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PERIODIC TRAVELING WAVES AND LOCATING OSCILLATING PATTERNS IN MULTIDIMENSIONAL DOMAINS

NICHOLAS D. ALIKAKOS, PETER W. BATES, AND XINFU CHEN

ABSTRACT. We establish the existence and robustness of layered, time-periodic solutions to a reaction-diffusion equation in a bounded domain in $\mathbb{R}^n$, when the diffusion coefficient is sufficiently small and the reaction term is periodic in time and bistable in the state variable. Our results suggest that these patterned, oscillatory solutions are stable and locally unique. The location of the internal layers is characterized through a periodic traveling wave problem for a related one-dimensional reaction-diffusion equation. This one-dimensional problem is of independent interest and for this we establish the existence and uniqueness of a heteroclinic solution which, in constant-velocity moving coodinates, is periodic in time. Furthermore, we prove that the manifold of translates of this solution is globally exponentially asymptotically stable.

1. INTRODUCTION

In this paper, we are concerned with two distinct, but strongly related problems involving bistable time-periodic nonlinearities: A traveling wave problem for a one-dimensional equation, and a multidimensional singular perturbation problem of reaction-diffusion type.

Reaction-diffusion equations with small diffusivity arise naturally in the modeling of many physical and biological phenomena. Often, as with phase transition and genotype models, two competing stable states are present. As a result of the bistability, patterns are generated wherein the solution in different regions of the domain takes on values close to one or the other of the two stable states. As a result of the small diffusivity (the singular perturbation nature of the equation) the patterns become well-defined, the solution changing sharply between the two stable states within a thin zone. This thin zone becomes a hypersurface as the diffusivity vanishes and is approximated by an intermediate level surface of the solution, which we call the interface.

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The interfaces which form the pattern evolve with time and we wish to understand their motion and the configurations which ultimately develop. We also wish to determine the fine structure of the transition across the interface. The transition occurs rapidly in the direction normal to the interface and so, in rescaled coordinates, is essentially one-dimensional.

At a given location the transition profile and its speed of propagation is governed by a traveling wave problem for a reaction-diffusion equation on the real line obtained through the rescaling. The traveling wave problem is of independent interest and, in the autonomous case, has a substantial history. However, the presence of periodic forcing, which has physical relevance, introduces significant difficulties and requires us to develop a new approach. This analysis occupies a large portion of this paper. We establish existence, uniqueness and global stability of traveling wave solutions for a general class of bistable time-periodic nonlinearities.

The traveling wave solutions to the one-dimensional problem are then employed to construct sub- and super-solutions to the higher-dimensional equation. The construction shows the development of interfaces and provides estimates for their speed of propagation, thereby establishing the existence of patterned solutions as asymptotic states.

Since the results for the multidimensional reaction-diffusion equation depend on results for the one-dimensional traveling wave problem, we first discuss the one-dimensional equation.

The traveling wave problem. We consider the asymptotic behavior, as $t \to \infty$, of the solutions of the following problem:

$$\text{(1.1)} \qquad \begin{cases} u_t - u_{zz} - f(u,t) = 0, & z \in \mathbb{R},\ t > 0, \\ u(z,0) = g(z), & z \in \mathbb{R}, \end{cases}$$

where $f(\cdot,t)$ is bistable, $f(u,\cdot)$ is T–periodic, i.e. $f(u,T+t) = f(u,t)$ for all $u,t \in \mathbb{R}$, and g is an arbitrary bounded function having certain asymptotic behavior as $z \to \pm\infty$. A typical example of f is the cubic potential $f = (1-u^2)(2u - \gamma(t))$ where $\gamma(\cdot) \in (-2,2)$ is T–periodic.

We claim that the long-time behavior of solutions of (1.1) is governed by periodic traveling wave solutions of (1.1a), that is, solutions which have the form

$$u(z,t) = U(z-ct,t), \qquad U(\cdot,t+T) = U(\cdot,t)$$

where c is some real number. In other words, a wave with speed c, which, when viewed from the standpoint of the moving coordinate frame (i.e. in $\xi := z - ct$), has a profile which oscillates periodically in time. We first establish the existence and uniqueness of such a solution for a class of non–linear potentials f. In the following theorems we assume that f satisfies the structure hypotheses:

(H1) There exists $T > 0$ such that $f(u,t) = f(u,t+T)$ for all $(u,t) \in \mathbb{R}^2$;

(H2) The period map $P(\alpha) := w(\alpha,T)$, where $w(\alpha,t)$ is the solution to

$$\text{(1.2)} \qquad w_t = f(w,t) \quad \forall t \in \mathbb{R}, \qquad w(\alpha,0) = \alpha \in \mathbb{R},$$

has exactly three fixed points $\alpha^-, \alpha^0, \alpha^+$ satisfying $\alpha^- < \alpha^0 < \alpha^+$. In addition, they are non–degenerate and $\alpha^\pm$ are stable, i.e.,

$$\text{(1.3)} \qquad \frac{d}{d\alpha}P(\alpha^\pm) < 1 < \frac{d}{d\alpha}P(\alpha^0).$$

Theorem 1.1. *Assume that $f(u,t) \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ satisfies (H1) and (H2) above. Then there exist a unique function $U(\xi,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and a unique constant $c \in \mathbb{R}$*

such that

$$\begin{cases} U_t - cU_\xi - U_{\xi\xi} - f(U,t) = 0 & \forall\,(\xi,t)\in\mathbb{R}^2, \\ U(\pm\infty,t) := \lim_{\xi\to\pm\infty} U(\xi,t) = w(\alpha^\pm,t) & \forall\, t\in\mathbb{R}, \\ U(\cdot,T) = U(\cdot,0), \quad U(0,0)=\alpha^0. & \end{cases} \tag{1.4}$$

In addition, (c,U) has the following properties:

1. *For each t, $U(\cdot,t)$ is monotonic; that is, $U_\xi(\cdot,\cdot)>0$ in $\mathbb{R}\times\mathbb{R}$;*
2. *U exponentially approaches its limits as $\xi\to\pm\infty$; in fact, there exist positive constants C and β such that*

$$|U(\pm\xi,t)-w(\alpha^\pm,t)| + |U_\xi(\pm\xi,t)| + |U_{\xi\xi}(\pm\xi,t)| \le Ce^{-\beta\xi} \qquad \forall\,\xi\ge 0,\ t\in\mathbb{R}.$$

We also relate the long-time behavior of the solution to (1.1) for a class of initial data g, with this periodic traveling wave.

Theorem 1.2. *Under the assumption that $f(u,t)\in C^{2,1}(\mathbb{R}\times\mathbb{R})$ satisfies (H1) and (H2) the mainfold $\mathcal{M} := \{U(z+\cdot,\cdot)\ :\ z\in\mathbb{R}^1\}$ is globally stable; that is, for every $g\in L^\infty(\mathbb{R}^1)$ satisfying*

$$\limsup_{z\to-\infty} g(z) < \alpha^0, \qquad \liminf_{z\to\infty} g(z) > \alpha^0, \tag{1.5}$$

the solution $u(z,t)$ to the initial value problem (1.1) satisfies

$$\|u(z,t) - U(z+z_g-ct,t)\|_{L^\infty(\mathbb{R}^1)} \le C_g e^{-\mu t} \qquad \forall t\ge 0 \tag{1.6}$$

where z_g and C_g are real numbers depending on g whereas μ is a positive number independent of g.

With the information provided by the above theorems we can now give a characterization of spatially robust patterns with oscillating amplitude generated by a periodic bistable reaction-diffusion equation with small diffusivity.

The singular perturbation problem. Consider the Neumann boundary value problem:

$$\begin{cases} u^\varepsilon_t - \varepsilon^2\Delta u^\varepsilon - F(x;u^\varepsilon,t) = 0, & x\in\Omega,\ t\in\mathbb{R}, \\ \frac{\partial}{\partial n}u^\varepsilon = 0, & x\in\partial\Omega,\ t\in\mathbb{R}. \\ u^\varepsilon(x,t+T) = u^\varepsilon(x,t), & x\in\Omega, t\in\mathbb{R}, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N\ge 1$) with C^1 boundary $\partial\Omega$. Here we assume that, for every $x\in\bar\Omega$, $F(x;\cdot,\cdot)$ satisfies the conditions placed upon f in the previous theorems. We are concerned with the existence of patterned structures given by solutions of (1.7), as $\varepsilon\searrow 0$. We establish the following theorem:

Theorem 1.3. *Assume that $F(x;u,t)\in C^{2,2,1}(\bar\Omega\times\mathbb{R}\times\mathbb{R})$ and that for each $x\in\bar\Omega$, $f(\cdot,\cdot) := F(x;\cdot,\cdot)$ satisfies (H1) and (H2) where T is independent of x. Denote by $\alpha^\pm(x)$ and $\alpha^0(x)$ the fixed points of the period map associated with $f(\cdot,\cdot)=F(x;\cdot,\cdot)$ given in (H2) and by $c(x)$ the corresponding traveling wave speed given by Theorem 1.1. Define, for each $\rho>0$,*

$$\Omega^\pm_\rho = \{x\in\bar\Omega\ :\ \pm c(x) > \rho\}. \tag{1.8}$$

Then, there exist positive constants μ and ε_0 such that for all $\varepsilon\in(0,\varepsilon_0]$, there exists a solution u^ε of (1.7) satisfying

$$\begin{cases} w(\alpha^-(x),t) - \varepsilon^\mu \le u^\varepsilon(x,t) \le w(\alpha^+(x),t) + \varepsilon^\mu & \forall\,(x,t)\in\bar\Omega\times[0,T], \\ |u^\varepsilon(x,t) - w(\alpha^\pm(x),t)| \le \varepsilon^\mu & \forall\,(x,t)\in\bar\Omega^\mp_{\varepsilon^\mu}\times[0,T], \end{cases} \tag{1.9}$$

where $w(\alpha,t) = w(x;\alpha,t)$ is the solution to (1.2).

In addition to such solutions existing we can say something about their local uniqueness, in an asymptotic sense. The second condition in (1.9) says that u^ε is "strongly patterned" in the sense that it has large amplitude transitions between the two stable states, and these transitions occur across a fairly narrow and well-defined region within Ω. The following theorem essentially says that any solution which is "weakly patterned" must actually have sharp interfaces as given in the previous theorem.

Theorem 1.4. *Under the assumptions above for any solution u^ε of (1.7), if there exist $y^+ \in \Omega^+_{\varepsilon^\mu}$ and $y^- \in \Omega^-_{\varepsilon^\mu}$ such that*

(1.10)
$$\begin{cases} u^\varepsilon(x,0) \le \alpha^0(x) - \varepsilon^\mu & \text{in } \{x \in \bar\Omega \ : \ |x-y^+| \le 6\varepsilon^{1/3}\}, \\ u^\varepsilon(x,0) \ge \alpha^0(x) + \varepsilon^\mu & \text{in } \{x \in \bar\Omega \ : \ |x-y^-| \le 6\varepsilon^{1/3}\}, \end{cases}$$

then u^ε has the property (1.9). (Here, we assume, for simplicity, that $\Omega^\pm_{\varepsilon^\mu}$ is connected. Otherwise we need the existence of $y^\pm$ in each component of $\Omega^\pm_{\varepsilon^\mu}$.)

An immediate consequence of this is

Corollary 1.5. *Assume that $\{u^\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is a family of solutions of (1.7) such that every u^ε satisfies (1.10) for some $y^\pm \in \Omega^\pm_{\varepsilon^\mu}$. Then*

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(x,t) = \begin{cases} w(\alpha^+(x),t), & (x,t) \in \Omega_0^- \times [0,T], \\ w(\alpha^-(x),t), & (x,t) \in \Omega_0^+ \times [0,T] \end{cases} \tag{1.11}$$

where the limit is uniform in $\Omega^\pm_\rho \times [0,T]$ for all $\rho > 0$.

Our technique of using sub- and super-solutions may also be used to show that the patterns are robust. In fact, if an initial function has an interface within $\partial\bar\Omega^\pm_{\varepsilon^\mu}$, then that interface must move until (1.9) holds (see Theorem 4.4 and Remark 4.5).

To put the above singular perturbation problem and the associated traveling wave problem into historical context we mention a few works which have had significant impact on developments in this field and have influenced us in particular. We do not however intend to provide an exhaustive list of related work.

There is a vast literature on the traveling wave problem (1.4) in the autonomous case (f independent of t), with the two prototypical nonlinearities being of Fisher-type ($u(1-u)$) or bistable ($u(1-u)(u-\gamma)$ with $0<\gamma<1$.) Such problems were introduced in the classic works of Fisher [15] and Kolmogorov-Petrovski-Piscounov [23] in 1937. There have been numerous contributions since, including the important contribution by Kanel [22] and the celebrated papers of Fife & McLeod [12, 13, 14] which settled most issues in great generality. Since that time there have of course been some refinements and many applications of these results. The new difficulty in our periodic case is that phase plane techniques are no longer available. The impact of this is most evident on the existence issue, but it also impinges on the techniques used to establish uniqueness. Periodicity also excludes variational techniques which, in the autonomous case, have been employed for establishing stability. Because of all these reasons, our approach had to be different from those mentioned above. In fact, our method is similar to that of Berestycki & Nirenberg [7]. The uniqueness part of our theorem above is very general; in particular it does not presuppose monotonicity of the wave.

We should mention that another class of periodic solutions to autonomous reaction–diffusion equations, rotating wave solutions, that is, those which are periodic both in space and time, have been studied by several authors. We refer to

Angenent & Fiedler [4], Gardner [17] and the references therein. That type of solution is quite different from those we study here and, as far as we can tell, unrelated to patterns in singularly perturbed reaction–diffusion equations.

The singular perturbation problem (1.7), but autonomous with respect to t, also has a substantial history. Again the nonlinearities are typically of the form $m(x)u(1-u)$ (Fisher) or $u(1-u)(u-\gamma(x))$ (bistable.) For the bistable case in a one-dimensional domain, Angenent, Mallet-Paret, and Peletier [5] characterized the limits of all stable solutions as $\varepsilon \to 0$. They showed that these may have layers only where $\gamma(x) = 1/2$ and that the transition must be in the appropriate direction according to the sign of $\gamma'(x)$. Stable solutions with any collection of such transitions exist. Independently, Fuji and Nishiura [16] obtained related results (see also [1] and [2]). The question of existence of stationary layered solutions to the autonomous bistable singular perturbation problem in higher dimensional domains was settled in the 70's in the paper of Fife & Greenlee [10] by employing, in a rigorous way, the method of formal asymptotic expansions. They obtained stationary solutions with interfaces tending to the locations where $\gamma(x) = 1/2$ as $\varepsilon \to 0$. Quite recently, del Pino [25] revisited the Fife–Greenlee problem and gave an elegant solution which, in particular, does not require the smoothness of the interface $\Gamma := \{x \in \bar{\Omega} \ : \ c(x) = 0\}$, and simultaneously allows this set to intersect $\partial\Omega$. Our treatment of (1.7) is close in spirit to del Pino's work, which we extend to the periodic setting. Although we do not prove this here, we suspect that the convergence in (1.11) is exponential in ε, as suggested by the approach of Bardi & Parthame [6].

To explain (1.11), we would like to mention the results of Chen [8] (in N dimensions) and Fife & Hsiao [11] (in one dimension): If $F(x; u, \varepsilon^{-1}\tau)$ does not vary in τ very rapidly ($\tau = \varepsilon t$), then starting with "roughly layered" initial data, the solution of (1.7a), (1.7b) becomes layered in $O(|\ln\varepsilon|)$ time (in the t–time); i.e., there exist regions $\Omega^{\pm}$ such that $u^{\varepsilon} \sim \alpha^{\pm}(x,0)$ in $\Omega^{\pm}$, whereas $\Gamma := \Omega \setminus (\Omega^{+} \cup \Omega^{-})$ is a thin region connecting the states $\alpha^{+}(x,0)$ and $\alpha^{-}(x,0)$. Here $\alpha^{+}(x,\tau), \alpha^{0}(x,\tau), \alpha^{-}(x,\tau)$ are the zeros of $F(x; \cdot, \varepsilon^{-1}\tau)$. Thereafter, the $\alpha^{0}(x,\tau)$ level–set of u^{ε} moves with normal velocity $\varepsilon c(x,\tau)$ (in the t–time) where $c(x,\tau)$ is the traveling wave speed for the autonomous potential $f(\cdot) = F(x; \cdot, \varepsilon^{-1}\tau)$. Since in the current case $c(x,\tau)$ oscillates rapidly (with period εT), Chen's result cannot be applied here. On the other hand, a certain homogenization should provide an "averaged" speed $\bar{c}(x)$. Our Theorem 1.1 suggests that the average speed can be obtained by solving (1.4). Therefore, the α^{0} level–set of u^{ε} will eventually settle down near $\{x \ : \ \bar{c}(x) = 0\}$; in other words, "layered" periodic solutions of (1.7) should have the property stated in Theorem 1.3.

For Fisher-type nonlinearities, $F(u,x,t) = m(x,t)h(u)$ where $m > 0$ is T–periodic, $h(0) = h(1) = 0, h'(1) < 0 < h'(0)$ and $h(u) > 0$ on $(0,1)$, related results were obtained by Dancer & Hess [9], following earlier work by Alikakos and Hess [3]. The main result in [9] is that if $\xi(x)$ is the average over the period of $m(x,t)$, then any family of T–periodic solutions u_{ε} with values in $(0,1)$ converges to the characteristic function of the set $\{x : \xi(x) > 0\}$ as $\varepsilon \to 0$. The proof is based on two abstract properties: strong monotonicity and the structure characterized by two equilibria connected by a family of sub- or supersolutions. The bistable case is considerably harder.

We would like to mention, in passing, two questions we consider of interest, although we did not pursue them in the present paper. One is the characterization of all solutions of (1.7), which satisfy (1.11) since our result suggests but does not prove local uniqueness of such solutions for $\varepsilon > 0$ but small. For one dimensional bounded Ω, if one assumes, in addition to the assumptions in Theorem 1.3, that $F_x > 0$, then it can be shown that when ε is sufficiently small, a non–trivial (not identically constant) and stable T–periodic solution of (1.7) is unique, therefore satisfies (1.11).

The other question concerns the stability of a general solution u^ε of (1.7). Stability is decided by the sign of the principle eigenvalue for the eigenvalue problem

$$\begin{cases} \varepsilon^2 \Delta h + F_u(x; u^\varepsilon, t)h = -\mu^\varepsilon h & \text{in } \Omega \times [0,T], \\ h(x,T) = h(x,0) & \text{in } \Omega, \\ \frac{\partial}{\partial n} h = 0 & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

Hess [21] developed the relevant theory for this kind of eigenvalue problem but here a more detailed knowledge of $F_u(x; u^\varepsilon, t)$ is needed.

Before we close this section, we would like to comment on the possibility of constructing solutions in closed form for the periodic problem (1.4) out of solutions of the autonomous problem when f has a particular form. Assume that

$$f(u,t) = p(u)(-p'(u) - \gamma(t)), \tag{1.12}$$

where $p \in C^3$ and $\gamma \in C^1$ satisfy $\gamma(\cdot + T) = \gamma(\cdot)$, $p(\pm 1) = 0$, and $p(\cdot) > 0$ in $(-1,1)$. Define (c, U) by

$$c = \frac{1}{T}\int_0^T \gamma(t)\,dt, \qquad U(\xi,t) = \Psi(\xi - a(t) + ct) \text{ for all } (\xi,t) \in \mathbb{R}^2,$$

where $a(t) = \int_0^t \gamma(\tau)\,d\tau$ and Ψ is determined by $\int_0^{\Psi(z)} \frac{du}{p(u)} = z$ for all $z \in \mathbb{R}$. Then one can easily verify that U solves (1.4). Note that if $p = 1 - u^2$ and $\gamma(t)$ is a constant function, then (1.4) is autonomous and the traveling wave solution so constructed is known as *Huxley's traveling wave* and one can compute it explicitly [18, p. 130]. It is even more remarkable that the profile of U is independent of $\gamma(\cdot)$. This is an algebraic fact specific to potentials of the form (1.12).

In order to see that the traveling wave so constructed is unique in the case that f is bistable, we now verify that f in (1.12) satisfies the assumption of Theorem 1.1 provided that $p''(\cdot) < 0$ in $(-1,1)$ and $\gamma(t) \in (-p'(-1), -p'(1))$ for all $t \in [0,T]$. To this end, we consider the more general form $f = p(u)q(u,t)$ where p and q are smooth functions satisfying $p(\pm 1) = 0$, $\pm p'(\pm 1) < 0$, $p > 0$ in $(-1,1)$, $p < 0$ in $(-\infty,-1) \cup (1,\infty)$, $q(\cdot, \cdot + T) = q(\cdot,\cdot)$, $\pm q(\pm u, t) > 0$ for $u \geq 1$ and all $t \in [0,T]$, and $q_u(u,t) > 0$ for all $(u,t) \in (-1,1) \times [0,T]$.

First we notice that, for this f, $w(\pm 1, t) \equiv \pm 1$ are exact solutions of (1.2). In addition, $P'(\pm 1) := w_\alpha(\pm 1, T) = \exp(\int_0^T p'(\pm 1) q(\pm 1, t)\,dt) < 1$. Hence, ± 1 are stable fixed points of the period map P.

Next we observe that $f < 0$ for $u \in (1,\infty)$ and $f > 0$ for $u \in (-\infty,-1)$, so that $P(\alpha) < \alpha$ for $\alpha > 1$ and $P(\alpha) > \alpha$ for $\alpha < -1$. That is, P has no fixed point in $(-\infty,-1) \cup (1,\infty)$.

Finally, for every $\alpha \in (-1,1)$, we have $w(\alpha,\cdot) \in (-1,1)$, and $P'(\alpha) = w_\alpha(\alpha,T) = \exp(\int_0^T p_u q dt + \int_0^T p q_u dt) = \frac{p(w(\alpha,T))}{p(\alpha)} \exp(\int_0^T p(w) q_u(w,t) dt) > \frac{p(w(\alpha,T))}{p(\alpha)}$. Hence, if $P(\alpha) := w(\alpha,T) = \alpha$, then $P'(\alpha) > 1$. That is, in $(-1,1)$, P has exactly one fixed point. Therefore, f satisfies (H1) and (H2) and so from Theorem 1.1 we know that the traveling wave constructed above for the case of (1.12) is the unique globally attracting solution of (1.4).

Similarly for the higher dimensional problem (1.7), if

$$F(x; u, t) = p(u)(-p'(u) - \gamma(x, t)) \tag{1.13}$$

where $p(u)$ and γ satisfy $p(\pm 1) = 0$, $\pm p'(\pm 1) < 0$, $p'' < 0$ on $(-1, 1)$, $\gamma(\cdot, t + T) = \gamma(\cdot, t)$ for all t, and $\gamma(x, t) \in (-p'(-1), -p'(1))$ for all $(x, t) \in \bar{\Omega} \times [0, T]$, then the traveling wave speed at point x is

$$c(x) = \tfrac{1}{T} \int_0^T \gamma(x, t)\, dt.$$

Consequently, in this special case, interfaces for solutions to (1.7) will move with normal velocity explicitly given to first order by $\varepsilon c(x)$.

2. Traveling wave problem

In this section, we shall prove the first theorem of the previous section, breaking it up into smaller parts. To begin, we show uniqueness of the traveling wave solution through a squeezing argument using sub– and super–solutions. These are built from horizontal and vertical translates of traveling waves, the amount of the translations evolving with time. Then we establish certain properties of solutions, not only to more fully understand their structure, but also as an aid to proving stability later. Following this we prove the existence of the traveling wave, first obtaining approximations on bounded intervals and then taking the limit as the interval expands to become the whole line.

2.1. **Uniqueness of traveling solutions.**

Theorem 2.1. *Under the conditions of Theorem 1.1, problem (1.4) admits at most one solution.*

Proof. Let (c, U) and $(\bar{c}, \bar{U})$ be any two solutions of (1.4). We shall prove that $c = \bar{c}$ and $U = \bar{U}$ in several steps. Without loss of generality, we assume that $\bar{c} \le c$.

Step 1. Set $M^{\pm} = \sup_{\xi \in \mathbb{R}}(\pm U(\xi, 0))$. Let $w(\alpha, t)$ be the function defined in (1.2). Then by the comparison principle, $w(-M^-, t) \le U(\xi, t) \le w(M^+, t)$ for all $\xi \in \mathbb{R}, t \ge 0$. Hence, by periodicity, $w(-M^-, kT + t) \le U(\xi, t) \le w(M^+, kT + t)$ for all $\xi \in \mathbb{R}$, $t > 0$, and all positive integers k. Since the Poincaré map $P(\alpha)$ is monotonic and has only three fixed points with $\alpha^{\pm}$ being stable, $P(\alpha) > \alpha$ for all $\alpha < \alpha^-$ and $P(\alpha) < \alpha$ for all $\alpha > \alpha^+$. It follows that $\lim_{k\to\infty} w(\pm M^{\pm}, t + kT) = w(\alpha^{\pm}, t) =: W^{\pm}(t)$. Therefore, we have that $W^-(t) \le U(\xi, t) \le W^+(t)$ for all $\xi \in \mathbb{R}$ and $t > 0$. Applying the strong maximum principle and using the periodicity of U and $W^{\pm}$, we then conclude that

$$W^-(t) < U(\xi, t) < W^+(t) \qquad \forall\, \xi \in \mathbb{R},\ t \in \mathbb{R}.$$

The same estimate holds also for $\bar{U}$.

Step 2. Define

(2.1)
$$\nu^{\pm} = -\tfrac{1}{T} \int_0^T f_u(W^{\pm}(t), t)\, dt, \qquad a^{\pm}(t) = \exp\left(\tfrac{\nu^{\pm} t}{2} + \int_0^t f_u(W^{\pm}(\tau), \tau)\, d\tau\right).$$

Since $P'(\alpha^\pm) = \exp(\int_0^T f_u(W^\pm(t),t)\,dt) < 1$, we have $\nu^\pm > 0$ and $a^\pm(T) = \exp(-\frac{\nu^\pm T}{2}) < 1$. Let

(2.2)

$$\begin{cases} \delta_0 := \dfrac{\sup\{\eta > 0 \ : \ |f_u(u,t) - f_u(W^\pm(t),t)| \le \frac{\nu^\pm}{2} \ \ \forall\, t \in [0,T],\ u \in [W^\pm(t)-\eta, W^\pm(t)+\eta]\}}{2\|a^+(\cdot)\|_{C^0([0,T])} + 2\|a^-(\cdot)\|_{C^0([0,T])}}, \\ \xi_0 := \inf\{\hat\xi \ge 1 \ : \ |U(\pm\xi,t) - W^\pm(t)| \le \frac{\delta_0}{2} \ \text{ for all } \ \xi \in [\hat\xi,\infty) \ \text{ and } \ t \in [0,T]\}. \end{cases}$$

Since $f \in C^{2,1}$ and one can show that $U(\pm\infty,t) = W^\pm(t)$ uniformly for $t \in [0,T]$, both δ_0 and ξ_0 are well–defined.

For every $\delta \in (0,\delta_0]$, define $U^\pm_\delta(\xi,t) = U(\xi,t) + \delta a^\pm(t)$. Then,

$$\begin{aligned} \mathcal{L}^c U^\pm_\delta &:= (U^\pm_\delta)_t - (U^\pm_\delta)_{\xi\xi} - c(U^\pm_\delta)_\xi - f(U^\pm_\delta,t) = \delta a^\pm_t + f(U,t) - f(U+\delta a^\pm,t) \\ &= \delta a^\pm[\tfrac{\nu^\pm}{2} + f_u(W^\pm(t),t) - \textstyle\int_0^1 f_u(U+\delta\theta a^\pm,t)\,d\theta] > 0 \end{aligned}$$

in $[\xi_0,\infty)\times[0,T]$ for the "+" sign and in $(-\infty,-\xi_0]\times[0,T]$ for the "−" sign. Hence, U^+_δ and U^-_δ are super–solutions of $\mathcal{L}^c U = 0$ in $[\xi_0,\infty)\times[0,T]$ and $(-\infty,-\xi_0]\times[0,T]$, respectively.

Step 3. Since $\bar U(\pm\infty,t) = W^\pm(t)$, there exists a large positive constant $\hat z_0$ such that

(2.3)

$$\bar U(\xi - z + (c-\bar c)t, t) \le \begin{cases} U(\xi,t) & \text{if } \ \xi \in [-\xi_0,\xi_0], \\ U(\xi,t) + \delta_0 & \text{if } \ \xi \notin [-\xi_0,\xi_0], \ \text{for all } \ t \in [0,T], z \ge \hat z_0. \end{cases}$$

Define

$$\delta_{\hat z_0} := \inf\{\delta > 0 \ : \ \bar U(\xi - z, 0) \le U(\xi,0) + \delta \ \ \forall z \ge \hat z_0, \xi \in \mathbb{R}\ \}. \tag{2.4}$$

Clearly, $\delta_{\hat z_0} \in [0,\delta_0]$. We claim $\delta_{\hat z_0} = 0$. In fact, noting $\mathcal{L}^c \bar U(\xi - z + (c-\bar c)t, t) = 0$, we can use (2.3) on $\{\xi_0\}\times[0,T]$ and (2.4) on $[\xi_0,\infty)\times\{0\}$ to compare $U^+_{\delta_{\hat z_0}}(\xi,t)$ and $\bar U(\xi - z + (c-\bar c)t,t)$ in $(\xi_0,\infty)\times[0,T]$ to obtain $\bar U(\xi - z + (c-\bar c)t, t) \le U^+_{\delta_{\hat z_0}}(\xi,t)$ for all $z \ge \hat z_0$ and all $(\xi,t) \in (\xi_0,\infty)\times[0,T]$. Since z can be an arbitrary number in $[\hat z_0,\infty)$ and $c \ge \bar c$, we then have that $\bar U(\xi - z, T) \le U^+_{\delta_{\hat z_0}}(\xi,T)$ for all $\xi \in [\xi_0,\infty)$ and $z \ge \hat z_0$. Using the periodicity of $\bar U$ and U, and the definition of $U^+_{\delta_{\hat z_0}}$, we then have that for all $z \ge \hat z_0$ and all $\xi \in [\xi_0,\infty)$, $\bar U(\xi - z, 0) \le U(\xi,0) + \delta_{\hat z_0} a^+(T)$. In a similar manner, we can show that for all $z \ge \hat z_0$ and all $\xi \in (-\infty,-\xi_0]$, $\bar U(\xi - z, 0) \le U(\xi,0) + \delta_{\hat z_0} a^-(T)$. Hence, from (2.3), $\bar U(\xi - z, 0) \le U(\xi,0) + \delta_{\hat z_0}\max\{a^+(T), a^-(T)\}$ for all $\xi \in \mathbb{R}$. Therefore, by the definition of $\delta_{\hat z_0}$, $\delta_{\hat z_0} \le \delta_{\hat z_0}\max\{a^+(T), a^-(T)\}$. Recalling that $a^\pm(T) < 1$, we must have $\delta_{\hat z_0} = 0$. In summary,

$$\bar U(\xi - z, 0) \le U(\xi, 0) \qquad \forall \xi \in \mathbb{R}, z \ge \hat z_0. \tag{2.5}$$

Step 4. From (2.5) and a comparison principle, we have $\bar U(\xi - \hat z_0 + (c-\bar c)t, t) \le U(\xi,t)$ for all $(\xi,t) \in \mathbb{R}\times[0,\infty)$. Consequently, by the periodicity, for every positive integer k,

$$\alpha^0 = U(0,0) = U(0,kT) \ge \bar U(-\hat z_0 + (c-\bar c)kT, kT) = \bar U(-\hat z_0 + (c-\bar c)kT, 0).$$

Sending k to ∞ and noticing that $\bar c \le c$ and $\bar U(\infty,0) = \alpha^+ > \alpha^0$, we conclude that $\bar c = c$.

Step 5. Define

$$z_0 := \inf\{\tilde z_0 \in \mathbb{R} \ : \ \bar U(\xi - z, 0) \le U(\xi,0)\ \forall z \ge \tilde z_0,\ \xi \in \mathbb{R}\ \}.$$

Then, by (2.5), z_0 is well–defined and is finite. We claim that $\bar{U}(\xi - z_0, 0) = U(\xi, 0)$ for all ξ. Assume that this is not true. Then by a strong maximum principle and periodicity (recalling that $\bar{c} = c$), $\bar{U}(\xi - z, t) < U(\xi, t)$ for all $(\xi, t) \in \mathbb{R}^2$ and all $z \in [z_0, \infty)$. Now let $\varepsilon > 0$ be a positive constant sufficiently small such that (2.3) holds for $\hat{z}_0 := z_0 - \varepsilon$. Then proceed as in Step 3 to conclude that $\bar{U}(\xi - z, 0) \leq U(\xi, 0)$ for all $z \geq \hat{z}_0$. Thus by the definition of z_0, $\hat{z}_0 \geq z_0$, which is impossible. Thus, we must have $\bar{U}(\xi - z_0, 0) = U(\xi, 0)$ for all $\xi \in \mathbb{R}$.

Step 6. We now show that z_0 in Step 5 is zero. In fact, from the definition of z_0 and strong maximum principle, we have that $\bar{U}(\xi - z, 0) < U(\xi, 0)$ for all $\xi \in \mathbb{R}$ and $z > z_0$. Since $U(\xi, 0) = \bar{U}(\xi - z_0, 0)$, we have $U(\xi + z_0 - z, 0) = \bar{U}(\xi - z, 0) < U(\xi, 0)$ for all $z > z_0$; that is, $U(\cdot, 0)$ is strictly monotonic. Observing that $U(z_0, 0) = \bar{U}(0, 0) = \alpha^0 = U(0, 0)$, we must have $z_0 = 0$. This completes the proof of Theorem 2.1.

2.2. **Basic properties of traveling wave solutions.**

Theorem 2.2. *Assume that (c, U) solves (1.4). Then the following holds:*

1. $U_\xi(\xi, t) > 0$ *for all* $\xi, t \in \mathbb{R}$*;*
2. *Define* $\nu^\pm$ *as in (2.1) and* $\beta_\pm := \frac{1}{2}(-c \mp \sqrt{c^2 + 4\nu^\pm})$*. Then there exists a constant* $C > 0$ *such that*

$$|U(\pm\xi, t) - W^\pm(t)| + |U_\xi(\pm\xi, t)| + |U_{\xi\xi}(\pm\xi, t)| \leq C e^{-|\beta_\pm|\xi}, \qquad \forall \xi \in [0, \infty),\ t \in [0, T].$$

Proof. (1) From Step 6 of the proof of Theorem 2.1, we know that $U_\xi \geq 0$. Hence, applying a strong maximum principle for the equation satisfied by U_ξ yields $U_\xi > 0$ in $\mathbb{R}^2$.

(2) Let M be a large constant to be determined. Define

$$v_M(\xi, t) = [W^+(t) - U(\xi + M, t)] \exp\Big(-\nu^+ t - \int_0^t f(W^+(\tau), \tau) d\tau \Big), \qquad (\xi, t) \in [0, \infty) \times [0, \infty).$$

By the definition of ν^+, v_M is T–periodic. Direct calculation yields

(2.6)
$$\mathcal{L}_M^c(v_M) := (v_M)_t - (v_M)_{\xi\xi} - c(v_M)_\xi + \nu^+ v_M + D v_M^2 = 0 \quad \text{in} \quad [0, \infty) \times [0, \infty),$$

where $D = D(M, \xi, t)$ is a function defined by

$$D(M, \xi, t) := \frac{f(U, t) - f(W^+, t) - f_u(W^+, t)(U - W^+)}{(U - W^+)^2} \exp\Big\{ \nu^+ t + \int_0^t f(W^+(\tau), \tau) d\tau \Big\}.$$

Since U is bounded by W^- and W^+, by Taylor's Theorem,

$$|D(M, \xi, t)| \leq D_0 := \sup_{t \in [0, T], s \in [W^-(t), W^+(t)]} \Big\{ \tfrac{1}{2} |f_{uu}(s, t)| \exp\Big(\nu^+ t + \int_0^t f(W^+(\tau), \tau) d\tau \Big) \Big\}.$$

We shall find super–solutions of (2.6) to estimate v_M.

Let $\delta_1 = \frac{\nu^+}{8D_0}$. For every $\delta_2 \in (0, \frac{\nu^+}{2D_0}]$ consider the function

$$V_{\delta_2}(\xi) := \delta_1 e^{\beta_+ \xi}(2 - e^{\beta_+ \xi}) + \delta_2, \quad \xi \geq 0.$$

For simplicity we write β for β_+ and ν for ν^+. Calculation shows

$$\mathcal{L}_M^c(V_{\delta_2}) = 2\delta_1 e^{\beta\xi}(-\beta^2 - c\beta + \nu) + \delta_1 e^{2\beta\xi}[4\beta^2 + 2c\beta - \nu + \delta_1 D(2 - e^{\beta\xi})^2] + \delta_2[\nu + 2D\delta_1 e^{\beta\xi}(2 - e^{\beta\xi}) + \delta_2 D].$$

Using the definition of β, we have that $-\beta^2 - c\beta + \nu = 0$ and $4\beta^2 + 2c\beta - \nu = 2\beta^2 + \nu > \nu$. Since $|D| \leq D_0$, it follows that

$$\mathcal{L}_M^c(V_{\delta_2}^+) > \delta_1 e^{2\beta\xi}[\nu - 4\delta_1 D_0] + \delta_2[\nu - 4D_0\delta_1 - \delta_2 D_0] \geq 0$$

by the definition of δ_1 and the constraint on δ_2. Therefore, for every $\delta_2 \in (0, \frac{\nu}{2D_0}]$, V_{δ_2} is a super–solution to (2.6) in $[0,\infty) \times [0,\infty)$.

Since $U(\infty, t) = W^+(t)$, there exists $M > 0$ such that $\sup_{\xi\in[0,\infty), t\in[0,T]} v_M(\xi, t) < \delta_1$. Define

$$\hat{\delta}_2 := \inf\{\delta_2 \in (0, \tfrac{\nu}{2D_0}] \ : \ V_{\delta_2}(\xi) \geq v_M(\xi, 0) \text{ in } [0,\infty)\}.$$

Then by the definition of M, $\hat{\delta}_2$ is well defined. In addition, noticing that $v_M(0,t) < \delta_1 \leq V_{\hat{\delta}_2}(0)$ for all $t \in [0,T]$, we can apply the comparison principle for the functions v_M and $V_{\hat{\delta}_2}$ in $[0,\infty) \times [0,T]$ to obtain that $v_M(\xi,t) < V_{\hat{\delta}_2}(\xi)$ for all $\xi \geq 0$ and all $t \in (0,T]$. Consequently, $v_M(\xi, 0) = v_M(\xi, T) < V_{\hat{\delta}_2}(\xi)$ for all $\xi \geq 0$. Since $v_M(\infty, t) = 0$, by the definition of $\hat{\delta}_2$, we must have $\hat{\delta}_2 = 0$. Hence,

$$v_M(\xi,t) \leq V_0(\xi) = \delta_1 e^{\beta\xi}(2 - e^{\beta\xi}) \leq 2\delta_1 e^{\beta\xi}, \qquad \xi \in [0,\infty).$$

It then follows by the definition of v_M that $0 < W^+(t) - U(\xi,t) < Ce^{\beta\xi}$ for some positive constant C and all $(\xi,t) \in [0,\infty) \times [0,T]$. Using local parabolic estimates for v_M, we can also derive that $|U_\xi| + |U_{\xi\xi}| \leq \hat{C}e^{\beta\xi}$ for all $\xi \in [0,\infty)$ and all $t \in [0,T]$. The case $\xi \leq 0$ can be similarly treated. The second assertion of the theorem thus follows.

2.3. **Existence of traveling waves in bounded domains.**

Let $M \geq 1$ be any fixed constant. Set $\Omega_M = (-M, M)$ and $Q_M = \Omega_M \times (0,T]$. For every constant $c \in \mathbb{R}$, consider the following initial boundary value problem:

$$(2.7) \qquad \begin{cases} \mathcal{L}^c(V) := V_t - cV_\xi - V_{\xi\xi} - f(V,t) = 0, & (\xi,t) \in Q_M, \\ V(\pm M, t) = W^\pm(t), & t \in [0,T], \\ V(\xi, 0) = g(\xi), & \xi \in \Omega_M, \end{cases}$$

where g is any element in the function class $\mathcal{X}_M$ defined by

$$\mathcal{X}_M := \{g \in C^0([-M,M]) \ : \ g(\pm M) = \alpha^\pm, g(0) = \alpha^0, g_\xi(\cdot) \geq 0 \text{ in } \Omega_M\}.$$

Lemma 2.3. *Let $M \geq 1$ be any fixed constant. The following hold:*

1. *For every $c \in \mathbb{R}$ and $g \in \mathcal{X}_M$, problem (2.7) admits a unique solution $V = V(g, c; \xi, t)$, and the solution satisfies*

$$W^-(t) < V(g,c;\xi,t) < W^+(t), \quad V_\xi(g,c;\xi,t) > 0, \quad V_c(g,c;\xi,t) > 0 \quad \forall\, (\xi,t) \in Q_M.$$

2. *There exist constants $C^+(M)$ and $C^-(M)$ such that $C^-(M) < C^+(M)$ and*

$$\inf_{g\in\mathcal{X}_M} V(g, C^+(M); 0, T) \geq \alpha^0, \qquad \sup_{g\in\mathcal{X}_M} V(g, C^-(M); 0, T) \leq \alpha^0.$$

Consequently, for any $g \in \mathcal{X}_M$, there exists a unique $c = C(M, g) \in \mathbb{R}$ such that $V(g, c; \ \cdot\ , T) \in \mathcal{X}_M$.

3. *There exists a unique* $g^M \in \mathcal{X}_M$ *such that* $V(g^M, C(M, g^M);\ \cdot\ , T) = g^M$*; namely, there exists a unique solution* (V^M, C^M) *to the following problem:*

(2.8)
$$\begin{cases} \mathcal{L}^{C^M}(V^M) := V^M_t - V^M_{\xi\xi} - C^M V^M_\xi - f(V^M, t) = 0 \quad \textit{in} \ \ Q_M, \\ V^M(\pm M, t) = W^\pm(t) \qquad \textit{in} \ \ [0, T], \\ V^M(\cdot, 0) = V^M(\cdot, T). \end{cases}$$

Proof. (1) Though f is nonlinear so that the solution of (2.7) may blow up, the property of g in $\mathcal{X}_M$ and a comparison principle yield the a priori estimate $W^-(t) < V(\xi, t) < W^+(t)$ for any $(\xi, t) \in Q_M$. Hence, (2.7) admits a unique solution $V = V(g, c; \xi, t)$. Since the a priori estimate implies that $V_\xi \geq 0$ at $\xi = \pm M$, the assumption $g_\xi \geq 0$ and the maximum principle for the equation satisfied by V_ξ then immediately yield $V_\xi > 0$ in Q_M. Notice that $V_c := \frac{\partial}{\partial c} V$ satisfies

$$(V_c)_t - (V_c)_{\xi\xi} - c(V_c)_\xi - f_u(V, t) V_c = V_\xi > 0 \qquad \text{in} \ \ Q_M$$

and $V_c = 0$ on the parabolic boundary of Q_M; it then follows that $V_c > 0$ in Q_M. This establishes the first assertion.

(2) Let $W(\xi, t)$ (depending on M) be any fixed function having the following properties:

$$W(\xi, 0) < \alpha^- \ \ \forall \xi \in [-M, M], \qquad W(\pm M, t) \leq W^\pm(t) \ \ \forall\, t \in [0, T],$$
$$W(0, T) = \alpha^0, \qquad W_\xi(\xi, t) > 0 \ \ \forall\, (\xi, t) \in [-M, M] \times [0, T].$$

Since $\alpha^- < \alpha^0 < \alpha^+$, such a function can be easily constructed. For example, pick any monotonic function $\zeta(\xi)$ satisfying $\zeta_\xi > 0$ in $[-M, M]$, $\zeta(\pm M) = \alpha^\pm$, $\zeta(0) = \alpha^0$. Then the function $W(\xi, t) := \zeta(\xi) - K(T - t)$ with sufficiently large K will satisfy all the properties needed. Define

$$C^+(M) := \sup_{(\xi,t) \in [-M,M] \times [0,T]} \frac{W_t - W_{\xi\xi} - f(W, t)}{W_\xi}.$$

Then one can verify that when $c = C^+(M)$, W is a sub–solution of (2.7a), (2.7b) and $W(\cdot, 0) < g$ for any $g \in \mathcal{X}_M$. Hence, by comparison, $W(\xi, t) \leq V(g, C^+(M); \xi, t)$ in Q_M for any $g \in \mathcal{X}_M$. Consequently, $\alpha^0 = W(0, T) \leq V(g, C^+(M); 0, T)$. This proves the existence of $C^+(M)$. The existence of $C^-(M)$ can be shown by a similar construction.

Recall that for any fixed $g \in \mathcal{X}_M$, $V(g, c; 0, T)$ is strictly monotonic in $c \in \mathbb{R}$. By the properties of $C^\pm(M)$, there exists a unique $C = C(M, g)$ such that $V(g, C; 0, T) = \alpha^0$. Moreover, recalling that $V_\xi(g, c; \cdot, T) > 0$ in Ω and $V(g, c; \pm M, T) = W^\pm(T) = \alpha^\pm$, we have that $V(g, C(M, g); \cdot, T) \in \mathcal{X}_M$. The second assertion of the lemma thus follows.

(3) For every $g \in \mathcal{X}_M$ define a mapping $\mathcal{T} : \mathcal{X}_M \to \mathcal{X}_M$ by

$$\mathcal{T}(g) = V(g, C(M, g); \cdot, T).$$

Then we know the following: a) $\mathcal{X}_M$ is a closed convex subset of $C^0([-M, M])$ and $\mathcal{T}$ maps $\mathcal{X}_M$ into itself; b) Since $V_c > 0$ and the solution $V(\cdot, c; \xi, t)$ depends on g continuously, $C(M, g)$ is continuous in g and consequently $\mathcal{T}$ is continuous from $\mathcal{X}_M$ to $\mathcal{X}_M$; c) By a parabolic estimate, $\mathcal{T}(\mathcal{X}_M)$ is a bounded set in $C^2([-M, M])$, so that $\mathcal{T}$ is compact. Therefore, by Schauder's fixed point theorem, there exists $g \in \mathcal{X}_M$ such that $\mathcal{T}(g) = g$.

Uniqueness of solutions of (2.8) follows the same moving plane technique introduced in Step 5 of the proof of Theorem 2.1 and is omitted. (In the current bounded domain situation, one does not need the lifting technique introduced in Steps 2 and 3 in the proof of Theorem 2.1.) This completes the proof of Lemma 2.3.

Now we shall find estimates for the solution to (2.8) which are independent of M so that we can take the limit as $M \to \infty$ to obtain a solution of (1.4). The basic idea is to use the following comparison principle.

Lemma 2.4. *Let $M \geq 1$ be any fixed constant and (V^M, C^M) be the solution to (2.8).*

1. *If $(\bar{V}, \bar{c})$ satisfies*

$$\begin{cases} \bar{V}_t - \bar{V}_{\xi\xi} - \bar{c}\bar{V}_\xi - f(\bar{V}, t) \leq 0, \quad (\xi, t) \in Q_M, \\ \bar{V}(\pm M, t) \leq W^{\pm}(t) \quad \forall\, t \in [0, T], \qquad \bar{V}(0,0) \geq \alpha^0, \\ \bar{V}(\xi, 0) \leq \bar{V}(\xi, T), \qquad \xi \in [-M, M], \end{cases} \tag{2.9}$$

   then $C^M \leq \bar{c}$.

2. *If $\hat{V}$ satisfies*

$$\begin{cases} \hat{V}_t - \hat{V}_{\xi\xi} - C^M \bar{V}_\xi - f(\hat{V}, t) \leq 0, \quad (\xi, t) \in [0, M] \times [0, T], \\ \hat{V}(M, t) \leq W^+(t), \;\; \hat{V}(0, t) \leq V^M(0, t), \quad \forall\, t \in [0, T], \\ \hat{V}(\xi, 0) \leq \max\{\alpha^0, \hat{V}(\xi, T)\}, \qquad \xi \in [0, M], \end{cases} \tag{2.10}$$

   then $\hat{V} \leq V^M$ in $[0, M] \times [0, T]$.

Proof. (1) Assume for contradiction that $C^M > \bar{c}$. Then, since $V^M_\xi > 0$ in Q_M,

$$\mathcal{L}^{\bar{c}}(V^M) := (V^M)_t - V^M_{\xi\xi} - \bar{c}V^M_\xi - f(V^M, t) = (C^M - \bar{c})V^M_\xi > 0 \qquad \text{in } \; Q_M.$$

Define

$$m_0 = \inf\{m \in (-2M, 2M) \; : \; V^M(\xi, 0) > \bar{V}(\xi - m, 0) \\ \text{in } \; (-M, M) \cap (m - M, m + M) \;\}.$$

Since $V^M(M, 0) = \alpha^+ > \bar{V}(-M, 0)$ and $V^M(0,0) = \alpha^0 \leq \bar{V}(0,0)$, $m_0 \in [0, 2M)$. In addition, there exists $\xi_0 \in \Omega^M_{m_0} := (m_0 - M, M)$ such that $V^M(\xi_0, 0) = \bar{V}(\xi_0 - m_0, 0)$. Notice that the boundary conditions of V^M and $\bar{V}$ imply that on the parabolic boundary of $\Omega^M_{m_0} \times (0, T]$, $V^M(\xi, t) \geq \bar{V}(\xi - m_0, t)$. Hence, applying a comparison principle to the functions $V^M(\xi, t)$ and $\bar{V}(\xi - m_0, t)$ in the domain $\Omega^M_{m_0} \times [0, T]$, we have that $V^M(\xi, T) > \bar{V}(\xi - m_0, T)$ for all $\xi \in \Omega^M_{m_0}$. But this is impossible since $V^M(\xi_0, T) = V^M(\xi_0, 0) = \bar{V}(\xi_0 - m_0, 0) \leq \bar{V}(\xi_0 - m_0, T)$. Hence, we must have $C^M \leq \bar{c}$.

(2) Define $m_0 = \inf\{m \geq 0 \; : \; V^M(\xi, 0) \geq \hat{V}(\xi - m, 0) \;\text{ in }\; [m, M]\}$. Using a comparison principle in $(m_0, M) \times (0, T]$, one can follow the idea in (1) to deduce that $m_0 = 0$.

Now we apply the first comparison principle in Lemma 2.4 to estimate C^M.

Lemma 2.5. *There exists $M_0 > 1$ such that for any $M \geq M_0$, the solution (V^M, C^M) of (2.8) satisfies the estimate*

$$|C^M| \leq 1 + \tfrac{1}{2}\sup\{(W^+(t) - W^-(t) + 2)|f_{uu}(u, t)| \; : \\ u \in [W^-(t) - 1, W^+(t) + 1], \; t \in [0, T]\}.$$

Proof. Let $\zeta(s) = \frac{1}{2}[1 + \tanh(\frac{s}{2})]$ so that $\zeta' = \zeta(1-\zeta)$ and $\zeta'' = \zeta'(1-2\zeta)$. Set $w_1(t) = W^+(t)$ and $w_2(t) = w(\alpha^- - \varepsilon_0, t)$ where ε_0 is a small constant such that $w_2(t) \geq W^-(t) - 1$ in $[0,T]$. Consider the function

$$\bar{V}(\xi,t) = w_1(t)\zeta(\xi+\xi_0) + w_2(t)[1-\zeta(\xi+\xi_0)]$$

where ξ_0 is a constant such that $\zeta(\xi_0) = \frac{\alpha^0 - \alpha^- + \varepsilon_0}{\alpha^+ - \alpha^- + \varepsilon_0}$. Since $w_1(T) = w_1(0)$ and $w_2(T) > w_2(0)$, $\bar{V}(\cdot,T) > \bar{V}(\cdot,0)$. Also, $\bar{V}(0,0) = \alpha^0$, $\bar{V}_\xi > 0$, $\bar{V}(\infty,0) = \alpha^+$, and $\bar{V}(-\infty,0) = \alpha^- - \varepsilon_0$.

Observe, by Taylor's expansion, that

$$\zeta f(w_1,t) + (1-\zeta)f(w_2,t) - f(\zeta w_1 + (1-\zeta)w_2, t) = \frac{1}{2}\zeta(1-\zeta)(w_1-w_2)^2 f_{uu}(\theta,t)$$

for some $\theta \in (w_2, w_1)$. Taking $\bar{c} = 1 + 2\sup\{|f_{uu}(u,t)| : u \in [w_2(t), w_1(t)], t \in [0,T]\}$, we have that, for all $(\xi,t) \in \mathbb{R}\times[0,T]$,

$$\begin{aligned}\mathcal{L}^{\bar{c}}(\bar{V}) &= [-\bar{c}\zeta' - \zeta''](w_1 - w_2) \\ &\quad + [\zeta f(w_1,t) + (1-\zeta)f(w_2,t) - f(\zeta w_1 + (1-\zeta)w_2,t)] \\ &= -\zeta(1-\zeta)(w_1-w_2)[\bar{c} + 1 - 2\zeta - \tfrac{1}{2}(w_1-w_2)f_{uu}(\theta,t)] < 0.\end{aligned}$$

Hence, by Lemma 2.4 (1), for all M satisfying $\zeta(-M) \leq \frac{\varepsilon_0}{\alpha^+ - \alpha^- - \varepsilon_0}$ (so that $\bar{V}(-M,0) \leq \alpha^-$), we have $C^M \leq \bar{c}$. Similarly, one can establish the lower bound of C^M, thereby completing the proof of the lemma.

2.4. **Existence of a periodic traveling wave.**

Theorem 2.6. *Problem (1.4) admits a unique solution (U,c), which can be obtained by taking the limit, as $M\to\infty$, in the solution (V^M, C^M) of (2.8).*

Proof. From Lemma 2.5, we know that $\{C^M\}_{M\geq M_0}$ is uniformly bounded. Hence, by parabolic estimates [24], $\sup_{M\geq M_0}\|V^M\|_{C^{2,1}(Q_M)}$ is uniformly bounded also. Therefore, we can select a subsequence $\{M_j\}_{j=1}^\infty$ such that as $j\to\infty$, $M_j\to\infty$, $C^{M_j}\to c^*$, and $V^{M_j}\to U^*$ (uniformly in any compact subset of $\mathbb{R}\times[0,T]$), where (c^*, U^*) satisfies the following equations:

$$\begin{cases} U_t^* - U_{\xi\xi}^* - c^*U_\xi^* - f(U^*,t) = 0 & \text{in } \mathbb{R}\times[0,T], \\ U^*(0,0) = \alpha^0, \quad U^*(\cdot,0) = U^*(\cdot,T) & \text{in } \mathbb{R}, \quad U_\xi^* \geq 0 \text{ in } \mathbb{R}\times[0,T]. \end{cases}$$

Thus, to show that (c^*, U^*) solves (1.4), we need only show that $U^*(\pm\infty,t) = W^\pm(t)$.

Assume for the moment that U^* is non–trivial; i.e., $U^*(\cdot,t) \not\equiv w(\alpha^0,t)$. Then $U_\xi^* \not\equiv 0$, so that by the condition $U_\xi^* \geq 0$ in $\mathbb{R}\times[0,T]$ and the strong maximum principle, we have $U_\xi^* > 0$ in $\mathbb{R}\times[0,T]$. Consequently, $U^*(\pm\infty,t) := \lim_{\xi\to\pm\infty} U^*(\xi,t)$ exist and $U^*(-\infty,0) < \alpha^0 < U^*(\infty,0)$. Since $U^*(\cdot,t)$ is monotonic, U_ξ^* and $U_{\xi\xi}^*$ approach zero weakly as $|\xi|\to\infty$. It then follows that $U^*(\infty,t)$ and $U^*(-\infty,t)$ are periodic solutions of $w_t = f(w,t)$. Hence, by the assumption on f, we must have $U^*(\pm\infty,t) = W^\pm(t)$. Since the solution of (1.4) is unique, we then know that the whole sequence (V^M, C^M) converges to (U^*, c^*) as $M\to\infty$.

Thus, to finish the proof, we need only show that U^* is non–trivial. Without loss of generality, we assume that $c^* \geq 0$. Also, we can assume that $U^*(0,t) \geq w(\alpha^0,t)$ for all $t\in[0,T]$ since otherwise, $U^*(0,t) \not\equiv w(\alpha^0,t)$ so that U^* is not trivial. Under these assumptions, we have that

$$\lim_{j\to\infty} C^{M_j} \geq 0, \qquad \lim_{j\to\infty}\min_{t\in[0,T]}\{V^{M_j}(0,t) - w(\alpha^0,t)\} \geq 0.$$

We shall use Lemma 2.4 (2) to show that U^* is non–trivial.

Since $w_\alpha = \exp(\int_0^t f_u) > 0$, $K = \max\{\frac{|w_{\alpha\alpha}(\alpha,t)|}{w_\alpha(\alpha,t)} + 1 : \alpha \in [\alpha^-, \alpha^+], t \in [0,T]\}$ is finite. Take $\delta = \min\left\{\frac{1}{16K}, \frac{\alpha^+-\alpha^0}{8}\right\}$. Let $\zeta(s) \in C^\infty([0,\infty))$ be a function such that

$$\begin{cases}\zeta(s) = \alpha^0 + (s+\sqrt{\delta})^2 - 2\delta & \text{if } s \in [0,\sqrt{\delta}],\\ 0 \le \zeta'(s) < 5\sqrt{\delta}, \quad \alpha^0 + 2\delta \le \zeta(s) < \alpha^0 + 7\delta, \quad |\zeta''(s)| \le 2 & \text{if } s \in [\sqrt{\delta},\infty).\end{cases}$$

For any $\delta_1 > 0$, let $\hat{w}(\alpha,t)$ be the solution to

$$\hat{w}_t = f(\hat{w},t) - \delta_1\Big(\max\{0, \hat{w} - w(\alpha^0+\delta,t)\}\Big)^3, \qquad \hat{w}(\alpha,0) = \alpha.$$

Clearly, $\hat{w}(\alpha,t) = w(\alpha,t)$ for all $\alpha \le \alpha^0+\delta$. Since $P(\alpha) > \alpha$ for all $\alpha \in (\alpha^0,\alpha^+)$, for every positive δ_1 sufficiently small, $\hat{w}(\alpha,T) > \alpha$ for all $\alpha \in (\alpha^0, \alpha^0+7\delta]$. In addition, by taking smaller δ_1 if necessary, we have that $\max_{\alpha\in[\alpha^0-\delta,\alpha^0+7\delta],t\in[0,T]} \frac{|\hat{w}_{\alpha\alpha}(\alpha,t)|}{\hat{w}_\alpha(\alpha,t)} \le K$. We henceforth fix such $\delta_1 > 0$. Also, we set $\delta_2 := \min_{t\in[0,T]}\{\hat{w}(\alpha^0+2\delta,t) - \hat{w}(\alpha^0+\delta,t)\}$.

Let ε be a small positive constant to be determined. Consider the function $\hat{V}(\xi,t) = \hat{w}(\zeta(\varepsilon\xi),t)$. One can calculate

$$\mathcal{L}^{C^{M_j}}(\hat{V}) = -\delta_1\Big(\max\{0, \hat{w}(\zeta,t) - \hat{w}(\alpha^0+\delta,t)\}\Big)^3 - \hat{w}_\alpha\Big(\varepsilon^2\zeta'' + \varepsilon C^{M_j}\zeta' + \varepsilon^2\frac{\hat{w}_{\alpha\alpha}}{\hat{w}_\alpha}(\zeta')^2\Big)$$

where $\hat{w}$ is evaluated at $(\zeta(\varepsilon\xi),t)$. We want to show $\mathcal{L}^{C^{M_j}}(\hat{V}) < 0$ in $[0,\infty)\times[0,T]$ by considering two cases: (i) $\zeta \ge \alpha^0 + 2\delta$; (ii) $\zeta < \alpha^0 + 2\delta$.

In the first case,

$$\mathcal{L}^{C^{M_j}}(\hat{V}) \le -\delta_1\delta_2^3 + \varepsilon\hat{w}_\alpha\Big(2\varepsilon - 5\sqrt{\delta}\min\{C^{M_j},0\} + 25\varepsilon K\delta\Big) \le -\delta_1\delta_2^3 - C\varepsilon < 0$$

if we take ε small enough.

In the second case, with ε fixed as above, let $s := \varepsilon\xi \in [0,\sqrt{\delta})$ so that $\zeta'' = 2$ and $\zeta' = 2(s+\sqrt{\delta}) < 4\sqrt{\delta}$. It then follows that

$$\mathcal{L}^{C^{M_j}}(\hat{V}) \le -\hat{w}_\alpha\varepsilon\Big(2\varepsilon + 4\min\{C^{M_j},0\}\sqrt{\delta} - 16\varepsilon K\delta\Big) < 0$$

if we take j large enough such that $C^{M_j} \ge -\frac{\varepsilon}{8\sqrt{\delta}}$.

In summary, there exist $\varepsilon > 0$ and $J > 0$ such that $\mathcal{L}^{C^{M_j}}(\hat{V}) < 0$ in $[0,\infty)\times[0,T]$ for all $j \ge J$.

Finally, observe that, for all $t \in [0,T]$, $\hat{V}(0,t) = \hat{w}(\zeta(0),t) = \hat{w}(\alpha^0-\delta,t) = w(\alpha^0-\delta,t) < V^{M_j}(0,t)$ if we take j large enough. Also, for any $M \in [1,\infty)$, $\hat{V}(M,t) \le \hat{w}(\alpha^0+7\delta,t) < w(\alpha^0+7\delta,t) < W^+(t)$. Furthermore, if $\hat{V}(\xi,0) > \alpha^0$, then $\zeta = \zeta(\varepsilon\xi) > \alpha^0$ so that $\hat{V}(\xi,0) = \hat{w}(\zeta(\varepsilon\xi),0) = \zeta(\varepsilon\xi) < \hat{w}(\zeta(\varepsilon\xi),T) = \hat{V}(\xi,T)$. Hence, by Lemma 2.4 (2), for all j large enough, $V^{M_j} \ge \hat{V}$ in $[0,M_j]\times[0,T]$. Consequently, $U^* \ge \hat{V}$ in $[0,\infty)\times[0,T]$, and therefore U^* cannot be trivial. This completes the proof of Theorem 2.6.

3. Stability of the traveling waves

In this section we study the asymptotic behavior, as $t \to \infty$, for the initial value problem (1.1) for a large class of initial conditions g, and we prove Theorem 1.2. The analysis can be naturally divided into two parts. In the first part one shows that a solution develops, after some time, a wave–like profile. In the second part, one shows that the solution converges exponentially in time to a translate of the traveling wave solution constructed in Section 2. The result in the second part is local in nature and can be deduced from very general facts on exponential stability with asymptotic phase of invariant manifolds, established long ago by Henry [18] in the context of reaction–diffusion equations, and known before that in the context of ODE's (Hale [19]). We will, for the convenience of the reader who may be unfamiliar with these ideas, present a different self–contained approach. On the other hand, the general abstract method in Henry [18], together with the necesssary spectral theory, is natural in this problem and we include that approach in the Appendix. One of the benefits of the abstract approach, besides conceptual clarity, is the identification of the best exponent μ in (1.6).

In the sequel, we shall denote by $U^g(\xi,t)$ the solution of

(3.1)
$$\begin{cases} \mathcal{L}^c(U^g) := U^g_t - cU^g_\xi - U^g_{\xi\xi} - f(U^g,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ U^g(\cdot,0) = g(\cdot) & \text{on } \mathbb{R}, \end{cases}$$

where c is the speed of the unique traveling wave solution of (1.4). Clearly, the solution u in (1.1) is given by $u(z,t) = U^g(z-ct,t)$. We denote $\|\cdot\| = \|\cdot\|_{L^\infty(\mathbb{R})}$.

We shall consider the evolution of a general "vaguely resembling front" (i.e., g satisfying (1.5)) in several stages.

3.1. **Short time evolution of "vaguely resembling fronts".**

Lemma 3.1. *Let (c,U) be the solution of (1.4) and let $U^g(\xi,t)$ be the solution of (3.1) for $g \in L^\infty(\mathbb{R})$.*

1. *If there exist constants $\alpha_1 \in (\alpha^+,\infty)$ and $\alpha_2 \in (\alpha^-,\alpha^0)$ such that*

(3.2)
$$g(\xi) \le \alpha_1 \quad \text{in } \mathbb{R}, \qquad g(\xi) \le \alpha_2 \quad \text{in } (-\infty,0),$$

then for any $\varepsilon > 0$, there exist a positive number $\hat z$ and a positive integer $\hat k$ such that

$$U^g(\xi,\hat kT) \le U(\xi+\hat z,0) + \varepsilon \qquad \forall\, \xi \in \mathbb{R}.$$

2. *If g satisfies (1.5), then for every $\varepsilon > 0$, there exist a positive number $\hat z = \hat z(\varepsilon,g)$ and a positive integer $\hat k = \hat k(\varepsilon,g)$ such that*

(3.3)
$$U(\xi-\hat z,0) - \varepsilon \le U^g(\xi,\hat kT) \le U(\xi+\hat z,0) + \varepsilon \qquad \forall\, \xi \in \mathbb{R}.$$

Proof. (1) Set $\zeta(s) = \frac12[1+\tanh\frac{s}{2}]$, $w_1(t) = w(2\alpha_1-\alpha_2,t)$, and $w_2(t) = w(\alpha_2,t)$ where $w(\alpha,t)$ is the solution of (1.2). Define

$$\hat c = c + 1 + \tfrac12 \sup\{(w_1(t)-w_2(t))|f_{uu}(\theta,t)| \ : \theta \in [w_2(t),w_1(t)], t\in[0,\infty)\},$$
$$V(\xi,t) = w_1(t)\zeta(\xi+\hat ct) + w_2(t)[1-\zeta(\xi+\hat ct)].$$

Then, by (3.2), $V(\cdot,0) \ge g(\cdot)$. The same computation as in the proof of Lemma 2.5 shows that $\mathcal{L}^c(V) > 0$ in $\mathbb{R}\times[0,\infty)$. A comparison principle then yields $U^g(\xi,t) \le V(\xi,t)$ in $\mathbb{R}\times[0,\infty)$. The first assertion of the lemma thus follows from the fact that $\lim_{k\to\infty} w_2(kT+t) = W^-(t)$, $\lim_{k\to\infty} w_1(kT+t) = W^+(t)$.

(2) The second assertion follows from (1) and a similar estimate on the lower bound of the solution.

Lemma 3.1 (2) reveals that a "vaguely resembling wave front" evolves into a "resembling wave front" (i.e., close to $W^\pm(t)$ for ξ near $\pm\infty$) after a certain time. We now study its subsequent evolution.

3.2. **Evolution of "resembling wave fronts".**

Lemma 3.2. 1. *There exist positive constants* $\varepsilon_0, K_0, \rho_0$ *such that if for some* $\varepsilon \in (0, \varepsilon_0]$ *and* $\hat{z} \in \mathbb{R}$

$$g(\cdot) \leq U(\cdot + \hat{z}, 0) + \varepsilon \quad \Big(or \;\; g(\cdot) \geq U(\cdot - \hat{z}, 0) - \varepsilon \Big),$$

then for all $t \geq 0$,

$$U^g(\cdot, t) \leq U(\cdot + \hat{z} + K_0\varepsilon, t) + K_0\varepsilon e^{-\rho_0 t} \quad \Big(or \;\; U^g(\cdot, t) \geq U(\cdot - \hat{z} - K_0\varepsilon, t) - K_0\varepsilon e^{-\rho_0 t} \Big).$$

2. *There exists a positive constant* K_1 *such that if* $\|g(\cdot) - U(\cdot, 0)\| \leq \varepsilon$ *for some* $\varepsilon \in (0, \varepsilon_0]$, *then*

$$\|U^g(\cdot, t) - U(\cdot, t)\| \leq K_1\varepsilon \qquad \forall\, t \geq 0.$$

Proof. We need only prove (1) since (2) is a direct consequence of (1). Without loss of generality, we assume that $\hat{z} = 0$.

Let $\nu^\pm$, $a^\pm(t)$, δ_0, ξ_0 be as in (2.1) and (2.2). Let $\zeta(s)$ be any $C^2(\mathbb{R})$ function satisfying

$$\zeta(s) = 1 \text{ in } [3, \infty), \quad \zeta(s) = 0 \text{ in } (-\infty, 0], \quad 0 \leq \zeta'(s) \leq 1 \text{ and } |\zeta''(s)| \leq 1 \text{ in } \mathbb{R}.$$

Define

(3.4)
$$A(\xi, t) = \zeta(\xi)a^+(t) + (1 - \zeta(\xi))a^-(t),$$

(3.5)
$$B(t) = \textstyle\int_0^t \max\{a^+(\tau), a^-(\tau)\}\, d\tau,$$

(3.6)
$$K = \Big(\nu^+ + \nu^- + 1 + |c| + 2\|f_u\|\Big) \Big/ \Big(\min_{t \in [0,T], \xi \in [-\xi_0, \xi_0]} U_\xi(\xi, t) \Big),$$

$$V(\xi, t) = U(\xi + K\varepsilon B(t), t) + \varepsilon A(\xi, t),$$

where $\|f_u\| = \max\{|f_u(u,t)| \;:\; t \in [0,T], u \in [W^-(t) - 1, W^+(t) + 1]\}$. Note that

$$a^\pm(t) \leq C \exp(-\frac{\nu^\pm t}{2}) \quad \text{for all } t \in [0, \infty),$$

where $C = \sup_{t \in [0,T]} \exp(\nu^\pm t + \int_0^t f_u(W^\pm, \tau)\, d\tau)$. It follows that as $t \to \infty$, $a^\pm(t)$ and $\|A(\cdot, t)\|_{C^0(\mathbb{R})}$ approach zero exponentially fast, and $B(t)$ is uniformly bounded. We take $\varepsilon_0 = \delta_0/(2KB(\infty))$. We want to show that $U^g(\cdot, \cdot) \leq V(\cdot, \cdot)$ in $\mathbb{R} \times [0, \infty)$.

When $t = 0$, $V(\cdot, 0) = U(\cdot, 0) + \varepsilon \geq g(\cdot) = U^g(\cdot, 0)$. Also, we can calculate

$$\mathcal{L}^c(V) = K\varepsilon B_t U_\xi + \varepsilon[A_t - cA_\xi - A_{\xi\xi} - Af_u(U + \varepsilon\theta A, t)] \qquad \text{for some } \theta(\xi, t) \in (0, 1).$$

Now we claim that $\mathcal{L}^c V \geq 0$ for all $\varepsilon \in (0, \varepsilon_0]$. We consider three cases: (i) $\xi \in [\xi_0, \infty)$, (ii) $\xi \in (-\infty, -\xi_0]$, and (iii) $\xi \in [-\xi_0, \xi_0]$.

In the first case, $\zeta = 1$, $A_\xi = A_{\xi\xi} = 0$, $B_t U_\xi > 0$, $|f_u(U + \varepsilon\theta A, t) - f_u(W^+(t), t)| \leq \frac{\nu^+}{2}$, and $A_t = A[\frac{\nu^+}{2} + f_u(W^+(t), t)]$. It then follows that $\mathcal{L}^c V \geq 0$ in $[\xi_0, \infty) \times [0, \infty)$. Similarly, $\mathcal{L}^c V \geq 0$ in the second case.

In the third case, i.e., $\xi \in [-\xi_0, \xi_0]$, we have that

$$|A_t - cA_\xi - A_{\xi\xi} - Af_u(U + \varepsilon\theta A, t)| \leq \max\{a^+(t), a^-(t)\}(\nu^+ + \nu^- + 2\|f_u\| + |c| + 1).$$

On the other hand, we have that

$$B_t U_\xi \geq \max\{a^+(t), a^-(t)\} \min\{U_\xi : t \in [0,T], \xi \in [-\xi_0, \xi_0]\}.$$

Hence, by the definition of K, $\mathcal{L}^c V \geq 0$ in $[-\xi_0, \xi_0] \times [0, \infty)$.

In conclusion, $\mathcal{L}^c V \geq 0$ in $\mathbb{R} \times [0, \infty)$. Therefore, by the comparison principle, $U^g \leq V$ in $\mathbb{R} \times [0, \infty)$. The assertion of the lemma thus follows.

The first part of Lemma 3.2 (1) shows that a "resembling wave front" preserves this structure uniformly for all $t \in (0, \infty)$. We now show that this forces convergence to a translate of the traveling wave.

3.3. **Nonlinear stability of the traveling wave.**

Lemma 3.3. *Assume that $g \in L^\infty(\mathbb{R})$ satisfies (1.5). Then there exists $z_g \in \mathbb{R}$, such that*

$$\lim_{t \to \infty} \|U^g(\cdot, t) - U(\cdot + z_g, t)\| = 0.$$

Proof. By Lemmas 3.1 and 3.2 (1), there exist a positive integer $\hat{k}$ and a large number $\hat{z}$ such that for all $(\xi, t) \in \mathbb{R} \times [\hat{k}T, \infty)$,

(3.7)
$$U(\xi - \hat{z} - K_0\varepsilon_0, t) - K_0\varepsilon_0 e^{-\rho_0 t} \leq U^g(\xi, t) \leq U(\xi + \hat{z} + K_0\varepsilon_0, t) + K_0\varepsilon_0 e^{-\rho_0 t}.$$

Notice that $\{U^g(\cdot, kT)\}_{k=1}^\infty$ is a bounded sequence in $C^1(\mathbb{R})$. Also notice that $U(\xi, t)$ approaches $W^\pm(t)$ as $\xi \to \infty$, uniformly in $t \in [0, T]$. Consequently there exist an integer sequence $\{k_j\}_{j=0}^\infty$ and a function $h(\xi)$ such that as $j \to \infty$, $k_j \to \infty$ and $\|U^g(\cdot, k_j T) - h(\cdot)\| \to 0$. In addition, $U(\cdot, t)$ satisfies

$$U(\xi - \hat{z} - K_0\varepsilon_0, 0) \leq h(\xi) \leq U(\xi + \hat{z} + K_0\varepsilon_0, 0) \qquad \forall \xi \in \mathbb{R}.$$

Set

$$z_g := \sup\{z \ : h(\cdot) \geq U(\cdot + z, 0) \text{ in } \ \mathbb{R}\}.$$

Clearly, $h \geq U(\cdot + z_g, 0)$. By translation if necessary, we can, without loss of generality, assume that $z_g = 0$. We claim that $h(\cdot) = U(\cdot, 0)$. Assume that this is not true. Then, by the strong maximum principle, $U^h(\cdot, T) > U(\cdot, 0)$. Let $\hat{\xi}$ be a large constant such that $\hat{\delta} := \sup_{|\xi| \geq \hat{\xi} - 1} U_\xi(\xi, 0) \leq \frac{1}{4K_0}$. Since $U^h(\cdot, T) > U(\cdot, 0)$, there exists $\varepsilon > 0$ such that $U^h(\xi, T) > U(\xi + \varepsilon, 0)$ for all $\xi \in [-\hat{\xi}, \hat{\xi}]$. Consequently,

$$\begin{aligned} &U^g(\cdot, (k_J + 1)T) - U(\cdot + \varepsilon, 0) \\ &\quad \geq -\|U^g(\cdot, k_J T + T) - U^h(\cdot, T)\| + U^h(\cdot, T) - U(\cdot + \varepsilon, 0) \\ &\quad \geq -C\|U^h(\cdot, k_J T) - h\| - \max_{|\xi| \geq \hat{\xi}} [U(\xi + \varepsilon, 0) - U(\xi, 0)] \geq -2\varepsilon\hat{\delta} \end{aligned}$$

if we take J large enough. Thus, by Lemma 3.2 (1),

$$U^g(\cdot, k_J T + T + t) \geq U(\cdot + \varepsilon - 2\varepsilon\hat{\delta}K_0, t) - 2K_0\varepsilon\delta e^{-\rho_0 t} \quad \forall t > 0.$$

Sending $t \to \infty$ along an appropriate sequence we deduce that

$$h(\cdot) \geq U(\cdot + \varepsilon - 2\varepsilon\hat{\delta}K_0, 0) \geq U(\cdot + \tfrac{1}{2}\varepsilon, 0)$$

by the definition of $\hat{\delta}$. But this contradicts the definition of z_g. Hence, we must have $h(\cdot) = U(\cdot + z_g, 0)$. The assertion of the lemma thus follows from Lemma 3.2 (2).

3.4. **Linear exponential stability of the traveling wave.**

To complete the proof of Theorem 1.2, we first establish the exponential stability of the linearized equation of (1.1) near the traveling wave (c, U).

For each $\psi \in L^\infty(\mathbb{R})$, we define $v(\psi;\xi,t)$ as the solution to the linear problem

$$\begin{cases} Lv := v_t - cv_\xi - v_{\xi\xi} - f_u(U(\xi,t),t)v = 0 & \text{in } \mathbb{R}\times(0,\infty),\\ v(\psi;\cdot,0) = \psi(\cdot) \quad \text{in } \mathbb{R}. \end{cases}$$

Notice that $LU_\xi = 0$.

Lemma 3.4. *There exists a constant C_1 such that, for all $g \in L^\infty(\mathbb{R})$,*

$$\|v(\psi;\cdot,t)\| \le C_1\|\psi\| \qquad \forall\, t \ge 0. \tag{3.8}$$

In addition, there exists $z_\psi \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \|v(\psi;\cdot,t) - z_\psi U_\xi(\cdot,t)\| = 0.$$

Proof. Let $A(\xi,t)$, $B(t)$, and K be as in (3.4)–(3.6). Define

$$\Psi(\xi,t) := KB(t)U_\xi(\xi,t) + A(\xi,t). \tag{3.9}$$

Performing the same calculation as in the proof of Lemma 3.2, we have that $L\Psi > 0$ in $\mathbb{R}\times[0,\infty)$. Since $\Psi(\xi,0) = 1$, a comparison principle shows that for any $\psi \in L^\infty(\mathbb{R})$,

$$|v(\psi;\xi,t)| \le \Psi(\xi,t)\|\psi\| \qquad \forall\,(\xi,t)\in\mathbb{R}\times[0,\infty). \tag{3.10}$$

Defining $C_1 := \sup_{t\ge0}\|\Psi(\cdot,t)\|$ yields estimate (3.8).

Since $\{v(\psi,\cdot,kT)\}_{k=1}^\infty$ is a bounded set in $C^1(\mathbb{R}\times[0,\infty))$, from (3.10) and the fact that

$$\lim_{z\to\infty,k\to\infty} \|\Psi(\cdot,\cdot)\|_{C^0(((-\infty,-z]\cup[z,\infty))\times[kT,\infty))} = 0,$$

there exist an integer sequence $\{k_j\}_{j=1}^\infty$ and a smooth function $h \in L^\infty(\mathbb{R})$ such that as $j\to\infty$, $k_j\to\infty$ and $\|v(\psi;\cdot,k_jT) - h(\cdot)\| \to 0$. In addition,

$$|h(\xi)| \le KB(\infty)\|\psi\|U_\xi(\xi,0) \qquad \forall\,\xi\in\mathbb{R}. \tag{3.11}$$

We now want to show that for some $z_\psi\in\mathbb{R}$, $h(\cdot) = z_\psi U_\xi(\cdot,0)$.

To this end, let

$$z_* := \sup\{z \;:\; h(\cdot) \ge zU_\xi(\cdot,0)\}.$$

Clearly, $|z_*| \le KB(\infty)\|\psi\|$. We want to show that $h = z_*U_\xi(\cdot,0)$. By working with $\psi - z_*U_\xi$ if necessary, we can without loss of generality, assume that $z_* = 0$, so that $h \ge 0$. We use a contradiction argument. Assume that $h \not\equiv 0$. Then, by the strong maximum principle, $v(h;\cdot,T) > 0$ so that there exists $\varepsilon > 0$ such that $v(h;\xi,T) \ge \varepsilon U_\xi(\xi,0)$ in $[-\hat\xi,\hat\xi]$ where $\hat\xi$ is a constant such that $KB(\infty)\sup_{|\xi|\ge\hat\xi} U_\xi(\xi,0) \le \frac14$. Consequently,

$$\begin{aligned} &v(\psi;\cdot,k_JT+T) - \varepsilon U_\xi(\xi,0)\\ &\quad\ge -\|v(\psi;\cdot,k_JT+T) - v(h;\cdot,T)\| + v(h;\cdot,T) - \varepsilon U_\xi(\xi,0)\\ &\quad\ge -C_1\|v(\psi;\cdot,k_JT) - h\| - \varepsilon\sup\nolimits_{|\xi|\ge\hat\xi} U_\xi(\xi,0)\\ &\quad\ge -2\varepsilon\sup\nolimits_{|\xi|\ge\hat\xi} U_\xi(\xi,0) \end{aligned}$$

for some J large enough. Consequently, by the comparison principle, for any $t\ge0$,

$$v(\psi;\cdot,k_JT+T+t) - \varepsilon U_\xi(\cdot,t) \ge -[2\varepsilon\sup\nolimits_{|\xi|\ge\hat\xi} U_\xi(\xi,0)]\Psi(\cdot,t).$$

Hence, taking $t = k_jT - k_JT - T$ and sending j to ∞, we have that

$$h(\cdot,t) - \varepsilon U_\xi(\cdot,0) \geq -[2\varepsilon \sup_{|\xi|\geq\hat{\xi}} U_\xi(\xi,0)]KB(\infty)U_\xi(\cdot,0) \geq -\frac{\varepsilon}{2}U_\xi(\cdot,0)$$

by the definition of $\hat{\xi}$. Therefore, $h(\cdot) \geq \frac{\varepsilon}{2}U_\xi(\cdot,0)$, which contradicts the definition of z_*. Hence, we must have $h \equiv 0$. The second assertion of the lemma thus follows from the first.

Now for each $\psi \in L^\infty(\mathbb{R})$, we define

$$\mathcal{T}\psi = v(\psi;\cdot,T) - z_1(\psi)U_\xi(\cdot,0) \qquad \text{where } z_1(\psi) := \int_{\mathbb{R}} v(\psi;\cdot,T)U_\xi(\cdot,0)\Big/\int_{\mathbb{R}} U_\xi^2(\cdot,0).$$

Since $LU_\xi = 0$, one can see that for any positive integer k,

$$\mathcal{T}^k = \overbrace{\mathcal{T}\cdots\mathcal{T}}^{k}\,\psi = v(\psi;\cdot,kT) - z_k(\psi)U_\xi(\cdot,0)$$
$$\text{where } z_k(\psi) := \int_{\mathbb{R}} v(\psi;\cdot,kT)U_\xi(\cdot,0)\Big/\int_{\mathbb{R}} U_\xi^2(\cdot,0).$$

Lemma 3.5. 1. *If $\{\psi_j\}_{j=1}^\infty$ is a bounded sequence in $C^0(\mathbb{R})$ and $\lim_{j\to\infty}\psi_j = 0$ uniformly in every compact subset of $\mathbb{R}$, then*

$$\lim_{j\to\infty,k\to\infty} \|\mathcal{T}^k\psi_j\| = 0.$$

2. *There exists a large integer k^* such that*

$$\lambda_{k^*} := \sup_{\|\psi\|=1} \|\mathcal{T}^{k^*}\psi\| < 1. \tag{3.12}$$

Proof. (1) For every $\varepsilon > 0$, let $\tilde{k}$ be a constant such that $\max\{a^+(\tilde{k}T), a^-(\tilde{k}T)\} \leq \varepsilon$. Since $\lim_{j\to\infty}\psi_j = 0$ uniformly in every compact subset of $\mathbb{R}$, there exists $J > 0$ such that

$$\sup_{j\geq J} \|v(\psi_j,\cdot,\cdot)\|_{C^0([-\xi_0,\xi_0]\times[0,\tilde{k}T])} \leq \min_{t\in[0,\tilde{k}T]} \{a^+(t), a^-(t)\}.$$

Hence, comparing $v(\psi_j;\xi,t)$ with $a^+(t)$ in $[\xi_0,\infty)\times[0,\tilde{k}T]$ and with $a^-(t)$ in $(-\infty,\xi_0]\times[0,\tilde{k}T]$, respectively, we have that

$$|v(\psi_j;\xi,t)| \leq \max\{a^+(t), a^-(t)\} \qquad \forall\, t\in[0,\tilde{k}T], |\xi| \geq \xi_0.$$

Hence, $\sup_{j\geq J}\|v(\psi_j;\cdot,\tilde{k}T)\| \leq \max\{a^+(\tilde{k}T), a^-(\tilde{k}T)\} \leq \varepsilon$. Consequently,

$$\sup_{j\geq J}\sup_{t\geq\tilde{k}T} \|v(\psi_j;\cdot,t)\| \leq C_1\varepsilon.$$

Since $|z_k(\psi)| \leq C\|v(\psi;\cdot,kT)\|$ for all integers k and all $\psi \in L^\infty$, we have that $\sup_{j\geq J}\sup_{k\geq\tilde{k}} \|\mathcal{T}^k\psi_j\| \leq (C_1 + CC_1)\varepsilon$. This proves the first assertion of the lemma.

(2) Assume that the second assertion is not true. Then there exists $\{\psi_k\}_{k=1}^\infty$ such that

$$\|\psi_k\| = 1, \qquad \|\mathcal{T}^k\psi_k\| \geq 1 - \frac{1}{2k} \qquad \forall\, k = 1,2,\cdots.$$

Since $|v(\psi_k;\cdot,kT)| \leq \Psi(\cdot,kT)$ and $|z_k(\psi_k)| \leq C\|v(\psi;\cdot,kT)\|$, we have $|\mathcal{T}^k\psi_k| \leq (1+C)\Psi(\cdot,kT)$. Also, noting that $\{\mathcal{T}^k\psi_k\}_{k=1}^\infty$ is a bounded sequence in $C^1(\mathbb{R})$, we can find a subsequence $\{k_j\}_{j=1}^\infty$ and a function h such that as $j\to\infty$, $k_j\to\infty$ and $\|\mathcal{T}^{k_j}\psi_{k_j} - h\| \to 0$. Clearly, we have that $\|h\| \geq 1$, $|h(\xi)| \leq (1+C)KB(\infty)U_\xi(\xi,0)$, and $\int_{\mathbb{R}} h(\cdot)U_\xi(\cdot,0) = 0$.

Notice that $\{\mathcal{T}\psi_{k_j}\}_{j=1}^{\infty}$ is a bounded sequence in $C^1(\mathbb{R})$, so we can select a subsequence, which we still denote by $\{k_j\}$ such that $\mathcal{T}\psi_{k_j} \to \hat{\psi}$ uniformly in every compact subset of $\mathbb{R}$, for some $\hat{\psi} \in C^1(\mathbb{R})$. Therefore,

$$\lim_{j\to\infty} \|\mathcal{T}^{k_j-1}\hat{\psi} - h\| \le \lim_{j\to\infty} \|\mathcal{T}^{k_j}\psi_{k_j} - h\| + \lim_{j\to\infty} \|\mathcal{T}^{k_j-1}(\mathcal{T}\psi_{k_j} - \hat{\psi})\| = 0$$

by the first assertion of the lemma. Since by Lemma 3.4,

$$\lim_{k\to\infty} \|v(\hat{\psi}; \cdot, kT) - \hat{z}U_\xi(\cdot, kT)\| = 0$$

for some $\hat{z} \in \mathbb{R}$, we must have $h = \hat{z}U_\xi(\cdot, 0)$, which, by the assumption $\int_{\mathbb{R}} hU_\xi = 0$, yields $h \equiv 0$. This contradicts the assumption that $\|h\| = 1$. Hence, there exists a positive integer k^* such that (3.12) holds.

3.5. **Nonlinear exponential stability of the traveling wave.**

We are now prepared to prove Theorem 1.2, restated as

Theorem 3.6. *There exists a positive constant $\mu > 0$ such that for every $g \in L^\infty(\mathbb{R})$ satisfying (1.5), the solution $U^g(\xi, t)$ of (3.1) satisfies, for some constants z_g and C_g,*

$$\|U^g(\cdot, t) - U(\cdot + z_g, t)\| \le C_g e^{-\mu t} \qquad \forall\, t \ge 0.$$

Proof. By Lemma 3.3, we can assume that $\|g(\cdot) - U(\cdot, 0)\|$ is sufficiently small for our purposes.

Assume that $\eta_0 := \|g - U(\cdot, 0)\| = \min_{z\in\mathbb{R}} \|g - U(\cdot + z, 0)\|$. Set $V(\cdot, t) = U^g(\cdot, t) - U(\cdot, t)$. Then, we have $LV = EV^2$ where

$$\|E\|_{L^\infty(\mathbb{R}\times[0,\infty))} \le \frac{1}{2} \sup\{|f_{uu}(\theta, t)| : \theta \in [W^-(t) - 1, W^+(t) + 1], t \in [0, T]\}.$$

Let k^* be as in Lemma 3.5. Then by parabolic estimates [24], there exists a constant $C(k^*)$ such that

$$\|V(\cdot, k^*T) - v(V(\cdot, 0); \cdot, k^*T)\| \le C(k^*)\|V^2\|_{L^\infty(\mathbb{R}\times[0,\infty))} \le C(k^*)K_1^2\eta_0^2$$

by Lemma 3.2 (2). Consequently, decomposing $v(V(\cdot, 0); \cdot, k^*T) = \mathcal{T}^{k^*}V(\cdot, 0) + z_{k^*}U_\xi$, we have

$$\begin{aligned} \|V(\cdot, k^*T) - z_{k^*}U_\xi(\cdot, 0)\| &\le \|V(\cdot, k^*T) - v(V(\cdot, 0); \cdot, k^*T)\| + \|\mathcal{T}^{k^*}V(\cdot, 0)\| \\ &\le (\lambda_{k^*} + C(k^*)K_1^2\eta_0)\eta_0. \end{aligned}$$

Noticing that $|z_{k^*}| \le CC_1\|V(\cdot, 0)\| \le CC_1\eta_0$, it then follows that

$$\begin{aligned} &\|U^g(\cdot, k^*T) - U(\cdot + z_{k^*}, 0)\| \\ &\le \|U(\cdot + z_{k^*}, 0) - U(\cdot, 0) - z_{k^*}U_\xi(\cdot, 0)\| + \|V(\cdot, k^*T) - z_{k^*}U_\xi(\cdot, 0)\| \\ &\le C\|U_{\xi\xi}(\cdot, 0)\||z_{k^*}|^2 + \|V(\cdot, k^*T) - z_{k^*}U_\xi(\cdot, 0)\| \\ &\le (\lambda_{k^*} + C(k^*)K_1^2\eta_0 + CC_1\eta_0)\eta_0 = (\lambda_{k^*} + C(k^*)K_1^2\eta_0 + CC_1\eta_0)\|g - U(\cdot, 0)\|. \end{aligned}$$

Let δ^* be a positive constant such that $\mu^* := \lambda_{k^*} + C(k^*)K_1^2\delta^* + CC_1\delta^* < 1$. Then for any g satisfying $\min_{z\in\mathbb{R}} \|g - U(\cdot + z, 0)\| \le \delta^*$, we have that

$$\min_{z\in\mathbb{R}} \|U^g(\cdot, k^*T) - U(\cdot + z, 0)\| \le \mu^* \min_{z\in\mathbb{R}} \|g - U(\cdot + z, 0)\|.$$

From this, one can easily derive the assertion of the theorem. Details are omitted.

4. The singular perturbation problem

Now we study the singular perturbation problem (1.7). For each fixed $x \in \bar\Omega$, we denote by $w(x;\alpha,t)$ the solution of (1.2) with $f(\cdot,\cdot) = F(x;\cdot,\cdot)$, by $\alpha^+(x)$, $\alpha^0(x)$, and $\alpha^-(x)$ the three fixed points of the period return map $P(x;\cdot) := w(x;\cdot,T)$, and by $W^\pm(x;t)$ and $W^0(x;t)$ the corresponding functions $w(x;\alpha^\pm(x),t)$ and $w(x;\alpha^0(x),t)$. Also, we denote by $(c(x),U(x,\xi,t))$ the traveling wave solution of (1.4) with $f(u,t) = F(x;u,t)$. For any $\rho > 0$, we define

$$\Omega_\rho^\pm := \{x \in \bar\Omega \ : \ \pm c(x) > \rho\}, \qquad \Omega_\rho := \{x \in \Omega \ : \ \mathrm{dist}(x,\partial\Omega) > \rho\}.$$

Finally, we denote by $B(x,R)$ the ball in $\mathbb{R}^N$ centered at x with radius R. For brevity, we write $\|v,w\|$ in place of $\max\{\|v\|,\|w\|\}$. Other similarly abbreviated notation will also be used.

In the sequel ε is a positive constant as small as we wish and C is a generic positive constant independent of ε.

We first consider the following initial value problem, for $u = u(g;x,t)$:

(4.1)
$$\begin{cases} \mathcal{L}^\varepsilon u := u_t - \varepsilon^2\Delta u - F(x;u,t) = 0, & (x,t) \in \Omega\times[0,\infty),\\ \frac{\partial}{\partial n}u = 0, & (x,t)\in\partial\Omega\times[0,\infty),\\ u(x,0) = g(x), & x\in\Omega. \end{cases}$$

We assume that g is bounded and write

$$\|g\|_{C^0(\Omega)} \le M_0 \tag{4.2}$$

where $M_0 \ge \max\{\|\alpha^+(x)\|_{C^0(\Omega)}, \|\alpha^-(x)\|_{C^0(\Omega)}\}$ is a fixed positive constant independent of ε. Set

$$M = 1 + \sup\{|w(x;\alpha,t)| \ : \ x\in\Omega, |\alpha| \le M_0, t\in[0,\infty)\}.$$

Since $P(x;\alpha) < \alpha$ when $\alpha > \alpha^+(x)$ and $P(x;\alpha) > \alpha$ when $\alpha < \alpha^-(x)$, M is finite. In the sequel, the norm of F is always taken on

$$Q = \bar\Omega \times [-M,M]\times[0,T].$$

We begin by constructing two super–solutions based on those constructed in [8]. The corresponding sub–solutions can be constructed in an analogous way but this is omitted here.

The first super–solution, obtained by modifying $w(x;g(x),t)$, deals with arbitrary initial data with no "sharp" layer; i.e., it deals with the time stage of "generation of interfaces" (cf. [8]).

For any small positive δ (which may depend on ε), let F^δ be a modification of F such that F^δ has the following properties:

(4.3)
$$\begin{cases} F^\delta(x;u,T+t) = F^\delta(x;u,t) & \forall (x,u,t)\in Q;\\ \delta \le F^\delta - F \le 2\delta & \forall(x,u,t)\in Q;\\ \frac{\partial}{\partial n}F^\delta = 0 & \forall(x,u,t)\in\partial\Omega\times[-M,M]\times[0,T];\\ \|F_u^\delta, \nabla_x F^\delta, F^\delta_{uu}, \delta D_x^2 F^\delta\|_{L^\infty(Q)} \le C \end{cases}$$

where C is independent of δ. Such F^δ can be obtained by modifying $F + \frac{3}{2}\delta$ near $(x,t)\in\partial\Omega\times[0,T]$.

Now let $w^\delta(x;\alpha,t)$ be the solution to

$$w_t^\delta = F^\delta(x;w,t), \qquad w^\delta(x;\alpha,0) = \alpha.$$

Denote by $\alpha_\delta^\pm(x), \alpha_\delta^0(x)$ the fixed points of $P^\delta(x;\cdot) := w^\delta(x;\cdot,T)$, preserved by the nondegeneracy condition (1.3), and by $W_\delta^\pm(x,t), W_\delta^0(x,t)$ the functions $w^\delta(x;\alpha_\delta^\pm(x),t)$ and $w^\delta(x;\alpha_\delta^0(x),t)$, respectively. Then one can verify the following estimates (but we leave this to the reader)

$$\begin{cases} \|w_\alpha^\delta, w_{\alpha\alpha}^\delta, \nabla_x w^\delta, \nabla_x w_\alpha^\delta, \delta\Delta_x w^\delta\|_{C^0(\bar\Omega\times[-M,M])} \le Ce^{\mu_1 t}; \\ \text{For any } \rho_1>0, |w^\delta(x;\alpha,t) - W_\delta^\pm(x,t)| \le \frac{C}{\rho_1}e^{-\mu_2 t} \qquad \text{if } \pm(\alpha-\alpha_\delta^0) > \rho_1; \\ \frac{\partial}{\partial n} w^\delta(x;\alpha,t) = 0 \qquad \forall\,(x,\alpha,t)\in\partial\Omega\times[-M,M]\times[0,T]; \\ C^{-1}\delta \le W_\delta^\pm(x,t) - W^\pm(x,t) \le C\delta \qquad \forall (x,t)\in\bar\Omega\times[0,T], \end{cases}$$

where C, μ_1, and μ_2 are positive constants independent of δ.

Lemma 4.1. *Assume that $\bar g \in C^2(\bar\Omega)$ satisfies $\frac{\partial}{\partial n}\bar g = 0$ on $\partial\Omega$ and $\|\bar g\|_{C^0(\Omega)} \le M_0$. Then for every integer $k>0$, the function*

$$V_1^\delta(\bar g; x,t) := w^\delta(x;\bar g(x),t), \quad (x,t)\in\bar\Omega\times[0,\infty)$$

is a super–solution to (4.1a), (4.1b) in $\Omega\times[0,kT]$ provided that

$$\delta \ge \varepsilon M_1 \exp(\tfrac{\mu_1 kT}{2})\sqrt{\delta\|\Delta\bar g\|_{C^0(\Omega)} + \delta\|\nabla\bar g\|^2_{C^0(\Omega)} + 1} \tag{4.4}$$

where M_1 is a constant independent of ε, k, δ, and $\bar g$.

Proof. Clearly, we have that $\frac{\partial}{\partial n}V_1^\delta = 0$ on $\partial\Omega\times[0,\infty)$. Also, we can calculate

$$\begin{aligned} \mathcal{L}^\varepsilon V_1^\delta &:= V_{1t}^\delta - \varepsilon^2\Delta V_1^\delta - F(x;V_1^\delta,t) \\ &= (F^\delta - F) - \varepsilon^2(w_\alpha^\delta\Delta\bar g + w_{\alpha\alpha}^\delta|\nabla\bar g|^2 + 2\nabla_x w_\alpha^\delta\nabla\bar g + \Delta_x w^\delta) \\ &\ge \delta - \varepsilon^2 Ce^{\mu_1 t}(\|\Delta g\| + \|\nabla g\|^2 + \delta^{-1}) > 0 \qquad \text{in } \Omega\times[0,kT] \end{aligned}$$

by the assumption on δ. The assertion of the lemma thus follows.

Using this lemma, we can show the following concerning the generation of interfaces:

Theorem 4.2. *There exist ε-independent positive constants $\mu\in(0,1/3)$ and K such that the following hold:*

1. *If $\|g\|_{C^0(\bar\Omega)} \le M_0$, then*

$$W^-(x,t) - \varepsilon^{2\mu} \le u(g;x,t) \le W^+(x,t) + \varepsilon^{2\mu} \qquad \forall\, x\in\bar\Omega,\ t\ge K|\ln\varepsilon|. \tag{4.5}$$

2. *If $\|g\|_{C^0(\bar\Omega)} \le M_0$ and for some $y\in\bar\Omega$*

$$g(x) \le \alpha^0(x) - \varepsilon^\mu \qquad \Big(or\ g(x) \ge \alpha^0(x) + \varepsilon^\mu\Big) \quad in\ \ B(y,6\varepsilon^{1/3})\cap\Omega, \tag{4.6}$$

then

$$|u(g;x,t) - W^-(x,t)| \le C\varepsilon^{2\mu} \quad \Big(or\ \ |u(g;x,t) - W^+(x,t)| \le C\varepsilon^{2\mu}\Big)$$
$$in\ \ (\Omega\cap B(y,5\varepsilon^{1/3}))\times[K|\ln\varepsilon|, K|\ln\varepsilon| + 2T].$$

Proof. We only establish the upper bounds. The lower bounds can be established in a similar manner.

(1) Set $\delta = M_1\varepsilon^{2/3}$, $k_1 = \frac{1}{3\mu_1 T}|\ln\varepsilon|$, and $\bar g \equiv M_0$. Then (4.4) holds. Hence, comparing $u(\cdot,\cdot)$ with $V_1^\delta(\bar g;\cdot,\cdot)$ in $\bar\Omega\times[0,k_1T]$, we obtain

$$u(g;x,t) \le V_1^\delta(\bar g;x,t) \qquad \forall\,(x,t)\in\bar\Omega\times[0,k_1T].$$

The first assertion for $t \in [k_1T - T, k_1T]$ follows from the fact that

$$V_1^\delta(g; x, k_1T - T + t) \le W_\delta^+(x,t) + Ce^{-\mu_2 k_1 T} \le W^+(x) + C\varepsilon^{2\mu}$$

where $\mu = \min\{1/4, \frac{\mu_2}{6\mu_1}\}$.

Observe that $V_1^\delta(\bar g; \cdot, T) < M_0$, so that $u(g; \cdot, T) < M_0$. Hence, a mathematical induction argument shows that $u(g; \cdot, mT + t) \le V_1^\delta(\bar g; \cdot, t)$ for all positive integers m and all $t \in [0, k_1T]$. The first assertion of the theorem thus follows.

(2) We refine the upper bound near y. We still take $k_1 = \frac{1}{3\mu_1 T}|\ln\varepsilon|$ but we take $\delta = 8M_1^2\varepsilon^{2/3}$ and a different $\bar g$. Let $\bar g$ be a modification of g such that

$$\begin{cases} \bar g \ge g \ \text{ in } \ \bar\Omega; \qquad \frac{\partial}{\partial n}\bar g = 0 \ \text{ on } \ \partial\Omega; \\ \bar g(x) \le \alpha_\delta^0(x) - \frac{1}{2}\varepsilon^\mu \ \text{ in } \ B(y^+, 5\varepsilon^{1/3}); \\ \|\Delta \bar g\|_{C^0(\Omega)} + \|\nabla \bar g\|^2_{C^0(\Omega)} \le 15\varepsilon^{-2/3}. \end{cases}$$

As before, (4.4) holds, so that $u(g; x, t) \le V_1^\delta(\bar g; x, t)$ in $\bar\Omega \times [0, k_1T]$. In particular, in $B(y, 5\varepsilon^{1/3}) \times [k_1T - 2T, k_1T]$,

$$\begin{aligned} u(g; x, t) &\le V_1^\delta(\bar g; x, k_1T - 2T) \le w^\delta(x; \alpha_\delta^0 - \tfrac{1}{2}\varepsilon^\mu, k_1T - 2T) \\ &\le \alpha_\delta^-(x) + C\varepsilon^{-\mu}e^{-\mu_2 k_1 T} \le W^-(x,t) + C\varepsilon^{2\mu}. \end{aligned}$$

The assertion of the theorem thus follows.

To see how the region $\{x : |u(g; x, t) - W^\pm(x,t)| \le \varepsilon^\mu\}$ expands, we now construct the second super–solution dealing with the "propagation of interfaces" (cf. [8]).

For any fixed $x \in \bar\Omega$, let $(c^\delta(x), U^\delta(x, \xi, t))$ be the traveling wave solution to

$$\begin{cases} U_t^\delta - c^\delta U_\xi - U_{\xi\xi}^\delta - F^\delta(x; U^\delta, t) = 0, & (\xi, t) \in \mathbb{R} \times [0, \infty), \\ U^\delta(x, \pm\infty, t) = W_\delta^\pm(x,t), & U^\delta(x, 0, t) = W_\delta^0(x,t), \\ U^\delta(x, \xi, t+T) = U^\delta(x, \xi, t), & (\xi, t) \in \mathbb{R} \times [0, \infty). \end{cases}$$

Then one can show, as in Section 2, the following:

$$\begin{cases} \|U^\delta(\cdot, \pm\xi, \cdot) - W_\delta^\pm\| + \|U_\xi^\delta, U_{\xi\xi}^\delta, \nabla_x U_\xi^\delta\|_{C^0(\Omega\times[0,T])} \le Ce^{-\mu_3\xi} \qquad \forall \xi > 0; \\ \|\nabla_x U^\delta, \delta\Delta_x U^\delta\|_{C^0(\Omega\times\mathbb{R}^2)} + \|\nabla_x c^\delta\|_{C^0(\Omega)} \le C, \\ \frac{\partial}{\partial n}U^\delta(x, \xi, t) = 0 \qquad \forall (x, \xi, t) \in \partial\Omega \times \mathbb{R}^2; \\ \|c^\delta(\cdot) - c(\cdot)\|_{C^0(\Omega)} \le C\delta. \end{cases}$$

Here C is a positive constant independent of δ.

Let ζ be a $C^2(\mathbb{R})$ function satisfying

$$\zeta(s) = s \ \text{ if } \ |s| \le 1, \quad \zeta(s) = \pm 3/2 \text{ if } \ \pm s > 2, \quad 0 \le \zeta' \le 1 \text{ and } \ |\zeta''| \le 2 \text{ on } \ \mathbb{R}.$$

For every $z \in \mathbb{R}^N$, we define

$$d(z, x) := \varepsilon^{1/3}\zeta\Big(\tfrac{|x-z|}{\varepsilon^{1/3}} - 3\Big). \tag{4.7}$$

One can directly verify that $d(z, x)$ has the following properties:

$$\begin{cases} d(z, x) = 3/2\varepsilon^{1/3} & \text{if } \ |x - z| \ge 5\varepsilon^{1/3}; \\ d(z, x) = -3/2\varepsilon^{1/3} & \text{if } \ |x - z| \le \varepsilon^{1/3}; \\ |\nabla_x d| = 1 & \text{if } \ |d| \le \varepsilon^{1/3}; \\ \varepsilon|\Delta_x d| + \varepsilon^{1/3}|\nabla_x d|^2 \le C\varepsilon^{1/3} & \text{in } \ \mathbb{R}^N. \end{cases}$$

We claim that

$$\frac{\partial}{\partial n} d(z; x) := \zeta' \frac{x - z}{|x - z|} \cdot n(x) \ge 0 \qquad \forall\, z \in \Omega_{\varepsilon^{1/3}}, \ x \in \partial\Omega, \tag{4.8}$$

where $n(x)$ is the outward unit normal to $\partial\Omega$ at x. In fact, when $|x-z| \geq 5\varepsilon^{1/3}$, $\zeta' = 0$, so that the claim automatically holds. When $|x-z| \leq 5\varepsilon^{1/3}$, since $\operatorname{dist}(z, \partial\Omega) \geq \varepsilon^{1/3}$ and $\partial\Omega \in C^1$, the line segment from z to x intersects $\partial\Omega$ only at the end point x, so that $(x-z)\cdot n(x) \geq 0$. Hence, (4.8) holds.

In the following, C is taken to be greater than the previous C's in this section and also greater than 6.

Lemma 4.3. *Let $M_2 > C^2$. For any small $\delta \geq \varepsilon^{1/3}$ and any $z_0 \in \Omega^+_{M_2\delta} \cap \Omega_{\varepsilon^{1/3}}$, the function*

$$V_2^\delta(z(t); x, t) := U^\delta\Big(x, \frac{d(z(t), x)}{\varepsilon}, t\Big)$$

is a super–solution of (4.1a),(4.1b) in the domain $\bar{\Omega} \times [0, \infty)$ provided that the function $z(t) : [0, \infty) \to \Omega_{\varepsilon^{1/3}}$ satisfies

$$|z_t(t)| \leq \varepsilon[c(z(t)) - C^2\delta], \qquad \forall t \geq 0, z(0) = z_0.$$

Proof. Since $z(t) \in \Omega_{\varepsilon^{1/3}}$ for all $t \geq 0$, from (4.8), $\frac{\partial}{\partial n} V_2^\delta(z; x, t) \geq 0$ on $\partial\Omega \times [0, \infty)$. Also,

$$\begin{aligned}
\mathcal{L}^\varepsilon V_2^\delta &= U_t^\delta + U_\xi^\delta(\varepsilon^{-1} d_t - \varepsilon \Delta_x d) - U_{\xi\xi}^\delta |\nabla_x d|^2 - 2\varepsilon \nabla_x d \nabla_x U_\xi^\delta - \varepsilon^2 \Delta_x U^\delta - F \\
&= F^\delta - F + U_\xi^\delta[\varepsilon^{-1} d_t + c^\delta - \varepsilon\Delta_x d] + U_{\xi\xi}^\delta[1 - |\nabla_x d|^2] \\
&\quad - 2\varepsilon \nabla_x d \nabla_x U_\xi^\delta - \varepsilon^2 \Delta_x U^\delta \\
&\geq \delta + U_\xi^\delta[c^\delta(x) - \varepsilon^{-1}|z_t| - C\varepsilon^{1/3}] - CU_{\xi\xi}^\delta \chi_{\{|d| \geq \varepsilon^{1/3}\}} - C\varepsilon[1 + \varepsilon/\delta].
\end{aligned}$$

If $|d| \geq \varepsilon^{1/3}$, then $|U_\xi^\delta|, |U_{\xi\xi}^\delta| \leq Ce^{-\mu_3|d|/\varepsilon} \leq \varepsilon^2$, so that $\mathcal{L}^\varepsilon V_1^\delta > 0$.

If $|d| \leq \varepsilon^{1/3}$, then $|x - z| \leq 4\varepsilon^{1/3}$, so that

$$c^\delta(x) - \varepsilon^{-1}|z_t| - C\varepsilon^{1/3} \geq [c(z) - C(\delta + |x - z|)] - [c(z) - C^2\delta] - C\varepsilon^{1/3} > 0,$$

which implies $\mathcal{L}^\varepsilon V_2^\delta > 0$. The assertion of the lemma thus follows.

In the next theorem μ and K are those constants given in Theorem 4.2.

Theorem 4.4. *Assume that $\|g\|_{C^0(\Omega)} \leq M_0$. Also assume that there exists $y \in \Omega^+_{\varepsilon^\mu}$ $\Big($or $y \in \Omega^-_{\varepsilon^\mu}\Big)$ such that (4.6) holds. Set $t_1 := K|\ln\varepsilon|$. Then for any $\tau \geq t_1$, a point $x \in \bar{\Omega}$ satisfies*

(4.9)

$$|u(g; x, \tau) - W^-(x, \tau)| \leq C\varepsilon^\mu \qquad \Big(\textit{or } \; |u(g; x, \tau) - W^+(x, \tau)| \leq C\varepsilon^\mu\Big)$$

provided that there exists a curve $z(\cdot) : [t_1, \tau] \to \Omega_{\varepsilon^{1/3}}$ such that

(i) $x \in B(z(\tau), 2\varepsilon^{1/3})$,

(ii) $|z_t| \leq \varepsilon[c(z(t)) - C^2\varepsilon^\mu]$ $\Big($or $|z_t| \leq -\varepsilon[c(z(t)) + C^2\varepsilon^\mu]\Big)$ for all $t \in [t_1, \tau]$.

Proof. For any $t \geq 0$, define $\Omega^+(t)$ by

$$\Omega^+(t) = \{\hat{z} \in \Omega_{\varepsilon^{1/3}} \ : \ u(g, \cdot, t) \leq V_2^{\varepsilon^\mu}(\hat{z}; \cdot, t) \text{ in } \ \bar{\Omega}\}.$$

Then, from Theorem 4.2, $y \in \Omega^+(t_1)$. In addition, by Lemma 4.3 and the comparison principle, $z(\tau) \in \Omega^+(\tau)$.

Observe that $V_2^{\varepsilon^\mu}(\hat{z}; x, t) \leq W^-(x, t) + C\varepsilon^\mu$ for all $x \in B(\hat{z}, 2\varepsilon^{1/3})$. Thus, we obtain the assertion of the theorem.

Remark 4.5. (1) The theorem asserts that the region $\{x \ : \ u(g;\cdot,t) \sim W^-\}$ expands with a normal velocity εc in the set where $c > 0$, and the region $\{u \sim W^+\}$ expands with a normal velocity $\varepsilon|c|$ in the set where $c < 0$.

(2) By replacing the function d defined in (4.7) by

$$d(z, x, r(t)) = \varepsilon^{1/3}\zeta(\frac{|x-z| - r(t)}{\varepsilon^{1/3}} - 3)$$

where $r(t)$ is any function satisfying $r_t < -\max_{B(z,r(t))} \varepsilon|c| - C^2\varepsilon^\mu$ and $B(z, r(0)) \subset \Omega_{\varepsilon^{1/3}}$, one can show that the set $\{u \sim W^-\}$ shrinks with a normal velocity no bigger than $\varepsilon[|c| + \varepsilon^\mu]$ in the set where $c < 0$. Similarly, one can show that the region $\{u \sim W^+\}$ shrinks with a normal velocity no bigger than $\varepsilon[c + \varepsilon^\mu]$ in the set where $c > 0$. Hence, we conclude that the interface between the set $\{u \sim W^+\}$ and the set $\{u \sim W^-\}$ moves with a normal velocity εc. For a more detailed discussion and rigorous mathematical statement, we refer interested readers to [8].

We now have enough to give the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let $y^+ \in \Omega^+_{\varepsilon^\mu} \cap \Omega_{\varepsilon^{1/3}}$ be arbitrary. Set $g(x) = V_2^{\varepsilon^\mu}(z(0); x, 0)$ where $z(t) = y^+ + \varepsilon^{1+2\mu} t\vec{e}$, $t \in [0, T]$, and $|\vec{e}| \le 1$. Taking all possible $\vec{e}$, we then obtain by Lemma 4.3, $u(g;\cdot,T) \le \min_{z \in B(y^+, \varepsilon^{1+2\mu}T)} V_2^{\varepsilon^\mu}(z; x, T) \le g$. It then follows from the comparison principle that $u(g;\cdot,T+t) \le u(g;\cdot,t)$ for all $t \ge 0$. Consequently, the family of smooth functions $\{u(g;\cdot,kT+\cdot)\}_{k=1}^\infty$ is pointwise monotonic on $\bar{\Omega} \times [0, 2T]$. Hence, there exists a periodic function $u^\varepsilon(x,t)$ such that as $k \to \infty$, $u(g;\cdot,kT+t) \to u^\varepsilon(\cdot,t)$ for all $t \in [0, 2T]$. Clearly, u^ε is a solution of (1.7). Notice that for every $x \in \Omega$ satisfying $|x - y^+| > 5\varepsilon^{1/3}$, $g(x) = V_2^\delta(y^+; x, 0) \ge \alpha^+ - C\delta \ge \alpha^0 + \varepsilon^\mu$, Theorem 4.2 then implies that u^ε satisfies (1.9). This establishes the theorem.

Theorem 1.4 now follows from Theorem 4.4.

Appendix A. Stability via spectral theory

In Section 2, we established the existence of a traveling wave solution $(c, U(\xi, t))$ for (1.1a). Since (1.1a) is translation invariant in z, this traveling wave solution provides a one dimensional manifold of special solutions to (1.1a):

$$\tilde{\mathcal{M}} = \{u(z,t) = U(z - ct - z_0, t) \ : \ z_0 \in \mathbb{R}\}.$$

In this appendix, we study the local, nonlinear stability of $\mathcal{M}$ in the class of solutions of (1.1). This appendix can be considered as an alternative to the analysis presented in Sections 3.4 and 3.5.

Traveling wave coordinates. We change to the traveling coordinates (ξ, t) where $\xi = z - ct$. Then equation (1.1a) takes the form

$$v_t - cv_\xi - v_{\xi\xi} - f(v,t) = 0, \quad \xi \in \mathbb{R},\ t > 0. \tag{A.1}$$

The original problem now can be recast as the stability of the manifold of stationary states:

$$\mathcal{M} := \{v(\xi,t) = U(\xi - \xi_0, 0) \ : \ \xi_0 \in \mathbb{R}\}$$

in the class of solutions of (A.1).

The period map. Let $\mathcal{X} \subset L^\infty(\mathbb{R})$ be a Banach space. For definiteness, we take $\mathcal{X} = C(\mathbb{R})$ with the $L^\infty(\mathbb{R})$ norm. Given $v_0(\cdot) \in \mathcal{X}$, we define the *period map* Π by

$$\Pi(v_0) = v(\cdot, T),$$

where $v(\xi,t)$ is the unique solution of (A.1) with initial condition $v(\cdot,0) = v_0(\cdot)$. Notice that any element in $\mathcal{M}$ is a fixed point of Π. Hence, $\mathcal{M}$ is an invariant manifold of Π. We are interested in the linearisation $d\Pi$ of Π about points in $\mathcal{M}$. Without loss of generality, we need only consider the point $U_0 = U(\cdot,0)$. One can easily show that $d\Pi(U_0)$ is given by the following recipe:

$$d\Pi(U_0)v = H(\cdot,T)$$

where H is the solution of

$$\text{(A.2)} \qquad \begin{cases} H_t - cH_\xi - H_{\xi\xi} - f_u(U_0,t)H = 0, & \xi \in \mathbb{R}, t > 0, \\ H(\cdot,0) = v(\cdot). \end{cases}$$

Notice that $v = U_\xi(\cdot,0)$ is an eigenfunction of $d\Pi(U_0)$ with eigenvalue 1. This is a simple geometric fact since $U_\xi(\cdot,0)$ is the tangent to the one dimensional invariant manifold $\mathcal{M}$ at $U(\cdot,0)$. Hence, if we can show that 1 is a simple eigenvalue of $d\Pi(U_0)$, and the rest of the spectrum of $d\Pi(U_0)$ is contained in a disk of radius $\tilde{\rho}(d\Pi(U_0))$ strictly less than 1, then it is well–known, and in a variety of contexts, that the manifold $\mathcal{M}$ is locally exponentially stable with asymptotic phase; namely, with the help of Lemma 3.3, we obtain the conclusion of Theorem 1.2. In addition, the exponent μ in (1.6) can be taken arbitrarily close to $-\ln(\tilde{\rho}(d\Pi(U_0)))$. We refer the reader to Henry [18, §9.2] and Hale & Massatt [20].

Hence, we need only analyze the spectrum of the operator $d\Pi(U_0)$.

Spectral analysis. From the above discussion, we can see that the following two lemmas are sufficient to establish the (local) exponential stability of $\mathcal{M}$, the reduction from global stability to local stability coming from Lemma 3.3.

Lemma A.1. *Let*

$$\nu^\pm = -\frac{1}{T}\int_0^T f_u(W^\pm(t),t)dt, \qquad \nu_0 = \min\{\nu^+,\nu^-\}.$$

Then the essential spectrum of $d\Pi(U_0)$ *is contained in the disk* $\{\lambda \in C \ : \ |\lambda| \le e^{-\nu_0 T}\}$. *Thus, if* λ *is in the spectrum of* $d\Pi(U_0)$ *and* $|\lambda| > e^{-\nu_0 T}$, *then* λ *is an eigenvalue, and for any* $r > e^{-\nu_0 T}$, *there are only a finite number of eigenvalues of* $d\Pi(U_0)$ *in* $\{\lambda \in C \mid |\lambda| \ge r\}$.

Lemma A.2. *Assume that* λ *is an eigenvalue of* $d\Pi(U_0)$ *with eigenfunction* v. *If* $v \notin span\{U_\xi(\cdot,0)\}$, *then* $|\lambda| < 1$.

Proof of Lemma A.1. Let $\zeta(\xi)$ be a $C^\infty(\mathbb{R})$ function satisfying $\zeta(\xi) = 0$ for $\xi \le -1$, $\zeta(\xi) = 1$ for $\xi \ge 1$, and $\zeta' \ge 0$ in $\mathbb{R}$. Consider an operator $\mathcal{K}$ defined, for every bounded v, by $\mathcal{K}v = \hat{H}(\cdot,T)$ where $\hat{H}(\xi,t)$ is the solution to

$$\begin{cases} \hat{H}_t - c\hat{H}_\xi - \hat{H}_{\xi\xi} + [\nu^+\zeta + \nu^-(1-\zeta)]\hat{H} = 0, & \xi \in \mathbb{R},\ t > 0, \\ \hat{H}(\cdot,0) = v(\cdot). \end{cases}$$

Since $\nu^+\zeta + (1-\zeta)\nu^- \ge \nu_0$, the maximum principle shows that $\|\hat{H}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le e^{-\nu_0 t}\|v\|_{L^\infty(\mathbb{R})}$ for all $t > 0$. In particular, $\|\mathcal{K}v\|_{L^\infty(\mathbb{R})} \le e^{-\nu_0 T}\|v\|_{L^\infty(\mathbb{R})}$. Therefore, the spectral radius of $\mathcal{K}$ is at most $e^{-\nu_0 T}$.

To connect the essential spectrum of $d\Pi(U_0)$ with that of $\mathcal{K}$, we make the following transformation for the solution of (A.2):

$$H(\xi,t) = \bar{H}(\xi,t)P(\xi,t)$$

where

$$P(\xi,t)=p^+(t)\zeta(\xi)+p^-(t)(1-\zeta(\xi)),\quad p^\pm(t)=\exp\Big(\int_0^t f_u(W^\pm(\tau),\tau)d\tau+\nu^\pm t\Big).$$

Note that $p^+(t), p^-(t), P(\cdot,t)$ are positive and periodic in t, and $p^\pm(T)=1$, $P(\cdot,T)\equiv 1$. It then follows that $d\Pi(U_0)v=H(\cdot,T)=\bar H(\cdot,T)$. Direct calculation shows that $\bar H$ satisfies

$$\bar H_t-\bar c(\xi,t)\bar H_\xi-\bar H_{\xi\xi}-\bar q\bar H=0$$

where

$$\begin{aligned}\bar c(\xi,t)&=c+\tilde c(\xi,t),\qquad \tilde c(\xi,t)=2(p^+-p^-)\zeta'/P,\\ \bar q&=f_u(U_0,t)+[c(p^+-p^-)\zeta'+(p^+-p^-)\zeta''-p_t^+\zeta-p_t^-(1-\zeta)]/P\\ &=-\nu^+\zeta-\nu^-(1-\zeta)+\tilde q(\xi,t),\\ \tilde q(\xi,t)&=\Big\{[f_u(U_0,t)-f_u(W^+(t),t)]p^+\zeta+[f_u(U_0,t)-f_u(W^-(t),t)]p^-(1-\zeta)\\ &\quad+\zeta(1-\zeta)(\nu^+-\nu^-)(p^--p^+)+(p^+-p^-)(c\zeta'+\zeta'')\Big\}/P.\end{aligned}$$

Notice that $\tilde c\equiv 0$ if $|\xi|\ge 1$ and $\tilde q$ approaches zero exponentially fast as $|\xi|\to\infty$. One can show that $\mathcal K-d\Pi(U_0)$ is compact from $\mathcal X$ into $\mathcal X$. The proof is straightforward provided that one is familiar with the fact that parabolic equations are smoothing. We omit the details.

Now by Weyl's well–known result, the essential spectrum of $d\Pi(y_0)$ is the same as that of $\mathcal K$. Hence, the radius of the essential spectrum of $d\Pi(U_0)$ is not bigger than $e^{-\nu_0 T}$.

Proof of Lemma A.2. Assume that λ is an eigenvalue with eigenfunction $v\in\mathcal X$ and $v\notin\operatorname{span}\{U_\xi(\cdot,0)\}$. Denote by H the solution of (A.2) with initial value v. Let

$$h(\xi,t)=e^{\mu t}H(\xi,t)\qquad\text{where}\quad \mu=-\frac{1}{T}\mathrm{Log}\lambda.$$

Then an easy calculation shows that (μ,h) satisfies

$$\text{(A.3)}\qquad \begin{cases} h_t-ch_\xi-h_{\xi\xi}-f_u(U_0,t)h=\mu h, & \xi\in\mathbb R,\ t>0,\\ h(\cdot,0)=h(\cdot,T).\end{cases}$$

Hence, (A.3) can be viewed as the spectral problem associated with the operator

$$\text{(A.4)}\qquad L:=\partial_t-c\partial_\xi-\partial_{\xi\xi}-f_u(U_0,t)$$

in an appropriate space of periodic functions.

The eigenvalue λ of the linearized period map $d\Pi(U_0)$ is called a *characteristic multiplier*, while the associated μ is called a *characteristic exponent*. Since Log is multi–valued, it is easy to see that if (μ,h) is a characteristic exponent/eigenfunction pair, so is $(\mu+\frac{2\pi \mathrm{i}n}{T}, he^{\frac{2\pi \mathrm{i}nt}{T}})$, where $\mathrm{i}=\sqrt{-1}$. Notice that all these exponents produce the same multiplier.

Clearly, to show that $|\lambda|<1$, we need only to show that $\mathrm{Re}(\mu)$, the real part of μ, is positive. Our proof is by contradiction. Assume that $\mu_1:=\mathrm{Re}(\mu)\le 0$. Consider the polar representation of h: $h=re^{\mathrm{i}\theta}$ where both r and θ are real and $r\ge 0$. In the set where r does not vanish, θ is well–defined and is smooth. Substituting this polar representation into (A.3) and taking the real part, we obtain

$$Lr=(\mu_1-\theta_\xi^2)r\le 0$$

on the set where $r>0$.

First we claim that $r \le MU_\xi$ for some M large enough. For this purpose, consider the periodic functions $Q^{\pm}(\xi,t) := e^{\gamma^{\pm}\xi}\exp(\int_0^t f_u(W^{\pm}(\tau),\tau)d\tau + \nu^{\pm}t)$ where $\gamma^{\pm} = (-c \pm \sqrt{c^2+2\nu^{\pm}})/2$. An easy calculation shows that, for some ξ_0 large enough,

$$\begin{aligned} LQ^{\pm} &= Q^{\pm}[f_u(W^{\pm}(t),t) - f_u(U_0,t) + \nu^{\pm} - c\gamma^{\pm} - (\gamma^{\pm})^2] \\ &= Q^{\pm}[f_u(W^{\pm}(t),t) - f_u(U_0,t) + \nu^{\pm}/2] > 0 \qquad \text{for all } |\xi| \ge \xi_0 \text{ and } \ t \in \mathbb{R}. \end{aligned}$$

Now let M_1 be a large constant such that $M_1U_\xi(\pm\xi_0,t) > r(\pm\xi_0,t)$ for all $t \in [0,T]$. We claim that $r < M_1U_\xi$ in $[\xi_0,\infty)\times[0,T]$. In fact, if this is not true, then, since $\nu^+ > 0$ we have $Q^+ \to \infty$ as $\xi \to \infty$, and so there exists $\delta \ge 0$ and $(\xi_1,t_1) \in (\xi_0,\infty)\times[0,T)$ such that $r \le M_1U_\xi + \delta Q^+$ in $[\xi_0,\infty)\times[0,T]$ and the equal sign holds at (ξ_1,t_1). Set $w = M_1U_\xi + \delta Q^+ - r$. Then $Lw > 0$ in $[\xi_0,\infty)\times[0,2T]\cap\{r \ne 0\}$. In addition, $w \ge 0$ in $[\xi_0,\infty)\times[0,2T]$ and $w > 0$ on $\{r = 0\}$. Hence, applying locally the Harnack inequality to each of the components where r does not vanish, we have that $w > 0$ in $[\xi_0,\infty)\times(t_1,t_1+T]$. This contradicts, by the periodicity of w, the assumption that $0 = w(x_1,t_1) = w(x_1,t_1+T)$. Hence, $r < M_1U_\xi$ in $[\xi_0,\infty)\times[0,T]$. Similarly, this inequality holds also on $(-\infty,-\xi_0]\times[0,T]$. Hence, there exists a positive M such that $r \le MU_\xi$ in $\mathbb{R}\times[0,T]$.

Now let M_0 be the minimum real number such that $r \le M_0U_\xi$ in $\mathbb{R}\times[0,T]$. Consider the case that $r \not\equiv M_0U_\xi$. Then applying locally Harnack's inequality in the set where r does not vanish, we obtain $r < M_0U_\xi$ in $\mathbb{R}\times[0,T]$. Consequently, there exists $\epsilon \in (0,M_0)$ such that $r < (M_0-\epsilon)U_\xi$ in $[-\xi_0,\xi_0]\times[0,T]$. Then as before, utilizing the function $Q^{\pm}$, we can conclude that $r < (M_0-\epsilon)U_\xi$ in $\mathbb{R}\times[0,T]$, which contradicts the definition of M_0. Hence, $r \equiv M_0U_\xi > 0$ in $\mathbb{R}\times[0,T]$. Consequently, $(\mu_1 - \theta_\xi^2)r \equiv 0$. Thus, $\mu_1 = 0$ and $\theta_\xi \equiv 0$. Using the θ equation, we then conclude that $\theta_t \equiv 0$ and hence θ is a constant function. That is, $h = re^{\mathrm{i}\theta}$ is a multiple of U_ξ, which contradicts the assumption that $h \notin \mathrm{span}\{U_\xi\}$. This contradiction shows that $\mathrm{Re}(\mu) > 0$, i.e., $|\lambda| < 1$. This completes the proof of Lemma A.2.

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Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1300

Department of Mathematics, University of Athens, Panestimiopolis, Greece 15784
E-mail address: alikakos@utk.edu, nalikako@atlas.uoa.gr

Department of Mathematics, Brigham Young University, Provo, Utah 84602
E-mail address: peter@math.byu.edu

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
E-mail address: xinfu+@pitt.edu