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# Periodic unfolding for lattice structures

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**Abstract**

This paper deals with the periodic unfolding for sequences defined on one dimensional lattices in $\mathbb{R}^N$. In order to transfer the known results of the periodic unfolding in $\mathbb{R}^N$ to lattices, the investigation of functions defined as interpolation on lattice nodes play the main role. The asymptotic behavior for sequences defined on periodic lattices with information until the first and until the second order derivatives are shown. In the end, a direct application of the results is given by homogenizing a 4th order Dirichlet problem defined on a periodic lattice.

**Keywords** Periodic unfolding method · Homogenization · Lattice graphs · Anisotropic sobolev spaces · Thin structures

**Mathematics Subject Classification** 35B27 · 35J86 · 47H05

## 1 Introduction

The work done in this paper is in the frame of homogenization of periodic structures through the method of unfolding, equivalent to the two-scale convergence, that has been broadly explained in [5]. The method itself has, among many others, found application in the homogenization for thin periodic structures like periodically perforated shells (see [8]) and textiles made of long curved beams (see [12, 13]).

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The classical theory of unfolding can be described as follows: given a small parameter $\varepsilon$ and a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, we consider the periodic paving of $\Omega$ made with cells of size $\varepsilon$. In [5, Section 1.4] it is extensively investigated the asymptotic behavior of sequences $\{\phi_\varepsilon\}_\varepsilon \in W^{1,p}(\Omega)$ such that either

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla\phi_\varepsilon\|_{L^p(\Omega)^N} \leq C \quad \text{or} \quad \|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla\phi_\varepsilon\|_{L^p(\Omega)^N} \leq C$$

and sequences in $W^{2,p}(\Omega)$ such that

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla\phi_\varepsilon\|_{L^p(\Omega)} + \|D^2\phi_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Additionally, the entirety of [6] has been devoted to the investigation of the so called "anisotropically bounded sequences", that are, sequences in $W^{1,p}(\Omega)$ with contrast on the gradient estimates with respect to the observed direction:

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \sum_{i=1}^{N_1} \|\partial_i\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \sum_{i=N_1+1}^{N} \|\partial_i\phi_\varepsilon\|_{L^p(\Omega)} \leq C.$$

The aim of the present is to get an equivalent formulation of the above results but for one-dimensional periodic lattice structures $\mathcal{S}_\varepsilon \subset \Omega$, spotting the obstacles we encountered and the tools we came up with to overcome them. Due to complexity reasons, we will consider as lattice structures the one-dimensional grids defined on each $\varepsilon$ cells and periodically repeated for each cell of $\Omega$. About previous works concerning the homogenization in the frame of lattice structures one can look, for an instance, into [1–3, 14–16].

The paper is organized as follows. In Sect. 2 we give the standard notation and tools of [5, 6] for the classical homogenization via unfolding method in periodic domains $\Omega \subset \mathbb{R}^N$. In Sect. 3, we recall the main results concerning the periodic unfolding for sequences bounded uniformly and anisotropically on $W^{1,p}(\Omega)$ but defined as $Q_1$ (or $N$-linear) interpolations on the vertexes of the $\varepsilon$ cells paving $\Omega$. The unfolding results for this class of functions will be crucial in the next sections, due to their interpolation properties. In Sect. 4, we first give a rigorous definition of one-dimensional lattice structure $\mathcal{S}_\varepsilon \subset \Omega$ and we build the unfolding operator for lattices and give its main properties. The main idea to transfer the periodic unfolding for lattices $\mathcal{S}_\varepsilon$ using the known results in $\mathbb{R}^N$ is explained in detail in Sect. 5: given a sequence $\{\phi_\varepsilon\}_\varepsilon$ bounded on $\mathcal{S}_\varepsilon$, we first decompose it into a sequence $\{\phi_{a,\varepsilon}\}_\varepsilon$, which is defined as a linear interpolation between lattice nodes, and remainder term $\{\phi_{0,\varepsilon}\}_\varepsilon$. Concerning $\{\phi_{a,\varepsilon}\}_\varepsilon$, we can uniquely extend it by $Q_1$ interpolation to the whole space, apply the unfolding results on $\mathbb{R}^N$ and restrict it back to the lattice itself, while on $\{\phi_{0,\varepsilon}\}_\varepsilon$ we can directly apply the one-dimensional unfolding since it is defined on straight segments of $\mathcal{S}_\varepsilon$. In this sense, Sect. 5 shows the asymptotic behavior of sequences in $W^{1,p}(\mathcal{S}_\varepsilon)$ bounded uniformly

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}},$$

as well as anisotropically:

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \sum_{i=1}^{N_1} \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} + \varepsilon \sum_{i=N_1+1}^{N} \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} \leq C\varepsilon^{\frac{1-N}{p}},$$

giving the sufficient assumptions on the sequences to ensure weak convergence in the space, as well as the rescaling factors according to the space dimension $N$ and the $L^p$ norm. Section 6 is devoted to find the asymptotic behavior of sequences uniformly bounded in $W^{2,p}(\mathcal{S}_\varepsilon)$:

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}^2\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}.$$

Here some more work is involved, since the derivation only makes sense in the lattice directions and thus gives no information on the mixed derivatives. Two approaches are considered: one follows closely the steps taken in Sect. 5 but with a decomposition on cubic interpolation on lattice nodes and remainder term, and extension by $Q_3$ interpolation (or $N$-cubic) to the whole space; the other one by using twice (on the sequence and on its gradient) the proved result for functions bounded in $W^{1,p}(\mathcal{S}_\varepsilon)$. As one can expect, such methods differ on the assumption strength and lead to different regularities of the unfolded limit fields. At last, in Sect. 7 we consider the fourth order Dirichlet problem defined on a lattice structure

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon) \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon \partial_{\mathbf{s}}^2 u_\varepsilon \, \partial_{\mathbf{s}}^2 \phi \, d\mathbf{s} = \int_{\mathcal{S}_\varepsilon} g_\varepsilon \, \partial_{\mathbf{s}}\phi \, d\mathbf{s} + \int_{\mathcal{S}_\varepsilon} f_\varepsilon \, \phi \, d\mathbf{s}, \quad \forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon). \end{cases}$$

Using the results in the previous sections, existence and uniqueness of the limit problem are shown and through the homogenization via unfolding, the cell problems and the macroscopic limit problem are found.

The present provides the main tools concerning the unfolding for lattice structures and gives a rigorous base for up-coming papers dealing with thin structures made from lattices. Among them, we would like to cite the homogenization via unfolding for stable lattice structures made of beams (see [10, 11]) and the upcoming unstable case [9], where it is additionally taken into consideration the problem of an anisotropically bounded sequence. More generally, such tools can be applied to many other problems related to partial differential equations on domains involving periodic grids, lattices, thin frames and glued fiber structures.

## 2 Preliminaries and notation

Let $\mathbb{R}^N$ be the euclidean space with usual basis $(\mathbf{e}_1, \ldots, \mathbf{e}_N)$ and $Y = (0, 1)^N$ the open unit parallelotope associated with this basis. For a.e. $z \in \mathbb{R}^N$, we set the unique decomposition $z = [z]_Y + \{z\}_Y$ such that

$$[z]_Y \doteq \sum_{i=1}^{N} k_i \mathbf{e}_i, \quad k = (k_1, \dots, k_N) \in \mathbb{Z}^N \quad \text{and} \quad \{z\}_Y \doteq z - [z]_Y \in Y.$$

Let $\{\varepsilon\}$ be a sequence of strictly positive parameters going to 0. We scale our paving by $\varepsilon$ writing

$$x = \varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\Big\{\frac{x}{\varepsilon}\Big\}_Y \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.1}$$

Let now $\Omega$ be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. We consider

$$\Xi_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^N \,\big|\, \varepsilon(\xi + Y) \subset \Omega\big\}$$

and set

$$\widehat{\Omega}_\varepsilon \doteq \operatorname{int}\Big\{\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\Big\}, \qquad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon. \tag{2.2}$$

We recall the definitions of classical unfolding operator and mean value operator.

**Definition 2.1** (see [5, Definition 1.2]) For every measurable function $\phi$ on $\Omega$, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as follows:

$$\mathcal{T}_\varepsilon(\phi) \doteq \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Note that such an operator acts on functions defined in $\Omega$ by operating on their restriction to $\widehat{\Omega}_\varepsilon$.

**Definition 2.2** (see [5, Definition 1.10]) For every measurable function $\widehat{\phi}$ on $L^1(\Omega \times Y)$, the mean value operator $\mathcal{M}_Y$ is defined as follows:

$$\mathcal{M}_Y(\widehat{\phi})(x) \doteq \frac{1}{|Y|}\int_Y \widehat{\phi}(x, y)dy, \qquad \text{for a.e. } x \in \Omega.$$

Let $p \in [1, +\infty]$. From [5, Propositions 1.8 and 1.11], we recall the properties of these operators:

$$\|\mathcal{T}_\varepsilon(\phi)\|_{L^p(\Omega\times Y)} \le |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega)} \quad \text{for every } \phi \in L^p(\Omega),$$
$$\|\mathcal{M}_Y(\widehat{\phi})\|_{L^p(\Omega)} \le |Y|^{-\frac{1}{p}} \|\widehat{\phi}\|_{L^p(\Omega\times Y)} \quad \text{for every } \widehat{\phi} \in L^p(\Omega \times Y).$$

Since we will deal with Sobolev spaces, we give hereafter some definitions:

$$W^{1,p}_{per}(Y) \doteq \big\{\phi \in W^{1,p}(Y) \,\big|\, \phi \text{ is periodic with respect to } y_i, i \in \{1, \dots, N\}\big\},$$
$$W^{1,p}_{per,0}(Y) \doteq \big\{\phi \in W^{1,p}_{per}(Y) \,\big|\, \mathcal{M}_Y(\phi) = 0\big\},$$
$$L^p(\Omega; W^{1,p}(Y)) \doteq \big\{\phi \in L^p(\Omega \times Y) \,\big|\, \nabla_y\phi \in L^p(\Omega \times Y)^N\big\}.$$

Now, let $(N_1, N_2)$ be in $\mathbb{N} \times \mathbb{N}^*$ such that $N = N_1 + N_2$. We split the space by setting

$$\mathbb{R}^{N_1} = \Big\{ x' \in \mathbb{R}^N \;\Big|\; x' = \sum_{i=1}^{N_1} x_i \mathbf{e}_i, \quad x_i \in \mathbb{R} \Big\},$$

$$\mathbb{R}^{N_2} = \Big\{ x'' \in \mathbb{R}^N \;\Big|\; x'' = \sum_{i=N_1+1}^{N} x_i \mathbf{e}_i, \quad x_i \in \mathbb{R} \Big\},$$

$$Y' = \Big\{ y' \in \mathbb{R}^N \;\Big|\; y' = \sum_{i=1}^{N_1} y_i \mathbf{e}_i, \quad y_i \in (0,1) \Big\},$$

$$Y'' = \Big\{ y'' \in \mathbb{R}^N \;\Big|\; y'' = \sum_{i=N_1+1}^{N} y_i \mathbf{e}_i, \quad y_i \in (0,1) \Big\}$$

and

$$\mathbb{Z}^{N_1} = \mathbb{Z}\mathbf{e}_1 \oplus \ldots \oplus \mathbb{Z}\mathbf{e}_{N_1}, \qquad \mathbb{Z}^{N_2} = \mathbb{Z}\mathbf{e}_{N_1+1} \oplus \ldots \oplus \mathbb{Z}\mathbf{e}_N.$$

One has

$$\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \qquad Y = Y' \oplus Y'', \qquad \mathbb{Z}^N = \mathbb{Z}^{N_1} \oplus \mathbb{Z}^{N_2}.$$

For every $x \in \mathbb{R}^N$ and $y \in Y$, we write

$$x = x' + x'' \in \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \qquad y = y' + y'' \in Y' \oplus Y''.$$

*From now on, however, we find easier to refer to such partition with the vectorial notation*

$$x = (x', x'') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \qquad y = (y', y'') \in Y' \times Y''.$$

Similarly to (2.1), we apply the paving to a.e. $x' \in \mathbb{R}^{N_1}$ and $x'' \in \mathbb{R}^{N_2}$ setting

$$x' = \varepsilon\Big[\frac{x'}{\varepsilon}\Big]_{Y'} + \varepsilon\Big\{\frac{x'}{\varepsilon}\Big\}_{Y'}, \qquad \text{with } \Big[\frac{x'}{\varepsilon}\Big]_{Y'} \in \mathbb{Z}^{N_1}, \quad \Big\{\frac{x'}{\varepsilon}\Big\}_{Y'} \in Y',$$
$$x'' = \varepsilon\Big[\frac{x''}{\varepsilon}\Big]_{Y''} + \varepsilon\Big\{\frac{x''}{\varepsilon}\Big\}_{Y''}, \qquad \text{with } \Big[\frac{x''}{\varepsilon}\Big]_{Y''} \in \mathbb{Z}^{N_2}, \quad \Big\{\frac{x''}{\varepsilon}\Big\}_{Y''} \in Y''.$$

**Definition 2.3** For every $\widehat{\phi} \in L^1(\Omega \times Y)$, the partial mean value operators are defined as follows:

$$\mathcal{M}_{Y'}(\widehat{\phi})(x, y'') \doteq \frac{1}{|Y'|}\int_{Y'} \widehat{\phi}(x, y', y'')dy', \qquad \text{for a.e. } (x, y'') \in \Omega \times Y'',$$
$$\mathcal{M}_{Y''}(\widehat{\phi})(x, y') \doteq \frac{1}{|Y''|}\int_{Y''} \widehat{\phi}(x, y', y'')dy'', \qquad \text{for a.e. } (x, y') \in \Omega \times Y'.$$

Denote

$$\begin{aligned}
L^p(\Omega,\nabla_{x'}) &\doteq \{\phi\in L^p(\Omega)\mid \nabla_{x'}\phi\in L^p(\Omega)^{N_1}\},\\
L^p(\Omega,\nabla_{x''}) &\doteq \{\phi\in L^p(\Omega)\mid \nabla_{x''}\phi\in L^p(\Omega)^{N_2}\},\\
L^p(\Omega,\nabla_{x'};W^{1,p}(Y'')) &\doteq \{\widetilde{\phi}\in L^p(\Omega\times Y'')\mid \nabla_{x'}\widetilde{\phi}\in L^p(\Omega\times Y'')^{N_1},\\
&\qquad \nabla_{y''}\widetilde{\phi}\in L^p(\Omega\times Y'')^{N_2}\},\\
L^p(\Omega,\nabla_{x''};W^{1,p}(Y')) &\doteq \{\widetilde{\phi}\in L^p(\Omega\times Y')\mid \nabla_{x''}\widetilde{\phi}\in L^p(\Omega\times Y')^{N_2},\\
&\qquad \nabla_{y'}\widetilde{\phi}\in L^p(\Omega\times Y')^{N_1}\},\\
L^p(\Omega\times Y'';W^{1,p}(Y')) &\doteq \{\widehat{\phi}\in L^p(\Omega\times Y)\mid \nabla_{y'}\widehat{\phi}\in L^p(\Omega\times Y)^{N_1}\},\\
L^p(\Omega\times Y';W^{1,p}(Y'')) &\doteq \{\widehat{\phi}\in L^p(\Omega\times Y)\mid \nabla_{y''}\widehat{\phi}\in L^p(\Omega\times Y)^{N_2}\}.
\end{aligned}$$

We endow these spaces with the respective norms:

$$\begin{aligned}
\|\cdot\|_{L^p(\Omega,\nabla_{x'})} &\doteq \|\cdot\|_{L^p(\Omega)}+\|\nabla_{x'}(\cdot)\|_{L^p(\Omega)^{N_1}},\\
\|\cdot\|_{L^p(\Omega,\nabla_{x''})} &\doteq \|\cdot\|_{L^p(\Omega)}+\|\nabla_{x''}(\cdot)\|_{L^p(\Omega)^{N_2}},\\
\|\cdot\|_{L^p(\Omega,\nabla_{x'};W^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega\times Y'')}+\|\nabla_{x'}(\cdot)\|_{L^p(\Omega\times Y'')^{N_1}}\\
&\quad +\|\nabla_{y''}(\cdot)\|_{L^p(\Omega\times Y'')^{N_2}},\\
\|\cdot\|_{L^p(\Omega,\nabla_{x''};W^{1,p}(Y'))} &\doteq \|\cdot\|_{L^p(\Omega\times Y')}+\|\nabla_{x''}(\cdot)\|_{L^p(\Omega\times Y')^{N_2}}\\
&\quad +\|\nabla_{y'}(\cdot)\|_{L^p(\Omega\times Y')^{N_1}},\\
\|\cdot\|_{L^p(\Omega\times Y'';W^{1,p}(Y'))} &\doteq \|\cdot\|_{L^p(\Omega\times Y)}+\|\nabla_{y'}(\cdot)\|_{L^p(\Omega\times Y)^{N_1}},\\
\|\cdot\|_{L^p(\Omega\times Y';W^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega\times Y)}+\|\nabla_{y''}(\cdot)\|_{L^p(\Omega\times Y)^{N_2}}.
\end{aligned}$$

## 3 Periodic unfolding in $\mathbb{R}^N$ for sequences defined as $Q_1$ interpolates

In this section we will first the class of function defined as $Q_1$ interpolates. The notion of $Q_1$ (also called $N$-linear) interpolation through a discrete approximation, as mentioned in [5, Section 1.6], is customary in the Finite Element Method and finds its origins in [4].

The periodic unfolding for this class of functions has two main advantages. The first is that less hypothesis are required for the sequences to ensure weak convergence. The second is that the convergences can be restricted to subspaces with lower dimension and it will be fundamental in the next sections, where lattice structures are taken into account.

Define the spaces

$$\begin{aligned}
Q^1(Y) \doteq &\{\phi\in W^{1,\infty}(Y)\,|\,\phi \text{ is the } Q_1 \text{ interpolation of its values}\\
&\text{on the vertexes of } \overline{Y}\},\\
Q^1_{per}(Y) \doteq &\{\phi\in Q^1(Y)\mid \phi \text{ is periodic with respect to } y_i,\, i\in\{1,\dots,N\}\},
\end{aligned}$$

$$Q^1_{per,0}(Y) \doteq \big\{\phi \in Q^1_{per}(Y) \;\big|\; \mathcal{M}_Y(\phi) = 0\big\}.$$

Denote

$$\widetilde{\Omega}_\varepsilon \doteq \mathrm{int}\Big\{\bigcup_{\xi\in\widetilde{\Xi}_\varepsilon} \varepsilon(\xi+\overline{Y})\Big\}, \qquad \widetilde{\Xi}_\varepsilon \doteq \Big\{\xi \in \mathbb{Z}^N \;\Big|\; \varepsilon(\xi+Y)\cap\Omega \neq \emptyset\Big\}. \tag{3.1}$$

Note that the covering $\widetilde{\Omega}_\varepsilon$ is now a connected open set and from (2.2) we have

$$\widehat{\Omega}_\varepsilon \subset \Omega \subset \widetilde{\Omega}_\varepsilon.$$

Hence, we need to extend the definition of the classical unfolding operator (2.1) to functions defined in the following neighborhood of $\Omega$:

$$\big\{x \in \mathbb{R}^N \;\big|\; \mathrm{dist}(x,\Omega) < \varepsilon \mathrm{diam}(Y)\big\}.$$

**Definition 3.1** For every measurable function $\phi$ on $\widetilde{\Omega}_\varepsilon$, the unfolding operator $\mathcal{T}^{ext}_\varepsilon$ is defined as follows:

$$\mathcal{T}^{ext}_\varepsilon(\phi) \doteq \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) \quad \text{for a.e. } (x,y) \in \widetilde{\Omega}_\varepsilon \times Y.$$

Every measurable function defined in $\Omega$ can be extend to $\widetilde{\Omega}_\varepsilon$ by setting it to 0 in $\widetilde{\Omega}_\varepsilon \cap (\mathbb{R}^N \setminus \overline{\Omega})$. Now, assume $\{\Phi_\varepsilon\}_\varepsilon$ to be a sequence uniformly bounded in $L^p(\widetilde{\Omega}_\varepsilon)$, $p \in (1,+\infty)$. Then, the sequence $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^p(\widetilde{\Omega}_\varepsilon \times Y)$ and thus in $L^p(\Omega \times Y)$. Hence, there exists a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\Phi} \in L^p(\Omega \times Y)$ such that

$$\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)_{|\Omega\times Y} \rightharpoonup \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega \times Y).$$

*For simplicity, we will omit the restriction and always write the above convergence as*

$$\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) \rightharpoonup \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega \times Y).$$

In this sense, all the results obtained in [5, 6] are easily transposed to this operator. Define the space of $Q_1$ interpolated functions on $\widetilde{\Omega}_\varepsilon$ by

$$Q^1_\varepsilon(\widetilde{\Omega}_\varepsilon) \doteq \Big\{\Phi \in W^{1,\infty}(\widetilde{\Omega}_\varepsilon) \;\Big|\; \Phi_{|\varepsilon\xi+\varepsilon Y} \in Q^1(\varepsilon\xi+\varepsilon Y) \;\; \text{for every}\, \xi \in \widetilde{\Xi}_\varepsilon\Big\}.$$

Due to the $Q_1$ interpolation character, for every function $\Phi \in Q^1_\varepsilon(\widetilde{\Omega}_\varepsilon)$ we remind that there exist a constant depending only on $p$ such that

$$\|\nabla\Phi\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le \frac{C}{\varepsilon}\|\Phi\|_{L^p(\widetilde{\Omega}_\varepsilon)}. \tag{3.2}$$

We have the following.

**Lemma 3.2** *Let $\{\Phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1_\varepsilon(\widetilde{\Omega}_\varepsilon)$ that satisfies*

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C,$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, $\widetilde{\Phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{per}(Y''))$, $\widehat{\Phi} \in L^p(\Omega\times Y''; Q^1_{per}(Y'))\cap L^p(\Omega; Q^1(Y))$, satisfying $\mathcal{M}_{Y'}(\widehat{\Phi}) = 0$ a.e. in $\Omega\times Y''$, such that*

$$\begin{aligned}
\Phi_{\varepsilon|\Omega} &\rightharpoonup \Phi \quad \text{weakly in} \quad L^p(\Omega, \nabla_{x'}),\\
\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) &\rightharpoonup \widetilde{\Phi} \quad \text{weakly in} \quad L^p(\Omega; Q^1(Y)),\\
\mathcal{T}^{ext}_\varepsilon(\nabla_{x'}\Phi_\varepsilon) &\rightharpoonup \nabla_{x'}\widetilde{\Phi} + \nabla_{y'}\widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1},\\
\frac{1}{\varepsilon}\big(\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) - \mathcal{M}_{Y'}\circ\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\big) &\rightharpoonup \nabla_{x'}\widetilde{\Phi}\cdot y'^c + \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y),
\end{aligned}$$

*where $\Phi = \mathcal{M}_{Y''}(\widetilde{\Phi})$ and $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$.*

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** First, since the sequence $\{\Phi_\varepsilon\}_\varepsilon$ belongs to $Q^1_\varepsilon(\widetilde{\Omega}_\varepsilon)$ we get (see (3.2))

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla_{x''}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C.$$

The constant does not depend on $\varepsilon$. The statement follows by [6, Lemma 4.3] and the fact that $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \subset L^p(\widetilde{\Omega}_\varepsilon; Q^1(Y))$. □

As a direct consequence, we have the following corollary.

**Corollary 3.3** *Let $\{\Phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1_\varepsilon(\widetilde{\Omega}_\varepsilon)$ satisfying*

$$\|\Phi_\varepsilon\|_{W^{1,p}(\widetilde{\Omega}_\varepsilon)} \le C,$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and functions $\Phi \in W^{1,p}(\Omega)$, $\widehat{\Phi} \in L^p(\Omega; Q^1_{per,0}(Y))$ such that*

$$\begin{aligned}
\Phi_{\varepsilon|\Omega} &\rightharpoonup \Phi \quad \text{weakly in} \quad W^{1,p}(\Omega),\\
\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) &\rightharpoonup \Phi \quad \text{weakly in} \quad L^p(\Omega; Q^1(Y)),\\
\mathcal{T}^{ext}_\varepsilon(\nabla\Phi_\varepsilon) &\rightharpoonup \nabla\Phi + \nabla_y\widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y)^N,\\
\frac{1}{\varepsilon}\big(\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) - \mathcal{M}_Y\circ\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\big) &\rightharpoonup \nabla\Phi\cdot y'^c + \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y),
\end{aligned}$$

*where $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$.*

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** The proof directly follow from Lemma 3.2 in the particular case $N_1 = N$ and $N_2 = 0$. As an equivalent proof, the statement follows by [5, Corollary 1.37 and Theorem 1.41] and the fact that $\{\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)\}_\varepsilon \in L^p(\widetilde{\Omega}_\varepsilon; Q^1(Y))$. □

## 4 The periodic lattice structure

We start by giving a rigorous definition of 1-dimensional periodic lattice structure in $\mathbb{R}^N$.

Let $i \in \{1, \dots, N\}$ and let $K_1, \dots, K_N \in \mathbb{N}^*$. Set

$$\mathbf{K} \doteq \prod_{i=1}^{N}\{0, \dots, K_i\} \subset \mathbb{N}^N, \quad \mathbf{K}_i \doteq \big\{k \in \mathbf{K} \mid k_i = 0\big\},$$
$$\widehat{\mathbf{K}} \doteq \prod_{i=1}^{N}\{0, \dots, K_i - 1\}, \qquad \widehat{\mathbf{K}}_i \doteq \big\{k \in \widehat{\mathbf{K}} \mid k_i = 0\big\}.$$

We denote $\mathcal{K}$ the set of points in the closure of the unit cell $\overline{Y}$ by

$$\mathcal{K} \doteq \left\{A(k) \in \mathbb{R}^N \;\middle|\; A(k) = \sum_{i=1}^{N}\frac{k_i}{K_i}\mathbf{e}_i, \quad k \in \mathbf{K}\right\} \subset \overline{Y}.$$

In this sense, the whole unit cell $\overline{Y}$ has the following split

$$\overline{Y} = \sum_{k\in\widehat{\mathbf{K}}} A(k) + \overline{Y}_K,$$

where $Y_K$ is the cell defined by

$$Y_K \doteq \prod_{i=1}^{N}\big(0, l_i\big), \quad l_i = \frac{1}{K_i}.$$

We denote $\mathcal{S}^{(i)}$ the set of all segments whose direction is $\mathbf{e}_i$ by

$$\mathcal{S}_c^{(i)} \doteq \bigcup_{k\in\mathbf{K}_i}\big[A(k), A(k) + \mathbf{e}_i\big], \qquad \mathcal{S}^{(i)} \doteq \bigcup_{k\in\widehat{\mathbf{K}}_i}\big[A(k), A(k) + \mathbf{e}_i\big]$$

Hence, the lattice structure in the unit cell $\overline{Y}$ is defined by

$$\mathcal{S}_c \doteq \bigcup_{i=1}^{N}\mathcal{S}_c^{(i)} \subset \overline{Y}, \qquad \mathcal{S} \doteq \bigcup_{i=1}^{N}\mathcal{S}^{(i)} \subset \overline{Y}.$$

Given $\Omega \subset \mathbb{R}^N$, we cover it as in (3.1) by a union of $\varepsilon$ cells. The periodic lattice structure is therefore defined by

$$\mathcal{S}_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} \big(\varepsilon\xi + \varepsilon\mathcal{S}_c\big) \subset \widetilde{\Omega}_\varepsilon, \qquad \mathcal{K}_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} \big(\varepsilon\xi + \varepsilon\mathcal{K}\big),$$
$$\mathcal{S}^{(i)}_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} \big(\varepsilon\xi + \varepsilon\mathcal{S}^{(i)}_c\big).$$

Denote $\mathbf{S}$ the running point of $\mathcal{S}$ and $\mathbf{s}$ that of $\mathcal{S}_\varepsilon$. That gives ( $i \in \{1, \dots, N\}$)

$$\mathbf{S} = A(k) + t\mathbf{e}_i \quad \text{in}\, \mathcal{S}^{(i)}, t \in [0, 1], k \in \widehat{\mathbf{K}}_i,$$
$$\mathbf{s} = \varepsilon\xi + \varepsilon A(k) + \varepsilon t\mathbf{e}_i \quad \text{in}\, \mathcal{S}^{(i)}_\varepsilon, t \in [0, 1], k \in \widehat{\mathbf{K}}_i,\ \xi \in \widetilde{\Xi}_\varepsilon.$$

Let $\mathcal{C}(\mathcal{S})$ and $\mathcal{C}(\mathcal{S}_\varepsilon)$ be the spaces of continuous functions defined on $\mathcal{S}$ and $\mathcal{S}_\varepsilon$ respectively. For $p \in [1, +\infty]$, we denote the spaces of functions defined on the lattice by $(i \in \{1, \dots, N\})$

$$\begin{aligned}
W^{1,p}(\mathcal{S}^{(i)}) &\doteq \big\{\phi \in L^p(\mathcal{S}^{(i)}) \;\big|\; \partial_{\mathbf{S}}\phi \in L^p(\mathcal{S}^{(i)})\big\},\\
W^{1,p}(\mathcal{S}^{(i)}_\varepsilon) &\doteq \big\{\phi \in L^p(\mathcal{S}^{(i)}_\varepsilon) \;\big|\; \partial_{\mathbf{s}}\phi \in L^p(\mathcal{S}^{(i)}_\varepsilon)\big\},\\
W^{1,p}(\mathcal{S}) &\doteq \big\{\phi \in \mathcal{C}(\mathcal{S}) \;\big|\; \partial_{\mathbf{S}}\phi \in L^p(\mathcal{S})\big\},\\
W^{1,p}(\mathcal{S}_\varepsilon) &\doteq \big\{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \;\big|\; \partial_{\mathbf{s}}\phi \in L^p(\mathcal{S}_\varepsilon)\big\}
\end{aligned}$$

and for $k \in \mathbb{N} \setminus \{0, 1\}$

$$\begin{aligned}
W^{k,p}(\mathcal{S}^{(i)}) &\doteq \big\{\phi \in W^{k-1,p}(\mathcal{S}^{(i)}) \;\big|\; \partial_{\mathbf{S}}\phi \in W^{k-1,p}(\mathcal{S}^{(i)})\big\},\\
W^{k,p}(\mathcal{S}^{(i)}_\varepsilon) &\doteq \big\{\phi \in W^{k-1,p}(\mathcal{S}^{(i)}_\varepsilon) \;\big|\; \partial_{\mathbf{s}}\phi \in W^{k-1,p}(\mathcal{S}^{(i)}_\varepsilon)\big\},\\
W^{k,p}(\mathcal{S}) &\doteq \big\{\phi \in \mathcal{C}(\mathcal{S}) \;\big|\; \partial_{\mathbf{S}}\phi_{|\mathcal{S}^{(j)}} \in W^{k-1,p}(\mathcal{S}^{(j)}),\ j \in \{1, \dots, N\}\big\},\\
W^{k,p}(\mathcal{S}_\varepsilon) &\doteq \big\{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \;\big|\; \partial_{\mathbf{s}}\phi_{|\mathcal{S}^{(j)}_\varepsilon} \in W^{k-1,p}(\mathcal{S}^{(j)}_\varepsilon),\ j \in \{1, \dots, N\}\big\}.
\end{aligned}$$

### 4.1 The unfolding operator for periodic lattices

We are now in the position to define an equivalent formulation of the unfolding operator and mean value operator (see Definition 2.1 and 2.2) for lattice structures.

**Definition 4.1** For every measurable function $\phi$ on $\mathcal{S}_\varepsilon$, the unfolding operator $\mathcal{T}^{\mathcal{S}}_\varepsilon$ is defined as follows:

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi)(x, \mathbf{S}) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon\mathbf{S}\Big) \quad \text{for a.e. } (x, \mathbf{S}) \in \widetilde{\Omega}_\varepsilon \times \mathcal{S}.$$

For every function $\widehat{\phi}$ on $L^1(\mathcal{S}^{(i)})$, $i \in \{1, \dots, N\}$, the mean value operator $\mathcal{M}_{\mathcal{S}^{(i)}}$ on direction $\mathbf{e}_i$ is defined as follows:

$$\mathcal{M}_{\mathcal{S}^{(i)}}(\widehat{\phi})(\mathbf{S}) \doteq \int_{A(k)}^{A(k)+\mathbf{e}_i} \widehat{\phi}(x, \mathbf{S}')d\mathbf{S}', \qquad \forall \mathbf{S} \in [A(k), A(k) + \mathbf{e}_i], \quad \forall k \in \widehat{\mathbf{K}}_i.$$

Observe that in the above definition of $\mathcal{T}_\varepsilon^{\mathcal{S}}$, the map $(x, \mathbf{S}) \longmapsto \varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon\mathbf{S}$ from $\widetilde{\Omega}_\varepsilon \times \mathcal{S}$ into $\mathcal{S}_\varepsilon$ is almost everywhere one to one. This is not the case if we replace $\mathcal{S}$ by $\mathcal{S}_c$.

Below, we give the main property of $\mathcal{T}_\varepsilon^{\mathcal{S}}$.

**Proposition 4.2** *For every $\phi \in L^p(\mathcal{S}_\varepsilon)$, $p \in [1, +\infty]$, one has*

$$\|\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)\|_{L^p(\widetilde{\Omega}_\varepsilon \times \mathcal{S})} \le \varepsilon^{\frac{N-1}{p}} |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\mathcal{S}_\varepsilon)}.$$

***Proof*** We start with $p = 1$. Let $\phi$ be in $L^1(\mathcal{S}_\varepsilon)$. We have

$$\begin{aligned}
\int_{\widetilde{\Omega}_\varepsilon \times \mathcal{S}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x, \mathbf{S})| dx d\mathbf{S} &= \int_{\widetilde{\Omega}_\varepsilon} \sum_{i=1}^N \int_{\mathcal{S}^{(i)}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x, \mathbf{S})| dx d\mathbf{S} \\
&= \sum_{\xi = \widetilde{\Xi}_\varepsilon} |\varepsilon\xi + \varepsilon Y| \sum_{i=1}^N \sum_{k \in \widehat{\mathbf{K}}_i} \int_0^1 |\phi\big(\varepsilon\xi + \varepsilon A(k) + \varepsilon t\big)| dt \\
&= \varepsilon^N |Y| \sum_{i=1}^N \sum_{k \in \widehat{\mathbf{K}}_i} \int_0^1 |\phi\big(\varepsilon\xi + \varepsilon A(k) + \varepsilon t\big)| dt \\
&\le \varepsilon^{N-1} |Y| \int_{\mathcal{S}_\varepsilon} |\phi(\mathbf{s})| d\mathbf{s}.
\end{aligned}$$

The case $p \in (1, +\infty)$ follows by definition of $L^p$ norm. The case $p = +\infty$ is trivial. □

## 5 Periodic unfolding for sequences defined on lattices with information on the first order derivatives

### 5.1 Asymptotic behavior of bounded sequences defined as $Q_1$ interpolated on lattice nodes

On $\mathcal{S}_\varepsilon$ (resp. $\mathcal{S}$) we define the space $Q^1(\mathcal{S}_\varepsilon)$ (resp. $Q^1(\mathcal{S})$) by

$$Q^1(\mathcal{S}_\varepsilon) \doteq \Big\{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \,\Big|\, \phi \text{ is affine between two contiguous points of } \mathcal{K}_\varepsilon\Big\},$$
$$\bigg(\text{resp. } Q^1(\mathcal{S}) \doteq \Big\{\phi \in \mathcal{C}(\mathcal{S}) \,\Big|\, \phi \text{ is affine between two contiguous points of } \mathcal{K}\Big\}\bigg).$$

Similarly we define the spaces $Q^1(\mathcal{S}'_\varepsilon)$, $Q^1(\mathcal{S}''_\varepsilon)$ and $Q^1(\mathcal{S}')$, $Q^1(\mathcal{S}'')$, $Q^1_{per}(\mathcal{S})$, $Q^1_{per}(\mathcal{S}')$, $Q^1_{per}(\mathcal{S}'')$ (see (5.5)).

A function belonging to $Q^1(\mathcal{S}_\varepsilon)$ is determined only by its values on the set of nodes $\mathcal{K}_\varepsilon$ and thus can be naturally extended to a function defined in $\widetilde{\Omega}_\varepsilon$.

**Definition 5.1** For every function $\psi \in Q^1(\mathcal{S}_\varepsilon)$, its extension $\mathfrak{Q}_\varepsilon(\psi)$ belonging to $W^{1,\infty}(\widetilde{\Omega}_\varepsilon)$ is defined by $Q_1$ interpolation on each parallelotope $\varepsilon\xi + \varepsilon A(k) + \varepsilon\overline{Y_K}$ belonging to $\varepsilon\xi + \varepsilon\overline{Y}$ for every $\xi \in \widetilde{\Xi}_\varepsilon$ and $k \in \widehat{\mathbf{K}}$.

Define the spaces

$$\begin{aligned} Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon) \doteq \Big\{ \Psi \in W^{1,\infty}(\widetilde{\Omega}_\varepsilon) \;\Big|\; & \Psi_{|\varepsilon\xi+\varepsilon A(k)+\varepsilon\overline{Y_K}} \text{ is the } Q_1 \text{ interpolate of its values} \\ & \text{on the vertexes of } \varepsilon\xi + \varepsilon A(k) + \varepsilon\overline{Y_K},\ \forall k \in \widehat{\mathbf{K}},\ \forall \xi \in \widetilde{\Xi}_\varepsilon \Big\}, \\ Q^1_{\mathcal{K}}(Y) \doteq \Big\{ \Psi \in W^{1,\infty}(Y) \;\Big|\; & \Psi_{|A(k)+\overline{Y_K}} \text{ is the } Q_1 \text{ interpolate of its values} \\ & \text{on the vertexes of } A(k) + \overline{Y_K},\ \forall k \in \widehat{\mathbf{K}} \Big\}. \end{aligned}$$

Similarly we define the spaces $Q^1_{\mathcal{K}}(Y')$, $Q^1_{\mathcal{K}}(Y'')$, $Q^1_{\mathcal{K},per}(Y)$, $Q^1_{\mathcal{K},per}(Y')$ and $Q^1_{\mathcal{K},per}(Y'')$.

By definition, the extension operator $\mathfrak{Q}_\varepsilon$ is both one to one and onto from $Q^1(\mathcal{S}_\varepsilon)$ to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$. Its inverse is given by the restriction $_{|\mathcal{S}_\varepsilon}$ from $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ to $Q^1(\mathcal{S}_\varepsilon)$.

Below, we show the main properties of this operator.

**Lemma 5.2** *For every $\psi \in Q^1(\mathcal{S}_\varepsilon)$, one has ($p \in [1, +\infty]$, $i \in \{1, \dots, N\}$)*

$$\|\mathfrak{Q}_\varepsilon(\psi)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{\frac{N-1}{p}} \|\psi\|_{L^p(\mathcal{S}_\varepsilon)},\ \|\partial_i \mathfrak{Q}_\varepsilon(\psi)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{\frac{N-1}{p}} \|\partial_{\mathbf{s}}\psi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}, \tag{5.1}$$

*where the constants do not depend on $\varepsilon$.*

***Proof*** We will only consider the case $p \in [1, +\infty)$, since the case $p = +\infty$ is trivial. First, remind that for every function $\phi$ defined as $Q_1$ interpolate of its values on the vertexes of the nodes in $\mathcal{K}$, we have ($i \in \{1, \dots, N\}$)

$$\begin{aligned} c\|\phi\|_{L^p(Y)} &\le \Big( \sum_{k\in\mathbf{K}} \big|\phi\big(A(k)\big)\big|^p \Big)^{1/p} \le C\|\phi\|_{L^p(\mathcal{S})}, \\ c\|\partial_{y_i}\phi\|_{L^p(Y)} &\le \big\|\partial_{\mathbf{s}}\phi\big\|_{L^p(\mathcal{S}^{(i)})}, \end{aligned} \tag{5.2}$$

where the constants do not depend on $p$.

We now prove $(5.1)_1$. For every $\psi \in Q^1(\mathcal{S}_\varepsilon)$, set $\Psi = \mathfrak{Q}_\varepsilon(\psi)$. From $(5.2)_1$ and an affine change of variables, we easily get

$$\int_{\widetilde{\Omega}_\varepsilon} |\Psi(x)|^p dx = \sum_{\xi\in\widetilde{\Xi}_\varepsilon} \int_{\varepsilon\xi+\varepsilon Y} |\Psi(x)|^p dx = \varepsilon^N \sum_{\xi\in\widetilde{\Xi}_\varepsilon} \int_Y |\Psi(\varepsilon\xi + \varepsilon y)|^p dy$$

$$\leq \varepsilon^N \sum_{\xi \in \widetilde{\Xi}_\varepsilon} \int_{\mathcal{S}} |\Psi(\varepsilon\xi + \varepsilon \mathbf{S})|^p d\mathbf{S} \leq \varepsilon^{N-1} \int_{\mathcal{S}_\varepsilon} |\Psi(\mathbf{s})|^p d\mathbf{s}$$

and thus (5.1)$_1$ holds since $\Psi_{|\mathcal{S}_\varepsilon} = \psi$.

We prove now (5.1)$_2$. Let $i$ be in $\{1, \dots, N\}$. From (5.2)$_2$ and an affine change of variables, we have

$$\int_{\widetilde{\Omega}_\varepsilon} |\partial_i \Psi(x)|^p dx = \sum_{\xi \in \widetilde{\Xi}_\varepsilon} \int_{\varepsilon\xi + \varepsilon Y} \Big|\frac{\partial}{\partial x_i} \Psi(x)\Big|^p dx = \varepsilon^{N-p} \sum_{\xi \in \widetilde{\Xi}_\varepsilon} \int_Y \Big|\frac{\partial}{\partial y_i} \Psi(\varepsilon\xi + \varepsilon y)\Big|^p dy$$
$$\leq \varepsilon^{N-p} \sum_{\xi \in \widetilde{\Xi}_\varepsilon} \int_{\mathcal{S}^{(i)}} |\partial_{\mathbf{s}} \Psi(\varepsilon\xi + \varepsilon \mathbf{S})\Big|^p d\mathbf{S} \leq \varepsilon^{N-1} \int_{\mathcal{S}^{(i)}_\varepsilon} \big|\partial_{\mathbf{s}} \Psi(\mathbf{s})\big|^p d\mathbf{s}.$$

And thus (5.1)$_2$ holds since $\Psi_{|\mathcal{S}^{(i)}_\varepsilon} = \psi_{|\mathcal{S}^{(i)}_\varepsilon}$. □

Note now that for every $\psi \in Q^1(\mathcal{S}_\varepsilon)$, the unfolding on the lattice is equivalent to first extending $\psi$ to $\Psi = \mathfrak{Q}_\varepsilon(\psi)$ (see Definition 5.1), then applying the unfolding results in $\mathbb{R}^N$ and lastly restricting the convergences to the lattice again, as the following commutative diagrams show ($i \in \{1, \dots, N\}$):

$$\begin{cases} \mathcal{T}^{\mathcal{S}}_\varepsilon(\psi) = \mathcal{T}^{\mathcal{S}}_\varepsilon(\Psi_{|\mathcal{S}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\Psi)_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}}, \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\psi) = \mathcal{T}^{\mathcal{S}}_\varepsilon\big(\partial_{\mathbf{s}}\Psi_{|\mathcal{S}^{(i)}_\varepsilon}\big) = \mathcal{T}^{ext}_\varepsilon\big(\partial_i \Psi\big)_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}}. \end{cases} \tag{5.3}$$

We can finally show the asymptotic behavior of sequences which belong to $Q^1(\mathcal{S}_\varepsilon)$ and we start with the following.

**Lemma 5.3** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ satisfying ($p \in (1, +\infty)$)*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}},$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$ such that*[1]

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})).^1 \tag{5.4}$$

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** The sequence $\{\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)\}_\varepsilon$ satisfies

$$\|\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)\|_{L^p(\widetilde{\Omega}_\varepsilon; W^{1,p}(\mathcal{S}))} \leq C \qquad \Longrightarrow \qquad \|\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)\|_{L^p(\Omega; W^{1,p}(\mathcal{S}))} \leq C.$$

[1] As for $\mathcal{T}_\varepsilon{}^{ext}$, this convergence must be understood

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)_{|\Omega \times \mathcal{S}} \rightharpoonup \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})).$$

It will be the same for all convergences involving the unfolding operator $\mathcal{T}^{\mathcal{S}}_\varepsilon$.

The constant does not depend on $\varepsilon$.

Hence, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$ such that convergence (5.4) holds. The periodicity of $\widehat{\phi}$ is proved as in [5, Theorem 1.36]. □

We consider now sequences whose gradient is anisotropically bounded on the lattice.

Accordingly to Sect. 2, we apply the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$ and define

$$\begin{aligned} &\mathcal{S}' \doteq \bigcup_{i=1}^{N_1} \mathcal{S}^{(i)}, \qquad \mathcal{S}'_c \doteq \bigcup_{i=1}^{N_1} \mathcal{S}^{(i)}_c, \qquad \mathcal{S}'_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} \big(\varepsilon\xi + \varepsilon\mathcal{S}'_c\big), \\ &\mathcal{S}'' \doteq \bigcup_{i=N_1+1}^{N} \mathcal{S}^{(i)}, \qquad \mathcal{S}''_c \doteq \bigcup_{i=N_1+1}^{N} \mathcal{S}^{(i)}_c, \qquad \mathcal{S}''_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} \big(\varepsilon\xi + \varepsilon\mathcal{S}''_c\big). \end{aligned} \tag{5.5}$$

We have the following.

**Lemma 5.4** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1(\mathcal{S}_\varepsilon)$ satisfying ($p \in (1, +\infty)$)*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}},$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, $\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{per}(\mathcal{S}''))$, $\widehat{\phi} \in L^p(\Omega; Q^1_{per}(\mathcal{S}))$, such that ($i \in \{1, \dots, N_1\}$)*

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega; Q^1(\mathcal{S})), \\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\widetilde{\phi} + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}), \\ &\frac{1}{\varepsilon}\Big(\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)\Big) \rightharpoonup \partial_i\widetilde{\phi}\,\mathbf{S}^c + \widehat{\phi} - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}}(\widehat{\phi}) \\ &\quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}), \end{aligned} \tag{5.6}$$

*where $\mathbf{S}^c \doteq \big(\mathbf{S} - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}}(\mathbf{S})\big) \cdot \mathbf{e}_i$.*[2] *The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** We extend the sequence $\{\phi_\varepsilon\}_\varepsilon$ to the sequence $\{\Phi_\varepsilon\}_\varepsilon = \{\mathfrak{Q}_\varepsilon(\phi_\varepsilon)\}_\varepsilon$ belonging to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$. By Lemma 5.2 and the $Q_1$ property (3.2), we get

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla_{x''}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C,$$

where the constant does not depend on $\varepsilon$.

By construction, the sequence $\{\Phi_\varepsilon\}_\varepsilon$ belongs to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ and thus $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon$ belongs to $L^p(\widetilde{\Omega}_\varepsilon; Q^1(Y))$.

[2] One has $\mathbf{S} = A(k) + t\mathbf{e}_i$ in the line $[A(k), A(k) + t\mathbf{e}_i]$, $t \in [0, 1]$, $k \in \widehat{\mathbf{K}}_i$. Hence $\mathbf{S}^c = t - 1/2$.

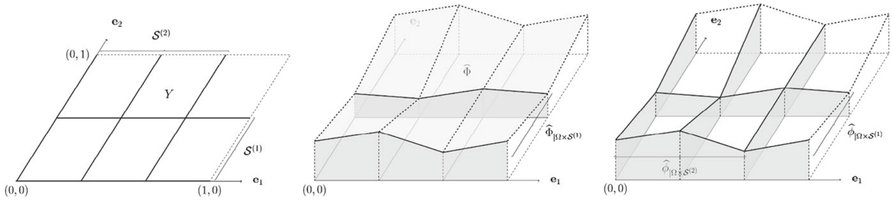


**Fig. 1** Construction of the periodic function $\widehat{\phi}$ for $N = 2$ and $(K_1, K_2) = (3, 2)$. On the left, the reference cell and the lattice $\mathcal{S} \doteq \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$ and the nodes $A(k)$, where $k$ belongs to $\mathbf{K} \doteq \{0, 1, 2, 3\} \times \{0, 1, 2\}$. On the center, the $Q_1$ interpolated on the lattice nodes $\widehat{\Phi}$ and its restriction to $\mathcal{S}^{(1)}$ (horizontal lines). On the right, the function $\widehat{\phi}$ given by $\widehat{\Phi}_{|\Omega\times\mathcal{S}^{(1)}}$ and the $Q_1$ interpolation along the segments in $\mathcal{S}^{(2)}$ (vertical lines)

Hence, Lemma 3.2 imply that there exist functions $\widetilde{\Phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{\mathcal{K},per}(Y''))$ and $\widehat{\Phi} \in L^p(\Omega \times Y''; Q^1_{\mathcal{K},per}(Y')) \cap L^p(\Omega; Q^1_{\mathcal{K}}(Y))$ satisfying $\mathcal{M}_{Y'}(\widehat{\Phi}) = 0$ a.e. in $\Omega \times Y''$, such that

$$
\begin{aligned}
\Phi_{\varepsilon|\Omega} \rightharpoonup \Phi \quad & \text{weakly in} \quad L^p(\Omega, \nabla_{x'}),\\
\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) \rightharpoonup \widetilde{\Phi} \quad & \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)),\\
\mathcal{T}_\varepsilon^{ext}(\nabla_{x'}\Phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\Phi} + \nabla_{y'}\widehat{\Phi} \quad & \text{weakly in} \quad L^p(\Omega \times Y)^{N_1},
\end{aligned}
$$

where $\Phi = \mathcal{M}_{Y''}(\widetilde{\Phi})$.

Using the relations (5.3), we can restrict the above convergences from $\Omega \times Y$ to the subset $\Omega \times \mathcal{S}$ (and from $\Omega \times Y'$, $\Omega \times Y''$ to $\Omega \times \mathcal{S}'$, $\Omega \times \mathcal{S}''$ respectively). Hence, $\widetilde{\phi} = \widetilde{\Phi}_{|\Omega\times\mathcal{S}}$ and thus $\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{per}(\mathcal{S}''))$. Now, let us consider $\widehat{\Phi}_{|\Omega\times\mathcal{S}'}$, we extend this function as an affine function between two contiguous nodes in $\mathcal{S}''$, this gives a function $\widehat{\phi}$ belonging to $L^p(\Omega; Q^1_{per}(\mathcal{S}))$ (see Fig. 1).

This proves convergences $(5.6)_{1,2}$, while $(5.6)_3$ is an immediate consequence of the Poincaré-Wirtinger inequality and $(5.6)_2$. □

Now, we show the asymptotic behavior of sequences in $Q^1(\mathcal{S}_\varepsilon)$ which are uniformly bounded in $W^{1,p}(\mathcal{S}_\varepsilon)$.

**Corollary 5.5** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1(\mathcal{S}_\varepsilon)$ satisfying ($p \in (1, +\infty)$)*

$$\|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}},$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in W^{1,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; Q^1_{per,0}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)*

$$
\begin{aligned}
\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightharpoonup \phi \quad & \textit{weakly in} \quad L^p(\Omega; Q^1(\mathcal{S})),\\
\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} \quad & \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}).
\end{aligned}
$$

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** The proof directly follows from Lemma 5.4 in the particular case $\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}'' = \emptyset$. □

### 5.2 Asymptotic behavior of sequences bounded anisotropically and uniformly in $W^{1,p}$

Denote ($p \in [1, +\infty]$, $i \in \{1, \dots, N\}$)

$$\mathcal{W}^{1,p}_{0,\mathcal{K}}(\mathcal{S}) = \{\phi \in W^{1,p}(\mathcal{S}) \mid \phi = 0 \ \text{ on } \mathcal{K}\},$$
$$\mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon) = \{\phi \in W^{1,p}(\mathcal{S}_\varepsilon) \mid \phi = 0 \ \text{ on } \mathcal{K}_\varepsilon\}.$$

Every function $\phi$ in $W^{1,p}(\mathcal{S})$ (resp. $\psi \in W^{1,p}(\mathcal{S}_\varepsilon)$) is defined on the set of nodes $\mathcal{K}$ (resp. $\mathcal{K}_\varepsilon$) and therefore can be decomposed as

$$\begin{aligned} \phi = \phi_a + \phi_0, \qquad \phi_a \in Q^1(\mathcal{S}), \quad \phi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{K}}(\mathcal{S}), \\ \big(\text{resp. } \psi = \psi_a + \psi_0, \qquad \psi_a \in Q^1(\mathcal{S}_\varepsilon), \quad \psi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)\big), \end{aligned} \tag{5.7}$$

where $\phi_a$, $\psi_a$ are affine functions defined as $Q_1$ interpolation on the nodes, and $\phi_0$, $\psi_0$ the reminder terms which are zero on every node.

**Lemma 5.6** *There exists a constant $C > 0$, which does not depend on $\varepsilon$, such that ($i \in \{1, \dots, N\}$)*

$$\begin{aligned} &\forall \phi \in W^{1,p}(\mathcal{S}), \qquad \|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)})} + \|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)})} \le C\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})}, \\ &\forall \psi \in W^{1,p}(\mathcal{S}_\varepsilon), \qquad \|\partial_{\mathbf{s}}\psi_a\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} + \|\partial_{\mathbf{s}}\psi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} \le C\|\partial_{\mathbf{s}}\psi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}, \\ &\qquad\qquad \|\psi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} \le C\varepsilon\|\partial_{\mathbf{s}}\psi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}. \end{aligned} \tag{5.8}$$

***Proof*** *Step 1*. First, we recall a simple result. Let $\psi$ be in the space $W^{1,p}(0,1)$ ($p \in [1, +\infty]$). Denote $\psi_a$ the affine function

$$\psi_a(t) \doteq \psi(0) + t\big(\psi(1) - \psi(0)\big), \qquad t \in [0,1].$$

One has

$$\|\psi_a'\|_{L^p(0,1)} \le \|\psi'\|_{L^p(0,1)}, \quad \|\psi - \psi_a\|_{L^p(0,1)} \le 2\|\psi'\|_{L^p(0,1)}. \tag{5.9}$$

*Step 2*. We prove the statements of the Lemma.

We start with $(5.8)_1$. By construction, $\mathcal{S}^{(i)}$ is the union of a finite number of segments whose extremities belong to $\mathbf{K}$. Hence, inequality $(5.9)_1$ and an affine change of variables leads to ($i \in \{1, \dots, N\}$)

$$\begin{aligned} &\|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)})} \le \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})}, \\ &\|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)})} \le \|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)})} + \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})} \le 2\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})} \end{aligned}$$

~~and thus (5.8)$_1$ is proved. Estimate (5.8)$_2$ follows by (5.8)$_1$ and an affine change of~~ variables, while (5.8)$_3$ follows by (5.8)$_2$ and again a change of variables.

The constant does not depend on $\varepsilon$ since $\mathcal{S}^{(i)}$ has a finite number of segments. □

We show now the asymptotic behavior of sequences that are anisotropically bounded.

**Lemma 5.7** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ satisfying ($p \in (1, +\infty)$)*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}, \tag{5.10}$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, $\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(\mathcal{S}''))$, $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$, such that ($i \in \{1, \dots, N_1\}$)*

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ &\quad \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\widetilde{\phi} + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}),\\ &\quad \frac{1}{\varepsilon}\Big(\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)\Big) \rightharpoonup \partial_i\widetilde{\phi}\,\mathbf{S}^c + \widehat{\phi} - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}}(\widehat{\phi})\\ &\quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}), \end{aligned} \tag{5.11}$$

*where $\mathbf{S}^c \doteq \big(\mathbf{S} - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}}(\mathbf{S})\big) \cdot \mathbf{e}_i$.*

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** Given $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$, we decompose $\phi_\varepsilon$ as in (5.7) and get

$$\phi_\varepsilon = \phi_{a,\varepsilon} + \phi_{0,\varepsilon}, \quad \phi_{a,\varepsilon} \in Q^1(\mathcal{S}_\varepsilon), \quad \phi_{0,\varepsilon} \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon).$$

By Lemma 5.6 and hypothesis (5.10) we have

$$\begin{aligned} \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} &\le C\varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}+1},\\ \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} &\le C\varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}},\\ \|\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}}\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{S}}\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} &\le C\varepsilon^{\frac{1-N}{p}}. \end{aligned} \tag{5.12}$$

where the constant does not depend on $\varepsilon$.

By estimates (5.12)$_{1,2}$ and [5, Theorem 1.36] applied on each line of $\mathcal{S}_\varepsilon$, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\phi}'_0 \in L^p(\Omega; \mathcal{W}^{1,p}_{0,\mathcal{K},per}(\mathcal{S}'))$ (where $\mathcal{W}^{1,p}_{0,\mathcal{K},per}(\mathcal{S}') \doteq \mathcal{W}^{1,p}_{0,\mathcal{K}}(\mathcal{S}') \cap \mathcal{W}^{1,p}_{per}(\mathcal{S}')$) and $\widehat{\phi}''_0 \in L^p(\Omega; \mathcal{W}^{1,p}_{0,\mathcal{K},per}(\mathcal{S}''))$ such that

$$\begin{aligned} \frac{1}{\varepsilon}\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}'_0 \quad &\text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}')),\\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}''_0 \quad &\text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}'')). \end{aligned}$$

By estimates (5.12)$_3$ and Lemma 5.4, there exist a subsequence, still denoted $\{\varepsilon\}$, and functions $\widetilde{\phi}_a \in L^p(\Omega, \nabla_{x'}; Q^1_{per}(\mathcal{S}''))$, $\widehat{\phi}_a \in L^p(\Omega; Q^1_{per}(\mathcal{S}))$ such that ($i \in \{1,\dots,N_1\}$)

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\phi_{a,\varepsilon}) \to \widetilde{\phi}_a \quad &\text{strongly in} \quad L^p(\Omega; Q^1(\mathcal{S})),\\ \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\partial_{\mathbf{s}}\phi_{a,\varepsilon}) \rightharpoonup \partial_i\widetilde{\phi}_a + \partial_{\mathbf{S}}\widehat{\phi}_a \quad &\text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

Hence ($i \in \{1,\dots,N_1\}$, $j \in \{N_1+1,\dots,N\}$)

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\phi_{\varepsilon}) \to \widetilde{\phi}_a \quad &\text{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}')),\\ \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\phi_{\varepsilon}) \rightharpoonup \widetilde{\phi}_a + \widehat{\phi}''_0 \quad &\text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}'')),\\ \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\partial_{\mathbf{s}}\phi_{\varepsilon}) \rightharpoonup \partial_i\widetilde{\phi}_a + \partial_{\mathbf{S}}(\widehat{\phi}_a + \widehat{\phi}'_0) \quad &\text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

Setting $\widetilde{\phi} \doteq \widetilde{\phi}_a + \widetilde{\phi}''_0$, we get that $\widetilde{\phi}$ belongs to $L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(\mathcal{S}''))$. Then, we set $\widehat{\phi} \doteq \widehat{\phi}_a + \widehat{\phi}'_0$, this function belongs to $L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$. Convergence (5.11)$_3$ is an immediate consequence of (5.11)$_2$. The proof is complete. □

As a direct consequence, it follows the asymptotic behavior of the uniformly bounded sequences.

**Corollary 5.8** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ satisfying ($p \in (1,+\infty)$)*

$$\|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}},$$

*where the constant does not depend on $\varepsilon$.*

*Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in W^{1,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(\mathcal{S}))$ such that ($i \in \{1,\dots,N\}$)*

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\phi_{\varepsilon}) \to \phi \quad &\textit{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ \mathcal{T}^{\mathcal{S}}_{\varepsilon}(\partial_{\mathbf{s}}\phi_{\varepsilon}) \rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} \quad &\textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** The proof directly follows from Lemma 5.7 in the particular case $\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}'' = \emptyset$. □

## 6 Periodic unfolding for sequences defined on lattices with information until the second order derivatives

The main problem that arises for functions in $W^{2,p}(\mathcal{S}_\varepsilon)$ is the lack of mixed derivatives. This comes from the fact that a function defined on the lattice segments can be derived twice, only in the segment directions. We overcome the problem in two different ways.

### 6.1 Unfolding via special $Q_3$ interpolation

Analogously to the previous section, we decompose a function into a remainder term and a cubic polynomial, this latter is extended to a special $Q_3$ (or $N$-cubic) interpolation to the whole space. Then, we use the periodic unfolding results for open subset in $\mathbb{R}^N$ and finally restrict these results to the lattice. However, to bound the extension, further assumptions on the original function must be applied.

First, we recall a basic result concerning the functions in $W^{2,p}(0,1)$.

**Lemma 6.1** *Let $\phi$ be in $W^{2,p}(0,1)$. There exist a unique decomposition*

$$\phi = \phi_p + \phi_0, \qquad (\phi_p, \phi_0) \in W^{2,p}(0,1)^2,$$

*where $\phi_p$ is the cubic polynomial defined by ($t \in [0,1]$)*

$$\phi_p(t) = \phi(0)(2t+1)(t-1)^2 + \phi(1)t^2(3-2t) + \phi'(0)t(t-1)^2 + \phi'(1)t^2(t-1)$$

*and $\phi_0$ is the remaining term satisfying*

$$\phi_0(0) = \phi_0(1) = \phi_0'(0) = \phi_0'(1) = 0. \tag{6.1}$$

*Moreover, there exists a constant $C > 0$, such that*

$$\begin{aligned} \forall \phi \in W^{2,p}(0,1), \quad \|\phi_p''\|_{L^p(0,1)} &\le C\|\phi''\|_{L^p(0,1)},\\ \|\phi_p'\|_{L^p(0,1)} &\le C\|\phi'\|_{W^{1,p}(0,1)},\\ \|\phi_p\|_{L^p(0,1)} &\le C\|\phi\|_{W^{2,p}(0,1)},\\ \|\phi_0\|_{W^{2,p}(0,1)} &\le C\|\phi''\|_{L^p(0,1)}. \end{aligned} \tag{6.2}$$

***Proof*** Given $\phi$ be in $W^{2,p}(0,1)$, it is clear that the decomposition is unique. Indeed, condition (6.1) implies that the function $\phi_p$ must satisfy

$$\phi_p(0) = \phi(0), \quad \phi_p(1) = \phi(1), \quad \phi_p'(0) = \phi'(0), \quad \phi_p'(1) = \phi'(1)$$

and therefore the 4 coefficients of the cubic polynomial are uniquely determined.

Now, we observe that

$$\begin{aligned} \phi_p'(t) &= \Big(\phi(1) - \phi(0) - \frac{1}{2}\big(\phi'(0) + \phi'(1)\big)\Big)6t(1-t) + \big(\phi'(1) - \phi'(0)\big)t + \phi'(0),\\ \phi_p''(t) &= \Big(\phi(1) - \phi(0) - \frac{1}{2}\big(\phi'(0) + \phi'(1)\big)\Big)6(1-2t) + \big(\phi'(1) - \phi'(0)\big). \end{aligned}$$

Then, we easily obtain the estimates $(6.2)_{1,2,3}$. Estimate $(6.2)_4$ follows by assumption (6.1), the Poincaré inequality applied twice and estimate $(6.2)_1$. ◻

Define the spaces ($p \in [1, +\infty]$)

$$\mathcal{W}^{2,p}_{0,\mathcal{K}}(\mathcal{S}) = \{\phi \in W^{2,p}(\mathcal{S}) \mid \phi = \partial_{\mathbf{S}}\phi = 0 \text{ on } \mathcal{K}\},$$
$$\mathcal{W}^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon) = \{\psi \in W^{2,p}(\mathcal{S}_\varepsilon) \mid \psi = \partial_{\mathbf{s}}\psi = 0 \quad \text{on } \mathcal{K}_\varepsilon\}.$$

Remind that for any $\phi \in W^{2,p}(\mathcal{S})$ (resp. $\psi \in W^{2,p}(\mathcal{S}_\varepsilon)$), its derivatives $\partial_{\mathbf{S}}\phi$ (resp. $\partial_{\mathbf{s}}\psi$) in direction $\mathbf{e}_i$ are functions belonging to $W^{1,p}(\mathcal{S}^{(i)})$ (resp. $W^{1,p}(\mathcal{S}^{(i)}_\varepsilon)$), for every $i \in \{1, \ldots, N\}$ and therefore defined on every node of the structure $\mathcal{S}$ (resp. $\mathcal{S}_\varepsilon$). Hence, they can be extended by $Q_1$ interpolation on the small segments of $\mathcal{S}^{(j)}$ (resp. $\mathcal{S}^{(j)}_\varepsilon$) for every $j \in \{1, \ldots, N\}$, $j \neq i$. We denote these extensions by $\partial_i\phi$ (resp. $\partial_i\psi$), for every $i \in \{1, \ldots, N\}$.

**Lemma 6.2** *For every $\phi \in W^{2,p}(\mathcal{S})$, there exist two functions $\Phi_p \in W^{2,p}(Y)$ and $\phi_0 \in \mathcal{W}^{2,p}_{0,\mathcal{K}}(\mathcal{S})$ such that*

$$\phi = \Phi_p + \phi_0 \quad \text{a.e. in } \mathcal{S}, \tag{6.3}$$

*where $\Phi_{p|\mathcal{S}}$ is a cubic polynomial on every small segment of $\mathcal{S}$.*

*Moreover, there exists a constant $C > 0$ such that*

$$\begin{aligned}
\|D^2\Phi_p\|_{L^p(Y)} &\leq C\sum_{i=1}^N \left\|\partial_{\mathbf{S}}(\partial_i\phi)\right\|_{L^p(\mathcal{S})},\\
\|\nabla\Phi_p\|_{L^p(Y)} &\leq C\Big(\|\partial_{\mathbf{S}}\phi\|_{L^p(\mathcal{S})} + \sum_{i=1}^N \left\|\partial_{\mathbf{S}}(\partial_i\phi)\right\|_{L^p(\mathcal{S})}\Big),\\
\|\Phi_p\|_{L^p(Y)} &\leq C\Big(\|\phi\|_{L^p(\mathcal{S})} + \|\partial_{\mathbf{S}}\phi\|_{L^p(\mathcal{S})} + \sum_{i=1}^N \left\|\partial_{\mathbf{S}}(\partial_i\phi)\right\|_{L^p(\mathcal{S})}\Big)
\end{aligned} \tag{6.4}$$

*and that*

$$\|\phi_0\|_{L^2(\mathcal{S})} + \|\partial_{\mathbf{S}}\phi_0\|_{L^2(\mathcal{S})} + \|\partial^2_{\mathbf{S}}\phi_0\|_{L^2(\mathcal{S})} \leq C\left\|\partial^2_{\mathbf{S}}\phi\right\|_{L^p(\mathcal{S})}. \tag{6.5}$$

***Proof*** We will only prove the case $N = 2$, since the extension to higher dimension is done by an analogous argumentation.

*Step 1.* A first result.

Denote $Q_0$, $Q_1$, $dQ_0$ and $dQ_1$ the following polynomial functions ($t \in [0, 1]$)

$$Q_0(t) = (2t+1)(t-1)^2, \quad dQ_0(t) = t(t-1)^2,$$
$$Q_1(t) = t^2(3-2t), \quad dQ_1(t) = t^2(t-1).$$

Let $\phi$ be a function continuous on $\partial Z$, $Z = (0, 1)^2$, and of class $W^{2,p}$ on every edge of $\overline{Z}$. We define the polynomial function $\Phi \in W^{2,\infty}(Z)$ by

$$\begin{aligned}
\Phi(t) = {}& \phi(0,0)P_{00}(t) + \phi(0,1)P_{0,1}(t) + \phi(1,0)P_{1,0}(t) + \phi(1,1)P_{1,1}(t)\\
&+ \partial_1\phi(0,0)d_1P_{00}(t) + \partial_1\phi(1,0)d_1P_{10}(t) + \partial_1\phi(0,1)d_1P_{01}(t) + \partial_1\phi(1,1)d_1P_{11}(t)\\
&+ \partial_2\phi(0,0)d_2P_{00}(t) + \partial_2\phi(0,1)d_2P_{01}(t) + \partial_2\phi(1,0)d_2P_{10}(t) + \partial_2\phi(1,1)d_2P_{11}(t)
\end{aligned}$$

where for all $t = (t_1, t_2) \in [0,1]^2$

$$
\begin{aligned}
P_{00}(t) &= Q_0(t_1)Q_0(t_2), & P_{01}(t) &= Q_0(t_1)Q_1(t_2),\\
P_{10}(t) &= Q_1(t_1)Q_0(t_2), & P_{11}(t) &= Q_1(t_1)Q_1(t_2),\\
d_1 P_{00} &= dQ_0(t_1)Q_0(t_2), & d_1 P_{10} &= dQ_1(t_1)Q_0(t_2),\\
d_1 P_{01} &= dQ_0(t_1)Q_1(t_2), & d_1 P_{11} &= dQ_1(t_1)Q_1(t_2),\\
d_2 P_{00} &= Q_0(t_1)dQ_0(t_2), & d_2 P_{01} &= Q_0(t_1)dQ_1(t_2),\\
d_2 P_{10} &= Q_1(t_1)dQ_0(t_2), & d_2 P_{11} &= Q_1(t_1)dQ_1(t_2).
\end{aligned}
$$

First, observe that the polynomial $\Phi$ can be rewritten as

$$
\begin{aligned}
\Phi(t) = &\big(\phi(0,0)Q_0(t_1) + \phi(1,0)Q_1(t_1) + \partial_1\phi(0,0)dQ_0(t_1) + \partial_1\phi(1,0)dQ_1(t_1)\big)Q_0(t_2)\\
&+ \big(\phi(0,1)Q_0(t_1) + \phi(1,1)Q_1(t_1) + \partial_1\phi(0,1)dQ_0(t_1) + \partial_1\phi(1,1)dQ_1(t_1)\big)Q_1(t_2)\\
&+ \big(\partial_2\phi(0,0)dQ_0(t_2) + \partial_2\phi(0,1)dQ_1(t_2)\big)Q_0(t_1)\\
&+ \big(\partial_2\phi(1,0)dQ_0(t_2) + \partial_2\phi(1,1)dQ_1(t_2)\big)Q_1(t_1).
\end{aligned}
$$

A straightforward calculation and Lemma (6.1) lead to

$$
\begin{aligned}
\|D^2\Phi\|_{L^p(Z)} \le\; & C\Big(\sum_{i=1}^{2}\|\partial^2_{ii}\phi\|_{L^p((\partial Z)_i)} + |\partial_2\phi(1,0) - \partial_2\phi(0,0)| + |\partial_2\phi(1,1) - \partial_2\phi(0,1)|\\
&+ |\partial_1\phi(0,1) - \partial_1\phi(0,0)| + |\partial_1\phi(1,1) - \partial_1\phi(1,0)|\Big)\\
\le\; & C\sum_{i=1}^{2}\|\partial_{\mathbf{S}}(\partial_i\phi)\|_{L^p((\partial Z)_i)}.
\end{aligned}
$$

where $(\partial Z)_1 = (0,1)\times\{0,1\}$ and $(\partial Z)_2 = \{0,1\}\times(0,1)$.

Observe also that ($i \in \{1,2\}$)

$$
\|\partial^2_{ii}\Phi\|_{L^p((\partial Z)_i)} \le C\|\partial^2_{\mathbf{S}}\phi\|_{L^p((\partial Z)_i)}. \tag{6.6}
$$

Then, we obtain

$$
\|\nabla\Phi\|_{L^p(Z)} \le C\big(\|\partial_{\mathbf{S}}\phi\|_{L^p(\partial Z)} + \|D^2\Phi\|_{L^p(Z)}\big).
$$

and thus

$$
\|\Phi\|_{L^p(Z)} \le C\big(\|\phi\|_{L^p(\partial Z)} + \|\nabla\Phi\|_{L^p(Z)}\big).
$$

*Step 2.* We prove the estimates (6.4) for $N = 2$.

In every small rectangle build on the nodes of $\mathcal{S}$ we extend $\phi$ as described in Step 1. That gives a function $\Phi_p \in W^{2,p}(Y)$ satisfying (6.4) for $N = 2$. Estimate (6.5) follows by applying the Poincaré inequality twice and the fact that (see (6.6))

$$
\|\partial^2_{\mathbf{S}}\phi_0\|_{L^p(\mathcal{S})} \le \sum_{i=1}^{2}\|\partial^2_{ii}\Phi\|_{L^p((\partial Z)_i)} + \|\partial^2_{\mathbf{S}}\phi\|_{L^p(\mathcal{S})} \le C\big\|\partial^2_{\mathbf{S}}\phi\big\|_{L^p(\mathcal{S})}.
$$

The proof is complete. □

We can finally show the asymptotic behavior of sequences bounded in $W^{2,p}(\mathcal{S}_\varepsilon)$, whose derivatives of the gradient extension from the lattice to the whole space are also bounded.

**Theorem 6.3** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$, $p \in (1, +\infty)$, satisfying*

$$\|\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \sum_{i=1}^{N} \left\|\partial_{\mathbf{s}}(\partial_i\phi_\varepsilon)\right\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}. \tag{6.7}$$

*Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\phi \in W^{2,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per}(\mathcal{S}))$ such that ($i \in \{1, \ldots, N\}$)*

$$\begin{aligned}
\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \to \phi \quad & \textit{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})),\\
\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \to \partial_i\phi \quad & \textit{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}^{(i)})),\\
\mathcal{T}_\varepsilon^{\mathcal{S}}\left(\partial^2_{\mathbf{s}}\phi_\varepsilon\right) \rightharpoonup \partial^2_{ii}\phi + \partial^2_{\mathbf{S}}\widehat{\phi} \quad & \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}).
\end{aligned}$$

***Proof*** Given $\{\phi_\varepsilon\}_\varepsilon \subset W^{2,p}(\mathcal{S}_\varepsilon)$, we decompose $\phi_\varepsilon(\varepsilon\xi + \varepsilon\mathbf{S})$, $\xi \in \widetilde{\Xi}_\varepsilon$, $\mathbf{S} \in \mathcal{S}$, as in (6.3) and get

$$\phi_\varepsilon = \Phi_{p,\varepsilon} + \phi_{0,\varepsilon}, \quad \Phi_{p,\varepsilon} \in W^{2,p}(\widetilde{\Omega}_\varepsilon), \quad \phi_{0,\varepsilon} \in \mathcal{W}^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon).$$

We first consider the sequence $\{\phi_{0,\varepsilon}\}_\varepsilon$ belonging to $\mathcal{W}^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. By estimate (6.5) together with an affine change of variables and (6.7), we have

$$\|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon^2\|\partial^2_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^2\|\partial^2_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}+2},$$

where the constant does not depend on $\varepsilon$. Hence, there exist a subsequence, still denoted $\{\varepsilon\}$, and a function $\widehat{\phi}_0 \in L^p(\Omega; \mathcal{W}^{2,p}_{0,\mathcal{K},per}(\mathcal{S}))$ such that

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}_0 \qquad \text{weakly in} \quad L^2(\Omega; W^{2,p}(\mathcal{S})). \tag{6.8}$$

Now we consider the sequence $\{\Phi_{p,\varepsilon}\}_\varepsilon$. By estimates (6.4) together with an affine change of variables and hypothesis (6.7) we have

$$\|\Phi_{p,\varepsilon}\|_{W^{2,p}(\widetilde{\Omega}_\varepsilon)} \leq C\varepsilon^{\frac{N-1}{p}}\left(\|\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \sum_{i=1}^{N}\left\|\partial_{\mathbf{s}}(\partial_i\phi_\varepsilon)\right\|_{L^2(\mathcal{S}_\varepsilon)}\right) \leq C.$$

Hence, by [5, Theorem 1.47], there exist a subsequence, still denoted $\{\varepsilon\}$, and functions $\phi \in W^{2,p}(\Omega)$ and $\widehat{\Phi}_p \in L^p(\Omega; W^{2,p}_{per}(Y))$ such that

$$\begin{aligned} \Phi_{p,\varepsilon|\Omega} \rightharpoonup \phi \quad & \text{weakly in} \quad W^{2,p}(\Omega), \\ \mathcal{T}_\varepsilon(\Phi_{p,\varepsilon}) \to \phi \quad & \text{strongly in} \quad L^p(\Omega; W^{2,p}(Y)), \\ \mathcal{T}_\varepsilon(\nabla\Phi_{p,\varepsilon}) \to \nabla\phi \quad & \text{strongly in} \quad L^p(\Omega; W^{1,p}(Y))^N, \\ \mathcal{T}_\varepsilon(D^2\Phi_{p,\varepsilon}) \rightharpoonup D^2\phi + D^2_y\widehat{\Phi}_p \quad & \text{weakly in} \quad L^p(\Omega \times Y)^{N\times N}. \end{aligned}$$

Note that the following relations hold ($i \in \{1, \ldots, N\}$):

$$\begin{cases} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{p,\varepsilon}) = \mathcal{T}^{\mathcal{S}}_\varepsilon(\Phi_{p,\varepsilon|\mathcal{S}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\Phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}}, \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}} = \mathcal{T}^{\mathcal{S}}_\varepsilon\big(\partial_{\mathbf{s}}\Phi_{p,\varepsilon|\mathcal{S}^{(i)}_\varepsilon}\big) = \mathcal{T}^{ext}_\varepsilon\big(\partial_i\Phi_{p,\varepsilon}\big)_{\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}}, \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}} = \mathcal{T}^{\mathcal{S}}_\varepsilon\big(\partial^2_{\mathbf{s}}\Phi_{p,\varepsilon|\mathcal{S}^{(i)}_\varepsilon}\big) = \mathcal{T}^{ext}_\varepsilon\big(\partial^2_i\Phi_{p,\varepsilon}\big)_{\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}}. \end{cases}$$

Hence, we can restrict the above convergences from $\Omega \times Y$ to the subsets $\Omega \times \mathcal{S}$ and $\Omega \times \mathcal{S}^{(i)}$, for every $i \in \{1, \ldots, N\}$.

Hence, there exist $\widehat{\phi}_p = \widehat{\Phi}_{p|\Omega\times\mathcal{S}} \in L^p(\Omega; W^{2,p}_{per}(\mathcal{S}))$ such that ($i \in \{1, \ldots, N\}$)

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{p,\varepsilon}) \to \phi \quad & \text{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})), \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{p,\varepsilon}) \to \partial_i\phi \quad & \text{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}^{(i)})), \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_{p,\varepsilon}) \rightharpoonup \partial^2_{ii}\phi + \partial^2_{\mathbf{s}}\widehat{\phi}_p \quad & \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}), \end{aligned}$$

where the strong convergences are preserved due to the polynomial character of the function $\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{p,\varepsilon})$ with respect to the second variable.

Hence, by the above convergences and (6.8) we get ($i \in \{1, \ldots, N\}$)

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi \quad & \text{strongly in} \quad L^2(\Omega; W^{2,p}(\mathcal{S})), \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \to \partial_i\phi \quad & \text{strongly in} \quad L^2(\Omega; W^{1,p}(\mathcal{S}^{(i)})), \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial^2_{ii}\phi + \partial^2_{\mathbf{s}}\big(\widehat{\phi}_p + \widehat{\phi}_0\big) \quad & \text{weakly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

Hence, the proof follows by setting $\widehat{\phi} \doteq \widehat{\phi}_p + \widehat{\phi}_0$, which belongs to $L^2(\Omega; W^{2,p}_{per}(\mathcal{S}))$. □

### 6.2 Unfolding via known results for sequences of functions uniformly bounded in $W^{1,p}$

We consider the sequences in $W^{2,p}(\mathcal{S}_\varepsilon)$ as sequences in $W^{1,p}(\mathcal{S}_\varepsilon)$ with partial derivatives belonging to $W^{1,p}(\mathcal{S}^{(i)}_\varepsilon)$, for each $i \in \{1, \ldots, N\}$. In this sense, we can apply the results obtained in Sect. 5. Even though no gradient extension is needed, the additional work must be done to show that the $N$ different limit functions, one for each partial derivative, are in fact a unique function restricted to each line.

From [7, Chapter 9], we recall that ($p \in (1, +\infty)$):

(i) if $u \in W^{1,p}(\Omega)$ satisfies $\Delta u \in L^p(\Omega)$ then $u \in W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ [3];

(ii) if $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary and if $u \in W^{1,p}_0(\Omega)$ satisfies $\Delta u \in L^p(\Omega)$ then $u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$.

Denote ($p \in [1, +\infty]$)

$$\mathcal{W}^{2,p}(\Omega) \doteq \Big\{\phi \in W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega) \;\Big|\; \partial^2_{ii} \in L^p(\Omega) \;\text{ for every }\; i \in \{1, \dots, N\}\Big\}.$$

We endow $\mathcal{W}^{2,p}(\Omega)$ with the following norm

$$\|\phi\|_{\mathcal{W}^{2,p}(\Omega)} \doteq \|\phi\|_{W^{1,p}(\Omega)} + \sum_{i=1}^N \|\partial^2_{ii}\phi\|_{L^p(\Omega)}.$$

**Theorem 6.4** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$, $p \in (1, +\infty)$, satisfying*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}. \tag{6.9}$$

*Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in \mathcal{W}^{2,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per,0}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)*

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) &\to \phi \quad \textit{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_i\phi \quad \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}),\\ \mathcal{T}^{\mathcal{S}}_\varepsilon\big(\partial^2_{\mathbf{s}}\phi_\varepsilon\big) &\rightharpoonup \partial^2_{ii}\phi + \partial^2_{\mathbf{S}}\widehat{\phi} \quad \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned} \tag{6.10}$$

*The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.*

***Proof*** *Step 1.* We prove convergences $(6.10)_{1,2}$.

By estimate (6.9), the sequence $\{\phi_\varepsilon\}_\varepsilon$ satisfies

$$\|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}$$

and thus by Corollary 5.8, there exist $\phi \in W^{1,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(\mathcal{S}))$ such that

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi \quad \text{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ &\mathcal{T}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}), \qquad i \in \{1, \dots, N\}. \end{aligned} \tag{6.11}$$

[3] In fact, we have $\rho D^2 u \in L^p(\Omega)^{N \times N}$ where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ for all $x \in \mathbb{R}^N$.

Now, we consider the sequences $\{\psi_\varepsilon^{(i)}\}_\varepsilon = \{\partial_{\mathbf{s}}\phi_{\varepsilon|\mathcal{S}_\varepsilon^{(i)}}\}_\varepsilon, i \in \{1, \dots, N\}$. From estimate (6.9) we have

$$\|\psi_\varepsilon^{(i)}\|_{W^{1,p}(\mathcal{S}_\varepsilon^{(i)})} \le C\varepsilon^{\frac{1-N}{p}}.$$

Since $\psi_\varepsilon^{(i)}$, $i \in \{1, \dots, N\}$, is defined on every node of $\mathcal{S}_\varepsilon$, we extend it as a function affine on every small segments in $\mathcal{S}_\varepsilon^{(j)}$, $j \in \{1, \dots, N\} \setminus \{i\}$. We still denote this extension $\psi_\varepsilon^{(i)}$. It satisfies

$$\|\psi_\varepsilon^{(i)}\|_{L^p(\mathcal{S}_\varepsilon)} + \left\|\partial_{\mathbf{s}}\psi_\varepsilon^{(i)}\right\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} + \varepsilon\left\|\partial_{\mathbf{s}}\psi_\varepsilon^{(i)}\right\|_{L^p(\mathcal{S}_\varepsilon^{[i]})} \le C\varepsilon^{\frac{1-N}{p}}, \quad \mathcal{S}_\varepsilon^{[i]} = \bigcup_{j=1,\, j\neq i}^{N} \mathcal{S}_\varepsilon^{(j)}.$$

Observe that a function defined and constant on every line of $\mathcal{S}^{(i)}$ can be extended to a function periodic on $\mathcal{S}$ and affine between two contiguous nodes of $\mathcal{S}^{(j)}$, where $j \in \{1, \dots, N\} \setminus \{i\}$. Lemma 5.7 gives a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\widetilde{\psi}^{(i)} \in L^p(\Omega, \partial_i; W^{1,p}_{per}(\mathcal{S}^{[i]}))$, $\widehat{\psi}^{(i)} \in L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$. Here, due to the above remark, we assume that $\boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}}(\widehat{\psi}^{(i)}) = 0$ a.e. in $\Omega \times \mathcal{S}^{(i)}$.

Thus, one has ($i \in \{1, \dots, N\}$)

$$\begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\psi_\varepsilon^{(i)}) &\rightharpoonup \widetilde{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\psi_\varepsilon^{(i)}) &\rightharpoonup \partial_i\widetilde{\psi}^{(i)} + \partial_{\mathbf{S}}\widehat{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

The above second convergence and $(6.11)_2$ yield

$$\partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} = \widetilde{\psi}^{(i)} \quad \text{a.e. in } \Omega \times \mathcal{S}^{(i)}, \quad i \in \{1, \dots, N\}.$$

Since $\widetilde{\psi}^{(i)}$ does not depend on $\mathbf{S}$ in $\mathcal{S}^{(i)}$ and $\widehat{\phi}$ is periodic with respect to $\mathbf{S}$ in $\mathcal{S}^{(i)}$ we have $\partial_i\phi = \widetilde{\psi}^{(i)}$ and $\partial_{\mathbf{S}}\widehat{\phi} = 0$ a.e. $\Omega \times \mathcal{S}^{(i)}$ for every $i \in \{1, \dots, N\}$.

Hence, $\widetilde{\psi}^{(i)}$ belongs to $L^p(\Omega, \partial_i)$ and thus $\partial_i\phi \in L^p(\Omega, \partial_i)$. Since $\Delta\phi \in L^p(\Omega)$, we have $\phi \in \boldsymbol{\mathcal{W}}^{2,p}(\Omega)$. Therefore, the following convergences hold:

$$\begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\to \phi \quad \text{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_i\phi \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}\big(\partial^2_{\mathbf{s}}\phi_\varepsilon\big) &\rightharpoonup \partial^2_{ii}\phi + \partial_{\mathbf{S}}\widehat{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

Moreover, we also have that, for each $i \in \{1, \dots, N\}$:

$$\frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) \rightharpoonup \partial^2_{ii}\phi\,\mathbf{S}^c + \widehat{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \tag{6.12}$$

*Step 2.* We prove the convergence $(6.10)_3$.

We have to prove the existence of $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per,0}(\mathcal{S}))$ such that

$$\begin{cases} \partial_{\mathbf{S}}\widehat{\phi} = \widehat{\psi}^{(1)} \quad \text{a.e. in } \ \Omega \times \mathcal{S}^{(1)}, \\ \qquad \vdots \\ \partial_{\mathbf{S}}\widehat{\phi} = \widehat{\psi}^{(N)} \quad \text{a.e. in } \ \Omega \times \mathcal{S}^{(N)}. \end{cases}$$

A necessary and sufficient condition to get existence of the function $\widehat{\phi}$ is (remind that $A(k + \mathbf{e}_i) = A(k) + l_i \mathbf{e}_i$)

$$\forall k \in \widehat{\mathbf{K}}, \quad \begin{aligned} &\int_{A(k)}^{A(k+\mathbf{e}_i)} \widehat{\psi}^{(i)}(\cdot, \mathbf{S})d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \widehat{\psi}^{(j)}(\cdot, \mathbf{S})d\mathbf{S} \\ &= \int_{A(k)}^{A(k+\mathbf{e}_j)} \widehat{\psi}^{(j)}(\cdot, \mathbf{S})d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \widehat{\psi}^{(i)}(\cdot, \mathbf{S})d\mathbf{S} \end{aligned} \tag{6.13}$$

a.e. in $\Omega$.

Since on a line belonging to $\mathcal{S}^{(i)}$, one has (see Lemma 5.4) $\mathbf{S}^c = t - \dfrac{1}{2}$, $t \in [0, 1]$, the above equality (6.13) is equivalent to $\forall k \in \widehat{\mathbf{K}}$,

$$\begin{aligned} &\int_{A(k)}^{A(k+\mathbf{e}_i)} \big(\partial^2_{ii}\phi\mathbf{S}^c + \widehat{\psi}^{(i)}(\cdot, \mathbf{S})\big)d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \big(\partial^2_{jj}\phi\mathbf{S}^c + \widehat{\psi}^{(j)}(\cdot, \mathbf{S})\big)d\mathbf{S} \\ &= \int_{A(k)}^{A(k+\mathbf{e}_j)} \big(\partial^2_{jj}\phi\mathbf{S}^c + \widehat{\psi}^{(j)}(\cdot, \mathbf{S})\big)d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \big(\partial^2_{ii}\phi\mathbf{S}^c + \widehat{\psi}^{(i)}(\cdot, \mathbf{S})\big)d\mathbf{S} \end{aligned} \tag{6.14}$$

a.e. in $\Omega$.

Convergence (6.12) gives (remind that $\partial^2_{ii}\phi$ does not depends on $\mathbf{S}$)

$$\forall k \in \widehat{\mathbf{K}} \qquad \begin{aligned} &\int_{A(k)}^{A(k+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big)d\mathbf{S} \\ &\to \int_{A(k)}^{A(k+\mathbf{e}_i)} \big(\partial^2_{ii}\phi\, \mathbf{S}^c + \widehat{\psi}^{(i)}\big)d\mathbf{S} \\ &= \partial^2_{ii}\phi \int_{k_i l_i}^{(k_i+1)l_i} \Big(t - \frac{1}{2}\Big)dt + \int_{A(k)}^{A(k+\mathbf{e}_i)} \widehat{\psi}^{(i)}(x, \mathbf{S})d\mathbf{S}.. \end{aligned}$$

Similarly, one has $(j \neq i)$

$$\begin{aligned} &\int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big)d\mathbf{S} \\ &\to \partial^2_{ii}\phi \int_{k_i l_i}^{(k_i+1)l_i} \Big(t - \frac{1}{2}\Big)dt + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \widehat{\psi}^{(i)}(x, \mathbf{S})\, d\mathbf{S} \end{aligned}$$

and same kind of results for the other two quantities.

Hence, to get (6.13), we have to prove that both quantities

$$\int_{A(k)}^{A(k+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big)d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(j)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big)d\mathbf{S} \tag{6.15}$$

and

$$\int_{A(k)}^{A(k+\mathbf{e}_j)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(j)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big)d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big)d\mathbf{S}. \tag{6.16}$$

admit the same limit or equivalently that the limit of their difference is 0.

First we note that

$$\begin{aligned}\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)d\mathbf{S} &= \frac{1}{\varepsilon}\int_{A(k)}^{A(k+\mathbf{e}_i)} \partial_{\mathbf{S}}\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)d\mathbf{S} \\ &= \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k+\mathbf{e}_i)\big) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k)\big)\Big) \quad \text{a.e. in } \widetilde{\Omega}_\varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)d\mathbf{S}\Big) \\ &\quad= \frac{1}{\varepsilon}\Big(\int_{A(k)}^{A(k+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)d\mathbf{S}\Big) \quad \text{a.e. in } \widetilde{\Omega}_\varepsilon.\end{aligned}$$

Now, recall that the function $\boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)$ is defined on $\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}$ and is constant on every line of $\mathcal{S}^{(i)}$. One has a.e. in $\widetilde{\Omega}_\varepsilon$

$$\begin{aligned}\boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &= \int_{A(k')}^{A(k')+\mathbf{e}_i} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\, d\mathbf{S} = \frac{1}{\varepsilon}\int_{A(k')}^{A(k')+\mathbf{e}_i} \partial_{\mathbf{S}}\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)d\mathbf{S} \\ &= \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k')+\mathbf{e}_i\big) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k')\big)\Big)\end{aligned}$$

on $\widetilde{\Omega}_\varepsilon \times [A(k'), A(k')+\mathbf{e}_i]$, $k' \in \widehat{\mathbf{K}}_i$.

Hence

$$\begin{aligned}&\int_{A(k)}^{A(k+\mathbf{e}_i)} \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)d\mathbf{S} \\ &\quad= \frac{l_i}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k')+\mathbf{e}_i\big) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k')\big)\Big) \quad \text{a.e. in } \widetilde{\Omega}_\varepsilon,\end{aligned}$$

where $k' \in \widehat{\mathbf{K}}_i$ is such that $k = k' + k_i \mathbf{e}_i$. Hence, we get

$$\frac{1}{\varepsilon}\Big(\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} - \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S}\Big)$$
$$= \frac{l_i}{\varepsilon^2}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k') + \mathbf{e}_i\big) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k')\big)$$
$$- \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k' + \mathbf{e}_j) + \mathbf{e}_i\big) + \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k' + \mathbf{e}_j)\big)\Big) \text{ a.e. in } \widetilde{\Omega}_\varepsilon$$

where $k' \in \widehat{\mathbf{K}}_i$ is such that $k = k' + k_i \mathbf{e}_i$.

Now, we can apply Lemma 8.1 and claim that the limit of the difference of the quantities in (6.15) and (6.16) is equal to 0. This proves (6.14) for every $k \in \widehat{\mathbf{K}}$. As a consequence, there exists a unique $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per,0}(\mathcal{S}))$ such that convergence $(6.10)_3$ holds. ☐

## 7 Application: homogenization of a fourth 4th order homogeneous Dirichlet problem on a periodic lattice structure

We can now give a direct application of the periodic unfolding for sequences in $H^2(\mathcal{S}_\varepsilon)$.

From now on, let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary. Let $\{A_\varepsilon\}_\varepsilon$ be the sequence of functions belonging to $L^\infty(\mathcal{S}_\varepsilon)$ defined by

$$A_\varepsilon(\mathbf{s}) \doteq A\Big(\Big\{\frac{\mathbf{s}}{\varepsilon}\Big\}\Big) \quad \text{for a.e. } \mathbf{s} \in \mathcal{S}_\varepsilon,$$

where $A$ belongs to $L^\infty(\mathcal{S})$ satisfies

$$\exists (c, C) \in (0, +\infty)^2 \quad \text{such that} \quad c \le A(\mathbf{S}) \le C \quad \text{for a.e. } \mathbf{S} \in \mathcal{S} \tag{7.1}$$

and let $\{g_\varepsilon\}_\varepsilon$ and $\{f_\varepsilon\}_\varepsilon$ be sequences in $L^2(\mathcal{S}_\varepsilon)$.

Set the space

$$H^1_0(\mathcal{S}_\varepsilon) \doteq \big\{\phi \in H^1(\mathcal{S}_\varepsilon) \;\big|\; \phi = 0 \text{ a.e. on } \partial\widetilde{\Omega}_\varepsilon \cap \mathcal{S}_\varepsilon\big\}.$$

By the Poincaré and Poincaré–Wirtinger inequalities, we have

$$\forall \phi \in H^1_0(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon), \qquad \|\phi\|_{L^2(\mathcal{S}_\varepsilon)} \le C\|\partial_{\mathbf{s}}\phi\|_{L^2(\mathcal{S}_\varepsilon)} \le C\|\partial^2_{\mathbf{s}}\phi\|_{L^2(\mathcal{S}_\varepsilon)},$$

where the constants do not depend on $\varepsilon$ (note that $\mathcal{M}_{\mathcal{S}^{(i)}}(\partial_{\mathbf{s}}\phi) = 0$ for every $i \in \{1, \dots, N\}$).

Consider the 4th order homogeneous Dirichlet problem in variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon) \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon \partial^2_{\mathbf{s}} u_\varepsilon \, \partial^2_{\mathbf{s}} \phi \, d\mathbf{s} = \int_{\mathcal{S}_\varepsilon} g_\varepsilon \, \partial_{\mathbf{s}} \phi \, d\mathbf{s} + \int_{\mathcal{S}_\varepsilon} f_\varepsilon \, \phi \, d\mathbf{s}, \quad \forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon). \end{cases} \tag{7.2}$$

The Lax−Milgram theorem implies that the problem (7.2) has a unique solution. Moreover, one has

$$\begin{aligned} c\|\partial^2_{\mathbf{s}} u_\varepsilon\|^2_{L^2(\mathcal{S}_\varepsilon)} &\le \|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \\ &\le C\big(\|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}\big)\|\partial^2_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}. \end{aligned}$$

Hence

$$\|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \le C\big(\|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}\big). \tag{7.3}$$

The constant does not depend on $\varepsilon$.

Below, we give the periodic homogenization via unfolding.

**Theorem 7.1** *Let $u_\varepsilon$ be the solution of problem (7.2) and $\{g_\varepsilon\}_\varepsilon$, $\{f_\varepsilon\}_\varepsilon$ be such that*

$$\begin{aligned} &\varepsilon^{\frac{1-N}{2}} \mathcal{T}^{\mathcal{S}}_\varepsilon(g_\varepsilon) \to g \quad \textit{strongly in} \quad L^2(\Omega \times \mathcal{S}), \\ &\varepsilon^{\frac{1-N}{2}} \mathcal{T}^{\mathcal{S}}_\varepsilon(f_\varepsilon) \to f \quad \textit{strongly in} \quad L^2(\Omega \times \mathcal{S}). \end{aligned} \tag{7.4}$$

*Then, there exist functions $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^2_{per,0}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)*

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(u_\varepsilon) \to u \quad \textit{strongly in} \quad L^2(\Omega; H^2(\mathcal{S})), \\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}} u_\varepsilon) \rightharpoonup \partial_i u \quad \textit{weakly in} \quad L^2(\Omega; H^1(\mathcal{S}^{(i)})), \\ &\mathcal{T}^{\mathcal{S}}_\varepsilon\big(\partial^2_{\mathbf{s}} u_\varepsilon\big) \to \partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u} \quad \textit{strongly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned} \tag{7.5}$$

*The couple $(u, \widehat{u})$ is the unique solution of problem*

$$\begin{cases} \displaystyle\sum_{i=1}^N \frac{1}{|\mathcal{S}|} \int_{\Omega \times \mathcal{S}^{(i)}} A\left(\partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u}\right)\left(\partial^2_{ii} \phi + \partial^2_{\mathbf{S}} \widehat{\phi}\right) dx d\mathbf{S} \\ \displaystyle= \int_\Omega G \cdot \nabla \phi \, dx + \int_\Omega F \, \phi \, dx, \\ \forall \phi \in H_0^1(\Omega) \cap H^2(\Omega) \quad \textit{and} \quad \forall \widehat{\phi} \in L^2(\Omega; H^2_{per,0}(\mathcal{S})) \end{cases} \tag{7.6}$$

*where*

$$G \doteq \sum_{i=1}^N \frac{1}{|\mathcal{S}|}\Big(\int_{\mathcal{S}^{(i)}} g(\cdot, \mathbf{S}) \, d\mathbf{S}\Big)\mathbf{e}_i, \qquad F \doteq \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} f(\cdot, \mathbf{S}) \, d\mathbf{S}.$$

***Proof*** The solution $u_\varepsilon$ of (7.2) satisfies (7.3). Due to the convergences (7.4) we have that

$$\|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{2}}.$$

The constant does not depend on $\varepsilon$.

Hence, up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, Theorem 6.4 gives functions $u \in H^1_0(\Omega)\cap H^2(\Omega)$ and $\widehat{u} \in L^p(\Omega; H^2_{per,0}(\mathcal{S}))$ such that the following convergences hold ($i \in \{1, \dots, N\}$):

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_\varepsilon(u_\varepsilon) &\to u \quad \text{strongly in} \quad L^2(\Omega; H^2(\mathcal{S})),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} u_\varepsilon) &\rightharpoonup \partial_i u \quad \text{weakly in} \quad L^2(\Omega; H^1(\mathcal{S}^{(i)})),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}\big(\partial^2_{\mathbf{s}} u_\varepsilon\big) &\rightharpoonup \partial^2_{ii} u + \partial^2_{\mathbf{S}}\widehat{u} \quad \text{weakly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned} \tag{7.7}$$

Now, we choose the test functions

- $\phi$ in $\mathcal{C}^\infty(\overline{\Omega}) \cap H^1_0(\Omega)$,
- $\Phi$ in $\mathcal{D}(\Omega)$ ,
- $\widehat{\phi}$ in $H^2_{per,0}(\mathcal{S})$.

Set

$$\phi_\varepsilon(x) \doteq \varepsilon^{\frac{1-N}{2}}\Big(\phi(\mathbf{s}) + \varepsilon^2\Phi(\mathbf{s})\widehat{\phi}\Big(\frac{\mathbf{s}}{\varepsilon}\Big)\Big), \qquad \text{a.e. } \mathbf{s} \in \mathcal{S}_\varepsilon.$$

Applying the unfolding operator to the sequence $\{\phi_\varepsilon\}_\varepsilon$, we get ($i \in \{1, \dots, N\}$)

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) &\to \phi \quad \text{strongly in} \quad L^2(\Omega; H^2(\mathcal{S})),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\to \partial_i\phi \quad \text{strongly in} \quad L^2(\Omega; H^1(\mathcal{S}^{(i)})),\\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial^2_{\mathbf{s}}\phi_\varepsilon) &\to \partial^2_{ii}\phi + \Phi\partial^2_{\mathbf{S}}\widehat{\phi} \quad \text{strongly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

Taking $\phi_\varepsilon$ as test function in (7.2), then transforming by unfolding and passing to the limit give (7.6) with $(\phi, \Phi\widehat{\phi})$. By density argumentation, we extend such results to all $\phi \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\widehat{\phi} \in L^2(\Omega; H^2_{per,0}(\mathcal{S}))$. Since the solution is unique, the whole sequences converge to their limit.

To conclude the proof, it is left to show that the third convergence in (7.7) is in fact strong. Taking $\phi = u_\varepsilon$ in (7.2), then transforming by unfolding and using the weak lower semicontinuity yield

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega\times\mathcal{S}^{(i)}} A\left|\partial^2_{ii}u+\partial^2_{\mathbf{S}}\widehat{u}\right|^2 dxd\mathbf{S}\\
&\quad\le \liminf_{\varepsilon\to 0}\sum_{i=1}^{N}\int_{\Omega\times\mathcal{S}}\mathcal{T}_\varepsilon(A_\varepsilon)\left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}u_\varepsilon)\right|^2 dxd\mathbf{S}\le \liminf_{\varepsilon\to 0}\varepsilon^{N-1}\sum_{i=1}^{N}\int_{\mathcal{S}_\varepsilon}A_\varepsilon\left|\partial^2_{\mathbf{s}}u_\varepsilon\right|^2 d\mathbf{s}\\
&\quad\le \limsup_{\varepsilon\to 0}\varepsilon^{N-1}\sum_{i=1}^{N}\int_{\mathcal{S}_\varepsilon}A_\varepsilon\left|\partial^2_{\mathbf{s}}u_\varepsilon\right|^2 d\mathbf{s}=\limsup_{\varepsilon\to 0}\varepsilon^{N-1}\Big(\int_{\mathcal{S}_\varepsilon}g_\varepsilon\,\partial_{\mathbf{s}}u_\varepsilon\,d\mathbf{s}+\int_{\mathcal{S}_\varepsilon}f_\varepsilon\,u_\varepsilon\,d\mathbf{s}\Big)\\
&\quad=|\mathcal{S}|\Big(\int_\Omega G\cdot\nabla\phi\,dx+\int_\Omega F\,\phi\,dx\Big)=\sum_{i=1}^{N}\int_{\Omega\times\mathcal{S}^{(i)}}A\left|\partial^2_{ii}u+\partial^2_{\mathbf{S}}\widehat{u}\right|^2 dxd\mathbf{S}.
\end{aligned}$$

Also observe that

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\sum_{i=1}^{N}\int_{\Omega\times\mathcal{S}}\mathcal{T}_\varepsilon(A_\varepsilon)\left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}u_\varepsilon)\right|^2 dxd\mathbf{S}&\le \limsup_{\varepsilon\to 0}\sum_{i=1}^{N}\int_{\Omega\times\mathcal{S}}\mathcal{T}_\varepsilon(A_\varepsilon)\left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}u_\varepsilon)\right|^2 dxd\mathbf{S}\\
&\le \limsup_{\varepsilon\to 0}\varepsilon^{N-1}\sum_{i=1}^{N}\int_{\mathcal{S}_\varepsilon}A_\varepsilon\left|\partial^2_{\mathbf{s}}u_\varepsilon\right|^2 d\mathbf{s}
\end{aligned}$$

From the above inequalities it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\int_{\mathcal{S}_\varepsilon}\mathcal{T}^{\mathcal{S}}_\varepsilon(A_\varepsilon)\left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}u_\varepsilon)\right|^2 dxd\mathbf{S}\\
&\quad=\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\int_{\mathcal{S}_\varepsilon}A_\varepsilon\left|\partial^2_{\mathbf{s}}u_\varepsilon\right|^2 d\mathbf{s}=\sum_{i=1}^{N}\int_{\Omega\times\mathcal{S}^{(i)}}A\left|\partial^2_{ii}u+\partial^2_{\mathbf{S}}\widehat{u}\right|^2 dxd\mathbf{S}.
\end{aligned}$$

Since the map $\Psi\in L^2(\Omega\times\mathcal{S})\longmapsto\sqrt{\int_{\Omega\times\mathcal{S}}A\,|\Psi|^2\,dxd\mathbf{S}}$ is a norm equivalent to the usual norm of $L^2(\Omega\times\mathcal{S})$, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{S}}\left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}u_\varepsilon)\right|^2 dxd\mathbf{S}=\int_{\Omega\times\mathcal{S}}\left|\partial^2_{ii}u+\partial^2_{\mathbf{S}}\widehat{u}\right|^2 dxd\mathbf{S}.$$

This, together with the fact that (7.7)$_3$ already converges weakly, ensures the strong convergence. □

We define the corrector $\widehat{\chi}_k$, $k\in\{1,\dots,N\}$, as the unique solution in $H^2_{per,0}(\mathcal{S})$ of the cell problem

$$\int_{\mathcal{S}}A\left(\mathbf{1}_{\mathcal{S}^{(k)}}+\partial^2_{\mathbf{S}}\widehat{\chi}_k\right)\partial^2_{\mathbf{S}}\widehat{w}\,d\mathbf{S}=0,\quad\forall\widehat{w}\in H^2_{per,0}(\mathcal{S}).\tag{7.8}$$

**Theorem 7.2** *The function $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique solution of the following homogenized problem:*

$$\begin{cases} \displaystyle\int_\Omega A^{hom}\, \partial^2 u \cdot \partial^2 \phi \, dx = \int_\Omega G \cdot \nabla \phi \, dx + \int_\Omega F\, \phi \, dx, \\ \forall \phi \in H_0^1(\Omega) \cap H^2(\Omega), \end{cases} \tag{7.9}$$

*where* $\partial^2 u \doteq \begin{pmatrix} \partial^2_{11} u \\ \vdots \\ \partial^2_{NN} u \end{pmatrix}$ *and* $\partial^2 \phi \doteq \begin{pmatrix} \partial^2_{11} \phi \\ \vdots \\ \partial^2_{NN} \phi \end{pmatrix}$.

*The homogenized matrix $A^{hom}$ is given by ($(i, j) \in \{1, \dots, N\}^2$)*

$$A^{hom}_{ij} \doteq \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A\left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_i\right)\left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_j\right) d\mathbf{S}. \tag{7.10}$$

***Proof*** Equation (7.6) with $\phi = 0$ leads to

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_{\Omega \times \mathcal{S}^{(i)}} A\left(\partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u}\right) \partial^2_{\mathbf{S}} \widehat{\phi} \, dx d\mathbf{S} = 0, \\ \forall \widehat{\phi} \in L^2(\Omega; H^2_{per,0}(\mathcal{S})), \end{cases}$$

from which we obtain the form of the cell problems (7.8) and thus the representation of $\widehat{u}$

$$\widehat{u}(x, \mathbf{S}) = \sum_{k=1}^N \partial^2_{kk} u(x)\, \widehat{\chi}_k(\mathbf{S}), \qquad \text{for a.e. } (x, \mathbf{S}) \in \Omega \times \mathcal{S}.$$

Replacing the above expression of $\widehat{u}$ in (7.6) and choosing

$$\widehat{\phi}(x, \mathbf{S}) = \sum_{k=1}^N \partial^2_{kk} \phi(x)\, \widehat{\chi}_k(\mathbf{S}), \qquad \text{for a.e. } (x, \mathbf{S}) \in \Omega \times \mathcal{S}$$

lead to the following left hand side of (7.6):

$$\begin{aligned} &\frac{1}{|\mathcal{S}|} \int_{\Omega \times \mathcal{S}} A\left(\sum_{i=1}^N \left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_i\right)\partial^2_{ii} u\right)\left(\sum_{j=1}^N \left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_j\right)\partial^2_{jj} \phi\right) dx d\mathbf{S} \\ &= \int_\Omega \sum_{i,j=1}^N \left(\frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A\left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_i\right)\left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_j\right) d\mathbf{S}\right) \partial^2_{ii} u \, \partial^2_{jj} \phi \, dx. \end{aligned}$$

Taking into account (7.8), the above expression becomes $\displaystyle\int_\Omega A^{hom}\, \partial^2 u \cdot \partial^2 \phi \, dx$ with the matrix $A^{hom}$ given by (7.10).

We prove now that $A^{hom}$ is coercive. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ be a vector with fixed entries. From (7.10) we first have

$$A^{hom}\boldsymbol{\xi}\cdot\boldsymbol{\xi} = \frac{1}{|\mathcal{S}|}\sum_{i,j=1}^{N}\int_{\mathcal{S}} A\left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}}\widehat{\chi}_i\right)\left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}}\widehat{\chi}_j\right) d\mathbf{S}\, \xi_i\, \xi_j$$
$$= \frac{1}{|\mathcal{S}|}\int_{\mathcal{S}} A\left(\widetilde{\boldsymbol{\xi}} + \partial^2_{\mathbf{S}}\widehat{\chi}_{\boldsymbol{\xi}}\right)^2 d\mathbf{S}$$

where

$$\widetilde{\boldsymbol{\xi}} \doteq \sum_{i=1}^{N}\xi_i \mathbf{1}_{\mathcal{S}^{(i)}}, \qquad \widehat{\chi}_{\boldsymbol{\xi}} = \sum_{k=1}^{N}\xi_k\,\widehat{\chi}_k, \quad \text{a.e. in } \mathcal{S} \text{ and for all } \boldsymbol{\xi}\in\mathbb{R}^N.$$

Then, by hypothesis (7.1) on $A$, we get

$$A^{hom}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \ge \frac{c}{|\mathcal{S}|}\left\|\widetilde{\boldsymbol{\xi}} + \partial^2_{\mathbf{S}}\widehat{\chi}_{\boldsymbol{\xi}}\right\|^2_{L^2(\mathcal{S})}.$$

By the periodicity of $\partial_{\mathbf{S}}\widehat{\chi}_{\boldsymbol{\xi}}$, for every $\boldsymbol{\xi}\in\mathbb{R}^N$ we get that

$$\left\|\widetilde{\boldsymbol{\xi}} + \partial^2_{\mathbf{S}}\widehat{\chi}_{\boldsymbol{\xi}}\right\|^2_{L^2(\mathcal{S})} = \left\|\widetilde{\boldsymbol{\xi}}\right\|^2_{L^2(\mathcal{S})} + \left\|\partial^2_{\mathbf{S}}\widehat{\chi}_{\boldsymbol{\xi}}\right\|^2_{L^2(\mathcal{S})} \ge \left\|\widetilde{\boldsymbol{\xi}}\right\|^2_{L^2(\mathcal{S})}$$
$$= \sum_{i=1}^{N}|\mathcal{S}^{(i)}||\xi_i|^2 \ge \min_k|\mathcal{S}^{(k)}|\sum_{i=1}^{N}|\xi_i|^2 = \left(\min_k|\mathcal{S}^{(k)}|\right)|\boldsymbol{\xi}|^2.$$

Thus the coercivity of $A^{hom}$ is proved since

$$A^{hom}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \ge c\,|\boldsymbol{\xi}|^2, \qquad \forall\boldsymbol{\xi}\in\mathbb{R}^N.$$

By the coercivity of $A^{hom}$ and the fact that $u \in H^1_0(\Omega)\cap H^2(\Omega)$, problem (7.9) admits a unique solution. □

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## 8 Appendix

**Lemma 8.1** *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$, $p \in (1, +\infty)$, satisfying*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}.$$

*For every $k' \in \widehat{\mathbf{K}}$ we define in $\widetilde{\Omega}_\varepsilon \times \widehat{\mathbf{K}}_i$ the piecewise constant function $\Phi_\varepsilon^{(i,j)}$, where $(i, j) \in \{1, \dots, N\}^2$, $i \neq j$, by*

$$\Phi_\varepsilon^{(i,j)}(\cdot, k') \doteq \begin{cases} \dfrac{l_i}{\varepsilon^2}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k') + \mathbf{e}_i\big) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k')\big) \\ \quad - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k' + \mathbf{e}_j) + \mathbf{e}_i\big) + \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot, A(k' + \mathbf{e}_j)\big)\Big) \\ \quad a.e.\ in\ \ \widetilde{\Omega}_\varepsilon \times \widehat{\mathbf{K}}_i, \\ \quad 0 \qquad a.e.\ in\ \ \big(\mathbb{R}^N \setminus \overline{\widetilde{\Omega}_\varepsilon}\big) \times \widehat{\mathbf{K}}_i. \end{cases}$$

*Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and a function $\phi$ in $W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ such that ($(i, j) \in \{1, \dots, N\}^2$, $i \neq j$, $k' \in \widehat{\mathbf{K}}_i$)*

$$\begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\to \phi \qquad \textit{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_j\phi \qquad \textit{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}^{(j)})), \\ \Phi_\varepsilon^{(i,j)}(\cdot, k') &\rightharpoonup -l_i l_j \partial^2_{ij}\phi \qquad \textit{weakly in} \quad W^{-1,p}(\mathbb{R}^N). \end{aligned} \tag{8.1}$$

***Proof*** There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and a function $\phi$ in the space $W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ such that convergences $(8.1)_{1,2}$ hold (see Theorem 6.4).

Now, let $\psi$ be in $W^{1,p'}(\mathbb{R}^N)$, one has

$$\begin{aligned} &\int_\Omega \psi(x)\, \Phi_\varepsilon^{(i,j)}(x, k')\, dx \\ &= \varepsilon^N \sum_{\xi \in \mathbb{Z}^N} \mathcal{M}_Y(\psi)(\varepsilon\xi) \frac{l_i}{\varepsilon^2}\Big(\phi_\varepsilon\big(\varepsilon\xi + \varepsilon A(k') + \varepsilon\mathbf{e}_i\big) - \phi_\varepsilon\big(\varepsilon\xi + \varepsilon A(k')\big) \\ &\quad - \phi_\varepsilon\big(\varepsilon\xi + \varepsilon A(k' + \varepsilon\mathbf{e}_j) + \varepsilon\mathbf{e}_i\big) + \phi_\varepsilon\big(\varepsilon\xi + \varepsilon A(k' + \mathbf{e}_j)\big)\Big) \\ &= \varepsilon^N l_i \sum_{\xi \in \mathbb{Z}^N} \frac{\mathcal{M}_Y(\psi)(\varepsilon\xi - \varepsilon\mathbf{e}_i) - \mathcal{M}_Y(\psi)(\varepsilon\xi)}{\varepsilon} \\ &\quad \cdot \frac{\phi_\varepsilon\big(\varepsilon\xi + \varepsilon A(k')\big) - \phi_\varepsilon\big(\varepsilon\xi + \varepsilon A(k' + \mathbf{e}_j)\big)}{\varepsilon} \\ &= l_i \int_\Omega \frac{\psi - \psi(\cdot - \varepsilon\mathbf{e}_i)}{\varepsilon}\Big(\int_{A(k')}^{A(k'+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\, d\mathbf{S}\Big) dx. \end{aligned}$$

Then, due to convergences (8.1)$_2$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \psi(x)\,\Phi_\varepsilon^{(i,j)}(x,k')\,dx = l_i\int_\Omega \partial_i\psi\Big(\int_{A(k')}^{A(k'+\mathbf{e}_j)}\partial_j\phi\,d\mathbf{S}\Big)dx = l_il_j\int_\Omega \partial_i\psi\,\partial_j\phi\,dx.$$

Hence, (8.1)$_3$ is proved. □

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