the continuity of $\hat{u}$. Thus we have proved that (1.4) is fulfilled.

Now we are going to prove the converse part of Lemma 1.1. Let $u \in \mathscr{D}^2_{0,2\pi}((T_1, T_2); \hat{S})$ be a generalized solution to (1.3). We prove that there exists a function $s \in \mathscr{D}^2_{2\pi}(S)$ such that (1.5) holds, again in several steps.

B 1) To the given function $u$ we define a function $\tilde{u}$ by $((t, x) \in (T_1, T_2) \times R \setminus \hat{S})$

$$(1.12) \qquad \tilde{u}(t, x) = u(t, x) - \hat{u}(t, x) \,,$$

where $\hat{u}$ is defined by (1.8) with $g(t, x) = \varepsilon\, F(u)\,(\varepsilon)\,(t, x)$. As the function $\hat{u}$ fulfils (1.9) we deduce immediately

$$(1.13) \qquad \iint_{(T_1, T_2) \times R} \tilde{u}\, \Box\varphi = 0$$

for every $\varphi \in \mathscr{C}_0^\infty((T_1, T_2) \times R)$. To complete the proof we must find a function $s \in \mathscr{D}_{2\pi}^2(S)$ such that $\tilde{u}(t, x) = s(x + t) - s(-x + t)$, $(t, x) \in (T_1, T_2) \times R \setminus \hat{S}$.

Let us denote by $I_k = (\alpha_k, \beta_k)$ $(k = 0, \pm 1, \pm 2, \ldots)$ the open components of $R \setminus S$ and let us suppose that $\beta_k = \alpha_{k+1}$. Further we use the notation

$$P(I_k) = \{(t, x) \in R^2;\ x - t \in I_k\}\,, \quad Q(I_l) = \{(t, x) \in R^2,\ x + t \in I_l\}\,,$$
$$R(I_k, I_l) \equiv R_{k,l} = ((T_1; T_2) \times R) \cap P(I_k) \cap Q(I_l)\,.$$

B 2) The function $\tilde{u}$ defined in B 1) is of class $\mathscr{C}^2$ on each $R_{k,l} \neq \emptyset$. This implies that there exist two functions $p_{k,l}, q_{k,l}$ defined on some intervals $I'_k \subset I_k$, $I'_l \subset I_l$ respectively (and with their second derivatives uniformly continuous) such that

$$\tilde{u}(t, x) = p_{k,l}(x - t) + q_{k,l}(x + t) \tag{1.14}$$

holds for every $(t, x) \in R_{k,l}$. The following two notations are useful in the sequel. Being $d$ a segment we denote by $d^0$ the same segment without its endpoints. For a set $M \subset R^2$ we denote by $\operatorname{cl} M$ its closure. Now we formulate an assertion which will be used later.

**Assertion.** *For every couple $(k, l)$ such that $\operatorname{cl}(R_{k,l}) \cap \operatorname{cl}(R_{k,l+1}) \neq \emptyset$ there exists $\gamma_{k,l} \in R$ fulfilling*

$$p_{k,l}(\xi) = p_{k,l+1}(\xi) + \gamma_{k,l} \tag{1.15}$$

*for $\xi$ from $\{\xi;\ \xi = x - t,\ (t, x) \in d_{k,l} \equiv ((\operatorname{cl} R_{k,l}) \cap (\operatorname{cl} R_{k,l+1}))^0\}$.*

We sketch the proof of this assertion. We choose $\varphi \in \mathscr{C}_0^\infty((T_1, T_2) \times R)$ such that $\operatorname{supp} \varphi$ is in a sufficiently small neighbourhood of $d_{k,l}$ and $d'_{k,l} \equiv d_{k,l} \cap \operatorname{supp} \varphi$ fulfils $\operatorname{cl}(d'_{k,l}) \subset d_{k,l}$. Then, using (1.13), (1.14), we obtain after the same arrangments as in A 4)

(1.16)
$$0 = \int_{d'_{k,l}} [p_{k,l}(x - t) + q_{k,l}(x + t) - p_{k,l+1}(x - t) - q_{k,l+1}(x + t)]\ (\mathrm{d}\varphi/\mathrm{d}s)\,\mathrm{d}s$$

where $\mathrm{d}\varphi/\mathrm{d}s$ is the derivative in the direction $d_{k,l}$. As $x + t$ is a constant for $(t, x) \in d_{k,l}$, (1.15) follows directly from (1.16).

B 3) Now we may easily define a $2\pi$-periodic function $\tilde{p} \in \mathscr{D}_{2\pi}^2(S)$ such that for every $(k, l)$ $(R_{k,l} \neq \emptyset)$ there exists $\gamma_{k,l} \in R$ for which $\tilde{p}(\xi) = \tilde{p}_{k,l}(\xi) + \gamma_{k,l}$ holds for every $\xi$ from the set $\{\xi;\ \xi = x - t,\ (t, x) \in R_{k,l}\}$. To this function $\tilde{p}$ we define a new function $\bar{u} \in \mathscr{D}_{0,2\pi}^2((T_1, T_2); \hat{S})$ by

$$\bar{u}(t, x) = \tilde{u}(t, x) - (\tilde{p}(x - t) - \tilde{p}(-x - t))\,, \quad (t, x) \in (T_1, T_2) \times R \setminus \hat{S}\,. \tag{1.17}$$

The function $\bar{u}$ fulfils

$$\iint_{(T_1, T_2) \times R} \bar{u}\ \Box\varphi = 0 \tag{1.18}$$

for every $\varphi \in \mathscr{C}_0^\infty((T_1, T_2) \times R)$. Moreover, by (1.14) there exists a function $q_1$ such that $\bar{u}(t, x) = q_1(x + t)$, $(t, x) \in R_{k,l}$, and a function $q_2$ such that $\bar{u}(t, x) = q_2(x + t)$, $(t, x) \in R(-I_l, -I_k)$ (where we put $-I = \{x \in R;\ -x \in I\}$). These functions $q_1, q_2, \bar{u}$ fulfil $q_1(x + t) = \bar{u}(t, x) = -\bar{u}(t, -x) = -q_2(-x + t)$. But this can hold only if $\bar{q}_1$ is a constant. So $\bar{u}$ is equal to a constant on every $R_{k,l} \neq \emptyset$.

B 4) We denote $\bar{u}|_{R_{k,l}} = \bar{u}_{k,l}$. Taking a function $\varphi \in \mathscr{C}_0^\infty((T_1, T_2) \times R)$ with its support in a neighbourhood of the point which is in the intersection of the closures of the nonvoid sets $R_{k,l}, R_{k,l+1}, R_{k+1,l}, R_{k+1,l+1}$ we deduce from (1.18)

$$\bar{u}_{k,l} + \bar{u}_{k+1,l+1} - \bar{u}_{k+1,l} - \bar{u}_{k,l+1} = 0\,. \tag{1.19}$$

B 5) Now we introduce two functions $\hat{p}, \hat{q}$ on $R - S$. The function $\hat{p}$ is equal to a constant on each $I_k$ and we put

$$\hat{p}(I_k) = \bar{u}_{k,l} - \bar{u}_{k-1,l}$$

for any $l$ such that both $R_{k,l}, R_{k-1,l}$ are nonvoid. (1.19) shows that this definition does not depend on $l$. Similarly, the function $\hat{q}$ is equal to a constant on each $I_l$ and we put

$$\hat{q}(I_l) = \bar{u}_{k,l} - \bar{u}_{k,l-1}$$

for any $k$. Both functions $\hat{p}, \hat{q}$ are $2\pi$-periodic. The oddness of the function $\bar{u}$ implies (we put $-I = \{x \in R;\ -x \in I\}$)

$$\hat{p}(I_k) = \bar{u}_{k,l} - \bar{u}_{k-1,l} = \bar{u}|_{R(-I_l, -I_{k-1})} - \bar{u}|_{R(-I_l, -I_k)} = \hat{q}(-I_{k-1})\,. \tag{1.20}$$

Let an integer $n$ be such that for certain $\gamma \in R$, $U_{\varkappa=1}^n I_\varkappa = (\gamma, \gamma + 2\pi) \setminus S$ holds. Then (1.20) yields

$$\sum_{\varkappa=1}^{n} \hat{p}(I_{k+\varkappa}) = \sum_{\varkappa=1}^{n} \hat{q}(I_{l+\varkappa})\,, \tag{1.21}$$

for every couple $(k, l)$ of integers.

Having in mind the definitions of $\hat{p}, \hat{q}$ we write

$$\bar{u}_{k+n,k+n} = \bar{u}_{l,k} + \sum_{\varkappa=1}^{n} (\hat{p}(I_{k+\varkappa}) + \hat{q}(I_{l+\varkappa}))\,.$$

Utilizing (1.21) and the $2\pi$-periodicity in $x$ of $\bar{u}$ we deduce that

$$\sum_{\varkappa=1}^{n} \hat{p}(I_{k+\varkappa}) = \sum_{\varkappa=1}^{n} \hat{q}(I_{l+\varkappa}) = 0 \tag{1.22}$$

holds for every couple $(k, l)$ of integers. Further we choose the couple $(k_1, l_1)$ such that $R_{k_1,l_1} \cap \{(t, x);\ t = 0\} \neq \emptyset$ and we define functions $p$ and $q$ respectively on $\bigcup_{\varkappa=1}^{\infty} I_{k_1+\varkappa}$ and $\bigcup_{\varkappa=1}^{\infty} I_{l_1+\varkappa}$ by $p(I_{k_1+m}) = \sum_{\varkappa=1}^{m} \hat{p}(I_{k_1+\varkappa})$, $q(I_{l_1+m}) = \sum_{\varkappa=1}^{m} \hat{q}(I_{l_1+\varkappa})$. According to (1.22) both functions $p$, $q$ may be extended onto $R \setminus S$ as $2\pi$-periodic functions. The definitions of $p, q$ easily imply $\bar{u}_{k,l} = p(I_k) + q(I_l)$, i.e.

$$(1.23) \qquad \bar{u}(t, x) = p(x - t) + q(x + t), \quad (t, x) \in (T_1, T_2) \times R \setminus \hat{S}.$$

The oddness in $x$ of the function $\bar{u}$ yields $q(\xi) = -p(-\xi)$. So (1.23) may be rewritten in the form $\bar{u}(t, x) = q(x + t) - q(-x + t)$. This last relation, (1.12) and (1.17) complete the proof of our Lemma.

Proof of Remark 1.2. The assertion contained in this remark is equivalent to the following one:

If for some $s \in \mathscr{D}^2_{2\pi}(S)$ the equation $s(x + t) - s(-x + t) = 0$ holds for all $(t, x) \in (T_1, T_2) \times R \setminus \hat{S}$, then there exists $\beta \in R$ such that $s(x) = \beta$ is fulfilled for all $x \in R \setminus S$. But this assertion is obvious.

Remark 1.3. If $S = \emptyset$, then the function $s$ mentioned in Lemma 1.1 and fulfilling (1.7) may be uniquely determined by the functions $u(0, x)$, $u_t(0, x)$. However, this assertion does not hold in the case $S \neq \emptyset$ as the following consideration shows. Let $s \in \mathscr{D}^2_{2\pi}(S)$, even and constant on each component be given. Then the function $u$ defined by $u(t, x) = s(x + t) - s(-x + t)$, $(t, x) \in R^2 \setminus \hat{S}$ is a generalized solution to $\Box u = 0$ and $u$ fulfils $u(0, x) = u_t(0, x) = 0$ for all $x \in R \setminus S$.

**Definition 1.5.** A function $u_\varepsilon \in \mathscr{D}^2_{\omega,2\pi}(R; \hat{S})$ which is a generalized solution to (1.3) is called a solution to the problem $(\mathscr{A}^\varepsilon_\omega)$.

According to Lemma 1.1, a function $u_\varepsilon \in \mathscr{D}^2_{0,2\pi}(R; \hat{S})$ is an $\omega$-periodic generalized solution to (1.3) if and only if for the corresponding $s_\varepsilon \in \mathscr{D}^2_{2\pi}(S)$ and for every $(t, x) \in R^2$ such that $(t, x) \notin \hat{S}$, $(t + \omega, x) \notin \hat{S}$,

$$s_\varepsilon(x + t + \omega) - s_\varepsilon(x + t) - s_\varepsilon(-x + t + \omega) + s_\varepsilon(-x + t) +$$
$$+ 2^{-1}\varepsilon \left( \int_0^{t+\omega} \int_{x-t-\omega+\tau}^{x+t+\omega-\tau} - \int_0^t \int_{x-t+\tau}^{x+t-\tau} \right) F(u_\varepsilon)(\varepsilon)(\tau, \xi)\, d\xi\, d\tau = 0$$

is fulfilled.

Utilizing the fact that the first integral after the transformation has the form $\int_{-\omega}^{t} \int_{x-t+\tau}^{x+t-\tau} F(u_\varepsilon)(\varepsilon)(\tau, \xi)\, d\xi\, d\tau$ and that (according to the oddness in $x$ of $F(u_\varepsilon)(\varepsilon)$)

$$\int_{-x+t-\tau}^{x-t+\tau} F(u_\varepsilon)(\varepsilon)(\tau, \xi)\, d\xi = 0$$

we can rewrite the above equation in the form

$$s_\varepsilon(x + t + \omega) - s_\varepsilon(x + t) + 2^{-1}\varepsilon \int_{-\omega}^{0} \int_{0}^{x+t} F(u_\varepsilon)(\varepsilon)(\tau, \xi - \tau)\, d\xi\, d\tau =$$

$$= s_\varepsilon(-x + t + \omega) - s_\varepsilon(-x + t) + 2^{-1}\varepsilon \int_{-\omega}^{0} \int_{0}^{-x+t} F(u_\varepsilon)(\varepsilon)(\tau, \xi - \tau)\, d\xi\, d\tau .$$

An easy consequence of this formula is

**Lemma 1.2.** *A generalized solution $u_\varepsilon \in \mathscr{D}^2_{0,2\pi}(R; \hat{S})$ to (1.3) is a solution to the problem $(\mathscr{A}^\varepsilon_\omega)$ if and only if there exists a constant $c \in R$ such that for the corresponding $s_\varepsilon$ and for every $x \in R$ fulfilling $x \notin S$, $x + \omega \notin S$,*

$$s_\varepsilon(x + \omega) - s_\varepsilon(x) + 2^{-1}\varepsilon \int_{-\omega}^{0} \int_{0}^{x} F(u_\varepsilon)(\varepsilon)(\tau, \xi - \tau)\, d\xi\, d\tau = c \tag{1.24}$$

*holds.*

Remark 1.4. For $S = \emptyset$, (1.24) is equivalent to

$$s'_\varepsilon(x + \omega) - s'_\varepsilon(x) + 2^{-1}\varepsilon \int_{-\omega}^{0} F(u_\varepsilon)(\varepsilon)(\tau, x - \tau)\, d\tau = 0\,, \quad x \in R\,.$$

The following lemma characterizes the dependence of the period $\omega(\varepsilon)$ on $\varepsilon$ for some solutions to the problem $(\mathscr{A}^\varepsilon_{\omega(\varepsilon)})$.

**Lemma 1.3.** *Let $\varepsilon_0 > 0$, $\omega' < -2\pi$ and a function $u_\varepsilon : [-\varepsilon_0, \varepsilon_0] \to \mathscr{D}^2_{0,2\pi}(R; \emptyset)$ be given. Suppose that $u_\varepsilon$ maps $[-\varepsilon_0, \varepsilon_0]$ into $\mathscr{D}^2_{0,2\pi}((\omega', 0); \emptyset)$ continuously and that $u_\varepsilon$ is a solution to the problem $(\mathscr{A}^\varepsilon_{\omega(\varepsilon)})$ where $\omega(\varepsilon) = 2\pi(1 + \nu(\varepsilon))$ with* $\lim_{\varepsilon \to 0} \nu(\varepsilon) = 0$. *According to Remark* 1.2 *let us choose to $u_\varepsilon$ a function $s_\varepsilon$ such that* (1.7) *is fulfilled. Further let us suppose that $s''_0$ is not zero identically on $R$.*

*Then $s_\varepsilon : [-\varepsilon_0, \varepsilon_0] \to \mathscr{D}^2_{2\pi}(S)$ is continuous and $\nu(\varepsilon) = \varepsilon\, \mu(\varepsilon)$ where the function $\mu(\varepsilon)$ is such that* $\lim_{\varepsilon \to 0} \mu(\varepsilon)$ *exists.*

Proof. The continuity of $s_\varepsilon$ is deduced from (1.5) by putting $-x + t = c$ and integrating with respect to $t$ over $[0, 2\pi]$. The functions $s_\varepsilon$, $u_\varepsilon$ fulfil (1.24) so that we can write

$$2\pi\, \varepsilon^{-1}\nu(\varepsilon) \int_0^1 s''_\varepsilon(x + 2\pi\alpha\nu(\varepsilon))\, d\alpha + 2^{-1} \int_{-2\pi - 2\pi\nu(\varepsilon)}^{0} F(u_\varepsilon)(\varepsilon)(\tau, x - \tau)\, d\tau = 0$$

for $\varepsilon$ with sufficiently small $|\varepsilon| > 0$ and $x \in R$ such that $s''_0(x) \neq 0$. Both the integral expressions in this equation have limits for $\varepsilon \to 0$, the former being different from zero. This proves the Lemma.

We summarize the results described above into

**Theorem 1.1.** *Let $\varepsilon_0 > 0$, $S \subset R$ with the property (1.1) be given.*

*There exist functions $u_\varepsilon : [-\varepsilon_0, \varepsilon_0] \to \mathscr{D}^2_{0,2\pi}(R; \hat{S})$, $\mu(\varepsilon) : [-\varepsilon_0, \varepsilon_0] \to R$ such that $u_\varepsilon$ is a solution to the problem $(\mathscr{A}^\varepsilon_{2\pi(1+\varepsilon\mu(\varepsilon))})$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ if and only if there exist $u_\varepsilon : [-\varepsilon_0, \varepsilon_0] \to \mathscr{D}^2_{0,2\pi}(R; \hat{S})$, $s_\varepsilon : [-\varepsilon_0, \varepsilon_0] \to \mathscr{D}^2_{2\pi}(S)$, $\mu(\varepsilon) : [-\varepsilon_0, \varepsilon_0] \to R$ and $c : [-\varepsilon_0, \varepsilon_0] \to R$ such that the equations*

$$G_1(u, s)(\varepsilon)(t, x) \equiv -u(t, x) + s(x + t) - s(-x + t) +$$

$$+ 2^{-1}\varepsilon \int_0^t \int_{x-t+\tau}^{x+t-\tau} F(u)(\varepsilon)(\tau, \xi)\, d\xi\, d\tau = 0 \quad for \quad (t, x) \in R^2 \setminus \hat{S},$$

$$G_2(u, s, \mu)(\varepsilon)(x) \equiv s(x + 2\pi\varepsilon\mu) - s(x) +$$

$$+ 2^{-1}\varepsilon \int_{-2\pi(1+\varepsilon\mu)}^{0} \int_0^x F(u)(\varepsilon)(\tau, \xi - \tau)\, d\xi\, d\tau = c(\varepsilon)$$

*for every $x \in R$ fulfilling $x \notin S$, $x + 2\pi\varepsilon\mu \notin S$ hold with $u = u_\varepsilon$, $s = s_\varepsilon$, $\mu = \mu(\varepsilon)$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Remark 1.5. If $S = \emptyset$, then the latter equation in Theorem 1.1 is equivalent to

$$s'(x + 2\pi\varepsilon\mu) - s'(x) + 2^{-1}\varepsilon \int_{-2\pi(1+\varepsilon\mu)}^{0} F(u)(\varepsilon)(\tau, x - \tau)\, d\tau \equiv 0\,.$$

Now for $\varepsilon \neq 0$ we define a new relation by

$$\hat{G}_2(u, s, \mu)(\varepsilon)(x) \equiv \varepsilon^{-1}(s'(x + 2\pi\varepsilon\mu) - s'(x)) +$$

$$+ 2^{-1} \int_{-2\pi(1+\varepsilon\mu)}^{0} F(u)(\varepsilon)(\tau, x - \tau)\, d\tau = 0$$

while $\hat{G}_2(u, s, \mu)(0)(x)$ is defined by

$$\hat{G}_2(u, s, \mu)(0)(x) \equiv 2\pi\mu s''(x) + 2^{-1} \int_{-2\pi}^{0} F(u)(0)(\tau, x - \tau)\, d\tau = 0\,.$$

Evidently, if $u_\varepsilon(t, x)$ is a solution to $(\mathscr{A}^\varepsilon_\omega)$, then $u_\varepsilon(t + h, x)$, $h$ arbitrary, is a solution of it as well. To remove this ambiguity in the determination of $u_\varepsilon$ we may require

(1.25) $$G_3(u)(\varepsilon) \equiv u_{tx}(0, 0) = 0\,.$$

In order to apply the implicit function theorem in its well known reading we had to find a B-space $\mathscr{B}_1$ of triples $(u, s, \mu)$ and a B-space $\mathscr{B}_2$ containing $G(\varepsilon)(\mathscr{B}_1)$, $G(\varepsilon) =$

$= (G_1(\varepsilon), \hat{G}_2(\varepsilon), G_3(\varepsilon))$, so that the operator $G$ with its G-derivative $G'_{(u,s,\mu)}(\varepsilon)$ would be continuous in $u$, $s$, $\mu$, $\varepsilon$, there would exist a solution $u_0$, $s_0$, $\mu_0$ of the limit equation $G(u, s, \mu)(0) = 0$ and at that point there would exist the inverse continuous operator $[G'_{(u,s,\mu)}(u_0, s_0, \mu_0)]^{-1}$. Unfortunately we have not succeeded in finding such a couple of spaces $\mathscr{B}_1$, $\mathscr{B}_2$, above all because of the presence of the first term in $\hat{G}_2$ whose crucial role (in other context), was already pointed out by J. P. Fink and W. S. Hall [4]. Indeed, if $s \in \mathscr{D}^2_{2\pi}(\emptyset)$ it may be shown that the expression $\varepsilon^{-1}(s'(x + 2\pi\varepsilon\mu) - s'(x)) - 2\pi\mu\, s''(x)$ in the sense of $\mathscr{D}^0_{2\pi}(\emptyset)$ is not continuous in $s$, $\mu$, $\varepsilon$ while on the other hand, if $s$ belongs to a class of still smoother functions there arise insurmountable difficulties concerning the existence of the inverse operator as the theory of ordinary differential equations indicates. At the same time it seems that if we pass in $\varepsilon^{-1}(s'(x + 2\pi\varepsilon\mu) - s'(x))$ to a limit for $\varepsilon \to 0$ at all then $2\pi\mu\, s''(x)$ is the only possible. J. P. Fink and W. S. Hall in the paper mentioned above evade this obstacle by not passing to the limit. Here we avoid it putting $\omega = 2\pi$ so that the term in question disappears.

**Corollary 1.1.** *Let* $\varepsilon_0 > 0, \omega' < -2\pi$, *the function* $\mu(\varepsilon) : [-\varepsilon_0, \varepsilon_0] \to R$, *continuous at* $\varepsilon = 0$, *and the function* $u_\varepsilon : [-\varepsilon_0, \varepsilon_0] \to \mathscr{D}^2_{\omega(\varepsilon),2\pi}(R; \emptyset)$ $(\omega(\varepsilon) = 2\pi(1 + \varepsilon\,\mu(\varepsilon)))$ *mapping continuously into* $\mathscr{D}^2_{0,2\pi}((\omega', 0); \emptyset)$ *at* $\varepsilon = 0$ *be given. Then the function* $u_\varepsilon$ *forms a solution to the problem* $(\mathscr{A}^\varepsilon_{\omega(\varepsilon)})$ *for* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *only if the function* $s_0 \in \mathscr{D}^2_{2\pi}(\emptyset)$ *determined by* $u_0(t, x) = s_0(x + t) - s_0(-x + t)$, $(t, x) \in R^2 \setminus \hat{S}$, *fulfils*

$$(1.26) \qquad 2\pi\,\mu(0)\, s''_0(x) + 2^{-1} \int_0^{2\pi} F(u_0)(0)(\tau, x - \tau)\, d\tau = 0$$

*for* $x \in R$.

**Definition 1.6.** We denote by $\mathscr{M}$ the set of all functions $f$ fulfilling (1.2) such that every solution $s_0 \in \mathscr{D}^2_{2\pi}(\emptyset)$ to (1.26) with $\mu(0) \neq 0$ is a constant function.

**Corollary 1.2.** *Let the assumptions of Corollary* 1.1 *be fulfilled. Further, suppose that* $f \in \mathscr{M}$, $\mu(0) \neq 0$ *and* $u_\varepsilon$ *is a solution to* $(\mathscr{A}^\varepsilon_{\omega(\varepsilon)})$ *for every* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

*Then the function* $u_\varepsilon$ *fulfils* $u_0 \equiv 0$. (*Consequently, if the other assumptions of Corollary* 1.2 *are preserved a solution* $u_\varepsilon$ *to* $(\mathscr{A}^\varepsilon_\omega)$ *not tending to* 0 *for* $\varepsilon \to 0$ *may exist only if* $\mu(0) = 0$.)

In the sequel, retaining the notation introduced in (1.6), we describe some functions which are elements of $\mathscr{M}$.

**Lemma 1.4.** *Let* $f = f(y_1, y_2)$ *fulfil* (1.2). *Then* $f \in \mathscr{M}$.

Proof. We denote $s'(x) = \sigma(x)$. Then the equation (1.26) assumes the form

$$2\pi\,\mu(0)\,\sigma'(x) + 2^{-1} \int_0^{2\pi} f(\sigma(x) - \sigma(-x + 2\tau), \sigma(x) + \sigma(-x + 2\tau))\, d\tau = 0\,.$$

Writing $\Phi(\zeta) = 2^{-1} \int_0^{2\pi} f(\zeta - \sigma(-x + 2\tau), \zeta + \sigma(-x + 2\tau))\, d\tau$ we can reduce the above relation to $2\pi\, \mu(0)\, \sigma'(x) + \Phi(\sigma(x)) = 0$. However, every function $\sigma \in \mathscr{D}^1_{2\pi}(\emptyset)$ fulfilling this equation is constant.

**Lemma 1.5.** *Let F and G be functions defined on R with continuous second derivatives such that F is odd and G is even. Then the function*

$$f(y, y_1, y_2) = F'(y)\, y_1 + G'(y)\, y_2$$

*is an element of* $\mathscr{M}$.

Proof. The integral in (1.26) has this form:

$$2^{-1} \int_0^{2\pi} F'(s(x) - s(-x + 2\tau))\,(s'(x) - s'(-x + 2\tau))\, d\tau +$$

$$+ 2^{-1} \int_0^{2\pi} G'(s(x) - s(-x + 2\tau))\,(s'(x) + s'(-x + 2\tau))\, d\tau\,.$$

We define two functions $\Phi$, $\Psi$ on $R$ by

$$\Psi'(\xi) = 2^{-1} \int_0^{2\pi} F'(\xi - s(\eta))\, d\eta\,, \quad \Phi'(\xi) = 2^{-1} \int_0^{2\pi} G'(\xi - s(\eta))\, d\eta\,.$$

Now we can write the relation (1.26) as $[2\pi\, \mu(0)\, s'(x) + \Psi(s(x)) + \Phi(s(x))]' = 0$. Again only a constant function $s \in \mathscr{D}^2_{2\pi}(\emptyset)$ can fulfil this relation.

Example 1.1. Lemma 1.5 implies that Corollary 1.2 may be applied to the equation

$$u_{tt} - u_{xx} = \varepsilon(1 - u^2)\, u_t\,.$$

## 2. EXAMPLES.

If a periodic solution to the problem $(\mathscr{A}^\varepsilon_\omega)$ keeps the period $\omega = 2\pi n$ ($n$ positive integer), the system in Theorem 1.1 reduces to

$$(2.1) \qquad G_1(u, s)(\varepsilon)(t, x) = -u(t, x) + s(x + t) - s(-x + t) +$$

$$+ 2^{-1}\varepsilon \int_0^t \int_{x-t+\tau}^{x+t-\tau} F(u)(\varepsilon)(\tau, \xi)\, d\xi\, d\tau = 0 \quad \text{for} \quad (t, x) \in R^2 \setminus \hat{S}\,,$$

$$G_2(u)(\varepsilon)(x) = \int_0^{2\pi} \int_0^x F(u)(\varepsilon)(\tau, \xi - \tau)\, d\xi\, d\tau = 0 \quad \text{for} \quad x \in R \setminus S\,,$$

where $F(u)(\varepsilon)$ is defined by (1.6). Let us note that if $S \neq \emptyset$, the normalization given by (1.25) is not appropriate and in the sequel will be replaced by keeping $S$ fixed during the whole resolution. It is clear that for a function $u$ fulfilling the equation

$$(2.2) \qquad G_4(u)(x) = \int_0^{2\pi} F(u)(\varepsilon)(\tau, x - \tau)\, d\tau = 0\,, \quad x \in R \setminus S\,,$$

also $G_2(u)(\varepsilon) = 0$ holds. So in the sequel we look for functions $u, s$ which fulfil

$$G_1(u, s)(\varepsilon)(t, x) = 0\,, \quad (t, x) \in (0, 2\pi n) \times R \setminus \hat{S}\,,$$
$$G_4(u)(\varepsilon)(x) = 0\,, \quad x \in R \setminus S\,.$$

If we find such two functions $u, s$ then by Theorem 1.1 the function $u$, extended as $2\pi n$-periodic function in $t$, represents a solution to the problem $(\mathscr{A}^{\varepsilon}_{2\pi n})$.

Further, we formulate an additional assumption on $f$. We say that a function $f$ fulfils the condition $(\mathscr{B}^k)$ if $f(x, y_0, y_1, y_2, \varepsilon)$ fufils (1.2) and the derivatives $\partial^p f / \partial x^i \partial^{j_0}_{y_0} \partial^{j_1}_{y_1} \partial^{j_2}_{y_2}$, $p = i + j_0 + j_1 + j_2$, $j_0 + j_1 + j_2 \geqq 1$, $p = 1, 2, \ldots, k$ are continuous on the set

$$\{(x, y_0, y_1, y_2, \varepsilon);\ x, y_0, y_1, y_2 \in R,\ \varepsilon \in [-\varepsilon_0, \varepsilon_0]\}\,.$$

Defining $Z : D^2_{2\pi}(S) \to D^2_{2\pi,2\pi}(R; \hat{S})$ by $(Zs)(t, x) = s(x + t) - s(-x + t)$, $(t, x) \in R^2 \setminus \hat{S}$ we state

**Theorem 2.1.** *Let the following assumptions be fulfilled:*

(i) *The positive integer $p$ equals 1 or 2.*

(ii) *The function $f(x, y_0, y_1, y_2, \varepsilon)$ fulfils $(\mathscr{B}^{4-p})$.*

(iii) *The set $S$ fulfils (1.1) and $\mathscr{D}^{2*}_{2\pi}(S)$, $\mathscr{D}^{3-p}_{2\pi *}(S)$, $\mathscr{D}^{2*}_{0,2\pi}((0, 2\pi n); \hat{S})$ are closed subspaces of $\mathscr{D}^2_{2\pi}(S)$, $\mathscr{D}^{3-p}_{2\pi}(S)$, $\mathscr{D}^2_{0,2\pi}((0, 2\pi n); \hat{S})$ respectively.*

(iv) *The equation*

$$\Gamma(\sigma_0)(x) \equiv \int_0^{2\pi n} f(x - \tau, \sigma_0(x) - \sigma_0(-x + 2\tau),$$
$$\sigma_0'(x) - \sigma_0'(-x + 2\tau),\ \sigma_0'(x) + \sigma_0'(-x + 2\tau), 0)\, d\tau = 0$$

*$(x \in R \setminus S)$ has a solution $\sigma_0 = \sigma_0^* \in \mathscr{D}^{2*}_{2\pi}(S)$, $\Gamma(\mathscr{D}^{2*}_{2\pi}(S)) \subset \mathscr{D}^{3-p}_{2\pi *}(S)$ and $Z\sigma_0^* \in \mathscr{D}^{2*}_{0,2\pi}((0, 2\pi n); \hat{S})$.*

(v)

$$G_1(\mathscr{D}^{2*}_{0,2\pi}((0, 2\pi n); \hat{S}), \mathscr{D}^{2*}_{2\pi}(S))(\varepsilon) \subset \mathscr{D}^{2*}_{0,2\pi}((0, 2\pi n); \hat{S})\,,$$
$$G_4(\mathscr{D}^{2*}_{0,2\pi}((0, 2\pi n); \hat{S}))(\varepsilon) \subset \mathscr{D}^{3-p}_{2\pi *}(S)\,.$$

(vi) *There exists*

$$[\Gamma'_{\sigma_0}(\sigma_0^*)]^{-1} \in L(\mathscr{D}^{3-p}_{2\pi*}(S), \mathscr{D}^{2*}_{2\pi}(S)) .$$

*Then for sufficiently small* $\varepsilon$ *the problem* $(\mathscr{A}^{\varepsilon}_{2\pi n})$ *has a unique solution* $u = u^*(\varepsilon) \in \mathscr{D}^2_{2\pi n,2\pi}(R; \hat{S})$ *such that* $u^*(0) = Z\sigma_0^*$ *and* $u^*(\varepsilon)$ *is continuous in* $\varepsilon$.

We apply this Theorem to three examples.

Example 2.1. Consider the problem $(\mathscr{A}^{\varepsilon}_{2\pi})$ with $f(u) = -\alpha u + \beta u^3$, $\beta \neq 0$, $\alpha/3\beta = \gamma > 0$. We verify the assumptions of Theorem 2.1 with $p = 1$ for an appropriate choice of functional spaces. The condition (ii) is evidently fulfilled. We will suppose from the very beginning that $\sigma$ fulfils $\sigma(x + \pi) = -\sigma(x)$ a.e. in $R$. Then the equation in (iv) assumes the form

$$\Gamma(\sigma_0)(x) = \sigma_0^3(x) + 3(I - \gamma)\sigma_0(x) = 0 \tag{2.3}$$

where

$$2\pi I = \int_0^{2\pi} \sigma_0^2(\xi)\,\mathrm{d}\xi . \tag{2.4}$$

Hence $\sigma_0(x)$ may be equal only to 0, $\pm\sqrt{(3(\gamma - I))}$. Fix a positive integer $m$ and put $S = \{jm^{-1}\pi;\ j \text{ integer}\}$. We denote by $\mathscr{D}^{2*}_{2\pi}(S) = \mathscr{D}^2_{2\pi*}(S)$ the closed subspace of $\mathscr{D}^2_{2\pi}(S)$ containing the functions $\sigma$ which fulfil

$$-\sigma(x + \pi) = \sigma(x) \quad \text{for} \quad x \in R \setminus S . \tag{2.5}$$

Now for every fixed $I < \gamma/3$ the equation (2.3) has solutions $\sigma_0(x) \in \mathscr{D}^{2*}_{2\pi}(S)$ of the following form: $\sigma_0(x)$ equals 0, $+\sqrt{(3(\gamma - I))}$ or $\sqrt{(3(\gamma - I))}$ on each interval $(jm^{-1}\pi, (j + 1)m^{-1}\pi)$, $j$ integer, $0 \leqq j \leqq m - 1$. Choose one of these solutions and denote it by $\sigma_I(x)$. We denote by $r$ the number of integers $j$, $0 \leqq j \leqq m - 1$ for which $\sigma_I((2j + 1)(2m)^{-1}\pi) \neq 0$. This function $\sigma_I$ satisfies (2.4) if

$$2\pi I = \int_0^{2\pi} \sigma_I^2(\xi)\,\mathrm{d}\xi = 6\pi r m^{-1}(\gamma - I) .$$

For $I = I^* = 3\gamma r m^{-1}(1 + 3rm^{-1})^{-1}$ the function $\sigma_I(x)$ is a solution of (2.3) and (2.4). Denoting this solution by $\sigma_0^*$ we find

$$[\sigma_0^*(x)]^2 = 3\gamma(1 + 3rm^{-1})^{-1} \quad \text{for} \quad x \in R \setminus S , \quad \sigma_0^*(x) \neq 0 . \tag{2.6}$$

Further we denote by $\mathscr{D}^{2*}_{0,2\pi}((0, 2\pi); \hat{S})$ the subspace of $\mathscr{D}^2_{0,2\pi}((0, 2\pi); S)$ containing the functions $u$ which fulfil $u(t, x) = u(t, \pi - x)$ $(t, x) \in (0, 2\pi) \times R \setminus \hat{S}$. The inclusions in (iv), (v) may be then readily verified.

Denote by $h$ an arbitrary element of $\mathscr{D}^2_{2\pi*}(S)$. Then the equation

$$(\Gamma'_\sigma(\sigma_0^*)\,\bar{\sigma})(x) \equiv 3([\sigma_0^*(x)]^2 + I^* - \gamma)\,\bar{\sigma}(x) + $$
$$+ 6\sigma_0^*(x)\,(2\pi)^{-1}\int_0^{2\pi} \sigma_0^*(\xi)\,\bar{\sigma}(\xi)\,\mathrm{d}\xi = h(x)\,, \quad (x \in R \setminus S)$$

has the solution

$$\bar{\sigma}(x) = 3^{-1}((\sigma_0^*(x))^2 + I^* - \gamma)^{-1}\,\{h(x) - \sigma_0^*(x)\,\pi^{-1}(1 + 3rm^{-1})^{-1}\,.$$
$$.\int_0^{2\pi} h(\xi)\,\sigma_0^*(\xi)\,((\sigma_0^*(\xi))^2 + I^* - \gamma)^{-1}\,\mathrm{d}\xi \in \mathscr{D}^{2*}_{2\pi}(S)$$

because

$$[\sigma_0^*(x)]^2 + I^* - \gamma = -\gamma(1 + 3rm^{-1})^{-1} \quad \text{for} \quad \sigma_0^*(x) = 0\,,$$
$$[\sigma_0^*(x)]^2 + I^* - \gamma = \phantom{-}2\gamma(1 + 3rm^{-1})^{-1} \quad \text{for} \quad \sigma_0^*(x) \neq 0\,.$$

Now we easily deduce that $[\Gamma'_\sigma(\sigma_0^*)]^{-1} \in L(\mathscr{D}^2_{2\pi*}(S), \mathscr{D}^{2*}_{2\pi}(S))$. In virtue of Theorem 2.1 we obtain

**Theorem 2.2.** *Let $f = -\alpha u + \beta u^3$, $\alpha > 0$, $\beta > 0$ in the problem $(\mathscr{A}^\varepsilon_{2\pi})$. Choose a solution $\sigma_0 = \sigma_0^* \in \mathscr{D}^{2*}_{2\pi}(S)$ to the equations (2.3), (2.4) of the type described above.*

*Then for sufficiently small $\varepsilon$ the problem $(\mathscr{A}^\varepsilon_{2\pi})$ has a unique solution $u = u^*(\varepsilon) \in \mathscr{D}^{2*}_{2\pi,2\pi}(R; \hat{S})$ continuous in $\varepsilon$ and such that $u^*(0) = Z\sigma_0^*$.*

Remark 2.1. Let us note that if the number $r$ occurring in (2.6) is equal to $m$ then the function $Z\sigma_0^*$ is a solution to $(\mathscr{A}^\varepsilon_{2\pi})$ as is elementarily verified. And because of the local uniqueness guaranteed by Theorem 2.2 applied in the proof it must coincide with $u^*(\varepsilon)$ from this theorem.

Example 2.2. Consider the problem $(\mathscr{A}^\varepsilon_{2\pi})$ with $f(u, u_t) = (-\gamma + u^2)\,u_t$. We verify the assumptions of Theorem 2.1 with $p = 2$. Let $S \subset R$, $\mathscr{D}^{2*}_{2\pi}(S)$, $\mathscr{D}^{2*}_{0,2\pi}((0, 2\pi); \hat{S})$ be the same sets as in Example 2.1. We denote by $\mathscr{D}^1_{2\pi*}(S)$ the set of all functions from $\mathscr{D}^1_{2\pi}(S)$ which fulfil (2.5). For $\sigma_0 \in \mathscr{D}^{2*}_{2\pi}(S)$ the equation in (iv) of Theorem 2.1 has the form

$$\Gamma(\sigma_0)(x) = (-\gamma + \sigma_0^2(x) + I)\,\sigma_0'(x) + $$
$$+ 2\,\sigma_0(x)\,(2\pi)^{-1}\int_0^{2\pi} \sigma_0(\xi)\,\sigma_0'(\xi)\,\mathrm{d}\xi = 0$$

where $2\pi I = \int_0^{2\pi} \sigma_0^2(\xi)\,\mathrm{d}\xi$. We choose a function $\sigma_0^* \in \mathscr{D}^{2*}_{2\pi}(S)$ constant on each component of $R \setminus S$ and such that

$$\Xi(x) \equiv -\gamma + [\sigma_0^*(x)]^2 + (2\pi)^{-1}\int_0^{2\pi} \sigma_0^{*2}(\xi)\,\mathrm{d}\xi > \delta > 0$$

for all $x \in R \setminus S$. This function $\sigma_0^*$ fulfils $\Gamma(\sigma_0^*) = 0$. The assumptions (ii), (iv) and (v) may be easily verified. The equation $\Gamma'_\sigma(\sigma_0^*)\,\bar{\sigma} = h \in \mathscr{D}^{1*}_{2\pi}(S)$ assumes the form

$$\Gamma'_\sigma(\sigma_0^*)\,\bar{\sigma}(x) = \Xi(x)\,\sigma'(x) + \pi^{-1}\,\sigma_0^*(x)\int_0^{2\pi}\sigma_0^*(\xi)\,\bar{\sigma}'(\xi)\,\mathrm{d}\xi = h(x)\,.$$

If we denote by $G$ the operator which maps every $g \in \mathscr{D}^1_{2\pi*}(S)$ into $\mathscr{D}^{2*}_{2\pi}(S)$ according to the formula

$$(Gg)(x) = \int_0^x g(\xi)\,\mathrm{d}\xi - 2^{-1}\int_0^\pi g(\xi)\,\mathrm{d}\xi$$

and

$$J^* = \left\{2\pi + 2\int_0^{2\pi}[\sigma_0^*(\xi)]^2\,\Xi^{-1}(\xi)\,\mathrm{d}\xi\right\}^{-1}\int_0^{2\pi}h(\xi)\,\sigma_0^*(\xi)\,\Xi^{-1}(\xi)\,\mathrm{d}\xi\,,$$

then $\bar{\sigma} = G(\Xi^{-1}(x)\,(h(x) - 2J^*\,\sigma_0^*(x))$ is an element of $\mathscr{D}^{2*}_{2\pi}(S)$ fulfilling $\Gamma'_\sigma(\sigma_0^*)\,\bar{\sigma} = h$. We denote this element by $Hh$. Then for every $h \in \mathscr{D}^1_{2\pi*}(S)$,

$$\Gamma'_\sigma(\sigma_0^*)\,Hh = h \tag{2.7}$$

is fulfilled.

The operator $H$ is a linear bounded operator from $\mathscr{D}^1_{2\pi*}(S)$ into $\mathscr{D}^{2*}_{2\pi}(S)$. The equation (2.7) substitutes the existence of the inverse operator to $\Gamma'_\sigma(\sigma_0^*)$ required in Theorem 2.1. However, with (2.7) the theorem remains valid if we do not require the uniqueness. So we can formulate

**Theorem 2.3.** *Let $f = (-\gamma + u^2)\,u_t$ in the problem $(\mathscr{A}^\varepsilon_{2\pi})$. Choose $\sigma_0^* \in \mathscr{D}^{2*}_{2\pi}(S)$ as is described above. Then for sufficiently small $\varepsilon$ the problem $(\mathscr{A}^\varepsilon_{2\pi})$ has a solution $u = u^*(\varepsilon) \in \mathscr{D}^2_{2\pi,2\pi}(R; \hat{S})$ continuous in $\varepsilon$ and such that $u^*(0) = Z\sigma_0^*$.*

Example 2.3. Consider the problem $(\mathscr{A}^\varepsilon_{2\pi})$ with $f(u_t) = -\alpha u_t + \beta u_t^3$, $\gamma = \alpha/3\beta > 0$. Let $S = \{x \in R;\ x = j\pi m^{-1}$ for an integer $j\}$, $m \geqq 2$ integer. We denote by $\mathscr{D}^{2*}_{2\pi}(S)$ the functions from $\mathscr{D}^2_{2\pi}(S)$ which are continuous and fulfil (2.5). The space $\mathscr{D}^1_{2\pi*}(S)$ is the same as in Example 2.2.

Further we denote by $\mathscr{D}^{2*}_{0,2\pi}((0, 2\pi); \hat{S})$ the functions from $\mathscr{D}^2_{0,2\pi}((0, 2\pi); \hat{S})$ which are continuous and fulfil $u(t, x) = u(t, \pi - x)$. Then the equation in (iv), Theorem 2.1 has the form

$$\Gamma(\sigma_0)(x) = \sigma_0'(x)\,((\sigma_0'(x))^2 + 3(I - \gamma)) = 0$$

where $2\pi I = \int_0^{2\pi}[\sigma_0'(\xi)]^2\,\mathrm{d}\xi$. For fixed $I \leqq \gamma$ we choose a function $\sigma_I \in \mathscr{D}^{2*}_{2\pi}(S)$ such that its derivative on each component of $R \setminus S$ is equal to 0 or $\pm\sqrt{(3(\gamma - I))}$. We denote by $r$ the number of integers $j$, $0 \leqq j \leqq m - 1$ for which $\sigma_I'((2j + 1)\,.\,(2m)^{-1}\,\pi) \neq 0$. Then $2\pi I = 2\pi r m^{-1}\,3(\gamma - I)$ and for

$$I = I^* = 3\gamma r m^{-1}(1 + 3rm^{-1})^{-1} \tag{2.8}$$

the function $\sigma_I$ which we denote by $\sigma_0^*$ fulfils $\Gamma(\sigma_0^*) = 0$. Now we verify the assumptions (vi) of Theorem 2.1. The operator equation $\Gamma'_\sigma(\sigma_0^*)\,\bar\sigma = h$ which reduces to

$$(\Gamma'_\sigma(\sigma_0^*)\,\bar\sigma)(x) \equiv 3((\sigma_0^{*\prime}(x))^2 + I^* - \gamma)\,\bar\sigma'(x) + $$
$$+ 6\sigma_0^{*\prime}(x)\,(2\pi)^{-1}\int_0^{2\pi}\sigma_0^{*\prime}(\xi)\,\bar\sigma'(\xi)\,\mathrm{d}\xi = h(x)\,,$$

has the solution $\bar\sigma$ which fulfils

$$\bar\sigma'(x) = 3^{-1}((\sigma_0^{*\prime}(x))^2 + I^* - \gamma)^{-1}\,\{h(x) - \sigma_0^{*\prime}(x)\,\pi^{-1}(1 + 3rm^{-1})^{-1}\,.$$
$$.\int_0^{2\pi} h(\xi)\,\sigma_0^{*\prime}(\xi)\,((\sigma_0^*(\xi))^2 + I^* - \gamma)^{-1}\,\mathrm{d}\xi\}\,,\quad x \in R \setminus S\,.$$

Denoting by $g$ the right hand side of this equation, we have $g \in \mathscr{D}^1_{2\pi *}(S)$, and $\bar\sigma(x) = \int_0^x g(\xi)\,\mathrm{d}\xi - 2^{-1}\int_0^\pi g(\xi)\,\mathrm{d}\xi \in \mathscr{D}^{2*}_{2\pi}(S)$. Now we easily deduce that $[\Gamma'_\sigma(\sigma_0^*)]^{-1} \in L(\mathscr{D}^1_{2\pi *}(S);\ \mathscr{D}^{2*}_{2\pi}(S))$. In virtue of Theorem 2.1 we obtain

**Theorem 2.4.** *Let $f = -\alpha u_t + \beta u_t^3$, $\alpha > 0$, $\beta > 0$ in the problem $(\mathscr{A}^\varepsilon_{2\pi})$. Let us choose $\sigma_0^* \in \mathscr{D}^{2*}_{2\pi}(S)$ as described above.*

*Then for sufficiently small $\varepsilon$ the problem $(\mathscr{A}^\varepsilon_{2\pi})$ has a unique solution $u = u^*(\varepsilon) \in \mathscr{D}^2_{2\pi,2\pi}(R;\hat S)$ continuous in $\varepsilon$ and such that $u^*(0) = Z\sigma^*$.*

Remark 2.2. If the number $r$ in (2.8) is equal to $m$, then the function $Z\sigma_0^*$ is a solution to $(\mathscr{A}^\varepsilon_{2\pi})$. (Cf. Remark 2.1.)

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