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# PERIODS, SUBCONVEXITY OF L-FUNCTIONS AND REPRESENTATION THEORY

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*Dedicated to Raoul Bott*

**Abstract**

We describe a new method to estimate the trilinear period on automorphic representations of $PGL_2(\mathbb{R})$. Such a period gives rise to a special value of the triple $L$-function. We prove a bound for the triple period which amounts to a subconvexity bound for the corresponding special value. Our method is based on the study of the analytic structure of the corresponding unique trilinear functional on unitary representations of $PGL_2(\mathbb{R})$.

## 1. Introduction

**1.1. Maass forms.** Let $\mathbb{H}$ denote the upper half plane equipped with the standard Riemannian metric of constant curvature $-1$. We denote by $dv$ the associated volume element and by $\Delta$ the corresponding Laplace–Beltrami operator on $\mathbb{H}$.

Fix a discrete group $\Gamma$ of motions of $\mathbb{H}$ and consider the Riemann surface $Y = \Gamma\backslash\mathbb{H}$. For simplicity, we assume that $Y$ is compact (the case of $Y$ of finite volume is discussed at the end of the introduction). According to the uniformization theorem, any compact Riemann surface $Y$ with the metric of constant curvature $-1$ is a special case of this construction.

Consider the spectral decomposition of the operator $\Delta$ in the space $L^2(Y, dv)$ of functions on $Y$. It is known that the operator $\Delta$ is nonnegative and has purely discrete spectrum; we will denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots$ the eigenvalues of $\Delta$. For these eigenvalues, we always use a natural form representation-theoretic point of view parametrization $\mu_i = \frac{1-\lambda_i^2}{4}$, where $\lambda_i \in \mathbb{C}$. We denote by $\phi_i = \phi_{\lambda_i}$, the corresponding eigenfunctions (normalized to have $L^2$-norm one).

In the theory of automorphic forms, the functions $\phi_{\lambda_i}$ are called automorphic functions or *Maass forms* (after Maass, [8]). The study of Maass forms plays an important role in analytic number theory, analysis

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and geometry. We are interested in their analytic properties and will present a new method of bounding some important quantities arising from the $\phi_i$.

A particular problem we are going to address in this paper belongs to an active area of research in the theory of automorphic functions studying an interplay between periods, special values of automorphic $L$-functions and representation theory. One of the central features of this interplay is the uniqueness of invariant functionals associated with corresponding periods. The discovery of this interplay goes back to classical works of Hecke and Maass.

It is well-known that the uniqueness principle plays a central role in the modern theory of automorphic functions (see [**11**]). The impact uniqueness has on the analytic behavior of periods and $L$-functions is yet another manifestation of this principle.

**1.2. Triple products.** For any three Maass forms $\phi_i,\ \phi_j,\ \phi_k$, we define the following *triple product* or *triple period*:

$$c_{ijk} = \int_Y \phi_i \phi_j \phi_k dv. \tag{1.1}$$

We would like to estimate the coefficient $c_{ijk}$ as a function of parameters $\lambda_i,\ \lambda_j,\ \lambda_k$. In particular, we would like to find bounds for these coefficients as one or more of the indices $i,\ j,\ k$ tend to infinity.

The bounds on the coefficient $c_{ijk}$ are related to bounds on automorphic $L$-functions as can be seen from the following beautiful formula of Watson (see [**15**]):

$$\left|\int_Y \phi_i \phi_j \phi_k dv\right|^2 = \frac{\Lambda(1/2, \phi_i \otimes \phi_j \otimes \phi_k)}{\Lambda(1, \phi_i, Ad)\Lambda(1, \phi_j, Ad)\Lambda(1, \phi_k, Ad)}. \tag{1.2}$$

Here, the $\phi_t$ are the so-called cuspidal Hecke–Maass functions of norm one on the Riemann surface $Y = \Gamma \setminus \mathbb{H}$ arising from the full modular group $\Gamma = SL_2(\mathbb{Z})$ or from the group of units of a quaternion algebra. The functions $\Lambda(s, \phi_i \otimes \phi_j \otimes \phi_k)$ and $\Lambda(s, \phi, Ad)$ are appropriate *completed* automorphic $L$-functions.

It was first discovered by Rankin and Selberg that the special cases of triple products as above give rise to automorphic $L$-functions (namely, they considered the case where one of Maass forms is replaced by an Eisenstein series). That allowed them to obtain analytic continuation and effective bounds for these $L$-functions and, as an application, to obtain first non-trivial bounds for Fourier coefficients of cusp forms towards Ramanujan conjecture. The relation (1.2) can be viewed as a far reaching generalization of the original Rankin–Selberg formula. The relation (1.2) was motivated by the work [**4**] by Harris and Kudla on a conjecture of Jacquet.

**1.3. Results.** In this paper, we consider the following problem. We fix two Maass forms $\phi = \phi_\tau$ and $\phi' = \phi_{\tau'}$ as above and consider the coefficients defined by the triple period:

$$c_i = \int_Y \phi\phi'\phi_i dv \tag{1.3}$$

as the $\phi_i$ run over an orthonormal basis of Maass forms.

Thus, we see from (1.2) that the estimates of the coefficients $c_i$ are essentially equivalent to the estimates of the corresponding $L$-functions. One would like to have a general method of estimating the coefficients $c_i$ and similar quantities. This problem was raised by Selberg in his celebrated paper [**13**].

The first non-trivial observation is that the coefficients $c_i$ have exponential decay in $|\lambda_i|$ as $i \to \infty$. Namely, as we have shown in [**2**], it is natural to introduce normalized coefficients

$$d_i = \gamma(\lambda_i)|c_i|^2. \tag{1.4}$$

Here, $\gamma(\lambda)$ is given by an explicit rational expression in terms of the standard Euler $\Gamma$-function (see [**2**]) and, for purely imaginary $\lambda$, it has an asymptotic $\gamma(\lambda) \sim \beta|\lambda|^2 \exp(\frac{\pi}{2}|\lambda|)$ when $|\lambda| \to \infty$ with some explicit $\beta > 0$. It turns out that the normalized coefficients $d_i$ have at most *polynomial growth* in $|\lambda_i|$, and hence the coefficients $c_i$ decay exponentially. This is consistent with (1.2) and general experience from the analytic theory of automorphic $L$-functions (see [**2**], [**15**]).

In [**2**], we proved the following mean value bound

$$\sum_{|\lambda_i| \leq T} d_i \leq AT^2, \tag{1.5}$$

for arbitrary $T > 1$ and some effectively computable constant $A$.

According to Weyl's law, the number of terms in this sum is of order $CT^2$. So this formula says that on average, the coefficients $d_i$ are bounded by some constant.

More precisely, let us we fix an interval $I \subset \mathbb{R}$ around point $T$ and consider the finite set of all Maass forms $\phi_i$ with parameter $|\lambda_i|$ inside this interval. Then, the average value of coefficients $d_i$ in this set is bounded by a constant *provided* the interval $I$ is long enough (i.e., of size $\approx T$).

Note that the best individual bound which we can get from this formula is $d_i \leq A|\lambda_i|^2$. For Hecke–Maass forms, this bound corresponds to the convexity bound for the corresponding $L$-function via Watson formula (1.2).

In this paper, we outline the proof of the following bound.

**Theorem 1.1.** *There exist effectively computable constants $B,\ b > 0$ such that, for an arbitrary $T > 1$, we have the following bound*

$$\sum_{|\lambda_i| \in I_T} d_i \leq BT^{5/3}, \tag{1.6}$$

*where $I_T$ is the interval of size $bT^{1/3}$ centered at $T$.*

Note that this theorem gives an individual bound $d_i \leq B|\lambda_i|^{5/3}$ (for $|\lambda_i| > 1$). Thanks to the Watson formula (1.2) and a lower bound of Iwaniec [**5**] on $L(1, \phi_{\lambda_i}, Ad)$ this leads to the following *subconvexity* bound for the triple $L$-function (for an exact relation between triple period and special values of $L$-functions, see [**15**]).

**Corollary 1.2.** *Let $\phi$ and $\phi'$ be fixed Hecke–Maass cusp forms. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the bound*

$$L(\tfrac{1}{2}\ , \phi \otimes \phi' \otimes \phi_{\lambda_i}) \leq C_\varepsilon |\lambda_i|^{5/3+\varepsilon} \tag{1.7}$$

*holds for any Hecke–Maass form $\phi_{\lambda_i}$.*

The convexity bound for the triple $L$-function corresponds to (1.7) with the exponent $5/3$ replaced by 2. We refer to [**6**] for a discussion of the subconvexity problem which is in the core of modern analytic number theory. We note that the above bound is the first subconvexity bound for an $L$-function of degree 8. All previous subconvexity results were obtained for $L$-functions of degree at most 4.

Recently, using ergodic theory methods, Venkatesh [**14**] obtained a subconvexity bound for the triple $L$-function in the level aspect (i.e., with respect to a tower of congruence subgroups $\Gamma(N)$ as $N \to \infty$).

We formulate a natural

**Conjecture 1.3.** For any $\varepsilon > 0$, we have $d_i \ll |\lambda_i|^\varepsilon$.

For Hecke–Maass forms on congruence subgroups, this conjecture is consistent with the Lindelöf conjecture for the triple $L$-functions (for more details, see [**2**] and [**15**]).

### 1.4. Remarks.

(1) Our results can be generalized to the case of a general finite co-volume lattice $\Gamma \subset G$. In this case, the spectral decomposition of the Laplace–Beltrami operator on $Y = \Gamma\backslash\mathbb{H}$ is given by a collection of eigenfunctions $\phi_s$ (including the Eisenstein series) where the parameter $s$ runs through some set $S$ with the Plancherel measure $d\mu$; for any function $u \in C_c^\infty(Y)$ the spectral decomposition takes the form $\int_S |< u, \phi_s >|^2 d\mu = ||u||^2_{L^2(Y)}$.

Let us fix two Maass *cusp forms* $\phi$ and $\phi'$ on $Y$. For every $s \in S$, we define the parameter $\lambda_s \in \mathbb{C}$ and the coefficient $d_s$ in the same way as before. In this case, we can prove the bound

$$\int_{S_T} d_s \, d\mu \leq BT^{5/3+\varepsilon}, \text{ where } S_T = \{s \in S \mid |\lambda_s| \in I_T\}.$$

(2) First results on the exact exponential decay of triple products for a general lattice $\Gamma$ were obtained by Good [**3**] and Sarnak [**12**] using ingenious analytic continuation of Maass form to the complexification of the Riemann surface $Y$ (for representation-theoretic approach to this method and generalizations see [**1**] and [**7**]). Our present method seems to be completely different and avoids analytic continuation.

## 2. The method

We describe now the general ideas behind our proof. It is based on ideas from representation theory (for a detailed account of the corresponding setting, see [**2**]). In what follows, we sketch the method of the proof with the complete details appearing elsewhere.

**2.1. Automorphic representations.** Let $G$ denote the group of all motions of $\mathbb{H}$. This group is naturally isomorphic to $PGL_2(\mathbb{R})$ and as a $G$-space, $\mathbb{H}$ is naturally isomorphic to $G/K$, where $K = PO(2)$ is the standard maximal compact subgroup of $G$.

By definition, $\Gamma$ is a subgroup of $G$. The space $X = \Gamma\backslash G$ with the natural right action of $G$ called an *automorphic space*. We will identify the Riemann surface $Y = \Gamma\backslash\mathbb{H}$ with $X/K$.

We start with the fact that every automorphic function $\phi$ (e.g., a Maass form) generates an automorphic representation of the group $G$; this means that, starting from $\phi$, we produce a smooth irreducible preunitary representation of the group $G$ in a space $V$ and its isometric realization $\nu : V \to C^\infty(X)$ in the space of smooth functions on $X$. If a Maass form $\phi$ has the eigenvalue $\mu = \frac{1-\lambda^2}{4}$, then the corresponding representation V is isomorphic to the representation of the principal series $V_\lambda$ when $\lambda \in i\mathbb{R}$, to the representation of complementary series $V_\lambda$ when $\lambda \in [0, 1)$ and to the trivial representation when $\lambda = 1$.

This means that we have a very explicit *model* of the abstract subspace $V \subset C^\infty(X)$ as the space of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$. Restricting to the unit circle $S^1 \subset \mathbb{R}^2$, we get realization of $V$ as the space of smooth even functions on the circle $S^1$ (see details in [**2**]). We will use this model to make explicit computations.

The triple product $c_i = \int_Y \phi\phi'\phi_i dv$ extends to a $G$-equivariant trilinear functional on the corresponding automorphic representations $l_i^{\rm aut} : V \otimes V' \otimes V_i \to \mathbb{C}$, where $V = V_\tau, V' = V_{\tau'}$ and $V_i = V_{\lambda_i}$.

Next, we use a general result from representation theory that such $G$-equivariant trilinear functional is unique up to a scalar ([**9**], [**10**]). This implies that the automorphic functional $l_i^{\rm aut}$ is proportional to some explicit *model* functional $l_{\lambda_i}^{\rm mod}$. In [**2**], we gave a description of such model functional $l_\lambda^{\rm mod} : V \otimes V' \otimes V_\lambda \to \mathbb{C}$ for any $\lambda$ using explicit realizations of representations $V$, $V'$ and $V_\lambda$ of the group $G$ in spaces of homogeneous functions; it is important that the model functional knows nothing about automorphic picture and carries no arithmetic information.

Thus, we can write $l_i^{\rm aut} = a_i \cdot l_{\lambda_i}^{\rm mod}$ for some constant $a_i$, and hence

$$c_i = l_i^{\rm aut}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) = a_i \cdot l_{\lambda_i}^{\rm mod}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}), \tag{2.1}$$

where $e_\tau,\ e_{\tau'},\ e_{\lambda_i}$ are K-invariant unit vectors in representations $V, V'$ and $V_{\lambda_i}$ corresponding to the automorphic forms $\phi$, $\phi'$ and $\phi_i$.

It turns out that the proportionality coefficient $a_i$ in (2.1) carries an important "automorphic" information while the second factor carries no arithmetic information and can be computed in terms of $\Gamma$-functions using explicit realizations of representations $V_\tau$, $V_{\tau'}$ and $V_\lambda$ (see Appendix in [**2**] where this computation is carried out). This second factor is responsible for the exponential decay, while the first factor $a_i$ has a polynomial behavior in parameter $\lambda_i$. An explicit computation shows that $|c_i|^2 = \frac{1}{\gamma(\lambda_i)}|a_i|^2$, and hence $d_i = |a_i|^2$ (where $\gamma(\lambda)$ was described in Section 1.3).

**2.2. Hermitian forms.** In order to estimate the quantities $d_i$, we consider the space $E = V_\tau \otimes V_{\tau'}$ and use the fact that the coefficients $d_i$ appear in the spectral decomposition of the following *geometrically defined* non-negative Hermitian form $H_\Delta$ on $E$ (for a detailed discussion, see [**2**]).

Consider the space $C^\infty(X \times X)$. The diagonal $\Delta : X \to X \times X$ gives rise to the restriction morphism $r_\Delta : C^\infty(X \times X) \to C^\infty(X)$. We define a non-negative Hermitian form $H_\Delta$ on $C^\infty(X \times X)$ by setting $H_\Delta = (r_\Delta)^*(P_X)$, where $P_X$ is the standard $L^2$ Hermitian form on $C^\infty(X)$ i.e.,

$$H_\Delta(w) = P_X(r_\Delta(w)) = \int_X |r_\Delta(w)|^2 d\mu_X$$

for any $w \in C^\infty(X \times X)$. We call the restriction of the Hermitian form $H_\Delta$ to the subspace $E \subset C^\infty(X \times X)$ the *diagonal* Hermitian form and denote it by the same letter.

We will describe the spectral decomposition of the Hermitian form $H_\Delta$ in terms of Hermitian forms corresponding to trilinear functionals. Namely, if $L$ is a pre-unitary representation of $G$ with $G$-invariant norm $|| \ ||_L$, then every $G$-invariant trilinear functional $l : V \otimes V' \otimes L \to \mathbb{C}$, defines a Hermitian form $H^l$ on $E$ by $H^l(w) = \sup_{||u||_L=1} |l(w \otimes u)|^2$.

Here is another description of this form (see [2]). Functional $l : V \otimes V' \otimes L \to \mathbb{C}$ gives rise to a $G$-intertwining morphism $T^l : E \to L^*$ which image lies in the smooth part of $L^*$. Then, the form $H^l$ is just the inverse image of the Hermitian form on $L^*$ corresponding to the inner product on $L$.

Consider the orthogonal decomposition $L^2(X) = (\oplus_i V_i) \oplus (\oplus_\kappa V_\kappa)$ where $V_i$ correspond to Maass forms and $V_\kappa$ correspond to representations of discrete series. Every subspace $L \subset L^2(X)$ defines a trilinear functional $l : E \otimes L \to \mathbb{C}$ and hence a Hermitian form $H^l$ on $E$. Hence, the decomposition of $L^2(X)$ gives rise to the corresponding decomposition $H_\Delta = \sum H_i^{\mathrm{aut}} + \sum H_\kappa^{\mathrm{aut}}$ of Hermitian forms (see [2]).

We denote by $H_\lambda$ the *model* Hermitian form corresponding to the *model* trilinear functional $l_\lambda^{\mathrm{mod}} : V \otimes V' \otimes V_\lambda \to \mathbb{C}$. From definition, we see that $H_i^{\mathrm{aut}} = d_i H_{\lambda_i}$ which leads us to

**Basic identity**

$$H_\Delta = \sum_i d_i H_{\lambda_i} + \sum_\kappa H_\kappa^{\mathrm{aut}}, \tag{2.2}$$

We will mostly use the fact that for every vector $w \in E$, this basic identity gives us an inequality

$$\sum_i d_i H_{\lambda_i}(w) \le H_\Delta(w) \tag{2.3}$$

which is an equality if the vector $r_\Delta(w)$ does not have projection on discrete series representations (for example, if the vector $w$ is invariant with respect to the diagonal action of $K$ on $E$).

We can use this inequality to bound coefficients $d_i$. Namely, for a given vector $w \in E$, we usually can compute the values $H_\lambda(w)$ by explicit computations in the model of representations $V, V', V_\lambda$. It is usually much more difficult to get reasonable estimates of the right-hand side $H_\Delta(w)$. In cases when we manage to do this, we get some bounds for the coefficients $d_i$.

**2.3. Mean-value estimates.** In [2], using the geometric properties of the diagonal form and explicit estimates of forms $H_\lambda$, we established the mean-value bound (1.5): $\sum_{|\lambda_i| \le T} d_i \le AT^2$ . Roughly speaking, the proof of this bound is based on the fact that while the value of the form $H_\Delta$ on a given vector $w \in E$ is very difficult to control, we can show that for many vectors $w$, the value $H_\Delta(w)$ can be bounded by $P_E(w)$, where $P_E$ is the Hermitian form which defines the standard unitary structure on $E$.

More precisely, consider the natural representation $\sigma = \pi \otimes \pi'$ of the group $G \times G$ on the space $E$. Then, for a given compact neighborhood $U \subset G \times G$ of the identity element, there exists a constant $C$ such that for any vector $w \in E$, the inequality $H_\Delta(\sigma(g)w) \leq CP_E(w)$ holds for at least half of the points $g \in U$. This follows from the fact that the average over $U$ of the quantity $H_\Delta(\sigma(g)w)$ is bounded by $CP_E(w)/2$.

This allows us for every $T \geq 1$, to find a vector $w \in E$ such that $H_\Delta(w) \leq CT^2$ while the inequality $H_\lambda(w) \geq c$ holds for all $|\lambda| \leq T$.

**2.4. Bounds for sums over shorter intervals.**

The main starting point of our approach to the subconvexity bound is the inequality (2.3) for Hermitian forms. For a given $T > 1$, we construct a test vector $w_T \in E$ such that the weight function $\lambda \mapsto H_\lambda(w_T)$ has a sharp peak near $|\lambda| = T$ (i.e., a vector satisfying the condition (2.5) below).

The problem is how to estimate effectively $H_\Delta(w_T)$. The idea is that the Hermitian form $H_\Delta$ is geometrically defined and, as a result, satisfies some non-trivial bounds, symmetries, etc. None of the explicit *model* Hermitian forms $H_\lambda$ satisfies similar properties. By applying these symmetries to the vector $w_T$, we construct a new vector $\tilde{w}_T$ and from the geometry of the automorphic space $X$, we deduce the bound $H_\Delta(w_T) \leq H_\Delta(\tilde{w}_T)$.

On the other hand, the weight function $H_\lambda(\tilde{w}_T)$ in the spectral decomposition $H_\Delta(\tilde{w}_T) = \sum d_i H_{\lambda_i}(\tilde{w}_T)$ for $\tilde{w}_T$ behaves quite differently from the weight function $H_\lambda(w_T)$ for $w_T$. Namely, the function $H_\lambda(\tilde{w}_T)$ behaves regularly (i.e., satisfies condition (2.6) below), while the weight function $H_\lambda(w_T)$ has a sharp peak near $|\lambda| = T$.

The regularity of the function $H_\lambda(\tilde{w}_T)$ coupled with the mean-value bound (1.5) allows us to prove a sharp upper bound on the value of $H_\Delta(\tilde{w}_T)$ by purely spectral considerations (in cases we consider, there is no contribution from discrete series). We do not see how to get such sharp bound by geometric considerations.

Using this bound for $H_\Delta(\tilde{w}_T)$ and the inequality $H_\Delta(w_T) \leq H_\Delta(\tilde{w}_T)$ we obtain a non-trivial bound for $H_\Delta(w_T)$ and, as a result, the desired bound for the coefficients $d_i$.

**2.5. Formulas for test vectors.** Let us describe the construction of vectors $w_T, \tilde{w}_T$. We assume for simplicity, that $V' \simeq \bar{V}$ – the complex conjugate representation; it is also an automorphic representation with the realization $\bar{\nu} : \bar{V} \to C^\infty(X)$. It is easy to see that the upper bound estimate that we need in the general case can be easily reduced to this special case.

We only consider the case of representations of the principal series, i.e., we assume that $V = V_\tau$, $V' = \bar{V} = V_{-\tau}$ for some $\tau \in i\mathbb{R}$; the case of representations of the complementary series can be treated similarly.

Let $\{e_n\}_{n\in 2\mathbb{Z}}$ be a $K$-type orthonormal basis in $V$. We denote by $\{e'_n = \bar{e}_{-n}\}$ the complex conjugate basis in $\bar{V}$.

For a given $T \geq 1$, we choose even $n$ such that $|T - 2n| \leq 10$ and set

$$w_T = e_n \otimes e'_{-n} \quad \text{and} \quad \tilde{w}_T = e_n \otimes e'_{-n} + e_{n+2} \otimes e'_{-n-2}.$$

With such a choice of test vectors, we have the following bounds.

**Geometric bound:**

$$H_\Delta(w_T) \leq H_\Delta(\tilde{w}_T) \tag{2.4}$$

**Spectral bounds:**

(i) There exist constants $b, c > 0$ such that

$$H_\lambda(w_T) \geq c|\lambda|^{-5/3} \text{ for } |\lambda| \in I_T \tag{2.5}$$

where $I_T$ is the interval of length $bT^{1/3}$ centered at point $T$.

(ii) There exists a constant $c'$ such that

$$H_\lambda(\tilde{w}_T) \leq \begin{cases} c'T^{-1}(1+|\lambda|)^{-1} & \text{for all } |\lambda| \leq 2T, \\ c'|\lambda|^{-3} & \text{for all } |\lambda| > 2T. \end{cases} \tag{2.6}$$

Using the bound (2.6), we can get a sharp estimate of $H_\Delta(\tilde{w})$. Namely, from (2.2), we conclude that $H_\Delta(\tilde{w}) = \sum d_i H_{\lambda_i}(\tilde{w})$ (since vectors $\tilde{w}_T$ are $\Delta K$-invariant, we do not have contribution from representations of discrete series).

The spectral bound (2.6) for $H_\lambda(\tilde{w})$ together with the mean-value bound (1.5) for coefficients $d_i$ imply that

$$H_\Delta(\tilde{w}_T) \leq D$$

for some explicit constant $D$.

Using the geometric inequality (2.4), we see that $H_\Delta(w_T) \leq D$. Using the spectral bound (2.5), we obtain

$$\sum_{|\lambda_i| \in I_T} d_i c T^{-5/3} \leq \sum_i d_i H_{\lambda_i}(w_T) \leq H_\Delta(w_T) \leq D.$$

From this, we deduce the bound (1.6) in Theorem 1.1.

**2.6. Proof of the geometric bound 2.4.** The inequality (2.4) easily follows from the pointwise bound on $X$ due to the fact that, in the automorphic realization, the vector $e_n \otimes e'_{-n}$ is represented by a function which restriction $u_n = r_\Delta(e_n \otimes e'_{-n})$ to the diagonal is non-negative

$$u_n(x) = \nu(e_n)(x) \cdot \bar{\nu}(e'_{-n})(x) = |\nu(e_n)(x)|^2 \geq 0.$$

From this, we see that

$$H_\Delta(w_T) = \int_X |u_n(x)|^2 d\mu_X \leq \int_X |u_n(x) + u_{n+2}(x)|^2 d\mu_X = H_\Delta(\tilde{w}_T).$$

**2.7. Sketch of proof of the spectral bounds (2.5) and (2.6).** We will use the explicit form of the kernel defining Hermitian forms $H_\lambda$ in the model realizations of representations $V$, $V'$ and $V_\lambda$. Namely, we use the standard realization of these representations in the space $C^\infty_{even}(S^1)$ of even functions on $S^1$ (see [**2**] and 2.1). Under this identification, the basis $\{e_n\}$ becomes the standard basis of exponents $\{e_n = e^{in\theta}\}$, where $0 \leq \theta < 2\pi$ is the standard parameter on $S^1$.

As was shown in [**2**], Section 5, in such realization, the invariant functional $l^{\mathrm{mod}}_\lambda$ on the space $V \otimes V' \otimes V_\lambda \simeq C^\infty((S^1)^3)$ is given by the following kernel on $(S^1)^3$

$$L_\lambda(\theta, \theta', \theta'') = |\sin(\theta - \theta')|^{\frac{-1+\lambda}{2}} |\sin(\theta - \theta'')|^{\frac{-1+2\tau-\lambda}{2}} |\sin(\theta' - \theta'')|^{\frac{-1-2\tau-\lambda}{2}},$$

where $V = V_\tau$, $V' = V_{-\tau}$ with $\tau \in i\mathbb{R}$. From this, it follows that the Hermitian forms $H_\lambda$ on $E \simeq C^\infty(S^1 \times S^1)$ are given by oscillatory integrals (over $(S^1)^4$) and the verification of conditions (2.5) and (2.6) is reduced to the stationary phase method.

In fact, we will use the values of $H_\lambda(w)$ only for $\Delta K$-invariant vectors $w \in E$. This considerably simplifies our computations since we can reduce them to two repeated integrations in one variable and use the stationary phase method in *one variable*.

Namely, let us fix a $\Delta K$-invariant vector $w \in E$. Then, the vector $T_\lambda(w) \in V_{-\lambda}$ is proportional to the standard $K$-invariant vector $e_0 \in V_{-\lambda} \simeq C^\infty_{\mathrm{even}}(S^1)$ (here, the operator $T_\lambda : E \to V_{-\lambda}$ corresponds to the model trilinear functional $l^{\mathrm{mod}}_\lambda$ as described in 2.2). This implies that $H_\lambda(w) = |T_\lambda(w)(0)|^2$. The value $T_\lambda(w)(0)$ is given by the following oscillating integral

$$T_\lambda(w)(0) = \langle w, K_\lambda \rangle = \int w(\theta, \theta') K_\lambda(\theta, \theta') d\theta d\theta',$$

where $K_\lambda(\theta, \theta') = L_\lambda(\theta, \theta', 0)$. Since the vector $w$ is $\Delta K$-invariant, it can be described by a function in one variable; namely, $w(\theta, \theta') = u(c)$ for $u \in C^\infty(S^1)$ and $c = (\theta - \theta')/2$. We have $\langle w, K_\lambda \rangle = \int u(c) k_\lambda(c) dc$, where the function $k_\lambda$ is obtained from $K_\lambda$ by averaging over $\Delta K$. Thus, for a $\Delta K$-invariant vector $w$, the estimates of $H_\lambda(w)$ are equivalent to estimates of the one-dimensional integral $\langle u, k_\lambda\rangle = \int u(c)k_\lambda(c)dc$.

The function $k_\lambda(c)$, which is obtained from $K_\lambda(\theta, \theta')$ via one-dimensional integration, is not an elementary function. However, using stationary phase method, we obtain the representation $k_\lambda(c) = |\lambda|^{-\frac{1}{2}} m_\lambda(c) + r_\lambda(c)$, where the main term $m_\lambda$ (given by contributions from non-degenerate stationary points of the phase in the corresponding integral) is an elementary function

$$m_\lambda(c) = \alpha(\lambda)|\sin(c)|^{-\frac{1}{2}-\frac{\lambda}{2}}|\cos(c)|^{-\frac{1}{2}+\frac{\lambda}{2}},$$

with $\alpha(\lambda) = (\pi)^{-1}e^{-i\frac{\pi}{4}}2^{-\frac{1}{2}+\frac{\lambda}{2}}$. The stationary phase method also gives a bound for the remainder term $||r_\lambda||_{L^1(S^1)} \leq a(1+|\lambda|)^{-3/2}$ for some constant $a$.

The vectors $w$ which we consider correspond to bounded functions $u(c)$. For such vectors, the estimate of $\langle u, k_\lambda\rangle$ is reduced to the estimate of $\langle u, m_\lambda\rangle = \int u(c)m_\lambda(c)dc$.

We deduce spectral bounds (2.5) and (2.6) by applying stationary phase method to integrals $\langle u_T, m_\lambda\rangle$ and $\langle \tilde{u}_T, m_\lambda\rangle$, where $u_T,\ \tilde{u}_T \in C^\infty(S^1)$ are functions corresponding to vectors $w_T,\ \tilde{w}_T \in E$.

The key fact responsible for the crucial bound in (2.5) is that for $T = |\lambda|$, the phase of the oscillating integral $\langle u_T, m_\lambda\rangle$ corresponding to the value $H_\lambda(w_T)$ has a *degenerate* critical point at $c = \pi/4$ with the *non-vanishing* amplitude at that point. For other values of $\lambda$, this phase has non-degenerate critical points.

Since this critical point is degenerate, the integral $\langle u_T, m_\lambda\rangle$ has a sharp peak at $|\lambda| = T$. The standard technique developed to analyze the asymptotic behavior of the Airy functions then gives the bound (2.5) for $|\lambda| \asymp T$.

On the other hand, for the oscillating integral $\langle \tilde{u}_T, m_\lambda\rangle$ corresponding to the value $H_\lambda(\tilde{w}_T)$, the phase is the same as for the integral $\langle u_T, m_\lambda\rangle$, but the amplitude has an additional factor $a(c) = 1 + e^{4ic}$ which was chosen in such a way that it vanishes at the degenerate critical point which develops at $|\lambda| = T$. As a result, this point does not give an additional contribution to this integral.

This is a classical situation for which the uniform bounds for the oscillating integrals are well-known (e.g., bounds on the Airy function and its derivative). From this, we deduce the bound in (2.6). In fact, we find that for $|\lambda| > T$, there are no critical points at all. This implies that for $|\lambda| > 2T$, we have a stronger bound $H_\lambda(\tilde{w}_T) \ll |\lambda|^{-N}$ for any $N > 1$ (compare to (2.6)).

For $|\lambda| \ll T$, we also consider singularities of the amplitude in the corresponding integrals in order to show that the low-lying spectrum contribution is bounded. This includes the contribution from representations of the complementary series and the trivial representation (in fact, we have to deal with the singularities of the amplitude for all values of $\lambda$).

The above arguments also prove the following result on the $L^4$-norm of $K$-types in irreducible automorphic representations of $PGL_2(\mathbb{R})$. This result is of independent interest.

**Theorem 2.1.** *For a fixed class one automorphic representation $\nu : V \to C^\infty(X)$, there exists $D > 0$ such that $||\nu(e_n)||_{L^4(X)} \leq D$ for all $n$.*

One would expect that a similar fact holds for representations of the discrete series as well.

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