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Periodicity of hermitian K-groups

by

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Abstract

Bott periodicity for the unitary and symplectic groups is fundamental to topological K-theory. Analogous to unitary topological K-theory, for algebraic K-groups with finite coefficients, similar results are consequences of the Milnor and Bloch-Kato conjectures, affirmed by Voevodsky, Rost and others. More generally, we prove that periodicity of the algebraic K-groups for any ring implies periodicity for the hermitian K-groups, analogous to orthogonal and symplectic topological K-theory.

The proofs use in an essential way higher KSC-theories, extending those of Anderson and Green. They also provide an upper bound for the higher hermitian K-groups in terms of higher algebraic K-groups.

We also relate periodicity to étale hermitian K-groups by proving a hermitian version of Thomason's étale descent theorem. The results are illustrated in detail for local fields, rings of integers in number fields, smooth complex algebraic varieties, rings of continuous functions on compact spaces, and group rings.

Key Words: Bott periodicity, higher hermitian K-groups, KSC-theories, étale cohomology.

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0. Introduction and statements of main results

By the fundamental work of Bott [11] it is known that the homotopy groups of classical Lie groups are periodic, of period 2 or 8. For instance, the general linear and symplectic groups satisfy the isomorphisms:

$$\pi_n(\mathrm{GL}(\mathbb{R})) \cong \pi_{n+8}(\mathrm{GL}(\mathbb{R}))$$

$$\pi_n(\mathrm{Sp}(\mathbb{C})) \cong \pi_{n+8}(\mathrm{Sp}(\mathbb{C}))$$

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$$\pi_n(\mathrm{GL}(\mathbb{C})) \cong \pi_{n+2}(\mathrm{GL}(\mathbb{C}))$$

These periodicity statements were interpreted by Atiyah, Hirzebruch and others in the framework of topological K-theory of a Banach algebra A: recall that there are isomorphisms

$$K_n^{\mathrm{top}}(A) \cong K_{n+p}^{\mathrm{top}}(A),$$

where $K_n^{\mathrm{top}}(A) = \pi_{n-1}(\mathrm{GL}(A))$ if $n > 0$ and $K_0^{\mathrm{top}}(A) = K(A)$ is the usual Grothendieck group. Here p is the period which is 2 or 8 according as A is complex or real. We refer to [37] and [50] for an overview of the subject, both algebraically and topologically.

A few years later, after higher algebraic K-theory was introduced by Quillen, an analogous periodicity statement was sought, of the form

$$K_n(A) \cong K_{n+p}(A),$$

where A is now a discrete ring. The first computations showed that a periodicity isomorphism of this form is far from true in basic examples. However, if we consider K-theory with finite coefficients, and n is at least a certain bound d, then some periodicity conjectures appeared feasible, at least for certain rings of a geometric nature. These conjectures were formulated for different prime power coefficient groups, and are essentially of the following type ($n \geq d$)

$$K_n(A;\mathbb{Z}/m) \cong K_{n+p}(A;\mathbb{Z}/m).$$

The relationship between the prime power m and the associated smallest period p is given by the following convention, which we maintain throughout the paper.

Convention 0.1 For $\mathbb{Z}/m$ coefficients, where $m = \ell^v$ with ℓ prime, the smallest period p is given by

$$p = \begin{cases} \sup\bigl(8, \ell^{v-1}\bigr) & \text{if } \ell = 2, \\ 2(\ell-1)\ell^{v-1} & \text{otherwise.} \end{cases}$$

Using techniques of algebraic geometry and a comparison theorem with étale K-theory, numerous examples listed below showed that these conjectures hold. In the case of a 2-power, the first three are particular cases of Theorem 2 in [52], based on the fundamental work of Voevodsky [62]. In the case of an odd prime power, the first four examples are consequences of the Bloch-Kato conjecture.

Before giving these examples, we define the mod 2 virtual étale cohomological dimension $\mathrm{vcd}_2(A)$ of a commutative ring A as the mod 2 étale cohomological dimension of $A \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_4]$ obtained by adjoining a primitive fourth root of unity to A. For convenience, if ℓ is odd, then $\mathrm{vcd}_\ell(A)$ denotes the mod ℓ étale cohomological dimension $\mathrm{cd}_\ell(A)$ of A. For ℓ fixed, here are the examples we consider.

1. Any field k of characteristic $\mathrm{char}(k) \neq \ell$ for which $\mathrm{vcd}_\ell(k) < \infty$. In this case, $d = \mathrm{vcd}_\ell(k) - 1$ if $\mathrm{vcd}_\ell(k) \neq 0$ and $d = 0$ otherwise.

2. The ring $\mathcal{O}_F[1/\ell]$ of ℓ-integers in any number field F. In this case, $d = \mathrm{vcd}_\ell(F) - 1 = 1$ (*cf.* [43] when $\ell = 2$).

3. Any finitely generated and regular $\mathbb{Z}[1/\ell]$-algebra A with finite mod ℓ virtual étale cohomological dimension. In this case $d = \sup\{\mathrm{vcd}_\ell(k(s)) - 1, 0\}$, where $k(s)$ is the residue field at any point $s \in \mathrm{Spec}(A)$. The same statement holds when replacing $\mathbb{Z}[1/\ell]$ by $\mathbb{Q}$ or by any other field k of characteristic $\neq \ell$.
The regularity assumption on A can be dispensed with when working with negative K-theory [4], [30], [31], [61]. As shown in [53, Theorem 4.5], this does not change the bound d.

4. Group rings $R[G]$, where G is finite and R is a ring of ℓ-integers in a number field, as shown in [66]. Here $d = 1$. For some explicit computations see [42].

5. The ring $C(X)$ of real or complex continuous functions on a compact space X, as shown in [18], [46]. In this case $d = 1$.

In these examples, the periodicity isomorphism between the groups $K_n(A; \mathbb{Z}/m)$ and $K_{n+p}(A; \mathbb{Z}/m)$ is defined by taking cup-product with a "Bott element" b_K. For $p = 2^{\nu-1}$ with $\nu \geq 4$, one can construct this element in the group $K_p(\mathbb{Z}; \mathbb{Z}/2p)$, such that its image in the topological K-group[1] $K_p(\mathbb{R}; \mathbb{Z}/2p) \cong \mathbb{Z}/2p$ is the class $\mathrm{mod} 2p$ of a generator in $K_p(\mathbb{R}) \cong \mathbb{Z}$. The cup-product alluded to above is a pairing

$$\cup : K_n(A; \mathbb{Z}/m) \times K_p(\mathbb{Z}; \mathbb{Z}/2p) \longrightarrow K_{n+p}(A; \mathbb{Z}/m).$$

We refer to Section 1 for precise definitions and the extension to odd prime powers.

As we can see in these examples, a key role is played by the infinite general linear group $\mathrm{GL}(A)$. However, it was already shown in the works of Bott and Borel [10], and also in topological applications, that other infinite series associated to classical Lie groups may be considered as well. More precisely, if we consider a ring with involution A and a sign of symmetry $\varepsilon = \pm 1$ generalizing the orthogonal ($\varepsilon = 1$) or symplectic ($\varepsilon = -1$) case, one defines higher hermitian K-groups, denoted in this paper ${}_\varepsilon KQ_n(A)$, in a parallel way to algebraic K-groups $K_n(A)$. These groups are associated to the infinite ε-orthogonal group ${}_\varepsilon O(A)$. A typical example is when A is commutative and $\varepsilon = -1$, in which case one recovers the infinite symplectic

[1] We shall write K_n instead of K_n^{top} when dealing with the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers with their usual topology, and likewise for spectra.

group on A. We refer to the survey paper [37] already mentioned above for precise definitions.

The main purpose of this paper is to show that a periodicity statement in algebraic K-theory implies a similar one in KQ-theory, when $1/2 \in A$. Since KQ-theory with coefficients $\mathbb{Z}/m$, with $m = 2^v$, is the most important and difficult case, we state the main theorems in this context, leaving the case of odd prime power coefficients to the end of this Introduction and to Section 5 of the main body of the paper.

For the first step in the argument, we introduce a parameter q that is essentially p, apart from a slight modification in the case $m = 16$. Specifically, we make the following convention.

Convention 0.2

$$q = \begin{cases} 8 & \text{if } m \leq 8, \\ 16 & \text{if } m = 16, \\ m/2 & \text{otherwise.} \end{cases}$$

In other words, $q = p$ except when $m = 16$, in which case $q = 2p$. It is meaningful to speak of periodicity maps raising dimension by q, since q is a multiple of p.

As a convenient notation, we write $\overline{KQ}$ (resp. $\overline{K}$) for the KQ-theory (resp. K-theory) with coefficients in $\mathbb{Z}/m$, the relationship between m and the period p being as in Convention 0.1. One of our main theorems is the following.

Theorem 0.3 *With the above definitions, assume that there exists an integer d such that the cup-product map*

$$\cup b_K : \overline{K}_n(A) \longrightarrow \overline{K}_{n+p}(A)$$

with the Bott element in $K_p(\mathbb{Z}; \mathbb{Z}/2p)$ is an isomorphism whenever $n \geq d$. Then, for $n \geq d + q - 1$, there is also an isomorphism

$$_\varepsilon\overline{KQ}_n(A) \cong {}_\varepsilon\overline{KQ}_{n+p}(A).$$

Surprisingly, the isomorphism between the KQ-groups is in general not given by cup-product with a Bott element (see Remark 4.8 in Section 4). This relates to the fact that hermitian K-theory possesses more than one Bott element, as we now describe. Whereas in algebraic K-theory universal Bott elements are to be found in the K-groups of the integers $\mathbb{Z}$, here, because we are working with rings containing $1/2$, our Bott elements are to be found in the hermitian K-groups of the ring of 2-integers $\mathbb{Z}' = \mathbb{Z}[1/2]$.

As in algebraic K-theory, using the methods of [5], in this paper we prove the existence of a "positive Bott element" b^+ in ${}_1KQ_p(\mathbb{Z}';\mathbb{Z}/2p)$ whose image in $K_p(\mathbb{Z}';\mathbb{Z}/2p) \cong K_p(\mathbb{Z};\mathbb{Z}/2p)$ is the Bott element in K-theory alluded to above.

On the other hand, one of the main differences between algebraic and hermitian K-theory in our context is the existence of another element[2] u in ${}_{-1}KQ_{-2}(\mathbb{Z}')$, which plays an important role in the fundamental theorem in hermitian K-theory [33]. We now define the *negative Bott element* b^- in hermitian K-theory to be the image of the element $u^{p/2}$ in the group ${}_1KQ_{-p}(\mathbb{Z}';\mathbb{Z}/2p)$.

To make the statement of Theorem 0.3 more precise, we note that the cup-product with the positive Bott element in ${}_1KQ_p(\mathbb{Z}';\mathbb{Z}/2p)$ determines a direct system of abelian groups

$$ {}_\varepsilon\overline{KQ}_n(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+p}(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+2p}(A) \longrightarrow \cdots . $$

Symmetrically, cup-product with the negative Bott element in ${}_1KQ_{-p}(\mathbb{Z}';\mathbb{Z}/2p)$ determines an inverse system of abelian groups

$$ \cdots \longrightarrow {}_\varepsilon\overline{KQ}_{n+2p}(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+p}(A) \longrightarrow {}_\varepsilon\overline{KQ}_n(A). $$

The theorem above can now be restated in a more precise form. (Recall that the overbar denotes $\mathbb{Z}/m$ coefficients.)

Theorem 0.4 *Let A be any ring (with $1/2 \in A$), m, p and q be 2-powers as in Conventions 0.1 and 0.2, and let $d \in \mathbb{Z}$, such that the cup-product with the Bott element b_K in $K_p(\mathbb{Z};\mathbb{Z}/2p)$ induces an isomorphism*

$$ \overline{K}_n(A) \xrightarrow{\cong} \overline{K}_{n+p}(A) $$

whenever $n \geq d$. Then, for $n \geq d$, there is an exact sequence

$$ \cdots \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_{n+1}(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{n+1+ps}(A) $$

$$ \to \varprojlim_\varepsilon \overline{KQ}_{n+ps}(A) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{n+ps}(A) $$

where θ^+ (respectively θ^-) is induced from the cup-product with the positive Bott element b^+ (resp. the negative Bott element b^-). Moreover, for $n \geq d+q-1$, there is a short split exact sequence

$$ 0 \to \varprojlim_\varepsilon \overline{KQ}_{n+ps}(A) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{n+ps}(A) \to 0. $$

[2] We recall that the negative K-groups of a regular noetherian ring (for instance $\mathbb{Z}$ or $\mathbb{Z}'$) are trivial.

It turns out that the inverse limit is not always trivial. This point is discussed in Section 2 (where the inverse limit vanishes) and Section 4 (where it does not).

However, for rings of geometric nature and of finite mod 2 virtual étale cohomological dimension, we conjecture that the inverse limit is trivial.

Definition 0.5 We say that a ring A is *hermitian regular* if $\varprojlim_{\varepsilon} \overline{KQ}_{n+ps}(A)$ and $\varprojlim^1{}_{\varepsilon} \overline{KQ}_{n+ps}(A)$ are trivial[3].

Remark 0.6 It should be noted that subsequent to the original submission of this paper at the beginning of February 2010, as a consequence of a more recent theorem of Hu, Kriz and Ormsby [25] in characteristic 0, the authors and M. Schlichting proved independently that a field of characteristic 0 that is of finite mod 2 virtual étale cohomological definition is hermitian regular. Furthermore, Schlichting extended this theorem for fields of characteristic $p > 0$ with the same cohomological properties. This affirms Conjecture 6.6 of the present paper, which implies in turn our Conjecture 0.14 and therefore considerably extends the number of examples of commutative rings (and schemes) that are hermitian regular. The details of the proofs will appear in a forthcoming joint paper of the authors and Schlichting [7]. A particular example quoted below is given by suitable rings of integers in a number field. In Theorem 0.10, we state the periodicity theorem in this case with an independent proof which will be given in Section 2. A more general theorem is as follows.

Theorem 0.7 *Let A be a ring which is hermitian regular and satisfies the hypothesis of the previous theorem for its K-groups. Then for $n \geq d$, the cup-product with the positive Bott element induces an isomorphism*

$$ {}_{\varepsilon}\overline{KQ}_n(A) \xrightarrow{\cong} {}_{\varepsilon}\overline{KQ}_{n+p}(A). $$

More generally, in order to fully exploit the spectrum approach and to improve the previous theorems, we may consider a pointed CW-complex X and define the group $K_X(A)$ as the group of homotopy classes of pointed maps from X to $\mathcal{K}(A)$, where $\mathcal{K}$ denotes the K-theory spectrum. If X is a pointed sphere S^n, we recover Quillen's K-group $K_n(A)$. For brevity, we shall also write $K_{X+t}(A)$ instead of $K_{X\wedge S^t}(A)$, and $K_{X-t}(A)$ instead of $K_X(S^t A)$, where $S^t A$ denotes the t-iterated suspension of A (see for instance [37] for the definition of the suspension and the basic definitions of various K-theories). We adopt the same conventions for hermitian K-theory, and also for algebraic or hermitian K-theory with coefficients, and finally for spectra.

[3] As a matter of fact, with our hypothesis about periodicity of the K-groups, we always have $\lim^1 = 0$, since the inverse system satisfies the Mittag-Leffler condition as we shall see in Sections 3 and 4.

The previous theorem can now be generalized as follows.

Theorem 0.8 *Let A be any ring (with $1/2 \in A$), m, p and q be 2-powers as in Conventions 0.1 and 0.2, and let $d \in \mathbb{Z}$, such that the cup-product with the Bott element b_K in $K_p(\mathbb{Z};\mathbb{Z}/2p)$ induces an isomorphism*

$$\overline{K}_n(A) \xrightarrow{\cong} \overline{K}_{n+p}(A)$$

whenever $n \geq d$. Then, if X is a $(d-1)$-connected space, there is an exact sequence

$$_\varepsilon\overline{KQ}_{X+1}(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{X+1+ps}(A)$$

$$\longrightarrow \varprojlim_\varepsilon \overline{KQ}_{X+ps}(A) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_X(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{X+ps}(A) \to \cdots.$$

If X is $(d+q-2)$-connected, there is a split short exact sequence

$$0 \to \varprojlim_\varepsilon \overline{KQ}_{X+ps}(A) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_X(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{X+ps}(A) \to 0.$$

Finally, if A is hermitian regular and if X is $(d-1)$-connected, the cup-product with the positive Bott element induces an isomorphism

$$_\varepsilon\overline{KQ}_X(A) \cong {}_\varepsilon\overline{KQ}_{X+p}(A).$$

Corollary 0.9 *For any $(d+q-2)$-connected space X and A as above (not necessarily hermitian regular), there is a periodicity isomorphism*

$$_\varepsilon\overline{KQ}_X(A) \cong {}_\varepsilon\overline{KQ}_{X+p}(A).$$

For suitable subrings A_S in a number field F, the previous results may be stated more precisely, by using the methods of [6]. The rings A_S, defined in Section 2 below, generalize both the ring of S-integers (when S is finite) and the number field F itself. (More general examples are considered in Section 6 and in [7].)

Theorem 0.10 *Let F be a totally real 2-regular number field as considered in [6]; also, let m and p be 2-powers as in Convention 0.1. Then, for all integers $n > 0$, the inverse limit $\varprojlim_\varepsilon \overline{KQ}_{n+ps}(A_S)$ is trivial (i.e. A_S is hermitian regular) and the "positive" Bott map*

$$\beta_n = \cup b^+ : {}_\varepsilon\overline{KQ}_n(A_S) \longrightarrow {}_\varepsilon\overline{KQ}_{n+p}(A_S)$$

is an isomorphism. More generally, if X is any connected CW-complex, the Bott map

$$\beta_X : {}_\varepsilon\overline{KQ}_X(A_S) \longrightarrow {}_\varepsilon\overline{KQ}_{X+p}(A_S)$$

is an isomorphism.

For completeness we mention the odd-primary analog of Theorem 0.4, which is proved in Section 5. Its applications are related to the Bloch-Kato conjecture as we mentioned at the beginning. We note that the hypothesis $1/2 \in A$ may be dropped in this case.

Theorem 0.11 *Let p and m be odd prime powers as in Convention 0.1. Let b_K be the associated Bott element in $K_p(\mathbb{Z};\mathbb{Z}/m)$ (see Section 1 for details). Now let A be any ring and assume that, whenever $n \geq d$, cup-product with b_K induces an isomorphism*

$$\overline{K}_n(A) \cong \overline{K}_{n+p}(A).$$

Then there exists a "mixed Bott element" b in ${}_1\overline{KQ}_p(\mathbb{Z}')$ such that for $n \geq d$, the cup-product with b induces an isomorphism between the related KQ-groups

$$\beta_n : {}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\cong} {}_\varepsilon\overline{KQ}_{n+p}(A).$$

More generally, if X is a $(d-1)$-connected CW complex, then the cup-product map with b induces an isomorphism

$$\beta_X : {}_\varepsilon\overline{KQ}_X(A) \xrightarrow{\cong} {}_\varepsilon\overline{KQ}_{X+p}(A).$$

In Section 6 we note that work in progress by Schlichting [56] allows us to extend our results from commutative rings to schemes S that are separated, noetherian and of finite Krull dimension. More precisely, following Jardine's method for algebraic K-theory [29] we define an "étale" $\overline{KQ}$-theory, denoted by ${}_\varepsilon\overline{KQ}^{\text{ét}}_n(S)$, where the coefficient groups are prime powers. The étale $\overline{KQ}$-theory shares many properties with the étale $\overline{K}$-theory introduced by Dwyer and Friedlander [15]. For example, there exists a comparison map

$$\sigma : {}_\varepsilon\overline{KQ}_n(S) \longrightarrow {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S).$$

For odd prime powers, there is an involution on the odd torsion group ${}_\varepsilon\overline{KQ}_n(S)$. Let ${}_\varepsilon\overline{KQ}^{\text{ét}}_n(S)_+$ and ${}_\varepsilon\overline{KQ}^{\text{ét}}_n(S)_-$ denote the corresponding eigenspaces. On the other hand, for any prime power (odd or even), the cup-product map with the Bott element b^+ defined above induces a direct system of groups, whose colimit we shall denote by ${}_\varepsilon\overline{KQ}_n(S)\big[\beta^{-1}\big]$, in the notation of [60]. Next, we state two theorems and a conjecture in this context.

Theorem 0.12 *With the coefficient group $\mathbb{Z}/\ell^\nu$, where ℓ is an odd prime, there is an isomorphism*

$${}_\varepsilon\overline{KQ}_n(S)\big[\beta^{-1}\big] \cong {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S)$$

for all n if $\mathrm{cd}_\ell(S) < \infty$. *Moreover, the comparison map* σ *induces an isomorphism*

$$ {}_\varepsilon\overline{KQ}_n(S)_+ \cong {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S) $$

for $n \geq \sup\{\mathrm{cd}_\ell(k(s)) - 1\}_{s\in S}$.

Recall that S is uniformly ℓ-bounded with bound d if for all residue fields $k(s)$ we have $\mathrm{cd}_\ell(k(s)) \leq d$. In the event that S is uniformly ℓ-bounded with bound d, then $\mathrm{cd}_\ell(S) \leq n + d$ where n denotes the Krull dimension of S; an elegant proof for this inequality is given in [39, Theorem 2.8].

At the prime 2 we prove the following theorem, reminiscent of the main results in [16] and in [60].

Theorem 0.13 *With the coefficient group* $\mathbb{Z}/2^\nu$, *there is an isomorphism*

$$ {}_\varepsilon\overline{KQ}_n(S)[\beta^{-1}] \cong {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S) $$

for all n if $\mathrm{vcd}_2(S) < \infty$. *Moreover, the comparison map*

$$ \sigma : {}_\varepsilon\overline{KQ}_n(S) \longrightarrow {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S) $$

is a split surjection for $n \geq \sup\{\mathrm{vcd}_2(k(s)) - 1\}_{s\in S} + q - 1$.

More generally, we make the following conjecture.

Conjecture 0.14 *With the coefficient group* $\mathbb{Z}/2^\nu$ *the map* σ *is bijective whenever* $n \geq \sup\{\mathrm{vcd}_2(k(s)) - 1\}_{s\in S}$.

Using algebro-geometric methods, in Theorem 6.5 below we show how to reduce this conjecture to the case of fields. As mentioned above, the characteristic 0 case was solved independently by the authors and M. Schlichting, while the positive characteristic case was solved by Schlichting. A proof of this conjecture in general will appear in a joint paper with Schlichting [7].

Let us now briefly discuss the contents of the paper.

In Section 1, for 2-power coefficients we carefully construct the Bott elements that play an important role in this work, as referred to above.

Section 2 is somewhat independent of the other sections. In particular, we prove a refined version of our theorems in the case A is the ring of integers in a totally real 2-regular number field. (This version is a particular case of the considerations in Section 6 for schemes. Assuming Conjecture 0.14, which will be proven in [7], Theorem 2.1 may be given an independent proof in a much more general framework.)

In Section 3, we introduce what we call "higher $\overline{KSC}$-theories". These theories in some sense measure the deviation of "negative" periodicity of the $\overline{KQ}$-groups. On the other hand, they are built by successive extensions of the $\overline{K}$-groups. Therefore, they are periodic if the $\overline{K}$-groups are periodic.

Section 4 is devoted to the proof of our main Theorems 0.4 and 4.5 (for arbitrary rings with 2 invertible and mod 2^ν coefficients). The proof is roughly divided into two steps as follows. In the first one, we prove a cruder periodicity statement for $n \geq d + q - 1$. In the second, we use the $\overline{KQ}$-spectrum and an argument about cohomology theories to prove the periodicity theorems in full generality. We conclude this section with an upper bound of the $\overline{KQ}$-groups in terms of the $\overline{K}$-groups.

In Section 5, we study the case of odd prime powers, which is paradoxically simpler in our framework. The main observation is that the $\overline{KQ}$-ring spectrum splits naturally as the product of two ring spectra, the first one being the "symmetric" part of the $\overline{K}$-theory spectrum.

Section 6 is more geometric in nature and generalizes the previous considerations (when A is commutative) to noetherian separated schemes of finite Krull dimension. Here we rely heavily on the fundamental theorem in hermitian K-theory proved in the scheme framework by Schlichting [56].

Finally, Sections 7 and 8 are devoted to selected applications: rings of integers in number fields, smooth complex algebraic varieties, and rings of continuous functions on compact spaces. Another application, to hermitian $\overline{KQ}$-theory of group rings, is a consequence of an appendix to this paper by C. Weibel [66].

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1. Bott elements in K- and KQ-theories

Let ℓ be a prime number and $\mathcal{S}^0/\ell^\nu$ the mod ℓ^ν Moore spectrum. In [1, §12], Adams constructed $\mathcal{KO}_*$-equivalences

$$A_{\ell^\nu} : \Sigma^p \mathcal{S}^0/\ell^\nu \longrightarrow \mathcal{S}^0/\ell^\nu.$$

The dimension shift p is $\sup\{8, 2^{\nu-1}\}$ if $\ell = 2$ and $2(\ell-1)\ell^{\nu-1}$ if ℓ is odd. As shown by Bousfield in [12, §4], work of Mahowald and Miller implies that a spectrum $\mathcal{E}$ is $\mathcal{KO}$-local if and only if its mod ℓ homotopy groups are periodic via A_ℓ for every prime ℓ. We shall refer to the periodicity manifested in $\mathcal{KO}$-local spectra as Bott periodicity. Note that $\mathcal{KO}$-localizations are the same as $\mathcal{KU}$-localizations [12, §4].

In general there are several choices of an element A_{ℓ^ν} as above if the only criterion is that it induces a $\mathcal{KO}$-isomorphism. We are interested in particular choices of elements pertaining to classical Bott periodicity. Let u denote a generator of the infinite cyclic group $\pi_2(BU)$. Then for $r \geq 1$ the Bott element u^{2r} in

$\pi_{4r}(BU)$ is independent of the choice of u. We denote by v the element of $\pi_{8r}(BO)$ mapping to the Bott element in $\pi_{8r}(BU)$ under the map induced by complexification $c : BO \to BU$.

The mod 2^ν Bott element in degree $8r > 0$ is the generator

$$\overline{v} = \mathrm{id}_{\mathcal{S}^0/2^\nu} \wedge v \in KO_{8r}(\mathcal{S}^0/2^\nu; \mathbb{Z}/2^\nu) = [\mathcal{S}^0/2^\nu, \mathcal{KO} \wedge \mathcal{S}^0/2^\nu]_{8r}.$$

The element A_{2^ν} is called an Adams periodicity operator if it maps to the mod 2^ν Bott element in degree p under the naturally induced $\mathcal{KO}$-Hurewicz map

$$\pi_*(\mathcal{S}^0/2^\nu; \mathbb{Z}/2^\nu) \to \mathcal{KO}_*(\mathcal{S}^0/2^\nu; \mathbb{Z}/2^\nu)$$

for $\mathcal{S}^0/2^\nu$. When $\ell \neq 2$, the definition of a mod ℓ^ν Bott element is the same as above, except that $\mathcal{KO}$ is replaced by $\mathcal{KU}$. Crabb and Knapp [14] have shown that there exist Adams periodicity operators for all ℓ and $\nu \geq 1$.

By smashing the unit map $\mathcal{S}^0 \to \mathcal{E}$ of a ring spectrum $\mathcal{E}$ with $\mathcal{S}^0/\ell^\nu$ and pushing forward the class in $\pi_p(\mathcal{S}^0/\ell^\nu; \mathbb{Z}/\ell^\nu)$ represented by the map A_{ℓ^ν}, one obtains a class in the group $\pi_p(\mathcal{E}; \mathbb{Z}/\ell^\nu)$ that we call a Bott element.

Next, for $m = 2p$, where $p = 2^{\nu-1}$ is a 2-power ≥ 8, we study mod m Bott elements in more detail for K- and KQ-theory in the example of $\mathbb{Z}'$. The case of an odd prime is dealt with in Section 5.

To begin, we shall consider "Bott elements" in $K_p(\mathbb{Z}'; \mathbb{Z}/m)$ and ${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/m)$, whose images in $K_p(\mathbb{R}; \mathbb{Z}/m)$ and ${}_1KQ_p(\mathbb{R}; \mathbb{Z}/m)$, respectively, are generators deduced from classical Bott periodicity for the real numbers (as KQ-modules). This is well-known for the algebraic K-groups; it is included here for the sake of completeness.

Bökstedt's square of algebraic K-theory spectra introduced in [9]

$$\begin{array}{ccc} \mathcal{K}(\mathbb{Z}')_\# & \longrightarrow & \mathcal{K}(\mathbb{R})^c_\# \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbb{F}_3)_\# & \longrightarrow & \mathcal{K}(\mathbb{C})^c_\# \end{array}$$

was verified to be homotopy cartesian by Rognes-Weibel in [49], [65], as a consequence of Voevodsky's proof of the Milnor conjecture. Here $_\#$ means 2-adic completions and ^{c} means connective cover. Smashing with $\mathcal{S}^0/2^\nu$ yields a homotopy cartesian square (an overbar indicates reduction mod m):

$$\begin{array}{ccc} \overline{\mathcal{K}}(\mathbb{Z}') & \longrightarrow & \overline{\mathcal{K}}(\mathbb{R})^c \\ \downarrow & & \downarrow \\ \overline{\mathcal{K}}(\mathbb{F}_3) & \longrightarrow & \overline{\mathcal{K}}(\mathbb{C})^c \end{array}$$

Denote by $\overline{K}$ the corresponding mod m homotopy groups. By Bott periodicity and the isomorphism $\overline{K}_{p-1}(\mathbb{Z}') \to \overline{K}_{p-1}(\mathbb{F}_3)$, there is a split short exact sequence

$$0 \longrightarrow \overline{K}_p(\mathbb{Z}') \longrightarrow \overline{K}_p(\mathbb{R}) \oplus \overline{K}_p(\mathbb{F}_3) \longrightarrow \overline{K}_p(\mathbb{C}) \longrightarrow 0.$$

On the other hand, Quillen's homotopy fibration

$$\Omega\overline{\mathcal{K}}(\mathbb{C}) \xrightarrow{\Psi^3-1} \Omega\overline{\mathcal{K}}(\mathbb{C}) \longrightarrow \overline{\mathcal{K}}(\mathbb{F}_3) \longrightarrow \overline{\mathcal{K}}(\mathbb{C}) \xrightarrow{\Psi^3-1} \overline{\mathcal{K}}(\mathbb{C})$$

yields an exact sequence

$$\overline{K}_{p+1}(\mathbb{C}) \xrightarrow{\cdot m} \overline{K}_{p+1}(\mathbb{C}) \longrightarrow \overline{K}_p(\mathbb{F}_3) \longrightarrow \overline{K}_p(\mathbb{C}) \xrightarrow{\cdot m} \overline{K}_p(\mathbb{C}),$$

and hence the isomorphisms

$$\overline{K}_p(\mathbb{F}_3) \cong {}_m\overline{K}_p(\mathbb{C}) \cong \mathbb{Z}/m.$$

Here ${}_nA$ denotes the kernel of the multiplication by n map on an abelian group A. Hence, diagram chasing shows there are isomorphisms

$$\overline{K}_p(\mathbb{Z}') \cong \overline{K}_p(\mathbb{R}) \quad \text{and} \quad \overline{K}_p(\mathbb{Z}') \cong \overline{K}_p(\mathbb{F}_3).$$

More precisely, there exists a Bott element b_K in $\overline{K}_p(\mathbb{Z}')$ mapping at the same time to a generator of $\overline{K}_p(\mathbb{R})$ and to a generator of $\overline{K}_p(\mathbb{F}_3)$.

We proceed in the same manner in order to explicate Bott elements in hermitian K-theory, having almost the exact same properties as their namesakes in algebraic K-theory. More precisely, we shall prove the following theorem:

Theorem 1.1 *Let $p \geq 8$ a 2-power and $m = 2p$. Then the group ${}_1KQ_p(\mathbb{Z}';\mathbb{Z}/m)$ is isomorphic to $\mathbb{Z}/m \oplus \mathbb{Z}/m \oplus \mathbb{Z}/2$. There is a Bott element b^+ in ${}_1KQ_p(\mathbb{Z}';\mathbb{Z}/m)$ that maps at the same time to a generator of $\mathbb{Z}/m$ in ${}_1KQ_p(\mathbb{F}_3;\mathbb{Z}/m) \cong \mathbb{Z}/m \oplus \mathbb{Z}/2$ and to a generator of ${}_1KQ_p(\mathbb{R};\mathbb{Z}/m)$, viewed as a module[4] over ${}_1KQ_0(\mathbb{R};\mathbb{Z}/m)$.*

Proof: In the following proof, we are going to use the results of [5, Theorem 6.1] and [6, Theorems 1.2, 1.5]. In [5], it is shown that the square of hermitian K-theory completed connective spectra

$$\begin{array}{ccc} {}_1\mathcal{KQ}(\mathbb{Z}')^c_\# & \longrightarrow & {}_1\mathcal{KQ}(\mathbb{R})^c_\# \\ \downarrow & & \downarrow \\ {}_1\mathcal{KQ}(\mathbb{F}_3)^c_\# & \longrightarrow & {}_1\mathcal{KQ}(\mathbb{C})^c_\# \end{array}$$

[4]The ring structure for KQ-theory with mod 2^ν coefficients is well-defined if $\nu \geq 4$.

is homotopy cartesian (Recall that ${}_1\mathcal{KQ}(\mathbb{C})$ is just $\mathcal{KO}$.) Reducing mod m yields another homotopy cartesian square:

$$\begin{array}{ccc} {}_1\overline{\mathcal{KQ}}(\mathbb{Z}')^c & \longrightarrow & {}_1\overline{\mathcal{KQ}}(\mathbb{R})^c \\ \downarrow & & \downarrow \\ {}_1\overline{\mathcal{KQ}}(\mathbb{F}_3)^c & \longrightarrow & {}_1\overline{\mathcal{KQ}}(\mathbb{C})^c \end{array}$$

This in turn gives rise to a short exact sequence

$$0 \longrightarrow {}_1\overline{KQ}_p(\mathbb{Z}') \longrightarrow {}_1\overline{KQ}_p(\mathbb{R}) \oplus {}_1\overline{KQ}_p(\mathbb{F}_3) \longrightarrow {}_1\overline{KQ}_p(\mathbb{C}) \longrightarrow 0, \qquad (1{:}1)$$

which splits since ${}_1\overline{KQ}_p(\mathbb{R})$ is a direct sum of two copies of ${}_1\overline{KQ}_p(\mathbb{C})$, say $G \oplus G$. The first copy of G, say G_1, is generated by the image of 1 under the Bott isomorphism ${}_1\overline{KQ}_0(\mathbb{R}) \cong {}_1\overline{KQ}_p(\mathbb{R})$ (see Appendix B in [6]). The splitting is given by the isomorphism between ${}_1\overline{KQ}_p(\mathbb{C})$ and the second copy of G, say G_2. Therefore, we get an isomorphism

$${}_1\overline{KQ}_p(\mathbb{Z}') \cong G_1 \oplus {}_1\overline{KQ}_p(\mathbb{F}_3).$$

In order to finish the proof of the theorem, we need to compute ${}_1\overline{KQ}_p(\mathbb{F}_3)$. By [19], there is a Bockstein exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow {}_1\overline{KQ}_p(\mathbb{F}_3) \longrightarrow \mathbb{Z}/m \longrightarrow 0. \qquad (1{:}2)$$

In order to resolve this extension problem, consider the map

$${}_1KQ_0(\mathbb{F}_3)/m = \mathbb{Z}/m \oplus \mathbb{Z}/2 \longrightarrow {}_1\overline{KQ}_p(\mathbb{F}_3)$$

given by cup-product with any element that maps to the generator of ${}_1\overline{KQ}_p(\mathbb{C}) \cong \mathbb{Z}/m$ under the Brauer lift ${}_1\overline{KQ}_p(\mathbb{F}_3) \to {}_1\overline{KQ}_p(\mathbb{C})$. This gives a splitting of the exact sequence (1:2), and therefore ${}_1\overline{KQ}_p(\mathbb{F}_3) \cong \mathbb{Z}/m \oplus \mathbb{Z}/2$. □

Remarks 1.2 By considering the forgetful map from the hermitian K sequence (1:1) to its algebraic K counterpart, one sees that the Bott element of ${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/m)$ maps to the Bott element in the corresponding algebraic K-theory group under the map induced by the forgetful functor. Moreover, all the results for $\mathbb{F}_3$ in the above also hold for any finite field $\mathbb{F}_t$ with t elements, provided $t \equiv \pm 3 \pmod 8$.

2. Proof of the periodicity theorem for totally real 2-regular number fields

Let A be the ring of 2-integers in a totally real 2-regular number field F with r real embeddings. In [6], we proved that the square of hermitian K-theory 2-completed

connective spectra

$$\begin{array}{ccc} {}_\varepsilon\mathcal{KQ}(A)^c_\# & \longrightarrow & \bigvee^r {}_\varepsilon\mathcal{KQ}(\mathbb{R})^c_\# \\ \downarrow & & \downarrow \\ {}_\varepsilon\mathcal{KQ}(\mathbb{F}_t)^c_\# & \longrightarrow & \bigvee^r {}_\varepsilon\mathcal{KQ}(\mathbb{C})^c_\# \end{array}$$

is homotopy cartesian (with t a carefully chosen odd prime and where $_\#$ denotes 2-adic completion). Therefore, the mod 2^ν reduction of this square, namely

$$\begin{array}{ccc} {}_\varepsilon\overline{\mathcal{KQ}}(A)^c & \longrightarrow & \bigvee^r {}_\varepsilon\overline{\mathcal{KQ}}(\mathbb{R})^c \\ \downarrow & & \downarrow \\ {}_\varepsilon\overline{\mathcal{KQ}}(\mathbb{F}_t)^c & \longrightarrow & \bigvee^r {}_\varepsilon\overline{\mathcal{KQ}}(\mathbb{C})^c \end{array}$$

is also homotopy cartesian, since ${}_\varepsilon KQ_{-1}(A) = 0$ by Lemmas 3.11 and 3.12 in [6]. Using this square, we deduce an enhanced version of our periodicity theorem.

Theorem 2.1 *For $n \geq 0$ and $p = \sup\{8, 2^{\nu-1}\}$ for $\nu \geq 1$, taking cup-product with the positive Bott element in ${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/2^\nu)$ induces an isomorphism*

$${}_\varepsilon KQ_n(A; \mathbb{Z}/2^\nu) \cong {}_\varepsilon KQ_{n+p}(A; \mathbb{Z}/2^\nu).$$

Proof: Cup-product with the Bott element in ${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/2^\nu)$ induces an isomorphism of ${}_\varepsilon\overline{KQ}$-groups for the rings $\mathbb{F}_t$, $\mathbb{R}$ and $\mathbb{C}$, where $\mathbb{F}_t$ is the finite field with t elements. This is due to $\mathcal{KO}$-localness of the corresponding hermitian K-theory spectra, and an induction on the order of the coefficient group based on the five lemma applied to the Bockstein exact sequence. Therefore, the result follows from the five lemma together with the homotopy cartesian square above. □

Remark 2.2 The isomorphism for $n = 0$ reflects the fact that 2-regularity implies that there is no nontrivial 2-torsion in the Picard group of A.

We note that the number ν was related to the choice of $t \equiv \pm 3 \pmod 8$ in Theorem 1.1 (for $A = \mathbb{Z}'$). However, the number t that makes the diagrams above homotopy cartesian (for A totally real 2-regular) is different in general.

Therefore, we can improve the previous result by replacing $m = 2^\nu$ by M, which is the number m multiplied by the 2-primary factor $m' = \left(\frac{t^2-1}{8}\right)_2$ of $(t^2-1)/8$ (compare with Lemma 2.9 in [6]). More precisely, we have the following proposition.

Proposition 2.3 *Let $p = \sup\{8, 2^{\nu-1}\}$ and consider the canonically induced map*

$${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/2^\nu) \longrightarrow {}_1KQ_p(A; \mathbb{Z}/2^\nu).$$

The image of the Bott element b in ${}_1KQ_p(A; \mathbb{Z}/2^\nu)$ is the reduction mod 2^ν *of a class $\overline{b}$* mod M*, with $M = m \cdot \left(\frac{t^2-1}{8}\right)_2$ as defined above.*

Proof: For brevity, we use the above notation, whereby $m = 2^\nu$ and $M = mm'$. As in the case of $\mathbb{Z}'$ considered in Section 1, we can write the following diagram of exact sequences (where $KQ = {}_1KQ$):

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & & KQ_p(\mathbb{R}^r;\mathbb{Z}/m') \oplus KQ_p(\mathbb{F}_t;\mathbb{Z}/m') & \to & KQ_p(\mathbb{C}^r;\mathbb{Z}/m') \to 0 \\
 & & \downarrow & & \downarrow \\
KQ_p(A;\mathbb{Z}/M) \to & & KQ_p(\mathbb{R}^r;\mathbb{Z}/M) \oplus KQ_p(\mathbb{F}_t;\mathbb{Z}/M) & \to & KQ_p(\mathbb{C}^r;\mathbb{Z}/M) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
KQ_p(A;\mathbb{Z}/m) \to & & KQ_p(\mathbb{R}^r;\mathbb{Z}/m) \oplus KQ_p(\mathbb{F}_t;\mathbb{Z}/m) & \to & KQ_p(\mathbb{C}^r;\mathbb{Z}/m) \to 0 \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

Chasing in this diagram shows the reduction map $KQ_p(A;\mathbb{Z}/M) \to KQ_p(A;\mathbb{Z}/m)$ is surjective. Therefore, the Bott element b in $KQ_p(A;\mathbb{Z}/m)$ is the reduction mod m of a class $\overline{b}$ mod M which we shall call an *exotic Bott element* (we do not claim, however, that $\overline{b}$ is unique). □

Theorem 2.4 *Let $\overline{b}$ be an exotic Bott element in the group ${}_1KQ_p(A;\mathbb{Z}/M)$ defined above. Then cup-product with $\overline{b}$ induces an isomorphism*

$$\overline{\beta} : {}_\varepsilon KQ_n(A;\mathbb{Z}/M') \xrightarrow{\cong} {}_\varepsilon KQ_{n+p}(A;\mathbb{Z}/M')$$

for every $n \geq 0$ and divisor M' of M.

Proof: We just copy the proof of Theorem 2.1, using the five lemma, since this periodicity statement holds for the rings $\mathbb{R}$,$\mathbb{C}$ and $\mathbb{F}_t$ (see the independent lemma below for the field $\mathbb{F}_t$). □

Lemma 2.5 *Let $\mathbb{F}_t$ be a finite field with t elements and let $p = \sup\{8, 2^{\nu-1}\}$ and $m = 2^\nu$. Then the image of the Bott element by the canonical map*

$${}_1KQ_p(\mathbb{Z}';\mathbb{Z}/m) \longrightarrow {}_1KQ_p(\mathbb{F}_t;\mathbb{Z}/m)$$

is the reduction mod m *of a class* mod M' *(with $m|M'$) if and only if*

$$(M')_2 \leq m \cdot ((t^2-1)/8)_2,$$

where $(i)_2$ is the 2-primary part of i.

Proof: We look at the following commutative diagram, with exact rows:

$$\begin{array}{ccccccc}
\longrightarrow & {}_1KQ_p(\mathbb{F}_t;\mathbb{Z}/M') & \longrightarrow & {}_1KQ_{p-1}(\mathbb{F}_t) & \xrightarrow{\cdot M'} & {}_1KQ_{p-1}(\mathbb{F}_t) \\
 & \downarrow \alpha_{M'} & & \downarrow \cdot M'2^{-\nu} & & \downarrow \mathrm{id} \\
\longrightarrow & {}_1KQ_p(\mathbb{F}_t;\mathbb{Z}/2^\nu) & \longrightarrow & {}_1KQ_{p-1}(\mathbb{F}_t) & \xrightarrow{\cdot 2^\nu} & {}_1KQ_{p-1}(\mathbb{F}_t)
\end{array}$$

The Bott element in the group ${}_1KQ_p(\mathbb{F}_t;\mathbb{Z}/2^\nu)$ maps nontrivially into the group ${}_1KQ_{p-1}(\mathbb{F}_t)$ and its image is divisible by $M'2^{-\nu}$. On the other hand, we know by [19] that ${}_1KQ_{p-1}(\mathbb{F}_t)$ is cyclic of order M, where M is the 2-primary part of $(t^{p/2}-1)$, which is also the 2-primary part of $(t^2-1)\cdot p/4$ by Lemma 2.7 in [6]. This number is also the 2-primary part of $2^\nu\cdot(t^2-1)/8$. Therefore, a simple diagram chase shows that $\alpha_{M'}$ is surjective if and only if $M'|M$. □

Now let us consider a nonzero prime ideal $\mathfrak{p}$ in A, and the quotient field $A/\mathfrak{p}$. There is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}' & \longrightarrow & A/\mathfrak{p} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/\mathfrak{p} \end{array}$$

where the right vertical arrow is the identity map. Since the Bott element in the KQ-group ${}_1KQ_p(A/\mathfrak{p};\mathbb{Z}/2^\nu)$ is the reduction mod2^ν of a class mod M, where M is a power of 2, we have an isomorphism

$${}_1KQ_p(A/\mathfrak{p};\mathbb{Z}/M)\cong\mathbb{Z}/M\oplus\mathbb{Z}/2$$

according to the computations of the KQ-theory of finite fields in [19] and Section 1. It follows that there is a periodicity isomorphism

$${}_\varepsilon KQ_n(A/\mathfrak{p};\mathbb{Z}/M')\cong{}_\varepsilon KQ_{n+p}(A/\mathfrak{p};\mathbb{Z}/M')$$

for $n\geq 0$ and any $M'|M$, given by the cup-product with an exotic Bott element. For the next two results, we recall from [6, Proposition 2.1] that F contains a unique dyadic prime (that is, prime ideal lying over the rational prime (2)). For any set S of valuations in F including the dyadic valuation and the infinite ones, we define A_S to consist of the elements in F whose valuations not in S are non-negative. Thus, when S is finite, A_S is just the ring of S-integers. When S comprises only the dyadic valuation and the infinite ones, $A_S=A$; while, when S comprises all valuations, $A_S=F$.

Theorem 2.6 *Let $p=\sup\{8,2^{\nu-1}\}$ for $\nu\geq 1$. Then, for $n>0$, cup-product with an exotic Bott element in ${}_1KQ_p(A;\mathbb{Z}/M)$ induces an isomorphism*

$$\overline{\beta}:{}_\varepsilon KQ_n(A_S;\mathbb{Z}/M')\xrightarrow{\cong}{}_\varepsilon KQ_{n+p}(A_S;\mathbb{Z}/M')$$

for any M' such that $2|M'|M$.

Proof: We use the homotopy fibration

$$\bigvee {}_{\varepsilon}\mathcal{U}(A/\mathfrak{p}) \longrightarrow {}_{\varepsilon}\mathcal{KQ}(A) \longrightarrow {}_{\varepsilon}\mathcal{KQ}(A_S)$$

noted in [24], where $\mathfrak{p}$ runs through all nonzero prime ideals in S. For the corresponding mod M' reductions (indicated as usual by an overbar) where $M'|M$, there is a homotopy fibration

$$\bigvee {}_{\varepsilon}\overline{\mathcal{U}}(A/\mathfrak{p}) \longrightarrow {}_{\varepsilon}\overline{KQ}(A) \longrightarrow {}_{\varepsilon}\overline{KQ}(A_S).$$

The maps in this fibration are compatible with cup-products with elements of $\overline{KQ}_*(A)$. The $\overline{U}$-theory spectra of finite fields are $\mathcal{KO}$-local as we showed more precisely above (this is a consequence of the same property for the $\overline{K}$ and $\overline{KQ}$-theories). From these facts, the five lemma implies the Bott periodicity isomorphism

$${}_{\varepsilon}KQ_n(A_S;\mathbb{Z}/M') \cong {}_{\varepsilon}KQ_{n+p}(A_S;\mathbb{Z}/M')$$

for $n > 0$, given by the cup-product with an exotic Bott element. □

Theorem 2.7 *Let A_S be as before, and let $\overline{b}$ be an exotic Bott element in the group ${}_1KQ_p(A;\mathbb{Z}/M)$. Then, for any connected CW-complex X, cup-product with $\overline{b}$ induces an isomorphism*

$$\overline{\beta} : {}_{\varepsilon}KQ_X(A_S;\mathbb{Z}/M') \cong {}_{\varepsilon}KQ_{X+p}(A_S;\mathbb{Z}/M')$$

for any M' such that $2|M'|M$. Moreover, when $A_S = A$, the previous isomorphism holds for any CW-complex, not necessarily connected.

Proof: By Theorem 2.6, the Bott map $\overline{\beta}$ is an isomorphism when X is a sphere S^n for $n \geq 1$. According to general facts about representable cohomology theories [8], it follows that $\overline{\beta}$ is also an isomorphism if X is a connected CW-complex, (finite or infinite, thanks to Milnor's $\lim^1$ exact sequence). If $A_S = A$, then the Bott map $\overline{\beta}$ is also an isomorphism when $X = S^0$. Therefore, the previous isomorphism holds also for not necessarily connected CW-complexes. □

3. Higher KSC-theories

The useful concept of topological K-theory based upon self conjugate vector bundles KSC was introduced by Anderson [2] and Green [20]. In [33, p. 281], for a ring A with involution, the spectrum $\mathcal{KSC}(A)$ was defined as the homotopy fiber of $1 - \tau$, where τ is the duality functor in algebraic K-theory

$$\tau : \mathcal{K}(A) \longrightarrow \mathcal{K}(A).$$

The importance of KSC-theory becomes evident from the homotopy fibration [33, p. 282]

$$\mathcal{KSC}(A) \longrightarrow \Omega_\varepsilon \mathcal{KQ}(A) \xrightarrow{\sigma^{(2)}} \Omega^{-1}{}_{-\varepsilon}\mathcal{KQ}(A),$$

which implies a long exact sequence (for legibility we omit the ring A in the notation)

$$\cdots \to {}_\varepsilon KQ_{n+2} \xrightarrow{s^{(2)}} {}_{-\varepsilon}KQ_n \to KSC_n \to {}_\varepsilon KQ_{n+1} \xrightarrow{s^{(2)}} {}_{-\varepsilon}KQ_{n-1} \to \cdots.$$

The morphism $s^{(2)}$ between the KQ-groups is the periodicity map made explicit in [33]. It is defined by taking cup-product with a generator of the free part of the group

$${}_{-1}KQ_{-2}(\mathbb{Z}') \cong {}_1W_0(\mathbb{Z}') \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

(Recall our assumption that $1/2 \in A$.) We should note that this cup-product induces a morphism between cohomology theories, and thence the associated KQ-spectra and $\overline{KQ}$-spectra, according to Brown's representability theorem.

It turns out that the KSC-groups measure the failure of negative Bott periodicity for the KQ-groups. To keep track of the degree shift we let $KSC^{(2)}$ (resp. $\mathcal{KSC}^{(2)}$) denote the KSC-groups (resp. KSC-spectrum).

There exist higher analogs of this spectrum corresponding to degree shifts by 4, 8 and higher 2-powers.

The next version, denoted[5] by ${}_\varepsilon\mathcal{KSC}^{(4)}(A)$, is the homotopy fiber of the composite map

$$\sigma^{(4)} : \Omega_\varepsilon \mathcal{KQ}(A) \xrightarrow{\sigma^{(2)}} \Omega^{-1}{}_{-\varepsilon}\mathcal{KQ}(A) \xrightarrow{\Omega^{(-2)}\sigma^{(2)}} \Omega^{-3}{}_\varepsilon\mathcal{KQ}(A).$$

Proposition 3.1 *There exists a homotopy fibration of spectra*

$${}_\varepsilon\mathcal{KSC}^{(2)}(A) \longrightarrow {}_\varepsilon\mathcal{KSC}^{(4)}(A) \longrightarrow \Omega^{-2}({}_\varepsilon\mathcal{KSC}^{(2)}(A))$$

and a long exact sequence (we again omit the ring A for convenience)

$$\cdots \to {}_\varepsilon KQ_{n+2} \xrightarrow{s^{(4)}} {}_\varepsilon KQ_{n-2} \to {}_\varepsilon KSC_n^{(4)} \to {}_\varepsilon KQ_{n+1} \xrightarrow{s^{(4)}} {}_\varepsilon KQ_{n-3} \to \cdots.$$

[5] *A priori*, this theory depends on ε. A proof of this statement may be found in Lemma 3.3 below. For KSC -theory with coefficients, we can also argue by contradiction as in Lemma 3.13.

Proof: This is just the observation that for two composable maps u and v, there is a homotopy fibration $\mathcal{F}(u) \longrightarrow \mathcal{F}(v \circ u) \longrightarrow \mathcal{F}(v)$, where $\mathcal{F}(f)$ denotes the homotopy fiber of some map f. □

Iterating, for $r > 4$ a 2-power, we proceed similarly and define ${}_\varepsilon\mathcal{KSC}^{(r)}(A)$ as the homotopy fiber of the map

$$\sigma^{(r)} : \Omega_\varepsilon \mathcal{KQ}(A) \longrightarrow \Omega^{-r+1}{}_\varepsilon\mathcal{KQ}(A)$$

where $\sigma^{(r)} = \Omega^{-r/2}\sigma^{(r/2)} \circ \sigma^{(r/2)}$. In the other direction, if we allow the convention ${}_\varepsilon\mathcal{KSC}^{(1)}(A) = \Omega K(A)$, then from the original definition of $\mathcal{KSC}$ above the following also holds for $r = 2$.

Proposition 3.2 *For a 2-power $r \geq 2$, there is a homotopy fibration of spectra*

$${}_\varepsilon\mathcal{KSC}^{(r/2)}(A) \longrightarrow {}_\varepsilon\mathcal{KSC}^{(r)}(A) \longrightarrow \Omega^{-r/2}{}_\varepsilon\mathcal{KSC}^{(r/2)}(A)$$

and an associated long exact sequence

$$\cdots \to {}_\varepsilon KQ_{n+2} \overset{s^{(r)}}{\to} {}_{\varepsilon'}KQ_{n+2-r} \to {}_\varepsilon KSC_n^{(r)} \to {}_\varepsilon KQ_{n+1} \overset{s^{(r)}}{\to} {}_{\varepsilon'}KQ_{n+1-r} \to \cdots$$

where $\varepsilon' = -\varepsilon$ if $r = 2$ and $\varepsilon' = \varepsilon$ if $r > 2$. □

Finally, we show that the higher KSC-theories depends on the sign of symmetry ε.

Lemma 3.3 *Let F be a finite field of characteristic $\neq 2$. Then the group ${}_1KSC_1^{(4)}(F)$ is isomorphic to $\mathbb{Z}/2$, while ${}_{-1}KSC_1^{(4)}(F) = 0$.*

Proof: Let us drop the field F for notational convenience. Then the group ${}_{-1}KSC_1^{(4)}$ fits into the exact sequence

$${}_{-1}KQ_3 \longrightarrow {}_{-1}KQ_{-1} \longrightarrow {}_{-1}KSC_1^{(4)} \longrightarrow {}_{-1}KQ_2 \longrightarrow {}_{-1}KQ_{-2}.$$

We have ${}_{-1}KQ_{-1} = 0$ by the same argument used in the proof of Lemma 3.11 in [6], where we should replace R_F by F. We also have ${}_{-1}KQ_2 = 0$ by a result of Friedlander [19]. Therefore, ${}_{-1}KSC_1^{(4)}(F) = 0$.

On the other hand, the group ${}_1KSC_1^{(4)}$ fits into the exact sequence

$${}_1KQ_3 \longrightarrow {}_1KQ_{-1} \longrightarrow {}_1KSC_1^{(4)} \longrightarrow {}_1KQ_2 \overset{\alpha}{\longrightarrow} {}_1KQ_{-2}.$$

For the same reason as above, we have ${}_1KQ_{-1} = 0$. We also have ${}_1KQ_2 = \mathbb{Z}/2$ by an analogous result of Friedlander [19]. The periodicity map α can be factored though the group ${}_{-1}KQ_0$ which is isomorphic to $\mathbb{Z}$. Therefore, $\mathrm{Ker}(\alpha) = \mathbb{Z}/2$ and the group ${}_1KSC^{(4)}$ is isomorphic to $\mathbb{Z}/2$. □

We can mimic the previous definitions by taking spectra or groups mod m, where m is related to p according to our convention 0.1. In that case, we shall write $\overline{\mathcal{KQ}}$ instead of $\mathcal{KQ}$, $\overline{KSC}$ instead of KSC, *etc.*

Proposition 3.4 *Let d be the number defined in the Introduction (i.e. the starting point of periodicity for the $\overline{K}$-groups). Then for any 2-power $r \geq 2$, the positive Bott map*

$$\sigma : {}_\varepsilon\overline{KSC}^{(r)}_n(A) \to {}_\varepsilon\overline{KSC}^{(r)}_{n+p}(A)$$

is an isomorphism if $n \geq d + r - 2$.

Proof: We argue by iteration on the 2-power r, using the following diagram of exact sequences from (3.2):

$$\begin{array}{ccccccc}
\overline{KSC}^{(r/2)}_{n+1-r/2} \to & \overline{KSC}^{(r/2)}_{n} \to & \overline{KSC}^{(r)}_{n} \to & \overline{KSC}^{(r/2)}_{n-r/2} \to & \overline{KSC}^{(r/2)}_{n-1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\overline{KSC}^{(r/2)}_{n++p+1-r/2} \to & \overline{KSC}^{(r/2)}_{n+p} \to & \overline{KSC}^{(r)}_{n+p} \to & \overline{KSC}^{(r/2)}_{n+p-r/2} \to & \overline{KSC}^{(r/2)}_{n+p-1}
\end{array} \tag{3:3}$$

Commutativity of this diagram follows from the fact that the vertical maps are induced by cup-product with the positive Bott element in KQ-theory as constructed in Section 1, and that all maps are KQ-module maps. By induction, we know that the vertical maps, with the possible exception of the middle one, are isomorphisms if $n - r/2 \geq d + r/2 - 2$, that is $n \geq d + r - 2$. We conclude thanks to the five lemma. □

Remark 3.5 A variant of this proposition is to consider a parameter space X instead of a sphere S^n. More precisely, by the method of proof of Theorem 2.7, the Bott map

$$\sigma : {}_\varepsilon\overline{KSC}^{(r)}_X(A) \to {}_\varepsilon\overline{KSC}^{(r)}_{X+p}(A)$$

is an isomorphism if the space X is $(d + r - 3)$-connected.

Next we consider the failure of positive p-periodicity in hermitian $\overline{K}$-theory. This is encoded in the homotopy fiber of the periodicity map given by the cup-product with the positive Bott element

$${}_\varepsilon\overline{\mathcal{KQ}}(A) \longrightarrow \Omega^p {}_\varepsilon\overline{\mathcal{KQ}}(A),$$

which we shall denote by $\mathcal{P}_\varepsilon\overline{\mathcal{KQ}}(A)$. In the same way, we denote by $P\overline{\mathcal{K}}(A)$ the homotopy fiber of the periodicity map in K-theory

$$\overline{\mathcal{K}}(A) \longrightarrow \Omega^p\overline{\mathcal{K}}(A).$$

According to our general assumptions, the homotopy groups $P\overline{K_n}(A)$ of $\mathcal{P}\overline{\mathcal{K}}(A)$ vanish if $n \geq d$. On the other hand, we can introduce, cf. [33], the homotopy fibers

$\mathcal{P}_\varepsilon\overline{\mathcal{U}}(A)$ and $\mathcal{P}_{-\varepsilon}\overline{\mathcal{V}}(A)$ of the hyperbolic and forgetful maps $\mathcal{P}\overline{\mathcal{K}}(A) \to \mathcal{P}_\varepsilon\overline{\mathcal{KQ}}(A)$ and $\mathcal{P}_{-\varepsilon}\overline{\mathcal{KQ}}(A) \to \mathcal{P}\overline{\mathcal{K}}(A)$, respectively.

Proposition 3.6 *There is a homotopy equivalence*

$$\mathcal{P}_{-\varepsilon}\overline{\mathcal{V}}(A) \simeq \Omega(\mathcal{P}_\varepsilon\overline{\mathcal{U}}(A)).$$

Moreover, the composition

$$\Omega^2(\mathcal{P}_\varepsilon\overline{\mathcal{KQ}}(A)) \longrightarrow \Omega(\mathcal{P}_\varepsilon\overline{\mathcal{U}}(A)) \simeq \mathcal{P}_{-\varepsilon}\overline{\mathcal{V}}(A) \longrightarrow \mathcal{P}_{-\varepsilon}\overline{\mathcal{KQ}}(A)$$

is induced by cup-product with the negative Bott element in the group ${}_{-1}KQ_{-2}(\mathbb{Z}')$.

Proof: The fundamental theorem of hermitian K-theory [33] exhibits an explicit homotopy equivalence (given both ways) between the spectra ${}_{-\varepsilon}\mathcal{V}(A)$ and $\Omega_\varepsilon\mathcal{U}(A)$.

Moreover, with the notation $\mathcal{KQ}(\mathbb{Z}') = {}_1\mathcal{KQ}(\mathbb{Z}') \vee {}_{-1}\mathcal{KQ}(\mathbb{Z}')$, these maps are $\mathcal{KQ}(\mathbb{Z}')$-module maps. Therefore, the reduction mod m of these spectra is also a homotopy equivalence. Since the composition

$$\Omega^2(\mathcal{P}_\varepsilon\overline{\mathcal{KQ}}(A)) \longrightarrow \Omega(\mathcal{P}_\varepsilon\overline{\mathcal{U}}(A)) \simeq \mathcal{P}_{-\varepsilon}\overline{\mathcal{V}}(A) \longrightarrow \mathcal{P}_{-\varepsilon}\overline{\mathcal{KQ}}(A)$$

is a $\mathcal{KQ}(\mathbb{Z}')$-module map, it is defined by the cup-product with the negative Bott element, as proved in [33]. □

Lemma 3.7 *If* $P\overline{K}_n(A) = 0$ *for* $n \geq d$*, then the negative Bott map*

$$P_\varepsilon\overline{KQ}_{n+2}(A) \to P_{-\varepsilon}\overline{KQ}_n(A)$$

is an isomorphism for $n \geq d$ *and is a monomorphism for* $n = d - 1$.

Proof: This follows from the diagram (A omitted) with exact rows

$$\begin{array}{ccccccc} P\overline{K}_{n+2} & \longrightarrow & P_\varepsilon\overline{KQ}_{n+2} & \longrightarrow & P_\varepsilon\bar{U}_{n+1} & \longrightarrow & P\overline{K}_{n+1} \\ & & & & \downarrow\cong & & \\ & & P\overline{K}_{n+1} & \longrightarrow & P_{-\varepsilon}\bar{V}_n & \longrightarrow & P_{-\varepsilon}\overline{KQ}_n \longrightarrow P\overline{K}_n \end{array}$$

the vertical isomorphism being a consequence of the fundamental theorem in hermitian K-theory. □

Iteration of this Bott map induces a further isomorphism

$$P_\varepsilon\overline{KQ}_{n+4}(A) \xrightarrow{\cong} P_\varepsilon\overline{KQ}_n(A).$$

The classical induction method [5, (3.5)], adapted to this case, enables us to prove the following theorem. We recall that the overbar over the KQ indicates reduction mod$m$, where m was defined in the Introduction. Strictly speaking, one has to take $m \geq 16$, in the theorem, so that $\overline{KQ}_*(\mathbb{Z}')$ is an associative ring — see Footnote 4. However, if $m < 16$, we can consider all these groups as modules over $KQ_*(\mathbb{Z}';\mathbb{Z}/16)$ and the Bott map still makes sense.

Theorem 3.8 *Assume that* $P\overline{K}_n(A) = 0$ *for* $n \geq d$. *Assume moreover that* $P_\varepsilon\overline{KQ}_n(A) = 0$ *for* $\varepsilon = \pm 1$ *and for* $n = d$ *and* $d+1$. *Then the cup-product with the Bott element in* ${}_1\overline{KQ}_p(\mathbb{Z}')$ *induces a morphism*

$$\beta_n : {}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\cong} {}_\varepsilon\overline{KQ}_{n+p}(A)$$

which is an isomorphism for $n \geq d+1$ *and a monomorphism for* $n = d$.

Proof: The 2-periodicity of the $P\overline{KQ}$-groups shown above (with a change of symmetry) implies that $P_\varepsilon\overline{KQ}_n(A) = 0$ for $n \geq d$ and $\varepsilon = \pm 1$. From the exact sequence

$$P_\varepsilon\overline{KQ}_n(A) \longrightarrow {}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\beta_n} {}_\varepsilon\overline{KQ}_{n+p}(A) \longrightarrow P_\varepsilon\overline{KQ}_{n-1}(A),$$

we deduce the required isomorphism (starting from $n = d+1$) and a monomorphism for $n = d$. □

Unfortunately, this strategy is not efficient to establish Bott periodicity because the starting point of the induction is not always valid (see the end of Section 4 for counterexamples and Section 2 for examples). Therefore, we are going to take another approach towards Bott periodicity. From now on, we often assume implicitly that the $\overline{K}$-groups are periodic starting in degree d. More precisely, the Bott map

$$\overline{K}_n(A) \longrightarrow \overline{K}_{n+p}(A)$$

is an isomorphism for $n \geq d$, which implies that $P\overline{K}_n(A) = 0$ in the same range. Our aim is to prove a similar periodicity assertion for KQ-theory, as we announced in the Introduction.

Let us investigate in detail the composition of the two "opposite" periodicity maps

$${}_\varepsilon\overline{\mathcal{KQ}}(A) \xrightarrow{u} \Omega^q {}_\varepsilon\overline{\mathcal{KQ}}(A) \xrightarrow{v} {}_\varepsilon\overline{\mathcal{KQ}}(A).$$

Proposition 3.9 *Let* m *and* q *be* 2*-powers as in Convention 0.2. Then the cup-product between the images of the negative and positive Bott elements in*

$${}_1KQ_{\pm q}(\mathbb{Z}';\mathbb{Z}/m)$$

is reduced to 0. *Therefore, the compositions* $v \circ u$ *and* $u \circ v$ *are nullhomotopic.*

Proof: Since, by *e.g.* [5, pp. 797, 799], ${}_1KQ_0(\mathbb{Z}';\mathbb{Z}/m)$ embeds in ${}_1KQ_0(\mathbb{R};\mathbb{Z}/m) \oplus {}_1KQ_0(\mathbb{F}_3;\mathbb{Z}/m)$, we consider separately the projections of the composites to each summand. In each case, the key point is that the negative Bott element is a power of an element in degree -2.

Projection to ${}_1KQ_0(\mathbb{R};\mathbb{Z}/m)$. We compute the cup-product of the two Bott elements, using the first step in [36, Lemma 1.1][6], that is, the twelve-term exact sequences of [33] for both $\mathbb{Z}'$ and the topological ring $\mathbb{R}$. As in [36, Lemma 1.1], they show that the map

$$\mathbb{Z}\oplus\mathbb{Z}/2\cong {}_1W_{-4}(\mathbb{Z}')\longrightarrow {}_1W_{-4}(\mathbb{R})\cong\mathbb{Z}$$

is given by $(w,\alpha)\mapsto 2z$ where z generates ${}_1W_{-4}(\mathbb{R})$. We now use some standard facts:

(i) there is a multiplicative isomorphism between $K_n(\mathbb{R})$ and ${}_1W_n(\mathbb{R})$ for all $n\in\mathbb{Z}$ [32, Théorème 2.3], and when n is a multiple of 8, each group is $\mathbb{Z}$, generator y_n say;

(ii) the cup-square z^2 of the generator z of $K_{-4}(\mathbb{R})\cong {}_1W_{-4}(\mathbb{R})$ is 4 times a generator y_{-8} of $K_{-8}(\mathbb{R})\cong\mathbb{Z}$;

(iii) the cup-square of any generator of the free part of ${}_1W_{-4}(\mathbb{Z}')$ projects to a generator of the free part of ${}_1W_{-8}(\mathbb{Z}')\cong\mathbb{Z}\oplus\mathbb{Z}/2$.

Let us write $q=2^{i+3}$ where $i\geq 0$. From these facts, under the map

$$\mathbb{Z}\oplus\mathbb{Z}/2\cong {}_1W_{-q}(\mathbb{Z}')\longrightarrow {}_1W_{-q}(\mathbb{R})\cong\mathbb{Z},$$

$(w,\alpha)^{2^{i+1}}$ is sent to $(2z)^{2^{i+1}}=2^{2^{i+1}}\cdot(4)^{2^i}y_{-q}=2^{q/2}y_{-q}$. Now consider the commuting diagram

$$\begin{array}{ccccc}
{}_1W_{-q}(\mathbb{Z}')\cong\mathbb{Z}\oplus\mathbb{Z}/2 & \overset{\cong}{\longleftarrow} & {}_1KQ_{-q}(\mathbb{Z}')\cong\mathbb{Z}\oplus\mathbb{Z}/2 & \longrightarrow & K_{-q}(\mathbb{Z}')=0\\
\downarrow & & \downarrow & & \downarrow\\
{}_1W_{-q}(\mathbb{R})\cong\mathbb{Z} & \longleftarrow & {}_1KQ_{-q}(\mathbb{R})\cong\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & K_{-q}(\mathbb{R})\cong\mathbb{Z}
\end{array}$$

Since the two lower horizontal maps correspond to the cokernel of the hyperbolic map (on the left) and to the signature map (on the right), they send a pair $(u,v)\in\mathbb{Z}\oplus\mathbb{Z}$ to $(u-v)y_{-q}\in {}_1W_{-q}(\mathbb{R})$ and $(u+v)y_{-q}\in K_{-q}(\mathbb{R})$ respectively. Now, by (iii) above, the element $1\in\mathbb{Z}\subset {}_1KQ_{-q}(\mathbb{Z}')$ may be taken as (up to sign) the projection of $(w,\alpha)^{2^{i+1}}$, and therefore maps both on the left to $\pm 2^{q/2}y_{-q}\in {}_1W_{-q}(\mathbb{R})$ and on the right to $0\in K_{-q}(\mathbb{R})$. Hence, the image $(u,v)\in {}_1KQ_{-q}(\mathbb{R})$ of 1 must have the form $\pm(2^{q/2-1},-2^{q/2-1})$.

[6] Erratum: in the statement of Lemma 1.1 of [36], one should replace $8y$ by $16y$ because in the proof the inclusion ${}_{-1}W^{\mathrm{top}}_{-6}\hookrightarrow {}_1W'^{\mathrm{top}}_{-8}$ is strict.

Therefore, since always $m \leq 2^{q/2-1}$ (the reason for the change from p to q), if we now take KQ-theory with coefficients in $\mathbb{Z}/m$, then the cup-product of the two Bott elements may be written

$$\sigma = (0,0,\eta) \in {}_1KQ_0(\mathbb{Z}';\mathbb{Z}/m) \cong \mathbb{Z}/m \oplus \mathbb{Z}/m \oplus \mathbb{Z}/2,$$

as calculated from the Bockstein exact sequence

$${}_1KQ_0(\mathbb{Z}') \longrightarrow {}_1KQ_0(\mathbb{Z}') \longrightarrow {}_1KQ_0(\mathbb{Z}';\mathbb{Z}/m) \longrightarrow {}_1KQ_{-1}(\mathbb{Z}') = 0$$

and Lemma 3.11 of [6]. Its image in ${}_1KQ_0(\mathbb{R};\mathbb{Z}/m) \cong \mathbb{Z}/m \oplus \mathbb{Z}/m$ is thus $\pm(2^{q/2-1}, -2^{q/2-1}) = (0,0)$.

Projection to ${}_1KQ_0(\mathbb{F}_3;\mathbb{Z}/m)$. To compute the cup-product γ of the images of the two Bott elements in ${}_1KQ_{\pm q}(\mathbb{F}_3;\mathbb{Z}/m)$, we exploit the definition of the negative Bott element as the iterated power of an element in ${}_{-1}KQ_{-2}(\mathbb{Z}')$. Therefore, γ is the image of the positive Bott element in ${}_1KQ_q(\mathbb{F}_3;\mathbb{Z}/m)$ under the following composition:

$$\begin{aligned} &{}_1KQ_q(\mathbb{F}_3;\mathbb{Z}/m) \to {}_{-1}KQ_{q-2}(\mathbb{F}_3;\mathbb{Z}/m) \to \cdots \to {}_1KQ_4(\mathbb{F}_3;\mathbb{Z}/m) \to \\ &\to {}_{-1}KQ_2(\mathbb{F}_3;\mathbb{Z}/m) \to {}_1KQ_0(\mathbb{F}_3;\mathbb{Z}/m) \end{aligned}$$

According to Friedlander [19], we have ${}_{-1}KQ_2(\mathbb{F}_3) = {}_{-1}KQ_1(\mathbb{F}_3) = 0$. Therefore, from another Bockstein exact sequence, ${}_{-1}KQ_2(\mathbb{F}_3;\mathbb{Z}/m) = 0$, and hence $\gamma = 0$.

Finally, for the last part of the proposition, we use well-known facts in cohomology theories mod 2^k [3, I. p. 75] to prove that the composite maps $v \circ u$ and $u \circ v$ are nullhomotopic. More specifically, the multiplication by 2^s on cohomology theories mod2^k is null-homotopic if $s \geq k$ and $s \geq 2$. □

For the next step, we need the following well-known Lemma (*cf.* [3, I. p. 75] again) which is a consequence of the splitting of the multiplication by m' on the spectrum S^0/m, where m and m' are 2-powers defined below.

Lemma 3.10 *Let h^* be a cohomology theory represented by a spectrum $\mathcal{S}$ and m be a 2-power. Let $h^*(-;\mathbb{Z}/m)$ be the associated cohomology theory represented by the spectrum $\mathcal{S}/m = \mathcal{S} \wedge S^0/m$. Finally, let $\mathcal{T}_{m'}$ be the homotopy fiber of the map*

$$\mathcal{S}/m \longrightarrow \mathcal{S}/m$$

defined by the multiplication by a 2-power m', where $m' \geq \sup\{4,m\}$. Then we have a canonical splitting

$$\mathcal{T}_{m'} \sim \mathcal{S}/m \times \Omega(\mathcal{S}/m).$$

Let us denote by $\mathcal{F}$ the generic homotopy fiber of the maps described before the lemma. There is a homotopy fibration

$$\mathcal{F}(u) \longrightarrow \mathcal{F}(v \circ u) \longrightarrow \mathcal{F}(v).$$

According to the above considerations, $\mathcal{F}(u)$ is the spectrum $\mathcal{P}_\varepsilon \overline{\mathcal{KQ}}(A)$, while $\mathcal{F}(v)$ is the spectrum $\Omega^{q-1}{}_\varepsilon \mathcal{KSC}^{(q)}(A)$. On the other hand, as a consequence of Proposition 3.9 and the previous lemma applied to the spectrum of hermitian K-theory, $\mathcal{F}(v \circ u)$ may be canonically identified with the product of spectra ${}_\varepsilon \overline{\mathcal{KQ}}(A) \times \Omega_\varepsilon \overline{\mathcal{KQ}}(A)$. Therefore, by taking homotopy groups of the previous fibration, we get the exact sequence

$$\overline{KSC}^{(q)}_{n+q} \to P\overline{KQ}_n \to \overline{KQ}_n \oplus \overline{KQ}_{n+1} \to \overline{KSC}^{(q)}_{n+q-1} \to P\overline{KQ}_{n-1}.$$

As a piece of convenient notation, set

$$\overline{KQ}_{n,n+1} = \overline{KQ}_n \oplus \overline{KQ}_{n+1}.$$

More generally, we shall also use the notation $\overline{KQ}_{X,X+1}$ for the direct sum $\overline{KQ}_X \oplus \overline{KQ}_{X+1}$.

Proposition 3.11 *We have the following two diagrams of exact sequences where the vertical maps are induced respectively by the cup-product with the negative or positive Bott element:*

$$\begin{array}{ccccc} P\overline{KQ}_n & \to & \overline{KQ}_{n,n+1} & \to & \overline{KSC}^{(q)}_{n+q-1} \\ \uparrow P\beta'_n & & \uparrow & & \uparrow \sigma'_n \\ P\overline{KQ}_{n+q} & \to & \overline{KQ}_{n+q,n+q+1} & \to & \overline{KSC}^{(q)}_{n+2q-1} \end{array}$$

and

$$\begin{array}{ccccc} P\overline{KQ}_n & \to & \overline{KQ}_{n,n+1} & \to & \overline{KSC}^{(q)}_{n+q-1} \\ \downarrow P\beta_n & & \downarrow & & \downarrow \sigma_n \\ P\overline{KQ}_{n+q} & \to & \overline{KQ}_{n+q,n+q+1} & \to & \overline{KSC}^{(q)}_{n+2q-1} \end{array}$$

In these diagrams, $P\beta'_n$ is an isomorphism if $n \geq d$ and a monomorphism if $n = d-1$. We also have $P\beta_n = 0$ if $n \geq d-1$. Finally, σ_n (resp. σ'_n) is an isomorphism (resp. the zero map) if $n \geq d-1$.

Proof: The second diagram is included in a bigger one with horizontal exact sequences of $\overline{KQ}(\mathbb{Z}')$-modules, where we recall that $\overline{KQ}(\mathbb{Z}') = {}_1\overline{KQ}(\mathbb{Z}') \oplus {}_{-1}\overline{KQ}(\mathbb{Z}')$:

$$\begin{array}{cccccc} \overline{KSC}^{(q)}_{n+q} & \to P\overline{KQ}_n & \to \overline{KQ}_{n,n+1} & \to \overline{KSC}^{(q)}_{n+q-1} & \to P\overline{KQ}_{n-1} \\ \downarrow & \downarrow P\beta_n & \downarrow & \downarrow & \downarrow \\ \overline{KSC}^{(q)}_{n+2q} & \to P\overline{KQ}_{n+q} & \to \overline{KQ}_{n+q,n+q+1} & \to \overline{KSC}^{(q)}_{n+2q-1} & \to P\overline{KQ}_{n+q-1} \end{array}$$

We claim that the second vertical map $P\beta_n$ is reduced to 0 if $n \geq d-1$. In order to see this, we consider the reverse map

$$P\beta'_n : P\overline{KQ}_{n+q} \longrightarrow P\overline{KQ}_n$$

given by the cup-product with the negative Bott element. This is a monomorphism for $n \geq d-1$ according to Lemma 3.7. Since the cup-product between the positive and the negative Bott elements is equal to 0 according to Proposition 3.9, we have necessarily $P\beta'_n = 0$. On the other hand, as we have seen in Proposition 3.4, the positive Bott map σ_n is an isomorphism

$$\overline{KSC}^{(q)}_{n+q-1} \cong \overline{KSC}^{(q)}_{n+2q-1}$$

for $n+q-1 \geq d+q-2$, *i.e.* for $n \geq d-1$.

The reverse map σ'_n is reduced to 0 since its composite with σ_n is trivial (note that all maps in these diagrams are $KQ(\mathbb{Z}')$-module maps).. □

Proposition 3.12 *If $n \geq d+q-1$, we have a split short exact sequence*

$$0 \to P_\varepsilon\overline{KQ}_n(A) \to {}_\varepsilon\overline{KQ}_{n,n+1}(A) \to {}_\varepsilon\overline{KSC}^{(q)}_{n+q-1}(A) \to 0.$$

Proof: Let us consider a bigger diagram, where we now choose $n \geq d+q-1$:

$$\begin{array}{ccccccc}
\overline{KSC}^{(q)}_n & \to & P\overline{KQ}_{n-q} & \to & \overline{KQ}_{n-q,n-q+1} & \to & \overline{KSC}^{(q)}_{n-1} \\
\sigma_{n-q+1}\downarrow\cong & & \downarrow 0 & & \downarrow \alpha_{n-q} & \swarrow\gamma & \sigma_{n-q}\downarrow\cong \\
\overline{KSC}^{(q)}_{n+q} & \to & P\overline{KQ}_n & \to & \overline{KQ}_{n,n+1} & \to & \overline{KSC}^{(q)}_{n+q-1} \\
\sigma_{n+1}\downarrow\cong & & \downarrow 0 & & \downarrow \alpha_n & & \sigma_n\downarrow\cong \\
\overline{KSC}^{(q)}_{n+2q} & \to & P\overline{KQ}_{n+q} & \to & \overline{KQ}_{n+q,n+q+1} & \to & \overline{KSC}^{(q)}_{n+2q-1}
\end{array}$$

We would like to insert a map $\gamma : \overline{KSC}^{(q)}_{n-1} \longrightarrow \overline{KQ}_{n,n+1}$ that renders this diagram commutative. For this, we consider the other composition

$$\Omega^q{}_\varepsilon\overline{\mathcal{KQ}}(A) \xrightarrow{v} {}_\varepsilon\overline{\mathcal{KQ}}(A) \xrightarrow{u} \Omega^q{}_\varepsilon\overline{\mathcal{KQ}}(A).$$

We get a map from $\Omega^{-q}(\mathcal{F}(v))$ to $\Omega^{-q}(\mathcal{F}(u\circ v))$ that induces the required map γ since $u\circ v$ is nullhomotopic. The commutativity of the above diagram with γ inserted is a consequence of the homotopy commutative square

$$\begin{array}{ccc}
\Omega^q{}_\varepsilon\overline{\mathcal{KQ}}(A) & \xrightarrow{v} & {}_\varepsilon\overline{\mathcal{KQ}}(A) \\
\downarrow \Omega^q(u) & & \downarrow u \\
\Omega^{2q}{}_\varepsilon\overline{\mathcal{KQ}}(A) & \xrightarrow{\Omega^q(v)} & \Omega^q{}_\varepsilon\overline{\mathcal{KQ}}(A)
\end{array}$$

where the vertical (resp. horizontal) maps are defined by the cup-product with the positive (resp. negative) Bott element. Therefore, we get the short split exact sequence

$$0 \longrightarrow P\overline{KQ}_n \longrightarrow \overline{KQ}_{n,n+1} \longrightarrow \overline{KSC}^{(q)}_{n+q-1} \longrightarrow 0,$$

which is an abbreviated formulation of our Proposition. □

Remark 3.13 The group ${}_\varepsilon\overline{KQ}_{n,n+1}$ therefore decomposes in two distinct ways as the direct sum of two groups, that is

$${}_\varepsilon\overline{KQ}_{n,n+1} = {}_\varepsilon\overline{KQ}_n \oplus {}_\varepsilon\overline{KQ}_{n+1}$$

and

$${}_\varepsilon\overline{KQ}_{n,n+1} \cong P_\varepsilon\overline{KQ}_n \oplus {}_\varepsilon\overline{KSC}^{(q)}_{n+q-1}.$$

In the appendix to [6] and also in the theory of stabilized Witt groups [38], we give many examples of rings A such that $K_n(A) = 0$ for all $n \in \mathbb{Z}$ and therefore ${}_\varepsilon\overline{KSC}^{(q)}_{n+q-1}(A) = 0$. This implies that our two direct sum decompositions are not the same in general, since we may choose A such that the two groups ${}_\varepsilon\overline{KQ}_n(A)$ and ${}_\varepsilon\overline{KQ}_{n+1}(A)$ are not 0. Moreover, this remark may be used to show that the higher ${}_\varepsilon\overline{KSC}$-theory depends *a priori* on the sign of symmetry ε. An example of this fact is $A = \mathbb{Z}'$, where we know that $P_\varepsilon\overline{KQ}_n(\mathbb{Z}') = 0$. On the other hand, from the table of the KQ-groups of $\mathbb{Z}'$ [5], it is easy to see that ${}_1\overline{KQ}_{n,n+1}(\mathbb{Z}') \neq {}_{-1}\overline{KQ}_{n,n+1}(\mathbb{Z}')$ in general.

For a better understanding of the periodicity statements we shall prove in full generality in Section 4, let us consider the case where the 2-primary abelian groups ${}_\varepsilon\overline{KQ}_n(A)$ are finite. The category of finite 2-primary abelian groups is of course well understood: its Grothendieck group (with respect to direct sums) is freely generated by the groups $\mathbb{Z}/2^k$. On the other hand, it follows from Proposition 3.12 that the groups ${}_\varepsilon\overline{KQ}_{n,n+1}(A) = {}_\varepsilon\overline{KQ}_n(A) \oplus {}_\varepsilon\overline{KQ}_{n+1}(A)$ are periodic of period q with respect to n for $n \geq d + q - 1$. More precisely, $P_\varepsilon\overline{KQ}_n$ is periodic for $n \geq d$ according to Lemma 3.7 and ${}_\varepsilon\overline{KSC}^{(q)}_{n+q-1}$ is periodic for $n + q - 1 \geq d + q - 2$, *i.e.* for $n \geq d - 1$, according to Proposition 3.4.

Let us write α_r for the class of the group ${}_\varepsilon\overline{KQ}_{r+d+q-1}$ in the Grothendieck group and put $\tau = \alpha_q - \alpha_0$. We have the identities $\alpha_q = \alpha_0 + \tau$, $\alpha_{q+1} = \alpha_1 - \tau$, $\alpha_{q+2} = \alpha_2 + \tau$, *etc.* In general, we may prove by induction on s the formula

$$\alpha_{r+qs} = \alpha_r + (-1)^r s\tau$$

when $r \geq 0$.

Proposition 3.14 *Let us assume that the cup-product with the Bott element induces an isomorphism*

$$K_n(A;\mathbb{Z}/m) \to K_{n+p}(A;\mathbb{Z}/m)$$

for $n \geq d$, and that the hermitian K-groups ${}_\varepsilon KQ_n(A;\mathbb{Z}/m)$ are finite for $n \geq d+q-1$. Then these groups are periodic with respect to n when $n \geq d+q-1$, of period q. In particular, we have $\tau = 0$ in the formulas above.

Proof: In the previous computation, let us write

$$\tau = \sum_{i=1}^{k} u_i - \sum_{j=1}^{k'} v_j$$

where u_i and v_i are classes of nonzero irreducible modules and where k and k' are chosen minimal. We have the two identities, valid for all $s \geq 0$:

$$\alpha_{qs} = \alpha_0 + s\tau$$

$$\alpha_{qs+1} = \alpha_1 - s\tau$$

From the former, for all such s, the module $s\sum_{j=1}^{k'} v_j$ is always a summand of ${}_\varepsilon KQ_{d+q-1}(A;\mathbb{Z}/m)$. Thus, $k' = 0$. Likewise, from the latter identity, $k = 0$. Hence, $\tau = 0$. □

Another example of a periodicity statement in hermitian K-theory is to consider the case where A is the ring with involution $B \times B^{\mathrm{op}}$. Here B^{op} is the opposite algebra of B, the involution on A being defined by $(b,b') \mapsto (b',b)$. It is easy to see that ${}_\varepsilon KQ_n(A) \cong K_n(B)$ and that the negative periodicity map ${}_\varepsilon KQ_n(A) \to {}_{-\varepsilon}KQ_{n-2}(A)$ is reduced to 0. Therefore, we have the isomorphisms

$${}_\varepsilon KSC_n(A) = {}_\varepsilon KSC_n^{(2)}(A) \cong K_{n+1}(B) \oplus K_n(B),$$

and, more generally

$${}_\varepsilon KSC_n^{(p)}(A) \cong K_{n+1}(B) \oplus K_{n-p+2}(B)$$

for p a 2-power. If we now reduce these theories mod m and assume the positive p-periodicity of the associated $\overline{K}$-groups (for $n \geq d$), then the last identity can also be written as

$$\begin{aligned} {}_\varepsilon\overline{KSC}{}_{n+p-1}^{(p)}(A) &\cong \overline{K}_{n+p}(B) \oplus \overline{K}_{n+1}(B) \\ &\cong \overline{K}_n(B) \oplus \overline{K}_{n+1}(B) \cong {}_\varepsilon\overline{KQ}_n(A) \oplus {}_\varepsilon\overline{KQ}_{n+1}(A), \end{aligned}$$

which is a particular case of (3.12), since $P_\varepsilon \overline{KQ}_n(A) = 0$. Note that the positive Bott map

$$ {}_\varepsilon KQ_n(A) \to {}_\varepsilon KQ_{n+p}(A) $$

is here an isomorphism for $n \geq d$.

In the spirit of Thomason, one may consider the periodized KQ-theory, which we shall denote by ${}_\varepsilon \overline{KQ}_n(A)\big[\beta^{-1}\big] = \varinjlim_\varepsilon \overline{KQ}_{n+ps}(A)$. We have the following theorem, quite similar to Connes' exact sequence relating cyclic and Hochschild homologies [13].

Proposition 3.15 *Let us take $n \geq d + r - 2$. Then (with the ring A omitted from notation) we have the exact sequence*

$$ \begin{aligned} \cdots \to {}_\varepsilon \overline{KQ}_{n+2}\big[\beta^{-1}\big] \to {}_{\varepsilon'} \overline{KQ}_{n+2-r}\big[\beta^{-1}\big] \to {}_\varepsilon \overline{KSC}^{(r)}_n \to {}_\varepsilon \overline{KQ}_{n+1}\big[\beta^{-1}\big] \\ \to {}_{\varepsilon'} \overline{KQ}_{n+1-r}\big[\beta^{-1}\big] \to \cdots \to {}_\varepsilon \overline{KQ}_{d-1+r}\big[\beta^{-1}\big] \to {}_{\varepsilon'} \overline{KQ}_{d-1}\big[\beta^{-1}\big], \end{aligned} $$

where $\varepsilon' = -\varepsilon$ if $r = 2$ and $\varepsilon' = \varepsilon$ if $r > 2$. Moreover, if also $r \geq q$, we have a splitting

$$ {}_\varepsilon \overline{KSC}^{(r)}_n(A) \cong {}_\varepsilon \overline{KQ}_{n+1}(A)\big[\beta^{-1}\big] \oplus {}_\varepsilon \overline{KQ}_{n+2}(A)\big[\beta^{-1}\big]. $$

Note that for $r = q$, this splitting is also proved in the beginning of the proof of Theorem 4.2.

Proof: The first part of the proposition is a direct consequence of Proposition 3.4 showing that the $\overline{KSC}^{(r)}$-groups are periodic for $n \geq d + r - 2$.

For the second part, we notice that the map between the $\overline{KQ}$-groups is 0, since the cup-product between the positive and negative Bott elements is 0, according to Proposition 3.9. Thus, the sequence decomposes into short exact sequences

$$ 0 \to {}_{\varepsilon'} \overline{KQ}_{n+2-r}(A)\big[\beta^{-1}\big] \longrightarrow {}_\varepsilon \overline{KSC}^{(r)}_n(A) \longrightarrow {}_\varepsilon \overline{KQ}_{n+1}(A)\big[\beta^{-1}\big] \to 0. $$

Now, the inversion of β yields an isomorphism

$$ {}_{\varepsilon'} \overline{KQ}_{n+2-r}(A)\big[\beta^{-1}\big] \cong {}_\varepsilon \overline{KQ}_{n+2}(A)\big[\beta^{-1}\big] $$

since $q | r$. Finally, the splitting of these sequences is as before a consequence of a general statement on cohomology theories. One has to replace n by a parameter space X, as we shall also do in the next section. □

Remarks 3.16 The most interesting cases of this proposition are when one has $r = 2$ or $r = q$. The proposition also shows that the groups ${}_\varepsilon \overline{KSC}^{(r)}_n(A)$ are isomorphic for $r \geq q$.

4. Proof of the periodicity theorems

Our aim in this Section is essentially to prove Theorems 0.4 and 0.7. We begin with a lemma showing that it suffices to consider only the parameter q of Convention 0.2, rather than the desired period p of Convention 0.1 when dealing with direct or inverse limits.

Lemma 4.1 *Let m be a 2-power and p,q be as in Conventions 0.1 and 0.2; that is,*

m	p	q
≤ 8	8	8
16	8	16
≥ 32	$m/2$	$m/2$

Then

$$\varinjlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m) = \varinjlim_{\varepsilon} KQ_{n+qs}(A;\mathbb{Z}/m)$$

and

$$\varprojlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m) = \varprojlim_{\varepsilon} KQ_{n+qs}(A;\mathbb{Z}/m).$$

Proof: We may focus on the exceptional case where $m = q = 16$ and $p = 8$. Here, by Theorem 1.1 there is a positive Bott element $b^+ \in {}_1\overline{KQ}_8(\mathbb{Z}')$, multiplication by which gives rise to the direct system of abelian groups

$${}_\varepsilon\overline{KQ}_n(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+8}(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+16}(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+24}(A) \longrightarrow \cdots.$$

The direct limit of its subsystem

$${}_\varepsilon\overline{KQ}_n(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+16}(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+32}(A) \longrightarrow \cdots$$

appears in the exact sequence of Theorem 4.2 below. However, since this subsystem is cofinal, its direct limit is precisely that of the original system. In other words, we may replace the term $\varinjlim_{\varepsilon}\overline{KQ}_{n+16s}(A)$ by $\varinjlim_{\varepsilon}\overline{KQ}_{n+8s}(A)$.

Since the negative Bott element originates in ${}_{-1}KQ_{-2}(\mathbb{Z}')$, a similar argument shows that the term $\varprojlim_{\varepsilon}\overline{KQ}_{n+16s}(A)$ may be replaced by $\varprojlim_{\varepsilon}\overline{KQ}_{n+8s}(A)$. □

We now start the proof of the periodicity theorems which will be a consequence of our considerations in Section 3.

Theorem 4.2 *Let A be a ring with involution such that $1/2 \in A$ and let m and p be 2-powers according to Convention 0.1. We assume the existence of an integer d, such that the cup-product with the Bott element in $K_p(\mathbb{Z};\mathbb{Z}/m)$ induces an isomorphism*

$$K_n(A;\mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A;\mathbb{Z}/m).$$

for $n \geq d$. For such n, there is an exact sequence

$$\cdots \xrightarrow{\theta_n^+} \varinjlim_{\varepsilon} KQ_{n+1+ps}(A;\mathbb{Z}/m) \to \varprojlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m)$$

$$\xrightarrow{\theta_n^-} {}_{\varepsilon}KQ_n(A;\mathbb{Z}/m) \xrightarrow{\theta_n^+} \varinjlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m),$$

which for $n \geq d+q-1$ gives a split short exact sequence

$$0 \to \varprojlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m) \xrightarrow{\theta_n^-} {}_{\varepsilon}KQ_n(A;\mathbb{Z}/m) \xrightarrow{\theta_n^+} \varinjlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m) \to 0.$$

Proof: For $n \geq d$, we consider the diagram of exact sequences of Propositions 3.11 and 3.12 (for convenience, we again drop the ring A and the sign of symmetry ε in the notation):

$$\begin{array}{ccccccccc}
\overline{KSC}^{(q)}_{n+q} & \to & P\overline{KQ}_n & \to & \overline{KQ}_{n,n+1} & \to & \overline{KSC}^{(q)}_{n+q-1} & \to & P\overline{KQ}_{n-1} \\
 & & P\beta_n \downarrow 0 & & \downarrow \alpha_n & & \sigma_n \downarrow \cong & & \\
0 & \to & P\overline{KQ}_{n+q} & \to & \overline{KQ}_{n+q,n+q+1} & \to & \overline{KSC}^{(q)}_{n+2q-1} & \to & 0
\end{array}$$

Since the direct limit of the $P\overline{KQ}_{n+qs}$ is equal to 0, we see that

$$\varinjlim \overline{KQ}_{n+qs,n+1+qs} \cong \overline{KSC}^{(q)}_{n+q-1} \cong \mathrm{Im}(\alpha_{n+q})$$

which has already been proven in Proposition 3.15.

We also have a reverse diagram of exact sequences

$$\begin{array}{ccccccc}
P\overline{KQ}_n & \to & \overline{KQ}_{n,n+1} & \to & \overline{KSC}^{(q)}_{n+q-1} \to & & \\
P\beta'_n \uparrow \cong & & \uparrow \alpha'_{n+q} & & \sigma'_n \uparrow 0 & & \\
0 \to P\overline{KQ}_{n+q} & \to & \overline{KQ}_{n+q,n+q+1} & \to & \overline{KSC}^{(q)}_{n+2q-1} \to 0 & &
\end{array} \qquad (4{:}4)$$

where the vertical maps are now induced by the cup-product with the negative Bott element. The first vertical map is an isomorphism, while the last one is reduced to 0, by Proposition 3.11.

From the splitting of exact sequences afforded by Proposition 3.12, we have

$$\varprojlim \overline{KQ}_{n+qs,n+1+qs} \cong \varprojlim P\overline{KQ}_{n+qs} \cong \mathrm{Im}(\alpha'_{n+q}) \cong P\overline{KQ}_n$$

by Lemma 3.7.

The first exact sequence in the first diagram above implies the exactness of the middle row of the diagram

$$\begin{array}{ccccc} \varprojlim \overline{KQ}_{n+1+qs} & \to & \overline{KQ}_{n+1} & \overset{\theta^+_{n+1}}{\to} & \varinjlim \overline{KQ}_{n+1+qs} \\ \downarrow & & \downarrow & & \downarrow \\ \varprojlim \overline{KQ}_{n+qs,n+1+qs} & \overset{\chi^-}{\to} & \overline{KQ}_{n,n+1} & \overset{\chi^+}{\to} & \varinjlim \overline{KQ}_{n+qs,n+1+qs} \\ \downarrow & & \downarrow & & \downarrow \\ \varprojlim \overline{KQ}_{n+qs} & \overset{\theta^-_n}{\to} & \overline{KQ}_n & \overset{\theta^+_n}{\to} & \varinjlim \overline{KQ}_{n+qs} \end{array} \tag{4:5}$$

Since the middle row is the direct sum of first and third rows, the exactness of the middle row implies the exactness of the third row. We apply the same argument for the left part of the required exact sequence.

Now suppose that $n \geq d + q - 1$. Then Proposition 3.12 implies that in the diagram above χ^+ is a split epimorphism. The result follows. □

Recall from Definition 0.5 in the Introduction that a ring A is *hermitian regular* if the inverse limits

$$\varprojlim \overline{KQ}_{n+ps}(A) \quad \text{and} \quad \varprojlim{}^1 \overline{KQ}_{n+ps}(A)$$

are reduced to 0 for all n.

Examples 4.3 We remark that the second condition (with $\lim^1$) is always fulfilled if A has a periodic $\overline{K}$-group after a certain range, since in Section 3 we have shown that the inverse system $\{\overline{KQ}_{n+ps}(A)\}$ satisfies the Mittag-Leffler property. We have also seen in Section 2 that suitable rings of integers in a number field are hermitian regular. On the other hand, according to Hu, Kriz and Ormsby [25] (resp. Schlichting), if k is a field of finite mod 2 virtual étale cohomological dimension and of characteristic 0 (resp. p), we have a homotopy equivalence

$${}_1\overline{KQ}(k) \simeq \overline{K}(k)^{h\mathbb{Z}/2}.$$

Since $\overline{K}_n(k) \cong \overline{K}_{n+p}(k)$ for n large enough by [52] and [62], the positive Bott map

$${}_1KQ_n(k) \longrightarrow {}_1KQ_{n+p}(k)$$

is also an isomorphism for n large enough. The same statement is true for the groups ${}_{-1}KQ_n$, as we see by relating the groups ${}_1KQ, {}_{-1}KQ$ and KSC (see Section 3). As a conclusion, the negative Bott map

$${}_\varepsilon\overline{KQ}_{n+p}(k) \longrightarrow {}_\varepsilon\overline{KQ}_n(k)$$

is trivial for n large enough, which implies that k is hermitian regular by Theorem 4.2. More generally, as a special case of the results in Section 6 and a planned joint paper with Schlichting [7], any commutative algebra A whose residual fields are all of finite mod 2 virtual étale cohomological dimension is hermitian regular. In [7], these results will be generalized in the scheme framework.

Remark 4.4 It is easy to see that the inverse systems of hermitian K-groups $\{{}_\varepsilon\overline{KQ}_{n+ps}(A)\}$ and Witt groups $\{{}_\varepsilon\overline{W}_{n+ps}(A)\}$ are equivalent since we have the following factorization of the negative Bott map:

$$ {}_\varepsilon\overline{KQ}_{n+8}(A) \to {}_\varepsilon\overline{W}_{n+4}(A) \to {}_\varepsilon\overline{KQ}_n(A) \to {}_\varepsilon\overline{W}_{n-4}(A). $$

Therefore, the lim and $\lim^1$ groups may as well be computed with higher Witt groups.

Theorem 4.5 *With the same hypotheses as in the previous theorem, let us assume moreover that the ring A is hermitian regular. Then, for $n \geq d$, the positive Bott map*

$$ {}_\varepsilon\overline{KQ}_n(A) \longrightarrow {}_\varepsilon\overline{KQ}_{n+p}(A) $$

is an isomorphism.

Proof: According to Theorem 4.2, it is enough to show that θ_d^+ is surjective. From the long exact sequence of Proposition 3.2, we obtain the map of exact sequences

$$ \begin{array}{ccccccc} \overline{KQ}_{d+q} & \overset{s_{d+q}}{\to} & \overline{KQ}_d & \to & \overline{KSC}^{(q)}_{d+q-2} & \to & \overline{KQ}_{d+q-1} \\ \downarrow & & \downarrow \theta_d^+ & & \downarrow\cong & & \downarrow \gamma \\ \varinjlim \overline{KQ}_{d+qs} & \overset{u}{\to} & \varinjlim \overline{KQ}_{d+qs} & \to & \overline{KSC}^{(q)}_{d+2q-2} & \to & \varinjlim \overline{KQ}_{d-1+qs} \end{array} \tag{4:6} $$

Observe from Theorem 4.2 that γ is injective because the inverse limits are reduced to 0. Since u is defined by the cup-product with the negative Bott element, it is reduced to 0. An elementary diagram chase now shows that θ_d^+ is surjective. □

Although for simplicity we have presented the arguments only in the case where X is a sphere, we observe that the groups obtained in the exact sequences of the previous theorems can be considered as cohomology theories with respect to pointed spaces X, if we replace the various spectra involved by their sufficiently connected associated spectra (so that the low-dimensional cohomology groups are trivial).

In order to pass from exactness of the sequence to split exactness as we did at the end of Section 3, we may appeal to a general fact about cohomology theories, that any surjective morphism like

$$ {}_\varepsilon\theta_X^+ : {}_\varepsilon\overline{KQ}_X(A) \longrightarrow \varinjlim{}_\varepsilon\overline{KQ}_{X+qs}(A) $$

always admits a section.

The following theorems are the analogs of the previous ones with a parameter space X.

Theorem 4.6 *Let A be a ring with involution such that $1/2 \in A$, m, p and q be 2-powers according to Convention 0.1 and 0.2. We assume the existence of an integer d such that for $n \geq d$ the cup-product with the Bott element in $K_p(\mathbb{Z};\mathbb{Z}/m)$ induces an isomorphism*

$$K_n(A;\mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A;\mathbb{Z}/m).$$

(a) *If X is $(d+q-2)$-connected, we have a split short exact sequence*

$$0 \to \varprojlim_{\varepsilon} KQ_{X+ps}(A;\mathbb{Z}/m) \xrightarrow{\theta_n^-} {}_\varepsilon KQ_X(A;\mathbb{Z}/m) \xrightarrow{\theta_n^+} \varinjlim_{\varepsilon} KQ_{X+ps}(A;\mathbb{Z}/m) \to 0.$$

As a consequence, the groups ${}_\varepsilon KQ_X(A;\mathbb{Z}/m)$ are "periodic" with respect to X of period p, more precisely

$${}_\varepsilon KQ_X(A;\mathbb{Z}/m) \cong {}_\varepsilon KQ_{X+p}(A;\mathbb{Z}/m).$$

In particular, if $n \geq d+q-1$, there is an isomorphism

$${}_\varepsilon KQ_n(A;\mathbb{Z}/m) \cong {}_\varepsilon KQ_{n+p}(A;\mathbb{Z}/m).$$

(b) *Moreover, if A is hermitian regular, the previous statements are still true if we replace the number q by* 1.

Corollary 4.7 *For A, X as in Theorem 4.6(a), and $n \geq d+q-1$, the positive Bott map*

$$\beta_n : {}_\varepsilon KQ_n(A;\mathbb{Z}/m) \longrightarrow {}_\varepsilon KQ_{n+p}(A;\mathbb{Z}/m)$$

has

(i) its image naturally isomorphic to the periodized KQ-theory, that is

$$\mathrm{Im}(\beta_n) \cong \varinjlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m),$$

and

(ii) its kernel and cokernel naturally isomorphic to $\varprojlim_{\varepsilon} KQ_{n+ps}(A;\mathbb{Z}/m)$. Consequently, if β_n is either injective or surjective then it is an isomorphism.

Moreover, if A is hermitian regular, the same statements remain true on replacing q by 1.

Proof: Here, we chase the following commutative diagram, where the vertical maps are given by the cup-product with the positive Bott element:

$$\begin{array}{ccccccccc} 0\to & \varprojlim_\varepsilon \overline{KQ}_{n+ps}(A) & \xrightarrow{\theta^-} & {}_\varepsilon\overline{KQ}_n(A) & \xrightarrow{\theta^+} & \varinjlim_\varepsilon \overline{KQ}_{n+ps}(A) & \to 0 \\ & \downarrow 0 & & \downarrow \beta_n & & \downarrow\cong & \\ 0\to & \varprojlim_\varepsilon \overline{KQ}_{n+p+ps}(A) & \xrightarrow{\theta^-} & {}_\varepsilon\overline{KQ}_{n+p}(A) & \xrightarrow{\theta^+} & \varinjlim_\varepsilon \overline{KQ}_{n+p+ps}(A) & \to 0 \end{array}$$

□

Remark 4.8 One may ask if the map θ_n^+ in Theorem 4.2 is an isomorphism in general for n sufficiently large. This is not the case however, as is shown in the Appendix C to [6] and in [38], where we give many examples of rings with trivial K-theory and nontrivial KQ-theory. It follows from the 12-term exact sequence of [33, p. 278], that for such rings θ_n^- is an isomorphism. From the short exact sequence of Theorem 4.2, θ_n^+ must therefore vanish, although the KQ-theory is nontrivial. Thus, θ_n^+ fails to be an isomorphism. Other examples may be found in the paper of Hu, Kriz and Ormsby [25] for commutative rings and schemes.

However, one may hope it is so for the examples of commutative rings A considered in the Introduction, which are of "geometric nature". See also Section 6 for an analogous conjecture in the category of schemes.

We finish this section with an application to the computation of the $\overline{KQ}$-groups in terms of the $\overline{K}$-groups when these groups are finite. (This result formally dates from the December 2010 resubmission of the paper.) For reading convenience, we again suppress the index $\varepsilon \in \{\pm 1\}$.

Theorem 4.9 *Let us assume the hypotheses of Theorem 4.6 and that for $n \geq d$ the $\overline{K}_n$-groups are finite, of order k_n. Then for $n \geq d$ the $\overline{KQ}_n$-groups are finite, of order* kq_n *subject to the inequality*

$$\mathrm{kq}_n + \mathrm{kq}_{n+1} \leq k_d + k_{d+1} + \cdots + k_{d+q-1}$$

and equality $\mathrm{kq}_n = \mathrm{kq}_{n+q}$.

Proof: According to (3:3), there is an exact sequence

$$\cdots \longrightarrow \overline{KSC}_n^{(r/2)} \longrightarrow \overline{KSC}_n^{(r)} \longrightarrow \overline{KSC}_{n-r/2}^{(r/2)} \longrightarrow \cdots.$$

In the case $r = 2$, the outer two groups are respectively $\overline{K}_{n+1}$ and $\overline{K}_n$, and so assumed finite when $n \geq d$. In general, if we denote by s_n^r the order of the group $\overline{KSC}_n^{(r)}$ when it is finite, we therefore have the inequality

$$s_n^r \leq s_{n-r/2}^{r/2} + s_n^{r/2}.$$

For instance, with $n \geq d$,

$$\begin{aligned} s^2_{n+1} \leq \quad & k_{n+1} + k_{n+2} \\ s^4_{n+3} \leq \quad & s^2_{n+1} + s^2_{n+3} \leq k_{n+1} + k_{n+2} + k_{n+3} + k_{n+4} \\ \vdots & \\ s^q_{n+q-1} \leq \quad & k_{n+1} + k_{n+2} + \cdots + k_{n+q} = k_d + \cdots + k_{d+q-1}, \end{aligned}$$

where the equality follows from the assumption of $\overline{K}$-periodicity. On the other hand, since the ring A is hermitian regular, according to Theorem 4.2 and its proof we have

$$s^q_{n+q-1} = \mathrm{kq}_n + \mathrm{kq}_{n+1}$$

for $n \geq d$. This gives the required inequality. Finally, the equality comes from Theorem 4.5. □

Example 4.10 If $m = 8$, we have $q = 8$. Therefore, for $n \geq d$, we have the inequality

$$\mathrm{kq}_n + \mathrm{kq}_{n+1} \leq k_d + k_{d+1} + \cdots + k_{d+7}.$$

In particular, if we assume that the groups ${}_\varepsilon KQ_n(A)$ and ${}_\varepsilon KQ_{n+1}(A)$ are finitely generated, and that for some r and ε the Witt group ${}_\varepsilon W_r(A)$ has an infinite free summand, then by periodicity of Witt groups after tensoring with $\mathbb{Q}$, one of the four groups ${}_\varepsilon\overline{KQ}_n$, ${}_\varepsilon\overline{KQ}_{n+1}$ ($\varepsilon \in \{\pm 1\}$) must be nonzero. It now follows that at least one of the groups $K_d(A), \ldots, K_{d+7}(A)$ must also be nonzero.

5. The case of odd prime power coefficients

For the sake of completeness, we should also study KQ-theory with odd prime power coefficients, which is much easier to handle, starting from known results in K-theory. As is well known, if ℓ is an odd prime, there is a remarkable Bott element b_K in the group $K_{2(\ell-1)\ell^{\nu-1}}(\mathbb{Z}; \mathbb{Z}/\ell^\nu)$ (see Section 1 of this paper). In particular, its image in the topological K-group

$$K_{2(\ell-1)\ell^{\nu-1}}(\mathbb{R}; \mathbb{Z}/\ell^\nu) \cong K_{2(\ell-1)\ell^{\nu-1}}(\mathbb{C}; \mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu$$

is the image of an integral Bott generator in $K_{2(\ell-1)\ell^{\nu-1}}(\mathbb{C}) \cong \mathbb{Z}$. According to Convention 0.1, we shall write $p = 2(\ell-1)\ell^{\nu-1}$ (for the period) and $m = \ell^\nu$ (for the order of the coefficient group).

Let us assume now that A is one of the examples of algebras described in the Introduction. For odd primes, we use the Bloch-Kato conjecture, which is now a theorem proven by Rost and Voevodsky, *cf.* [54], [55], [63], [21], [58], [59],

and [64]. This implies that the cup-product with the Bott element b induces an isomorphism

$$K_n(A;\mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A;\mathbb{Z}/m)$$

whenever $n \geq d$ for the type of rings A considered in the Introduction.

In order to extend our previous results to hermitian K-theory with odd prime power coefficients, it is convenient to describe more geometrically elements of the K-groups and KQ-groups. This description in terms of "virtual" flat bundles is given in detail in Appendix 1 of [35]. For instance, an element of ${}_\varepsilon KQ_n(A)$ can be described as a flat A-bundle E over an homology sphere of dimension n, provided with a nondegenerate quadratic form q. With this language, we can easily define an involution on the KQ-groups: it is induced by the correspondence

$$(E,q) \mapsto (E,-q).$$

Let us now consider the groups ${}_\varepsilon KQ_n(A)' = {}_\varepsilon KQ_n(A) \otimes_{\mathbb{Z}} \mathbb{Z}'$. The tensor product of virtual A-bundles (when A is commutative) induces a ring structure on the direct sum of all these groups (with $\varepsilon = \pm 1$). On the other hand, the previous involution enables us to split each group ${}_\varepsilon KQ_n(A)'$ as a direct sum ${}_\varepsilon KQ_n(A)'_+ \oplus {}_\varepsilon KQ_n(A)'_-$.

Lemma 5.1 *The sum decomposition of the ${}_\varepsilon KQ_n(A)'$ described above is a ring product decomposition. In other words, the cup-product map between elements of KQ'_+ and KQ'_- is reduced to* 0.

Proof: Since we make 2 invertible in the KQ'-groups involved, one may think of an element z of KQ'_+ as a sum $(E,q) + (E,-q)$ and an element z' of KQ'_- as a difference $(E',q') - (E',-q')$, where q and q' are ε- and ε'-quadratic forms. The product $z \cdot z'$ is therefore (with $q \otimes q'$ an $\varepsilon\varepsilon'$-quadratic form)

$$(E \otimes E', q \otimes q') + (E \otimes E', -q \otimes q') - (E \otimes E', q \otimes -q') - (E \otimes E', -q \otimes -q'),$$

which is of course zero. □

Corollary 5.2 *The ring product decomposition of the direct sum of the groups ${}_\varepsilon KQ_n(A)'$ induces a ring product decomposition of the direct sum of the groups $KQ_n(A;\mathbb{Z}/m)$ when m is an odd prime power* > 3.

Proof: The corollary follows from general arguments about cohomology theories mod$m$ [3]. □

Remark 5.3 One also has an involution on the K-groups induced by the duality functor, as was noticed already in Section 3. If we perform the tensor product by $\mathbb{Z}[1/2]$ or we take coefficients in the group $\mathbb{Z}/m$ with m odd, the symmetric part

is in bijective correspondence with the symmetric part of the corresponding KQ-group. This correspondence is induced by the forgetful functor or the hyperbolic functor [33].

Remark 5.4 Before introducing the Bott elements in this situation, it is worth mentioning that the fundamental theorem in hermitian K-theory holds for arbitrary rings (we no longer assume that $1/2 \in A$) when we localize away from 2. The details may be found in [36, Lemma 1.1]. More precisely, in this case the symmetric part $\mathcal{KQ}(A)'_+$ of the spectrum of $\mathcal{KQ}(A)'$ is the symmetric part of the spectrum $\mathcal{K}(A)'$, whereas the antisymmetric part $\mathcal{KQ}(A)'_-$ has periodic homotopy groups of period 4, which are the higher Witt groups. This remark also applies when we take mod m coefficients with m odd.

Let b_+ denote the Bott element in ${}_1KQ_p(\mathbb{Z};\mathbb{Z}/m)_+ = {}_1\overline{KQ}_p(\mathbb{Z})_+$ corresponding to the usual Bott element b_K in

$$K_p(\mathbb{Z};\mathbb{Z}/m) = K_p(\mathbb{Z};\mathbb{Z}/m)_+ = \overline{K}_p(\mathbb{Z})_+.$$

The following Theorem is now obvious, since $\overline{K}_+ \cong \overline{KQ}_+$.

Theorem 5.5 *Let A be any ring such that the cup-product map with the K-theory Bott element b_K induces an isomorphism*

$$\overline{K}_n(A) \cong \overline{K}_{n+p}(A)$$

for $n \geq d$. Then, taking cup-product with b_+ also induces an isomorphism

$${}_\varepsilon\overline{KQ}_n(A)_+ \cong {}_\varepsilon\overline{KQ}_{n+p}(A)_+$$

for the same values of n. □

On the other hand, there is another "Bott element" b' that lies in ${}_1\overline{KQ}_p(\mathbb{Z})_- = {}_1W_p(\mathbb{Z};\mathbb{Z}/m)$, which is the higher Witt group mod m. It is the image mod m of a suitable power of $u \in {}_{-1}W_2(\mathbb{Z})$ constructed in [33] and [36, Theorem 1.4].

Theorem 5.6 *For an odd prime ℓ, let $p = 2(\ell-1)\ell^{\nu-1}$ and $m = \ell^\nu$ as in (0.1). Let A be any ring such that its K-theory* mod m *is periodic for $n \geq d$, the periodicity being given by the cup-product with the Bott element b_K. As usual, denote by $\overline{KQ}$ the KQ-groups* mod m. *Then, for $n \geq d$, there is an isomorphism*

$${}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\theta_{+-}} \varinjlim {}_\varepsilon\overline{KQ}_{n+ps}(A)$$

where θ_{+-} is induced by the cup-product with the sum $b_+ + b'$ of the two previously defined Bott elements in the group

$${}_1\overline{KQ}_p(\mathbb{Z}) = {}_1\overline{KQ}_p(\mathbb{Z})_+ \oplus {}_1\overline{KQ}_p(\mathbb{Z})_-.$$

Proof: If we consider the direct sum

$$_\varepsilon\overline{KQ}_n(A) = {}_\varepsilon\overline{KQ}_n(A)_+ \oplus {}_\varepsilon\overline{KQ}_n(A)_-,$$

then the cup-product with this element b' is trivial on the summand ${}_\varepsilon\overline{KQ}_n(A)_+$, and induces an isomorphism from ${}_\varepsilon\overline{KQ}_n(A)_-$ to ${}_\varepsilon\overline{KQ}_{n+p}(A)_-$ after [32, Théorème 3.9]. In a parallel way, the cup-product with b_+ is trivial on the term ${}_\varepsilon\overline{KQ}_n(A)_-$ and induces an isomorphism from ${}_\varepsilon\overline{KQ}_n(A)_+$ to ${}_\varepsilon\overline{KQ}_{n+p}(A)_+$, as shown in Corollary 5.2. The theorem follows. □

Remark 5.7 By choosing a "negative Bott element" in ${}_1\overline{KQ}_{-p}(\mathbb{Z})_-$, we obtain an equivalent version of the previous theorem, in analogy with Theorem 0.4, as the following short split exact sequence:

$$0 \to \varprojlim_\varepsilon \overline{KQ}_{n+ps}(A) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_n(A) \xrightarrow{\theta^+} \varinjlim_\varepsilon \overline{KQ}_{n+ps}(A) \to 0.$$

In this exact sequence the inverse system (resp. direct system) is given by taking cup-product with the negative (resp. positive) Bott element in ${}_1\overline{KQ}_{-p}(\mathbb{Z})_-$ (resp. ${}_1\overline{KQ}_p(\mathbb{Z})_+$).

Remark 5.8 For simplicity, we have assumed the ring A commutative in order to define internal cup-products. However, a closer look at the arguments shows that we have in fact used "external" cup-products of the type

$$\overline{KQ}(A) \times \overline{KQ}(\mathbb{Z}) \longrightarrow \overline{KQ}(A)$$

Therefore, the previous theorem extends easily to noncommutative rings, such as group rings.

6. Generalization to schemes and étale theories

The proof and the statement of Theorem 0.4 apply verbatim to schemes for which 2 is invertible. This follows since the fundamental theorem in hermitian K-theory has been generalized by Schlichting – see his work in progress on exact categories with weak equivalences and duality [56]. (Of course, in the context of CW-spectra, weak equivalences are in fact homotopy equivalences.) In this section we view this generalization in the context of the étale descent problem for hermitian K-theory.

Let S be a regular and separated noetherian scheme of finite Krull dimension (in the interest of generalizing these assumptions the inclined reader may compare with [53] and [56]). Throughout, for a fixed prime ℓ, we assume that $\mathcal{O}_S$ is a sheaf of $\mathbb{Z}[1/\ell]$-modules. In particular, for the important case $\ell = 2$, 2 is invertible in S.

For the definition of the hermitian K-theory spectrum ${}_\varepsilon\mathcal{KQ}(S)$, and the forgetful and hyperbolic maps between the algebraic and hermitian K-theory of S, we refer to Schlichting's work [56]. In particular, as for rings, we may form the fibers ${}_\varepsilon\mathcal{V}(S)$ and ${}_\varepsilon\mathcal{U}(S)$ of the forgetful and hyperbolic maps, respectively. The generalization of the fundamental theorem to schemes in [56] shows that there is a homotopy equivalence

$$ {}_{-\varepsilon}\mathcal{V}(S) \simeq \Omega\,{}_\varepsilon\mathcal{U}(S) \tag{6:7} $$

such that the composite map

$$ \Omega^2\,{}_\varepsilon\mathcal{KQ}(S) \to \Omega\,{}_\varepsilon\mathcal{U}(S) \to {}_{-\varepsilon}\mathcal{V}(S) \to {}_{-\varepsilon}\mathcal{KQ}(S) $$

is given by the cup-product with the negative Bott element in ${}_{-1}KQ_{-2}(\mathbb{Z}')$.

With these results in hand, the proof of Theorem 0.4 carries over to the setting of schemes. That is, if cup-product with the Bott element in $K_p(\mathbb{Z};\mathbb{Z}/m)$, m and p being 2-powers linked by our Convention 0.1, induces an isomorphism

$$ K_n(S;\mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(S;\mathbb{Z}/m) $$

for $n \geq d$, then, for $n \geq d+q-1$ there is a split short exact sequence

$$ 0 \to \varprojlim{}_\varepsilon\overline{KQ}_{n+ps}(S) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_n(S) \xrightarrow{\theta^+} \varinjlim{}_\varepsilon\overline{KQ}_{n+ps}(S) \to 0. $$

We expect that the map θ^+ is an isomorphism in many cases of geometric interest. (For $n \geq d$, and not just $\geq d+q-1$: compare with Theorem 4.5.) This question is closely related to the so-called étale descent problem for hermitian K-theory and explicit computations, which are our main concerns in this section.

Jardine introduced in [28] the Bott periodic étale hermitian K-theory spectra of S with mod ℓ^ν-coefficients

$$ {}_\varepsilon\mathcal{KQ}^{\text{ét}}/\ell^\nu(S). $$

More precisely, ${}_\varepsilon\mathcal{KQ}^{\text{ét}}/\ell^\nu(S)$ is the $\mathcal{KO}$- or equivalently $\mathcal{KU}$-localization of Jardine's étale hermitian K-theory. By construction of the Bott periodic étale theory, there exists an induced mod ℓ^ν comparison map

$$ \Gamma_S : {}_\varepsilon\mathcal{KQ}/\ell^\nu(S) \longrightarrow {}_\varepsilon\mathcal{KQ}^{\text{ét}}/\ell^\nu(S) $$

obtained by taking global sections of a globally fibrant model for the presheaf ${}_\varepsilon\mathcal{KQ}/\ell^\nu()$ on some sufficiently large étale site of S.

Similarly, the mod ℓ^ν étale self-conjugate K-theory $\mathcal{KSC}^{\text{ét}}/\ell^\nu(S)$ of S is defined by taking a globally fibrant model of the presheaf $\mathcal{KSC}/\ell^\nu()$. Later in this section we shall make use of specific fibrant models.

Recall that the notation vcd_ℓ stands for the mod ℓ virtual cohomological dimension. When ℓ is odd, then vcd_ℓ coincides with the usual mod ℓ cohomological dimension cd_ℓ. Although cd_2 is infinite in the examples of $\mathbb{Z}[1/2]$ and $\mathbb{R}$, the number vcd_2 is finite in both cases. For a proof of the next result we refer to [40, Proposition 6.1].

Lemma 6.1 *If* $\mathrm{vcd}_\ell(S) < \infty$ *then the étale hypercohomology presheaf*

$$\mathbb{H}^\bullet_{\text{ét}}(-, {}_\varepsilon\mathcal{KQ}/\ell^\nu())$$

is a globally fibrant model for ${}_\varepsilon\mathcal{KQ}/\ell^\nu(-)$. *The same fibrancy result holds for the presheaf* $\mathcal{KSC}/\ell^\nu(-)$. □

The étale descent problem for self-conjugate K-theory can be solved easily using the solution for algebraic K-theory [52]. For a point $s \in S$, let $k(s)$ denote the corresponding residue field.

Theorem 6.2 *The comparison map*

$$\mathcal{KSC}/\ell^\nu(S) \longrightarrow \mathcal{KSC}^{\text{ét}}/\ell^\nu(S)$$

is a weak equivalence on $\sup\{\mathrm{vcd}_\ell(k(s)) - 2\}_{s\in S}$*-connected covers.*

Hence, if $\mathrm{vcd}_\ell(S) < \infty$ *then there is a weak equivalence*

$$L_{\mathcal{KU}}\mathcal{KSC}/\ell^\nu(S) \longrightarrow \mathcal{KSC}^{\acute{e}t}/\ell^\nu(S) \simeq \mathbb{H}^\bullet_{\acute{e}t}(S, L_{\mathcal{KU}}\mathcal{KSC}/\ell^\nu()).$$

Proof: There exists a naturally induced commutative diagram of fiber sequences of presheaves of spectra

$$\begin{array}{ccccc} \mathcal{KSC}/\ell^\nu(-) & \longrightarrow & \mathcal{K}/\ell^\nu(-) & \longrightarrow & \mathcal{K}/\ell^\nu(-) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{KSC}^{\text{ét}}/\ell^\nu(-) & \longrightarrow & \mathcal{K}^{\text{ét}}/\ell^\nu(-) & \longrightarrow & \mathcal{K}^{\text{ét}}/\ell^\nu(-) \end{array}$$

□

Similarly to Theorem 6.2, we do not expect that the comparison map Γ_S is a weak equivalence in general, but in many cases of interest it should be a weak equivalence on some connected cover.

Let ℓ be an odd prime, and let the subscript $+$ as in K_+ denote the symmetric part of K-theory. Then the forgetful and hyperbolic functors induce isomorphisms between the symmetric parts of the algebraic and hermitian K-groups of S at ℓ. This allows us to infer the following result by referring to [52] and to Corollary 6.8 for the symmetry of étale hermitian K-theory at odd primes.

Theorem 6.3 *Let $\ell \neq 2$. The comparison map*

$$ {}_\varepsilon\mathcal{KQ}/\ell^\nu(S)_+ \longrightarrow {}_\varepsilon\mathcal{KQ}^{ét}/\ell^\nu(S)_+ \simeq {}_\varepsilon\mathcal{KQ}^{ét}/\ell^\nu(S) $$

is a weak equivalence on $\sup\{\mathrm{vcd}_\ell(k(s))-2\}_{s\in S}$*-connected covers.*

Hence, if $\mathrm{vcd}_\ell(S) < \infty$ *then there is a weak equivalence*

$$ L_{\mathcal{KU}}\,{}_\varepsilon\mathcal{KQ}/\ell^\nu(S)_+ \longrightarrow {}_\varepsilon\mathcal{KQ}^{ét}/\ell^\nu(S) \simeq \mathbb{H}^\bullet_{ét}(S, L_{\mathcal{KU}}\,{}_\varepsilon\mathcal{KQ}/\ell^\nu()). $$

□

At $\ell = 2$, we prove Theorem 0.13 stated in the Introduction. In the following, let $n \geq \sup\{\mathrm{vcd}_2(k(s))-1\}_{s\in S} + q - 1$. Inverting the positive Bott element in the direct sum decomposition

$$ {}_\varepsilon\overline{KQ}_{n,n+1}(S) \cong P\,{}_\varepsilon\overline{KQ}_n(S) \oplus {}_\varepsilon\overline{KSC}^{(p)}_{n+p-1}(S) $$

yields

$$ {}_\varepsilon\overline{KQ}_{n,n+1}(S)[\beta^{-1}] \cong {}_\varepsilon\overline{KSC}^{(p)}_{n+p-1}(S)[\beta^{-1}]. $$

In order to simplify the right hand side of this isomorphism, we first use the fact that if $\mathrm{vcd}_2(S) < \infty$ then there is a weak equivalence

$$ \overline{\mathcal{KSC}}^{(r)}(S)[\beta^{-1}] \longrightarrow \overline{\mathcal{KSC}}^{(r)^{\text{ét}}}(S). $$

Then, by étale descent for self-conjugate K-theory shown in Theorem 6.2, the induced comparison map

$$ \overline{\mathcal{KSC}}^{(r)}(S) \longrightarrow \overline{\mathcal{KSC}}^{(r)^{\text{ét}}}(S) $$

is a weak equivalence on $\sup\{\mathrm{vcd}_2(k(s))+r-4\}_{s\in S}$-connected covers. Hence, there is an isomorphism

$$ {}_\varepsilon\overline{KSC}^{(p)}_{n+p-1}(S)[\beta^{-1}] \cong {}_\varepsilon\overline{KSC}^{(p)}_{n+p-1}(S). $$

As a result,

$$ {}_\varepsilon\overline{KQ}_{n,n+1}(S) = {}_\varepsilon\overline{KQ}_n(S) \oplus {}_\varepsilon\overline{KQ}_{n+1}(S) $$

maps by a split surjection onto its Bott localization

$$ {}_\varepsilon\overline{KQ}_{n,n+1}(S)[\beta^{-1}] \cong {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S) \oplus {}_\varepsilon\overline{KQ}^{\text{ét}}_{n+1}(S). $$

By looking at one component at a time, we deduce that there is a split surjection

$$ {}_\varepsilon\overline{KQ}_n(S) \longrightarrow {}_\varepsilon\overline{KQ}_n(S)[\beta^{-1}] $$

and an isomorphism

$$ {}_\varepsilon\overline{KQ}_n(S)[\beta^{-1}] \cong {}_\varepsilon\overline{KQ}^{\text{ét}}_n(S) $$

for all n. □

Our next result is a local-global comparison theorem. It will be a consequence of the homotopical setup due to Jardine, see *e.g.* [29], and the rigidity theorem for hermitian K-theory of henselian pairs proven by the second author in [34, Theorem 4], *cf.* the unpublished work [27] for an approach using homotopy theory of simplicial presheaves. The specific result we use is as follows.

Theorem 6.4 *([34]) Let (A, I) be a henselian pair with $q \in A^\times \cap \mathbb{Z}$ such that I is invariant by the involution on A, and $\lambda + \overline{\lambda} = 1$ for some $\lambda \in A$ if q is even. Then the map of rings with involutions $A \to A/I$ induces an isomorphism*

$$ {}_\varepsilon KQ_n(A; \mathbb{Z}/q) \xrightarrow{\cong} {}_\varepsilon KQ_n(A/I; \mathbb{Z}/q) $$

for all ε and $n \geq 0$.

We note that the sharper bound for the connected covers in the theorem below (relative to that in Theorem 0.13) equals the one shown for algebraic K-theory in [52].

Theorem 6.5 *Suppose that $\Gamma_{k(s)}$ is a weak equivalence on $(\mathrm{vcd}_2(k(s)) - 2)$-connected covers for every residue field $k(s)$ of S. Then the comparison map*

$$ \Gamma_S : {}_\varepsilon\mathcal{KQ}/2^\nu(S) \longrightarrow {}_\varepsilon\mathcal{KQ}^{\acute{e}t}/2^\nu(S) $$

is a weak equivalence on $\sup\{\mathrm{vcd}_2(k(s)) - 2\}_{s \in S}$-connected covers. □

Proof: There is a functorially induced commutative diagram with the mod 2 comparison map displayed on top:

$$ \begin{array}{ccc} {}_\varepsilon\mathcal{KQ}/2(S) & \longrightarrow & \mathbb{H}^\bullet_{\text{ét}}(S, L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/2()) \\ \downarrow & & \downarrow \\ \mathbb{H}^\bullet_{\text{Nis}}(S, {}_\varepsilon\mathcal{KQ}/2()) & \longrightarrow & \mathbb{H}^\bullet_{\text{Nis}}(S, \mathbb{H}^\bullet_{\text{ét}}(S, L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/2())) \end{array} $$

We claim that the vertical maps are weak equivalences. By the Nisnevich descent theorem in [56], this holds for the left hand side. For the right hand side, the étale topology is finer than the Nisnevich one; so, the direct image functor maps $\mathbb{H}^\bullet_{\text{ét}}(S, L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/2())$ to a globally fibrant object on the Nisnevich site of S. We claim that the mod 2 comparison map is a stalkwise weak equivalence on the given connected cover for the Nisnevich topology. In fact, let A be a Hensel local ring

with residue field k and consider the functorially induced commutative diagram

$$\begin{array}{ccc} {}_\varepsilon\mathcal{KQ}/2(A) & \longrightarrow & {}_\varepsilon\mathcal{KQ}/2(k)) \\ \downarrow & & \downarrow \\ \mathbb{H}^\bullet_{\text{ét}}(A, L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/2()) & \longrightarrow & \mathbb{H}^\bullet_{\text{ét}}(k, L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/2()) \end{array}$$

Combining the previous theorem concerning rigidity for hermitian K-theory and the equivalence between the étale sites of A and k, this reduces the stalkwise weak equivalence to the assumed case of fields. It follows that the lower horizontal map in Diagram (6) is a stalkwise weak equivalence on the same connected covers between globally fibrant objects, and hence it is a pointwise weak equivalence on $\sup\{vcd(k(s)-2\}_{s\in S}$-connected covers. □

Motivated by the local hypotheses of Theorem 6.5, we make the following forecast of the outcome of the étale descent problem for hermitian K-theory.

Conjecture 6.6 *Suppose that k is a field of characteristic* $\neq 2$. *Then the comparison map*

$$\Gamma_k : {}_\varepsilon\mathcal{KQ}/2^\nu(k) \longrightarrow {}_\varepsilon\mathcal{KQ}^{\acute{e}t}/2^\nu(k)$$

is a weak equivalence on ($\mathrm{vcd}_2(k)-2$)*-connected covers.*

Conjecture 6.6, in conjunction with Theorem 6.5, predicts that, in many cases of interest, hermitian K-theory is Bott periodic on some connected cover. Our earlier results on Bott periodicity can be taken as oblique evidence for this conjecture. For $n \geq \mathrm{vcd}_2(k)+q-1$, recall the exact sequence of KQ-groups with 2-power coefficients:

$$0 \to \varprojlim{}_\varepsilon\overline{KQ}_{n+ps}(k) \xrightarrow{\theta^-} {}_\varepsilon\overline{KQ}_n(k) \xrightarrow{\theta^+} \varinjlim{}_\varepsilon\overline{KQ}_{n+ps}(k) \to 0.$$

By Bott periodicity, Conjecture 6.6 implies that the inverse limit is trivial, *i.e.*

$$\varprojlim{}_\varepsilon\overline{KQ}_{n+ps}(k) = 0.$$

In other words, the field k should be hermitian regular (Definition 0.5). Conversely, if the inverse limit is trivial, then there is an isomorphism

$$\theta^+ : {}_\varepsilon\overline{KQ}_n(k) \xrightarrow{\cong} \varinjlim{}_\varepsilon\overline{KQ}_{n+ps}(k)$$

for $n \geq \mathrm{vcd}_2(k)-1$, according to our Theorem 4.5. As noted in the Introduction, a proof of the above conjecture is to appear in a joint paper with Schlichting [7].

Lemma 6.1 can be motivated by the conditionally convergent right half-plane cohomological descent spectral sequence established by Thomason [60]:

$${}_\varepsilon E_2^{p,q} = H^p_{\acute{e}t}(S, \widetilde{\pi}_q L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/\ell^\nu()) \Longrightarrow \pi_{q-p}\mathbb{H}^\bullet_{\acute{e}t}(S, L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/\ell^\nu()). \qquad (6{:}8)$$

Here the coefficient sheaf indicated by $\widetilde{\pi}_*$ is the étale sheafification of the presheaf of stable homotopy groups $\pi_* L_{\mathcal{KU}}{}_\varepsilon\mathcal{KQ}/\ell^\nu(\,)$. The concept of "conditional convergence" for spectral sequences was introduced by Boardman in [8]. A useful consequence is that the descent spectral sequence (6:8) converges strongly provided that there exists only a finite number of nontrivial differentials. Thus, the spectral sequence (6:8) is strongly convergent if S has finite mod ℓ étale cohomological dimension. (This will be the case in all the examples we consider.) The d_r-differential in (6:8) has bidegree $(r, 1-r)$.

In order to identify the étale stalks of ${}_\varepsilon\mathcal{KQ}/\ell^\nu(\,)$, and consequently the E_2-page of (6:8), *cf.* [28], [57, Theorem 2.6], we invoke the Rigidity Theorem 6.4 together with the homotopy equivalences

$$ {}_\varepsilon\mathcal{KQ}/\ell^\nu(A) \simeq {}_\varepsilon\mathcal{KQ}/\ell^\nu(\mathbb{C}) \simeq \begin{cases} \mathcal{K}/\ell^\nu(\mathbb{R}) & \varepsilon = 1 \\ \Omega^4\mathcal{K}/\ell^\nu(\mathbb{R}) & \varepsilon = -1 \end{cases} \tag{6:9} $$

for a strict Hensel local ring A. The above is very similar to the case of algebraic K-theory, where the étale sheaf associated to the presheaf

$$ U \mapsto \pi_n\mathcal{K}/\ell^\nu(U) $$

is the Tate twisted sheaf of roots of unity $\mu_{\ell^\nu}^{\otimes k}$ when $n = 2k$ is even, and trivial when n is odd. For $\mathcal{KSC}$ and ${}_\varepsilon\mathcal{KQ}$ at ℓ we have:

Corollary 6.7 *([57]) The étale sheaf associated to the presheaf*

$$ U \mapsto \pi_n(\mathcal{KSC}/\ell^\nu(U)) $$

is given by:

n mod 4	$\ell = 2$	$\ell \neq 2$
$4k$	$\mu_{2^\nu}^{\otimes 2k}$	$\mu_{\ell^\nu}^{\otimes 2k}$
$4k+1$	$\mu_2^{\otimes 2k+1}$	0
$4k+2$	$\mu_2^{\otimes 2k+1}$	0
$4k+3$	$\mu_{2^\nu}^{\otimes 2k+2}$	$\mu_{\ell^\nu}^{\otimes 2k+2}$

Corollary 6.8 *([28], [57]) The étale sheaf associated to the presheaf*

$$ U \mapsto \pi_n({}_\varepsilon\mathcal{KQ}/\ell^\nu(U)) $$

is given as follows.

1. *For $\ell = 2$ by:*

n mod 8	$\varepsilon=1,\nu=1$	$\varepsilon=-1,\nu=1$	$\varepsilon=1,\nu>1$	$\varepsilon=-1,\nu>1$
$8k$	$\mu_2^{\otimes 4k}$	$\mu_2^{\otimes 4k}$	$\mu_{2^\nu}^{\otimes 4k}$	$\mu_{2^\nu}^{\otimes 4k}$
$8k+1$	$\mu_2^{\otimes 4k+1}$	0	$\mu_2^{\otimes 4k+1}$	0
$8k+2$	$\mu_4^{\otimes 4k+1}$	0	$(\mu_2^{\otimes 4k+1})^{\oplus 2}$	0
$8k+3$	$\mu_2^{\otimes 4k+1}$	0	$\mu_2^{\otimes 4k+1}$	0
$8k+4$	$\mu_2^{\otimes 4k+2}$	$\mu_2^{\otimes 4k+2}$	$\mu_{2^\nu}^{\otimes 4k+2}$	$\mu_{2^\nu}^{\otimes 4k+2}$
$8k+5$	0	$\mu_2^{\otimes 4k+3}$	0	$\mu_2^{\otimes 4k+3}$
$8k+6$	0	$\mu_4^{\otimes 4k+3}$	0	$(\mu_2^{\otimes 4k+3})^{\oplus 2}$
$8k+7$	0	$\mu_2^{\otimes 4k+3}$	0	$\mu_2^{\otimes 4k+3}$

2. *For $\ell\neq 2$ and $\varepsilon=\pm 1$ by $\mu_{\ell^\nu}^{\otimes 2k}$ if $n=4k$, and trivial otherwise.*

Remark 6.9 In Corollary 6.8, the (4,2)-periodicity in the change of symmetry between the $\varepsilon=1$ and $\varepsilon=-1$ cases in the table for $\ell=2$ is given by cup-product with a generator of ${}_{-1}KQ_4(\mathbb{C};\mathbb{Z}/2^\nu)$. The case $\ell\neq 2$ is similar. In degrees $8k+2$, recall that $\mathbb{R}P^2$ is a mod 2 Moore space and $\widetilde{KO}(\mathbb{R}P^2)\cong\mathbb{Z}/4$ generated by the tangent bundle, while the universal coefficient sequence splits for $\nu>1$.

As a consequence of Lemma 6.1 and Corollary 6.8, we conclude that if $\mathrm{vcd}_2(S)<\infty$ then there exist conditionally convergent cohomological spectral sequences

$${}_1E_2^{p,q}=\left\{\begin{array}{ll} H^p_{\text{ét}}(S,\mu_2^{\otimes 4k}) & q=8k\\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+1}) & q=8k+1\\ H^p_{\text{ét}}(S,\mu_4^{\otimes 4k+1}) & q=8k+2\\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+1}) & q=8k+3\\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+2}) & q=8k+4\\ 0 & q\equiv 5,6,7\ (\mathrm{mod}\ 8)\end{array}\right\}\Longrightarrow {}_1KQ^{\text{ét}}_{q-p}/2(S),$$

and

$${}_{-1}E_2^{p,q}=\left\{\begin{array}{ll} H^p_{\text{ét}}(S,\mu_2^{\otimes 4k}) & q=8k\\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+2}) & q=8k+4\\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+3}) & q=8k+5\\ H^p_{\text{ét}}(S,\mu_4^{\otimes 4k+3}) & q=8k+6\\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+3}) & q=8k+7\\ 0 & q\equiv 1,2,3\ (\mathrm{mod}\ 8)\end{array}\right\}\Longrightarrow {}_{-1}KQ^{\text{ét}}_{q-p}/2(S).$$

And for $\nu \geq 2$, the descent spectral sequences take the forms

$$ {}_1E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S, \mu_{2^\nu}^{\otimes 4k}) & q = 8k \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 4k+1}) & q = 8k+1 \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 4k+1})^{\oplus 2} & q = 8k+2 \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 4k+1}) & q = 8k+3 \\ H^p_{\text{ét}}(S, \mu_{2^\nu}^{\otimes 4k+2}) & q = 8k+4 \\ 0 & q \equiv 5,6,7 \pmod 8 \end{array} \right\} \Longrightarrow {}_1KQ^{\text{ét}}_{q-p}/2^\nu(S), $$

and

$$ {}_{-1}E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S, \mu_{2^\nu}^{\otimes 4k}) & q = 8k \\ H^p_{\text{ét}}(S, \mu_{2^\nu}^{\otimes 4k+2}) & q = 8k+4 \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 4k+3}) & q = 8k+5 \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 4k+3})^{\oplus 2} & q = 8k+6 \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 4k+3}) & q = 8k+7 \\ 0 & q \equiv 1,2,3 \pmod 8 \end{array} \right\} \Longrightarrow {}_{-1}KQ^{\text{ét}}_{q-p}/2^\nu(S). $$

For $\ell \neq 2$ and $\text{cd}_\ell(S) < \infty$, the descent spectral sequence takes the form

$$ {}_\varepsilon E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S, \mu_{\ell^\nu}^{\otimes \frac{q}{2}}) & q \equiv 0 \pmod 4 \\ 0 & q \not\equiv 0 \pmod 4 \end{array} \right\} \Longrightarrow {}_\varepsilon KQ^{\text{ét}}_{q-p}/\ell^\nu(S). $$

Note that the E_2-pages are independent of the symmetry ε. This is not surprising since on symmetric parts K-theory mod ℓ^ν maps by a weak equivalence to hermitian K-theory mod ℓ^ν.

The following results are concerned with Bousfield ℓ-adic completions (denoted by #) of the self-conjugate and hermitian K-theory spectra. Bousfield introduced this notion in [12]. First we shall tabulate the corresponding étale sheaves. Let $\mathbb{Z}_\ell^{\otimes k}$ denote the kth Tate twist of the ℓ-adic integers.

In the example of self-conjugate K-theory the étale sheaves are periodic in the following sense.

Corollary 6.10 *The étale sheaf associated to the presheaf*

$$ U \mapsto \pi_n(\mathcal{KSC}(U)_\#) $$

of ℓ-adically completed self-conjugate K-theory is given by:

For hermitian K-theory the étale sheaves are periodic in the following sense.

n mod 4	$\ell = 2$	$\ell \neq 2$
$4k$	$\mathbb{Z}_2^{\otimes 2k}$	$\mathbb{Z}_\ell^{\otimes 2k}$
$4k+1$	$\mu_2^{\otimes 2k+1}$	0
$4k+2$	0	0
$4k+3$	$\mathbb{Z}_2^{\otimes 2k+2}$	$\mathbb{Z}_\ell^{\otimes 2k+2}$

Corollary 6.11 *The étale sheaf associated to the presheaf*

$$U \mapsto \pi_n({}_\varepsilon\mathcal{KQ}(U)_\#)$$

of ℓ-adically completed hermitian K-theory is given as follows.

1. *For $\ell = 2$ by:*

n mod 8	$\varepsilon = 1$	$\varepsilon = -1$
$8k$	$\mathbb{Z}_2^{\otimes 4k}$	$\mathbb{Z}_2^{\otimes 4k}$
$8k+1$	$\mu_2^{\otimes 4k+1}$	0
$8k+2$	$\mu_2^{\otimes 4k+1}$	0
$8k+3$	0	0
$8k+4$	$\mathbb{Z}_2^{\otimes 4k+2}$	$\mathbb{Z}_2^{\otimes 4k+2}$
$8k+5$	0	$\mu_2^{\otimes 4k+3}$
$8k+6$	0	$\mu_2^{\otimes 4k+3}$
$8k+7$	0	0

2. *For $\ell \neq 2$ and $\varepsilon = \pm 1$ by $\mathbb{Z}_\ell^{\otimes 2k}$ if $n = 4k$, and trivial otherwise.*

In the following, étale cohomology is continuous étale cohomology [15], [26].

As a result of the previous corollaries, the descent spectral sequences for the 2-completed étale self-conjugate étale K-theory and hermitian K-theory of S take the forms

$${}_1E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S, \mathbb{Z}_2^{\otimes 2k}) & q = 4k \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 2k+1}) & q = 4k+1 \\ H^p_{\text{ét}}(S, \mathbb{Z}_2^{\otimes 2k+2}) & q = 4k+3 \\ 0 & q = 4k+2 \end{array} \right\} \Longrightarrow KSC^{\text{ét}}_{q-p}(S)_\#,$$

$$ {}_1E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k}) & q = 8k \\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+1}) & q = 8k+1 \\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+1}) & q = 8k+2 \\ H^p_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+2}) & q = 8k+4 \\ 0 & q \equiv 3,5,6,7 \text{ (mod 8)} \end{array} \right\} \Longrightarrow {}_1KQ^{\text{ét}}_{q-p}(S)_{\#}, $$

and

$$ {}_{-1}E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k}) & q = 8k \\ H^p_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+2}) & q = 8k+4 \\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+3}) & q = 8k+5 \\ H^p_{\text{ét}}(S,\mu_2^{\otimes 4k+3}) & q = 8k+6 \\ 0 & q \equiv 1,2,3,7 \text{ (mod 8)} \end{array} \right\} \Longrightarrow {}_{-1}KQ^{\text{ét}}_{q-p}(S)_{\#}. $$

For $\ell \neq 2$ the descent spectral sequences take the forms

$$ {}_1E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k}) & q = 4k \\ H^p_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+2}) & q = 4k+3 \\ 0 & q \equiv 1,2 \text{ (mod 4)} \end{array} \right\} \Longrightarrow KSC^{\text{ét}}_{q-p}(S)_{\#}, $$

and

$$ {}_\varepsilon E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes \frac{q}{2}}) & q \equiv 0 \text{ (mod 4)} \\ 0 & q \not\equiv 0 \text{ (mod 4)} \end{array} \right\} \Longrightarrow {}_\varepsilon KQ^{\text{ét}}_{q-p}(S)_{\#}. $$

Our next objective is to compute ℓ-adically completed étale self-conjugate and hermitian K-groups in terms of étale cohomology groups. To this end we need some more notation.

Let $A \bullet B$ denote an abelian group extension of B by A, so that there exists a short exact sequence

$$ 0 \to A \to A \bullet B \to B \to 0. $$

Lemma 6.12 *If* $\operatorname{cd}_2(S) = 2$ *and* $H^0_{\acute{e}t}(S,\mathbb{Z}_2^{\otimes i}) = 0$ *for* $i > 0$ *then the 2-completed étale hermitian K-groups of S are computed up to extensions in the following table.*

Remark 6.13 The assumption on the vanishing of $H^0_{\text{ét}}(S,\mathbb{Z}_2^{\otimes i})$ for $i > 0$ in Lemma 6.12 is a commonplace and holds for the examples considered in Section 7. We note, however, that the assumptions in Lemma 6.12 are not satisfied for the field of real numbers, and for number fields with at least one real embedding.

n mod 8	${}_1KQ^{\text{ét}}_n(S)_\#$	${}_{-1}KQ^{\text{ét}}_n(S)_\#$
$8k>0$	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+1})\bullet H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+1})$	0
$8k+1$	$H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+1})\bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+1})$	0
$8k+2$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+2})\bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+1})$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+2})$
$8k+3$	$H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+2})$	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+3})\bullet H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+2})$
$8k+4$	0	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+3})\bullet H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+3})$
$8k+5$	0	$H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+3})\bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+3})$
$8k+6$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+4})$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+4})\bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+3})$
$8k+7$	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+5})\bullet H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+4})$	$H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+4})$

n mod 4	${}_\varepsilon KQ^{\text{ét}}_n(S)_\#$
$4k>0$	$H^4_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+2})\bullet H^0_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k})$
$4k+1$	$H^7_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+4})\bullet H^3_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+2})$
$4k+2$	$H^6_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+4})\bullet H^2_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+2})$
$4k+3$	$H^5_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+4})\bullet H^1_{\text{ét}}(S,\mathbb{Z}_\ell^{\otimes 2k+2})$

Lemma 6.14 *If ℓ is an odd prime and $\mathrm{cd}_\ell(S)\le 7$ the ℓ-completed étale hermitian K-groups of S are computed up to extensions in the following table.*

Corollary 6.11 and the corresponding result for algebraic K-theory imply the next result by inspection.

Corollary 6.15 *The étale sheaf associated to the presheaf*

$$U\mapsto \pi_n({}_\varepsilon\mathcal{V}(U)_\#)$$

of ℓ-adically completed hermitian V-theory is given as follows.

1. *For $\ell=2$ by:*
2. *For $\ell\neq 2$ by $\mathbb{Z}_\ell^{\otimes 2k+1}$ if $n=4k+1$, and trivial otherwise.*

The previous corollary allows us to immediately identify the E_2-page of the descent spectral sequence for ℓ-adically completed étale hermitian V-groups in terms of étale cohomology. As a consequence, imposing a commonplace restriction on the étale cohomological dimension yields the following computation.

Lemma 6.16 *If $\mathrm{cd}_2(S)=2$ and $H^0_{\acute{e}t}(S,\mathbb{Z}_2^{\otimes i})=0$ for $i>0$, then the 2-completed étale V-groups of S are computed up to extensions in the following table.*

The previous computations are supplemented by more specialized examples in the next section. For the earliest étale K-theory computations we refer the reader

n mod 8	$\varepsilon = 1$	$\varepsilon = -1$
$8k$	0	0
$8k+1$	$\mathbb{Z}_2^{\otimes 4k+1}$	$\mathbb{Z}_2^{\otimes 4k+1}$
$8k+2$	$\mu_2^{\otimes 4k+1}$	0
$8k+3$	$\mu_2^{\otimes 4k+2}$	0
$8k+4$	0	0
$8k+5$	$\mathbb{Z}_2^{\otimes 4k+3}$	$\mathbb{Z}_2^{\otimes 4k+3}$
$8k+6$	0	$\mu_2^{\otimes 4k+3}$
$8k+7$	0	$\mu_2^{\otimes 4k+4}$

n mod 8	${}_1V_n^{\text{ét}}(S)_\#$	${}_{-1}V_n^{\text{ét}}(S)_\#$
$8k \geq 0$	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+1}) \bullet H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+1})$	$H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+1})$
$8k+1$	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+2}) \bullet H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+1})$	0
$8k+2$	$H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+2}) \bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+1})$	0
$8k+3$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+3}) \bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+2})$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+3})$
$8k+4$	$H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+3})$	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+3}) \bullet H^1_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+3})$
$8k+5$	0	$H^2_{\text{ét}}(S,\mu_2^{\otimes 4k+4}) \bullet H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+3})$
$8k+6$	0	$H^1_{\text{ét}}(S,\mu_2^{\otimes 4k+4}) \bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+3})$
$8k+7$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+5})$	$H^2_{\text{ét}}(S,\mathbb{Z}_2^{\otimes 4k+5}) \bullet H^0_{\text{ét}}(S,\mu_2^{\otimes 4k+4})$

to [15] and [60]. It is worthwhile to point out that the difference between the étale K-theory, in *loc. cit.*, and the étale hermitian K-theory computations in this paper is reminiscent of the situation for the classical Atiyah-Hirzebruch spectral sequences based on complex and real topological K-theory. This analogy is evident on the level of étale stalks by comparison with the complex and real K-theories of a point.

7. Applications to finite fields, local and global fields

In this section we point out some computational consequences of the above results. The examples are geometric in nature and relate to finite fields, and to local and global number fields. Our main interest and focus are on the 2-primary computations.

Example 7.1 In what follows, we apply Lemma 6.14 to some classes of examples. Throughout, ℓ is an odd prime number.

1. If S is a d-dimensional smooth complex variety then $\text{cd}_\ell(S) \leq 2d$. Lemma 6.14 computes the group ${}_\varepsilon KQ_n^{\text{ét}}(S)_\#$ up to extensions if S is of dimension

at most 3. For curves and surfaces there are no undetermined extensions. Detailed computations of the algebraic K-theory of S were worked out in [44] and [45].

2. If S is a smooth curve over a number field then $\mathrm{cd}_\ell(S) = 4$. In this case, for $n > 0$, there are no undetermined extensions in the computation of ${}_\varepsilon KQ^{\text{ét}}_n(S)_\#$ given in Lemma 6.14. For detailed computations of the algebraic K-theory of S we refer to [51].

3. The ring $\mathcal{O}_F[1/\ell]$ of ℓ-integers in any number field F has cohomological dimension $\mathrm{cd}_\ell(\mathcal{O}_F[1/\ell]) = \mathrm{cd}_\ell(F) = 2$. In particular, ${}_\varepsilon KQ^{\text{ét}}_n(\mathcal{O}_F[1/\ell])_\#$ is trivial when $0 < n \equiv 0,1 \pmod 4$ and finite when $n \equiv 2 \pmod 4$. The same cohomological dimension bound holds for local number fields and their valuation rings, *e.g.* the field of ℓ-adic numbers.

Let F be a field of characteristic $\neq 2$ and ζ_r be a primitive r th root of unity. For $i \in \mathbb{Z}$, let $w_i(F)$ be the maximal 2-power 2^n such that the exponent of the Galois group of $F(\zeta_{2^n})/F$ divides i. If F contains ζ_4 and $i = 2^\lambda k$ with k odd, then $w_i(F) = 2^{r+\lambda}$ where r is maximal such that F contains a primitive 2^r-root of unity. If i is odd, $w_i(\mathbb{Q}(\sqrt{-1})) = 4$, while if $\sqrt{-1} \notin F$ then $w_i(F) = 2$. In all our examples the number $w_i(F)$ is finite.

Using Lemma 6.12 and the étale cohomology groups of finite fields, we tabulate the 2-completed étale hermitian K-groups of $\mathbb{F}_t$ for t odd. Our findings are in agreement with Friedlander's computation of the hermitian K-groups of $\mathbb{F}_t$ in [19].

Example 7.2 Let $\mathbb{F}_t$ be a finite field with an odd number of elements t. The 2-completed étale hermitian K-groups of $\mathbb{F}_t$ are computed in the following table.

n mod 8	${}_1KQ^{\text{ét}}_n(\mathbb{F}_t)_\#$	${}_{-1}KQ^{\text{ét}}_n(\mathbb{F}_t)_\#$
$8k > 0$	$\mathbb{Z}/2$	0
$8k+1$	$(\mathbb{Z}/2)^2$	0
$8k+2$	$\mathbb{Z}/2$	0
$8k+3$	$\mathbb{Z}/w_{4k+2}(\mathbb{F}_t)$	$\mathbb{Z}/w_{4k+2}(\mathbb{F}_t)$
$8k+4$	0	$\mathbb{Z}/2$
$8k+5$	0	$(\mathbb{Z}/2)^2$
$8k+6$	0	$\mathbb{Z}/2$
$8k+7$	$\mathbb{Z}/w_{4k+4}(\mathbb{F}_t)$	$\mathbb{Z}/w_{4k+4}(\mathbb{F}_t)$

The extension problem for ${}_1KQ^{\text{ét}}_{8k+1}(\mathbb{F}_t)_\#$ can be resolved using the homotopy fibration [33]

$$\mathcal{KSC}(\mathbb{F}_t) \longrightarrow \Omega_\varepsilon \mathcal{KQ}(\mathbb{F}_t) \xrightarrow{\sigma^{(2)}} \Omega^{-1}{}_{-\varepsilon}\mathcal{KQ}(\mathbb{F}_t).$$

The group $KSC_{8k}(\mathbb{F}_t)_\#$ has order 2, *cf.* Example 7.12, and is a direct summand of ${}_1KQ_{8k+1}(\mathbb{F}_t)_\#$. This also resolves the extension problem in degree $8k+5$ for $\varepsilon = -1$.

Example 7.3 Lemma 6.12 applies to every dyadic local field F, *i.e.* every finite extension of the 2-adic numbers $\mathbb{Q}_2$. If the field extension degree $[F:\mathbb{Q}_2] = d$, then $H^0_{\text{ét}}(F, \mathbb{Z}_2^{\otimes i}) = \delta_{i0}\mathbb{Z}_2$, $H^1_{\text{ét}}(F, \mathbb{Z}_2^{\otimes 1}) = \mathbb{Z}_2^{d+1} \oplus \mathbb{Z}/2$, $H^2_{\text{ét}}(F, \mathbb{Z}_2^{\otimes 1}) = \mathbb{Z}_2$,

$$H^1_{\text{ét}}(F, \mathbb{Z}_2^{\otimes i}) = \begin{cases} \mathbb{Z}_2^d \oplus \mathbb{Z}/2 & i > 1 \text{ odd}, \\ \mathbb{Z}_2^d \oplus \mathbb{Z}/w_i(F) & i \text{ even}, \end{cases}$$

and

$$H^2_{\text{ét}}(F, \mathbb{Z}_2^{\otimes i}) = \begin{cases} \mathbb{Z}/w_{i-1}(F) & i > 1 \text{ odd}, \\ \mathbb{Z}/2 & i \text{ even}. \end{cases}$$

The 2-completed étale hermitian K-groups of F are computed up to extensions in the following table.

n mod 8	${}_1KQ^{\text{ét}}_n(F)_\#$	${}_{-1}KQ^{\text{ét}}_n(F)_\#$
$8k > 0$	$\mathbb{Z}/2 \bullet (\mathbb{Z}/2)^{d+2}$	0
$8k+1$	$(\mathbb{Z}/2)^{d+2} \bullet \mathbb{Z}/2$	0
$8k+2$	$\mathbb{Z}/2 \bullet \mathbb{Z}/2$	$\mathbb{Z}/2$
$8k+3$	$\mathbb{Z}_2^d \oplus \mathbb{Z}/w_{4k+2}(F)$	$\mathbb{Z}/2 \bullet (\mathbb{Z}_2^d \oplus \mathbb{Z}/w_{4k+2}(F))$
$8k+4$	0	$\mathbb{Z}/2 \bullet (\mathbb{Z}/2)^{d+2}$
$8k+5$	0	$(\mathbb{Z}/2)^{d+2} \bullet \mathbb{Z}/2$
$8k+6$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \bullet \mathbb{Z}/2$
$8k+7$	$\mathbb{Z}/2 \bullet (\mathbb{Z}_2^d \oplus \mathbb{Z}/w_{4k+4}(F))$	$\mathbb{Z}_2^d \oplus \mathbb{Z}/w_{4k+4}(F)$

For a non-dyadic local field, *i.e.* a finite extension of the p-adic numbers $\mathbb{Q}_p$ for some odd prime p, the 2-completed étale hermitian K-groups are comprised of finite groups in positive degrees. The étale cohomology computation leading to this conclusion is given in [41, Proposition 7.3.10].

Example 7.4 The 2-completed étale hermitian K-groups of a finite extension F of $\mathbb{Q}_p$ for p odd are computed up to extensions in the following table.

A totally imaginary number field F is called 2-*regular* if the 2-Sylow subgroup of $K_2(\mathcal{O}_F)$ is trivial. The Gaussian numbers $\mathbb{Q}(\sqrt{-1})$ is an example of such a number field. For the étale cohomology of $\mathcal{O}_F[1/2]$ the 2-regular assumption implies that $H^2_{\text{ét}}(\mathcal{O}_F[1/2], \mathbb{Z}_2^{\otimes i})$ is trivial for $i \neq 0, 1$ [48, Proposition 2.2]. Moreover, $H^1_{\text{ét}}(\mathcal{O}_F[1/2], \mathbb{Z}_2^{\otimes i})$ identifies with $\mathbb{Z}_2^c \oplus \mathbb{Z}/w_i(F)$ for $i \neq 0$, and $H^1_{\text{ét}}(\mathcal{O}_F[1/2], \mu_2^{\otimes i}) \cong (\mathbb{Z}/2)^{c+1}$ where c denotes the number of pairs of complex

n mod 8	${}_1KQ^{\text{ét}}_n(F)_\#$	${}_{-1}KQ^{\text{ét}}_n(F)_\#$
$8k>0$	$\mathbb{Z}/2\bullet(\mathbb{Z}/2)^2$	0
$8k+1$	$(\mathbb{Z}/2)^2\bullet\mathbb{Z}/2$	0
$8k+2$	$\mathbb{Z}/2\bullet\mathbb{Z}/2$	$\mathbb{Z}/2$
$8k+3$	$\mathbb{Z}/w_{4k+2}(F)$	$\mathbb{Z}/2\bullet\mathbb{Z}/w_{4k+2}(F)$
$8k+4$	0	$\mathbb{Z}/2\bullet(\mathbb{Z}/2)^2$
$8k+5$	0	$(\mathbb{Z}/2)^2\bullet\mathbb{Z}/2$
$8k+6$	$\mathbb{Z}/2$	$\mathbb{Z}/2\bullet\mathbb{Z}/2$
$8k+7$	$\mathbb{Z}/2\bullet\mathbb{Z}/w_{4k+4}(F)$	$\mathbb{Z}/w_{4k+4}(F)$

embeddings of the number field F. By way of example, the number $w_i(\mathbb{Q}(\sqrt{-1})) = 2^{2+(i)_2}$ for all i , where $(i)_2$ is the 2-adic valuation of i.

With these preliminaries in hand, we are ready to state the following computation.

Example 7.5 Let F be a totally imaginary 2-regular number field with c pairs of complex embeddings. The 2-completed étale hermitian K-groups of its ring of 2-integers $\mathcal{O}_F[1/2]$ are computed up to extensions in the following table.

n mod 8	${}_1KQ^{\text{ét}}_n(\mathcal{O}_F[1/2])_\#$	${}_{-1}KQ^{\text{ét}}_n(\mathcal{O}_F[1/2])_\#$
$8k>0$	$(\mathbb{Z}/2)^{c+1}$	0
$8k+1$	$(\mathbb{Z}/2)^{c+1}\bullet\mathbb{Z}/2$	0
$8k+2$	$\mathbb{Z}/2$	0
$8k+3$	$\mathbb{Z}_2^c\oplus\mathbb{Z}/w_{4k+2}(F)$	$\mathbb{Z}_2^c\oplus\mathbb{Z}/w_{4k+2}(F)$
$8k+4$	0	$(\mathbb{Z}/2)^{c+1}$
$8k+5$	0	$(\mathbb{Z}/2)^{c+1}\bullet\mathbb{Z}/2$
$8k+6$	0	$\mathbb{Z}/2$
$8k+7$	$\mathbb{Z}_2^c\oplus\mathbb{Z}/w_{4k+4}(F)$	$\mathbb{Z}_2^c\oplus\mathbb{Z}/w_{4k+4}(F)$

Remark 7.6 We expect that ${}_1KQ^{\text{ét}}_{8k+1}(\mathcal{O}_F[1/2])_\#$ and ${}_{-1}KQ^{\text{ét}}_{8k+5}(\mathcal{O}_F[1/2])_\#$ are elementary abelian 2-groups of rank equal to $c+2$.

In the following discussion of étale V-theory we shall specialize Lemma 6.16 to the previous examples of finite fields, local fields and totally imaginary 2-regular number fields. As for hermitian étale K-theory, it turns out that the étale V-groups of dyadic and non-dyadic local number fields are completely different; although in some degrees we are only able to compute these groups up to extensions, we can conclude that the former allow free summands in some degrees while the latter are always finite abelian groups.

Our computation of the 2-completed étale V-groups of finite fields is in agreement with Hiller's results for V-groups in [22].

Example 7.7 Let $\mathbb{F}_t$ be a finite field with an odd number of elements t. The 2-completed étale V-groups of $\mathbb{F}_t$ are computed in the following table.

n mod 8	${}_1V_n^{\text{ét}}(\mathbb{F}_t)_\#$	${}_{-1}V_n^{\text{ét}}(\mathbb{F}_t)_\#$
$8k \geq 0$	$\mathbb{Z}/w_{4k+1}(\mathbb{F}_t)$	$\mathbb{Z}/w_{4k+1}(\mathbb{F}_t)$
$8k+1$	$\mathbb{Z}/2$	0
$8k+2$	$(\mathbb{Z}/2)^2$	0
$8k+3$	$\mathbb{Z}/2$	0
$8k+4$	$\mathbb{Z}/w_{4k+3}(\mathbb{F}_t)$	$\mathbb{Z}/w_{4k+3}(\mathbb{F}_t)$
$8k+5$	0	$\mathbb{Z}/2$
$8k+6$	0	$(\mathbb{Z}/2)^2$
$8k+7$	0	$\mathbb{Z}/2$

The extension problem in degree $8k+2$ can be resolved using that ${}_1KQ_{8k+2}(\mathbb{F}_t)_\#$ has order 2 and is a direct summand of ${}_1V_{8k+2}(\mathbb{F}_t)_\#$. Likewise, this also resolves the extension problem in degree $8k+6$ for $\varepsilon = -1$.

Next we turn to local number fields. We find it convenient to distinguish between dyadic and non-dyadic local fields.

Example 7.8 The 2-completed étale V-groups of a dyadic local number field F of degree d are computed up to extensions in the following table.

n mod 8	${}_1V_n^{\text{ét}}(F)_\#$	${}_{-1}V_n^{\text{ét}}(F)_\#$
$8k \geq 0$	$\mathbb{Z}/2 \bullet (\mathbb{Z}_2^d \oplus \mathbb{Z}/2)$	$\mathbb{Z}_2^d \oplus \mathbb{Z}/2$
$8k+1$	$\mathbb{Z}/2 \bullet (\mathbb{Z}/2)^{d+2}$	0
$8k+2$	$(\mathbb{Z}/2)^{d+2} \bullet \mathbb{Z}/2$	0
$8k+3$	$\mathbb{Z}/w_{4k+2}(F) \bullet \mathbb{Z}/2$	$\mathbb{Z}/w_{4k+2}(F)$
$8k+4$	$\mathbb{Z}_2^d \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \bullet (\mathbb{Z}_2^d \oplus \mathbb{Z}/2)$
$8k+5$	0	$\mathbb{Z}/2 \bullet (\mathbb{Z}/2)^{d+2}$
$8k+6$	0	$(\mathbb{Z}/2)^{d+2} \bullet \mathbb{Z}/2$
$8k+7$	$\mathbb{Z}/w_{4k+4}(F)$	$\mathbb{Z}/w_{4k+4}(F) \bullet \mathbb{Z}/2$

If i is even, the number $w_i(\mathbb{Q}_2) = 2^{2+(i)_2}$.

For non-dyadic local number fields the étale V-groups turn out to be torsion abelian groups.

Example 7.9 The 2-completed étale V-groups of a non-dyadic local number field F are computed up to extensions in the following table.

n mod 8	${}_1V_n^{\text{ét}}(F)_\#$	${}_{-1}V_n^{\text{ét}}(F)_\#$
$8k \geq 0$	$\mathbb{Z}/2 \bullet \mathbb{Z}/2$	$\mathbb{Z}/2$
$8k+1$	$\mathbb{Z}/2 \bullet (\mathbb{Z}/2)^2$	0
$8k+2$	$(\mathbb{Z}/2)^2 \bullet \mathbb{Z}/2$	0
$8k+3$	$\mathbb{Z}/w_{4k+2}(F) \bullet \mathbb{Z}/2$	$\mathbb{Z}/w_{4k+2}(F)$
$8k+4$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \bullet \mathbb{Z}/2$
$8k+5$	0	$\mathbb{Z}/2 \bullet (\mathbb{Z}/2)^2$
$8k+6$	0	$(\mathbb{Z}/2)^2 \bullet \mathbb{Z}/2$
$8k+7$	$\mathbb{Z}/w_{4k+4}(F)$	$\mathbb{Z}/w_{4k+4}(F) \bullet \mathbb{Z}/2$

Our last example concerning étale V-theory deals with totally imaginary 2-regular number fields. We refer to the discussion prior to Example 7.5 for some of the salient features of these number fields.

Example 7.10 Let F be a totally imaginary 2-regular number field with c pairs of complex embeddings. The 2-completed étale V-groups of its ring of 2-integers $\mathcal{O}_F[1/2]$ are computed up to extensions in the following table.

n mod 8	${}_1V_n^{\text{ét}}(\mathcal{O}_F[1/2])_\#$	${}_{-1}V_n^{\text{ét}}(\mathcal{O}_F[1/2])_\#$
$8k \geq 0$	$\mathbb{Z}_2^c \oplus \mathbb{Z}/w_{4k+1}(F)$	$\mathbb{Z}_2^c \oplus \mathbb{Z}/w_{4k+1}(F)$
$8k+1$	$(\mathbb{Z}/2)^{c+1}$	0
$8k+2$	$(\mathbb{Z}/2)^{c+1} \bullet \mathbb{Z}/2$	0
$8k+3$	$\mathbb{Z}/2$	0
$8k+4$	$\mathbb{Z}_2^c \oplus \mathbb{Z}/w_{4k+3}(F)$	$\mathbb{Z}_2^c \oplus \mathbb{Z}/w_{4k+3}(F)$
$8k+5$	0	$(\mathbb{Z}/2)^{c+1}$
$8k+6$	0	$(\mathbb{Z}/2)^{c+1} \bullet \mathbb{Z}/2$
$8k+7$	0	$\mathbb{Z}/2$

Remark 7.11 We expect that ${}_1V_{8k+2}^{\text{ét}}(\mathcal{O}_F[1/2])_\#$ and ${}_{-1}V_{8k+6}^{\text{ét}}(\mathcal{O}_F[1/2])_\#$ are elementary abelian 2-groups of rank equal to $c+2$.

The next examples concern self-conjugate algebraic K-theory.

Example 7.12 The 2-completed KSC-groups of a finite field $\mathbb{F}_t$ of odd characteristic are given in the following table.

Recall that τ denotes the duality functor in algebraic K-theory. The map

$$\pi_n(1-\tau) : K_n(\mathbb{F}_t)_\# \longrightarrow K_n(\mathbb{F}_t)_\#$$

is multiplication by 2 if $n \equiv 1 \pmod 4$ and trivial otherwise.

n mod 4	$KSC_n(\mathbb{F}_t)_\#$
$4k > 0$	$\mathbb{Z}/2$
$4k+1$	$\mathbb{Z}/2$
$4k+2$	$\mathbb{Z}/w_{2k+2}(\mathbb{F}_t)$
$4k+3$	$\mathbb{Z}/w_{2k+2}(\mathbb{F}_t)$

Example 7.13 Let F be a totally imaginary 2-regular number field with c pairs of complex embeddings. The 2-completed KSC-groups of its ring of 2-integers $\mathcal{O}_F[1/2]$ are given in the following table.

n mod 4	$KSC_n(\mathcal{O}_F[1/2])_\#$
$4k > 0$	$(\mathbb{Z}/2)^{c+1}$
$4k+1$	$\mathbb{Z}/2$
$4k+2$	$\mathbb{Z}_2^c \oplus \mathbb{Z}/w_{2k+2}(F)$
$4k+3$	$\mathbb{Z}_2^c \oplus \mathbb{Z}/w_{2k+2}(F)$

The map

$$\pi_n(1-\tau) : K_n(\mathcal{O}_F[1/2])_\# \longrightarrow K_n(\mathcal{O}_F[1/2])_\#$$

is multiplication by 2 if $n \equiv 1 \pmod 4$ and trivial otherwise. There is an exact sequence (with $A = \mathcal{O}_F[1/2]$)

$$0 \to KSC_{2n+1}(A)_\# \to K_{2n+1}(A)_\# \to K_{2n+1}(A)_\# \to KSC_{2n}(A)_\# \to 0.$$

In the examples above, the assertions concerning $\pi_n(1-\tau)$ follow by inspection, using the computations of $K_n(\mathbb{F}_t)_\#$ [47] and $K_n(\mathcal{O}_F[1/2])_\#$ [48, Theorem 3.1].

A systematic approach to the KSC-computations is to first compute the descent spectral sequence for étale KSC-theory obtained from Corollary 6.10, and then invoke Theorem 6.2. In general, if $\mathrm{cd}_2(S) < \infty$, this approach gives "in sufficiently high degrees" a strongly convergent cohomological spectral sequence

$${}_1E_2^{p,q} = \left\{ \begin{array}{ll} H^p_{\text{ét}}(S, \mathbb{Z}_2^{\otimes 2k}) & q = 4k \\ H^p_{\text{ét}}(S, \mu_2^{\otimes 2k+1}) & q = 4k+1 \\ H^p_{\text{ét}}(S, \mathbb{Z}_2^{\otimes 2k+2}) & q = 4k+3 \\ 0 & q = 4k+2 \end{array} \right\} \text{ “}\Longrightarrow\text{” } KSC_{q-p}(S)_\#.$$

8. Applications to group rings, complex varieties and commutative Banach algebras

Let G be a finite group and R be the ring of S-integers in a number field. Let m be a prime power which is prime to the order of G. In [66], Weibel proves a periodicity theorem for the higher algebraic K-theory of the group ring $A = R[G]$, with coefficients in $\mathbb{Z}/m$. More precisely, the cup-product with the Bott element in K-theory induces an isomorphism

$$K_i(A;\mathbb{Z}/m) \cong K_{i+p}(A;\mathbb{Z}/m)$$

for $i > 0$. Here, the integers m and p are linked according to our Convention 0.1. We can apply our periodicity theorems of Section 4 in order to show that, for $1/2 \in A$ and any involution on A, for instance the one induced by $g \mapsto g^{-1}$, we have a split short exact sequence for m a power of 2 and $i > q - 1$.

$$0 \longrightarrow \varprojlim{}_{\varepsilon} KQ_{i+ps}(A;\mathbb{Z}/m) \xrightarrow{\theta_n^-} {}_{\varepsilon}KQ_i(A;\mathbb{Z}/m) \xrightarrow{\theta_n^+} \varinjlim{}_{\varepsilon} KQ_{i+ps}(A;\mathbb{Z}/m) \longrightarrow 0.$$

Here the number q is given by our convention 0.2. We note that Weibel's theorem is also true for $i \geq 0$ if we replace the number ring R by a local field.

In particular, the KQ-groups are also periodic, *i.e.*

$${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m) \cong {}_{\varepsilon}KQ_{i+p}(A;\mathbb{Z}/m)$$

at least for $i > q - 1$. In fact, a more careful analysis forces us to distinguish two cases according to the parity of m.

If m is even, the condition that the order to G is prime to m implies that G is of odd order. According to the famous theorem of Feit and Thompson [17], this implies that G is solvable. Therefore, for 2-primary coefficients, we have a periodicity statement only for a special class of solvable groups.

If m is odd, we already know, without the hypothesis $1/2 \in A$, that the group ${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)$ splits into the direct sum

$${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m) \cong {}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_+ \oplus {}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_-.$$

In this direct sum decomposition, the group ${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_-$ is the higher Witt group with $\mathbb{Z}/m$ coefficients and we have the periodicity isomorphism

$${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_- \cong {}_{-e}KQ_{i+2}(A;\mathbb{Z}/m)_-$$

for all values $i \in \mathbb{Z}$. On the other hand, the group ${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_+$ may be identified with $K_i(A;\mathbb{Z}/m)_+$, the symmetric part of $K_i(A;\mathbb{Z}/m)$ with respect to

the involution given by the duality. Since this involution is compatible with the Bott map, Weibel's theorem [66] implies another periodicity isomorphism

$$ {}_\varepsilon KQ_i(A;\mathbb{Z}/m)_+ \cong {}_\varepsilon KQ_{i+p}(A;\mathbb{Z}/m)_+ $$

but only if $i > 0$. Summarizing, we have proved the following theorem.

Theorem 8.1 *Let G be a finite group and let R be the ring of S-integers in a number field. If m is a 2-power and if G is of odd order, then we have a periodicity isomorphism*

$$ {}_\varepsilon KQ_i(R[G];\mathbb{Z}/m) \cong {}_\varepsilon KQ_{i+p}(R[G];\mathbb{Z}/m) $$

if $1/2 \in R$ and if $i > q-1$. On the other hand, if m is an odd prime power and if G is an arbitrary finite group whose order is prime to m, we have the same periodicity isomorphism, with only the restriction that $i > 0$.

Remark 8.2 As in Section 6, we may conjecture that, in the case where m is a 2-power, the inverse limit

$$ \varprojlim_\varepsilon KQ_{i+ps}(A;\mathbb{Z}/m) $$

is reduced to 0. In other words, we conjecture that the ring A is hermitian regular according to Definition 0.5. This will imply that the positive Bott map

$$ {}_\varepsilon KQ_i(R[G];\mathbb{Z}/m) \longrightarrow {}_\varepsilon KQ_{i+p}(R[G];\mathbb{Z}/m) $$

is an isomorphism for $i > 0$ according to Theorem 4.5.

Let us now turn our attention to a smooth complex variety S of dimension n. As we briefly mentioned in Section 6, the étale dimension of S is $2n$. As a consequence of Artin-Grothendieck theory, it is well-known that the Betti cohomology of S with coefficients $\mathbb{Z}/m$ is isomorphic to the mod m étale cohomology. The same result is valid for any cohomology theory by the method initiated by Dwyer and Friedlander [15]. For instance, the mod m étale K-theory of S coincides with the mod m complex topological K-theory of Atiyah and Hirzebruch. By the same argument, the mod m étale ${}_1KQ$-theory coincides with the mod m K-theory of complex vector bundles provided with a nondegenerate symmetric bilinear form. This theory is well understood and is detailed for instance in Appendix B to [6]: it is the usual mod m topological real K-theory. In the same way, the mod m étale ${}_{-1}KQ$-theory coincides with the mod m K-theory of complex vector bundles provided with a nondegenerate antisymmetric bilinear form. This theory is also well understood: it is the usual mod m topological symplectic K-theory. In both cases, we shall write ${}_\varepsilon\overline{KQ}_n^{\mathrm{top}}(S)$, with $\varepsilon = 1, -1$ if we consider symmetric or antisymmetric bilinear forms respectively.

Theorem 8.3 *Let S be a smooth complex variety of dimension n. Then the mod m étale ${}_\varepsilon KQ$-theory of S coincides with the mod m topological K-theory of its complex points, real or symplectic according to ε. Moreover, the canonical map*

$$ {}_\varepsilon\overline{KQ}_i(S) \longrightarrow {}_\varepsilon\overline{KQ}_i^{\text{ét}}(S) \cong {}_\varepsilon\overline{KQ}_i^{\text{top}}(S(\mathbb{C})) $$

is split surjective[7] *when $i \geq 2n+q-1$. Moreover, for odd prime power coefficients, it is an isomorphism for $i \geq 2n-1$, with an identification of ${}_\varepsilon\overline{KQ}_i^{\text{top}}(S)$ with $\overline{K}_i^{\text{top}}(S)_+$, the symmetric part of $\overline{K}_i^{\text{top}}(S)$ with respect to the involution induced by the duality functor.*

Proof: This theorem is mostly a consequence of the general results in Section 6. What remains to be shown is that ${}_\varepsilon\overline{KQ}_i(S)_-$ is zero for odd prime power coefficients: this is a consequence of the fact that -1 has a square root in $\mathbb{C}$. Therefore, the classical Witt group and also the higher Witt groups have only 2-torsion. □

Let us now consider a real or complex commutative Banach algebra A. It is a theorem of Fisher [18] and Prasolov [46] that the natural map

$$ K_i^{\text{alg}}(A;\mathbb{Z}/m) \longrightarrow K_i^{\text{top}}(A;\mathbb{Z}/m) $$

is an isomorphism for $i \geq 1$. In particular, the groups $K_i^{\text{alg}}(A;\mathbb{Z}/m)$ are periodic of period 2 if A is complex and of period 8 if A is real. In this context, it is natural to state the following conjecture.

Conjecture 8.4 *Let A be a real or complex commutative Banach algebra with involution. Then the natural map*

$$ {}_\varepsilon KQ_i^{\text{alg}}(A;\mathbb{Z}/m) \longrightarrow {}_\varepsilon KQ_i^{\text{top}}(A;\mathbb{Z}/m) $$

is an isomorphism for $i \geq 1$.

Applying the general arguments in this paper, we can prove a theorem that would also be a consequence of this conjecture, namely the periodicity of the groups ${}_\varepsilon KQ_i^{\text{alg}}(A;\mathbb{Z}/m)$, which we simply write ${}_\varepsilon KQ_i(A;\mathbb{Z}/m)$. More precisely, the theorem of Fisher and Prasolov implies that the Bott map $K_i(A;\mathbb{Z}/m) \to K_{i+p}(A;\mathbb{Z}/m)$ is an isomorphism for $i \geq 1$, with m and p being related by our Convention 0.1. From Theorem 0.13, we therefore deduce the following periodicity pattern for the groups ${}_\varepsilon KQ_i(A;\mathbb{Z}/m)$.

[7] In [7], we show that in fact this canonical map is an isomorphism when $i \geq 2n-1$.

Theorem 8.5 *Let A be a real or complex commutative Banach algebra with involution. Then we have an isomorphism*

$$ {}_{\varepsilon}KQ_i(A;\mathbb{Z}/m) \cong {}_{\varepsilon}KQ_{i+p}(A;\mathbb{Z}/m) $$

for $i \geq q$, where m, p and q are 2-powers related by our Conventions 0.1 and 0.2. □

As a matter of fact, if m is an odd prime power, we can prove a much better result. For, we know already by our general theory that the subgroup ${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_-$ is periodic of period 4 for all values of i. Moreover, ${}_{\varepsilon}KQ_i(A;\mathbb{Z}/m)_+$ is isomorphic to $K_i(A;\mathbb{Z}/m)_+$, the symmetric part of K-theory which (as a direct consequence of the theorem of Fisher and Prasolov) is periodic of period 4 if A is complex or real. Summarizing, we get the following more precise theorem for m an odd prime power.

Theorem 8.6 *Let A be a real or complex commutative Banach algebra with involution and let m be an odd prime power. Then, for $i \geq 1$, we have an isomorphism given by the cup-product with a Bott element*

$$ {}_{\varepsilon}KQ_i(A;\mathbb{Z}/m) \xrightarrow{\cong} {}_{\varepsilon}KQ_{i+4}(A;\mathbb{Z}/m). $$

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