PERMUTOHEDRA, ASSOCIAHEDRA, AND BEYOND

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Abstract. The volume and the number of lattice points of the permutohedron P_n are given by certain multivariate polynomials that have remarkable combinatorial properties. We give 3 different formulas for these polynomials. We also study a more general class of polytopes that includes the permutohedron, the associahedron, the cyclohedron, the Stanley-Pitman polytope, and various generalized associahedra related to wonderful compactifications of De Concini-Procesi. These polytopes are constructed as Minkowski sums of simplices. We calculate their volumes and describe their combinatorial structure. The coefficients of monomials in $\mathrm{Vol}\, P_n$ are certain positive integer numbers, which we call the mixed Eulerian numbers. These numbers are equal to the mixed volumes of hypersimplices. Various specializations of these numbers give the usual Eulerian numbers, the Catalan numbers, the numbers $(n+1)^{n-1}$ of trees (or parking functions), the binomial coefficients, etc. We calculate the mixed Eulerian numbers using certain binary trees. Many results are extended to an arbitrary Weyl group.

1. Permutohedron

Let $x_1, \ldots, x_{n+1}$ be real numbers. The *permutohedron* $P_n(x_1, \ldots, x_{n+1})$ is the convex polytope in $\mathbb{R}^{n+1}$ defined as the convex hull of all permutations of the vector $(x_1, \ldots, x_{n+1})$:

$$P_n(x_1, \ldots, x_{n+1}) := \mathrm{ConvexHull}((x_{w(1)}, \ldots, x_{w(n+1)}) \mid w \in S_{n+1}),$$

where S_{n+1} is the symmetric group. Actually, this is an n-dimensional polytope that lies in some hyperplane $H \subset \mathbb{R}^{n+1}$. More generally, for a Weyl group W, we can define the *weight polytope* as a convex hull of a Weyl group orbit:

$$P_W(x) := \mathrm{ConvexHull}(w(x) \mid w \in W),$$

where x is a point in the weight space on which the Weyl group acts.

For example, for $n = 2$ and distinct x_1, x_2, x_3, the permutohedron $P_2(x_1, x_2, x_3)$ (type A_2 weight polytope) is the hexagon shown below. If some of the numbers x_1, x_2, x_3 are equal to each other then the permutohedron degenerates into a triangle, or even a single point.

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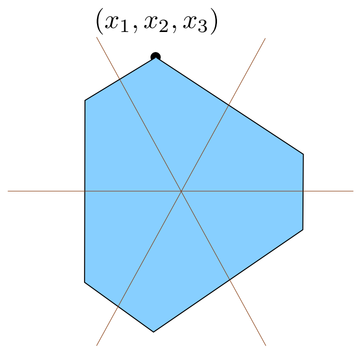
$P_2(x_1, x_2, x_3) =$



Question: What is the volume of the permutohedron $V_n := \mathrm{Vol}\, P_n$?

Since P_n does not have the full dimension in $\mathbb{R}^{n+1}$, one needs to be careful with definition of the volume. We assume that the volume form Vol is normalized so that the volume of a unit parallelepiped formed by generators of the integer lattice $\mathbb{Z}^{n+1} \cap H$ is 1. More generally, we can ask the following question.

Question: What is the number of lattice points $N_n := P_n \cap \mathbb{Z}^{n+1}$?

We will see that both V_n and N_n are polynomials of degree n in the variables $x_1, \ldots, x_{n+1}$. The polynomial V_n is the top homogeneous part of N_n. The *Ehrhart polynomial* of P_n is $E(t) = N_n(tx_1, \ldots, tx_n)$, and V_n is its top degree coefficient.

We will give 3 totally different formulas for these polynomials.

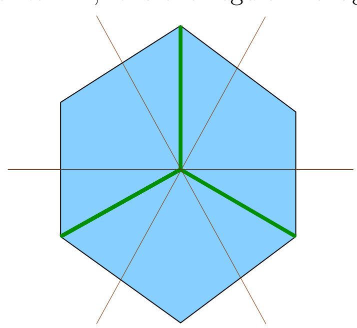
Let us first mention the special case $(x_1, \ldots, x_{n+1}) = (n+1, \ldots, 1)$. The polytope

$$P_n(n+1, n, \ldots, 1) = \mathrm{ConvexHull}((w(1), ..., w(n+1)) \mid w \in S_{n+1})$$

is *the most symmetric* permutohedron. It is invariant under the action of the symmetric group S_{n+1}. For example, for $n = 2$, it is the regular hexagon:

regular hexagon

subdivided into 3 rhombi



The polytope $P_n(n+1, \ldots, 1)$ is a *zonotope,* i.e., Minkowski sum of line intervals. It is well known that

- $V_n(n+1, \ldots, 1) = (n+1)^{n-1}$ is the number of trees on $n+1$ labelled vertices. Indeed, $P_n(n+1, \ldots, 1)$ can be subdivided into parallelepipeds of unit volume associated with trees. This follows from a general result about zonotopes.
- $N_n(n+1, \ldots, 1)$ is the number of forests on $n+1$ labelled vertices.

In general, for arbitrary $x_1, \ldots, x_{n+1}$, the permutohedron $P_n(x_1, \ldots, x_{n+1})$ is not a zonotope. We cannot easily calculate its volume by subdividing it into parallelepipeds.

2. First Formula

Theorem 2.1. *Fix any distinct numbers* $\lambda_1, \ldots, \lambda_{n+1}$ *such that* $\lambda_1 + \cdots + \lambda_{n+1} = 0$. *We have*

$$V_n(x_1, \ldots, x_{n+1}) = \frac{1}{n!} \sum_{w \in S_{n+1}} \frac{(\lambda_{w(1)} x_1 + \cdots + \lambda_{w(n+1)} x_{n+1})^n}{(\lambda_{w(1)} - \lambda_{w(2)})(\lambda_{w(2)} - \lambda_{w(3)}) \cdots (\lambda_{w(n)} - \lambda_{w(n+1)})}.$$

Notice that all λ_i's in the right-hand side are canceled after the symmetrization.

More generally, let W be the Weyl group associated with a rank n root system, and let $\alpha_1, \ldots, \alpha_n$ be a choice of simple roots.

Theorem 2.2. *Let* λ *be any regular weight. The volume of the weight polytope is*

$$\operatorname{Vol} P_W(x) = \frac{f}{|W|} \sum_{w \in W} \frac{(\lambda, w(x))^n}{(\lambda, w(\alpha_1)) \cdots (\lambda, w(\alpha_n))},$$

where the volume is normalized so that the volume of the parallelepiped generated by the simple roots α_i *is 1, and* f *is the index of the root lattice in the weight lattice.*

Idea of Proof. Let us use Khovansky-Pukhlikov's method [PK]. Instead of counting the number of lattice points in P, calculate the sum $\Sigma[P]$ of formal exponents e^a over lattice points $a \in P \cap \mathbb{Z}^n$. We can work with unbounded polytopes. For example, for a simplicial cone C, the sum $\Sigma[C]$ is given by a simple rational expression. Any polytope P can be explicitly presented as an alternating sum of simplicial cones $\Sigma[P] = \Sigma[C_1] \pm \Sigma[C_2] \pm \cdots$ over vertices of P.

Applying this method to the weight polytope, we obtain the following claim.

Theorem 2.3. *For a dominant weight* μ, *the sum of exponents over lattice points of the weight polytope* $P_W(\mu)$ *equals*

$$\Sigma[P_W(\mu)] := \sum_{a \in P_W(\mu) \cap (L+\mu)} e^a = \sum_{w \in W} \frac{e^{w(\mu)}}{(1 - e^{-w(\alpha_1)}) \cdots (1 - e^{-w(\alpha_n)})},$$

where L *be the root lattice.*

Compare this claim with Weyl's character formula! If we replace the product over simple roots α_i in the right-hand side of Theorem 2.3 by a similar product over all positive roots, we obtain exactly Weyl's character formula.

Theorem 2.2 and its special case Theorem 2.1 and be deduced from Theorem 2.3 in the same way as Weyl's dimension formula can be deduced from Weyl's character formula. □

Remark 2.4. The sum of exponents $\Sigma[P_W(\mu)]$ over lattice points of the weight polytope is obtained from the character $ch\, V_\mu$ of the irreducible representation V_μ of the associated Lie group by replacing all nonzero coefficients in $ch\, V_\mu$ with 1. For example, in type A, $ch\, V_\mu$ is the Schur polynomial s_μ and $\Sigma[P_n(\mu)]$ is obtained from the Schur polynomial s_μ by erasing the coefficients of all monomials.

3. Second Formula

Let us use the coordinates $y_1, \ldots, y_{n+1}$ related to $x_1, \ldots, x_{n+1}$ by

$$\begin{cases} y_1 = -x_1 \\ y_2 = -x_2 + x_1 \\ y_3 = -x_3 + 2x_2 - x_1 \\ \cdots \\ y_{n+1} = -\binom{n}{0} x_n + \binom{n}{1} x_{n-1} - \cdots \pm \binom{n}{n} x_1 \end{cases}$$

Write $V_n = \mathrm{Vol}\ P_n(x_1, \ldots, x_{n+1})$ as a polynomial in the variables $y_1, \ldots, y_{n+1}$.

Theorem 3.1. *We have*

$$V_n = \frac{1}{n!} \sum_{(S_1, \ldots, S_n)} y_{|S_1|} \cdots y_{|S_n|},$$

where the sum is over ordered collections of subsets $S_1, \ldots, S_n \subset [n+1]$ such that, for any distinct $i_1, \ldots, i_k$, we have $|S_{i_1} \cup \cdots \cup S_{i_k}| \geq k+1$.

This theorem implies that $n!\, V_n$ is a polynomial in $y_2, \ldots, y_{n+1}$ with positive integer coefficients. For example, $V_1 = \mathrm{Vol}\,([(x_1, x_2), (x_2, x_1)]) = x_1 - x_2 = y_2$ and $2V_2 = 6\, y_2^2 + 6\, y_2\, y_3 + y_3^2$.

Remark 3.2. The condition on subsets $S_1, \ldots, S_n$ in Theorem 3.1 is very similar to the condition in Hall's marriage theorem. One just needs to replace the inequality $\geq k+1$ with $\geq k$ to obtain Hall's marriage condition.

It is not hard to prove the following analog of Hall's theorem.

Dragon marriage problem: *There are $n+1$ brides and n grooms living in a medieval village. A dragon comes to the village and takes one of the brides. We are given a collection G of pairs of brides and grooms that can marry each other. Find the condition on G that ensures that, no matter which bride the dragon takes, it will be possible to match the remaining brides and grooms.*

Proposition 3.3. (Dragon marriage theorem) *Let $S_1, \ldots, S_n \subset [n+1]$. The following three conditions are equivalent:*

1. *For any distinct $i_1, \ldots, i_k$, we have $|S_{i_1} \cup \cdots \cup S_{i_k}| \geq k+1$.*
2. *For any $j \in [n{+}1]$, there is a system of distinct representatives in $S_1, \ldots, S_n$ that avoids j. (This is a reformulation of the dragon marriage problem.)*
3. *One can remove some elements from S_i's to get the edge set of a spanning tree in K_{n+1}.*

Theorem 3.1 can be extended to a larger class of polytopes discussed below.

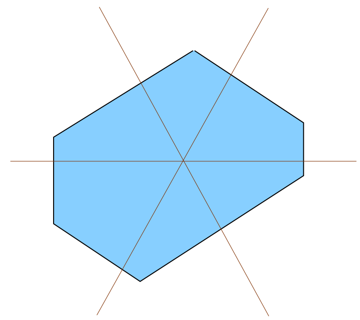
4. Generalized permutohedra

Let $\Delta_{[n+1]} = \mathrm{ConvexHull}(e_1, \ldots, e_{n+1})$ be the standard coordinate simplex in $\mathbb{R}^{n+1}$. For a subset $I \subset [n+1]$, let $\Delta_I = \mathrm{ConvexHull}(e_i \mid i \in I)$ denote the face of the coordinate simplex $\Delta_{[n+1]}$. Let $\mathbf{Y} = \{Y_I\}$ be a collection of parameters $Y_I \geq 0$ for all nonempty subsets $I \subset [n+1]$. Let us define the *generalized permutohedron* $P_n(Y)$ as the Minkowski sum of the simplices Δ_I scaled by factors Y_I:

$$P_n(\mathbf{Y}) := \sum_{I \subset [n+1]} Y_I \cdot \Delta_I \qquad \text{(Minkowski sum)}$$

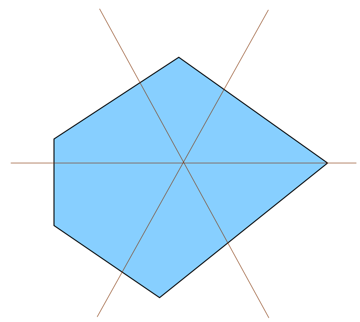
Generalized permutohedra are obtained from usual permutohedra by moving their faces while preserving all angles. So, instead of n degrees of freedom in usual permutohedra, we have $2^{n+1}-2$ degrees of freedom in generalized permutohedra.

a generalized permutohedron



this is also

a generalized permutohedron



The combinatorial type of $P_n(\mathbf{Y})$ depends only on the set of $B \subset 2^{[n+1]}$ of I's for which $Y_I \neq 0$. Here are some interesting examples of generalized permutohedra.

- If $Y_I = y_{|I|}$, i.e., the variables Y_I are equal to each other for all subsets of the same cardinality, then $P_n(\mathbf{Y})$ is the usual permutohedron P_n.
- If $B = \{\{i, i+1, \dots, j\} \mid 1 \leq i \leq j \leq n\}$ is the set of consecutive intervals, then $P_n(\mathbf{Y})$ is the *associahedron*, also known as the *Stasheff polytope*. The polytope $P_n(\mathbf{Y})$ can be equivalently defined as the Newton polytope of $\prod_{1\leq i\leq j\leq n+1}(x_i + x_{i+1} + \cdots + x_j)$. This is exactly *Loday's realization* of the associahedron, see [L].
- If B is the set of cyclic intervals, then $P_n(\mathbf{Y})$ is a *cyclohedron.*
- If B is the set of connected subsets of nodes of a Dynkin diagram, then $P_n(\mathbf{Y})$ the polytope related to De Concini-Procesi's *wonderful compactification,* see [DP], [DJS].
- If $B = \{[i] \mid i = 1, \dots, n+1\}$ is the complete flag of initial intervals, then $P_n(\mathbf{Y})$ is the *Stanley-Pitman polytope* from [SP].

Theorem 3.1 can be extended to generalized permutohedra, as follows.

Theorem 4.1. *The volume of the generalized permutohedron $P_n(\mathbf{Y})$ is given by*

$$\operatorname{Vol} P_n(\mathbf{Y}) = \frac{1}{n!} \sum_{(S_1,\dots,S_n)} Y_{S_1} \cdots Y_{S_n},$$

where the sum is over ordered collections of subsets $S_1, \dots, S_n \subset [n+1]$ such that, for any distinct $i_1, \dots, i_k$, we have $|S_{i_1} \cup \dots \cup S_{i_k}| \geq k+1$.

Theorem 4.2. *The number of lattice points in the generalized permutohedron $P_n(\mathbf{Y})$ is*

$$P_n(\mathbf{Y}) \cap \mathbb{Z}^{n+1} = \frac{1}{n!} \sum_{(S_1,\dots,S_n)} \{Y_{S_1} \cdots Y_{S_n}\},$$

where the summation is over the same collections $(S_1, \dots, S_n)$ as before, and

$$\left\{\prod_I Y_I^{a_I}\right\} := (Y_{[n+1]}+1)^{\{a_{[n+1]}\}} \prod_{I \neq [n+1]} Y_I^{\{a_I\}}, \text{ where } Y^{\{a\}} = Y(Y+1)\cdots(Y+a-1).$$

In other words, the formula for the number of lattice points in $P_n(\mathbf{Y})$ is obtained from the formula for the volume by replacing usual powers in all terms by raising powers.

These formulas generalize formulas from [SP] for the volume and the number of lattice points in the Stanley-Pitman polytope. In this case, collections $(S_1, \dots, S_n)$ of *initial intervals* $S_i = [s_i]$ that satisfy the dragon marriage condition, see Proposition 3.3, are in one-to-one correspondence with *parking functions* $(s_1, \dots, s_n)$. The volume $P_n(\mathbf{Y})$ is given by the sum over parking functions.

5. Nested families and generalized Catalan numbers

In this section, we describe the combinatorial structure of the class of generalized permutohedra $P_n(\mathbf{Y})$ such that the set $B \subset 2^{[n+1]}$ of subsets I with nonzero Y_I satisfies the following connectivity condition:

(B1) If $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.

All examples of generalized permutohedra mentioned in the previous section satisfy this additional condition. Without loss of generality we will also assume that

(B2) B contains all singletons $\{i\}$, for $i \in [n+1]$.

Indeed, the Minkowski sum of a polytope with $\Delta_{\{i\}}$, which is a single point, is just a parallel translation of the polytope.

Sets $B \subset 2^{[n+1]}$ that satisfy conditions (B1) and (B2) are called *building sets*. Note that condition (B1) implies that all maximal by inclusion elements in B are pairwise disjoint.

Let us say that a subset N in a building set B is a *nested family* if it satisfies the following conditions:

(N1) For any $I, J \in N$, we have either $I \subseteq J$, or $J \subseteq I$, or I and J are disjoint.
(N2) For any collection of $k \geq 2$ disjoint subsets $I_1, \dots, I_k \in N$, their union $I_1 \cup \cdots \cup I_k$ is not in B.
(N3) N contains all maximal elements of B.

Let $\mathcal{N}(B)$ be the poset of all nested families in B ordered by reverse inclusion.

Theorem 5.1. *The poset of faces of the generalized permutohedron $P_n(\mathbf{Y})$ ordered by inclusion is isomorphic to $\mathcal{N}(B)$.*

This claim was independently discovered by E. M. Feichtner and B. Sturmfels [FS].

For a graph G on the set of vertices $[n+1]$, let B_G be the collection of nonempty subsets $I \subset [n+1]$ such that the induced graph $G|_I$ is connected. Then B_G satisfies conditions (B1) and (B2) of a building set. The generalized permutohedron associated with B_G is combinatorially equivalent to the *graph associahedron* constructed by Carr and Devadoss [CD] using blow-ups. In this case, it is enough to require condition (N2) only for pairs of subsets, in the definition of a nested family.

Remark 5.2. Since our generalized permutohedra include the associahedron, one can also call them *generalized associahedra*. However this name is already reserved for a different generalization of the associahedron studied by Fomin, Chapoton, and Zelevinsky [FCZ].

Let $f_B(q)$ be the *f-polynomial* of the generalized permutohedron:

$$f_B(q) = \sum_{i=0}^{n} f_i \, q^i = \sum_{N \in \mathcal{N}(B)} q^{n+1-|N|},$$

where f_i is the number of i-dimensional faces of $P_n(\mathbf{Y})$.

Let us say that a building set B is *connected* if it has a unique maximal element. Each building set B is a union of pairwise disjoint connected building sets, called the *connected components* of B. For a subset $S \subset [n+1]$, the *induced building set* is defined as $B|_S = \{I \in B \mid I \subset S\}$. In the case of building sets B_G associated with graphs G, notions of connected components and induced building sets correspond to similar notions for graphs.

Theorem 5.3. *The f-polynomial $f_B(q)$ is determined by the following recurrence relations:*

(1) *If B consists of a single singleton, then $f_B(q) = 1$.*
(2) *If B has connected components $B_1, \ldots, B_k$, then*

$$f_B(q) = f_{B_1}(q) \cdots f_{B_k}(q).$$

(3) *If $B \subset 2^{[n+1]}$ is a connected building and $n \geq 1$, then*

$$f_B(q) = \sum_{S \subsetneq [n+1]} q^{n-|S|} f_{B|_S}(q).$$

Define the *generalized Catalan number*, for a building set B, as the number $C(B) = f_B(0)$ of vertices of the corresponding generalized permutohedron $P_n(\mathbf{Y})$. These numbers are given by the recurrence relations similar to the relations in Theorem 5.3. (But in (3) we sum only over subsets $S \subset [n+1]$ of cardinality n).

For a graph G, let $C(G) = C(B_G)$. Let T_{pqr} be the graph that has a central node with 3 attached chains of with p, q and r vertices. For example, T_{111} is the Dynkin diagram of type D_4. The above recurrence relations imply the following expression for the generating function for generalized Catalan numbers:

$$\sum_{p,q,r \geq 0} C(T_{pqr}) \, x^p y^q z^r = \frac{C(x)\, C(y)\, C(z)}{1 - x\, C(x) - y\, C(y) - z\, C(z)},$$

where $C(x) = \sum_{n \geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the usual Catalan numbers.

Proposition 5.4. *For the Dynkin diagram of type A_n, we have $C(A_n) = C_n = \frac{1}{n+1}\binom{2n}{n}$ is the usual Catalan number. For the extended Dynkin diagram of type $\hat{A}_n$, we have $C(\hat{A}_n) = \binom{2n}{n}$. For the Dynkin diagram of type D_n, the corresponding Catalan number is*

$$C(D_n) = 2\, C_n - 2\, C_{n-1} - C_{n-2}.$$

Remark 5.5. One can define the generalized Catalan number for any Lie type. However this number does not depend on multiplicity of edges in the Dynkin diagram. Thus Lie types B_n and C_n give the same (usual) Catalan number as type A_n.

6. Mixed Eulerian numbers

Let us return to the usual permutohedron $P_n(x_1, \ldots, x_{n+1})$. Let us use the coordinates $z_1, \ldots, z_n$ related to $x_1, \ldots, x_{n+1}$ by

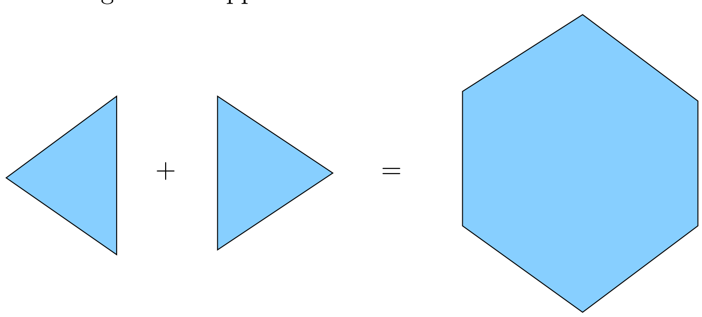
$$z_1 = x_1 - x_2,\ z_2 = x_2 - x_3,\ \cdots,\ z_n = x_n - x_{n+1}$$

This coordinate system is canonically defined for an arbitrary Weyl group as the coordinate system in the weight space given by the fundamental weights.

The permutohedron P_n can be written as the Minkowski sum

$$P_n = z_1\,\Delta_{1n} + z_2\,\Delta_{2n} + \cdots + z_n\,\Delta_{nn}$$

of the *hypersimplices* $\Delta_{kn} = P_n(1, \ldots, 1, 0, \ldots, 0)$ (with k 1's). For example, the hexagon can be express as the Minkowski sum of the hypersimplices Δ_{12} and Δ_{22}, which are two triangles with opposite orientations:



This implies that the volume of P_n can be written as

$$\text{Vol}\ P_n = \sum_{c_1,\ldots,c_n} A_{c_1,\ldots,c_n} \frac{z_1^{c_1}}{c_1!} \cdots \frac{z_n^{c_n}}{c_n!},$$

where the sum is over $c_1, \ldots, c_n \geq 0$, $c_1 + \cdots + c_n = n$, and

$$A_{c_1,\ldots,c_n} = \text{MixedVolume}(\Delta_{1n}^{c_1}, \ldots, \Delta_{nn}^{c_n}) \in \mathbb{Z}_{>0}$$

is the mixed volume of hypersimplices. In particular, $n!\,V_n$ is a positive integer polynomial in $z_1, \ldots, z_n$. Let us call the integers $A_{c_1,\ldots,c_n}$ the *Mixed Eulerian numbers.*

Examples: The mixed Eulerian numbers are marked in bold.

$$V_1 = \mathbf{1}\,z_1;$$
$$V_2 = \mathbf{1}\,\tfrac{z_1^2}{2} + \mathbf{2}\,z_1z_2 + \mathbf{1}\,\tfrac{z_2^2}{2};$$
$$V_3 = \mathbf{1}\,\tfrac{z_1^3}{3!} + \mathbf{2}\,\tfrac{z_1^2}{2}z_2 + \mathbf{4}\,z_1\tfrac{z_2^2}{2} + \mathbf{4}\,\tfrac{z_2^3}{3!} + \mathbf{3}\,\tfrac{z_1^2}{2}z_3 + \mathbf{6}\,z_1z_2z_3 + \\ + \mathbf{4}\,\tfrac{z_2^2}{2}z_3 + \mathbf{3}\,z_1\tfrac{z_3^2}{2} + \mathbf{2}\,z_2\tfrac{z_3^2}{2} + \mathbf{1}\,\tfrac{z_3^3}{3!}.$$

Theorem 6.1. *Mixed Eulerian numbers have the following properties:*

(1) $A_{c_1,\ldots,c_n}$ *are positive integers defined for* $c_1, \ldots, c_n \geq 0$, $c_1 + \cdots + c_n = n$.
(2) $\sum \frac{1}{c_1!\cdots c_n!} A_{c_1,\ldots,c_n} = (n+1)^{n-1}$.
(3) $A_{0,\ldots,0,n,0,\ldots,0}$ *(n in the k-th position) is the usual Eulerian number* A_{kn}, *i.e., the number of permutations in* S_n *with* k *descents.*

(4) $A_{0,\dots,0,i,n-i,0,\dots,0}$ *(with k 0's in front of $i, n-i$) is equal to the number of permutations $w \in S_{n+1}$ with k descents and $w(n+1) = i+1$.*
(5) $A_{1,\dots,1} = n!$
(6) $A_{k,0,\dots,0,n-k} = \binom{n}{k}$
(7) $A_{c_1,\dots,c_n} = 1^{c_1} 2^{c_2} \cdots n^{c_n}$ *if $c_1 + \cdots + c_i \geq i$, for $i = 1, \dots, n$. There are exactly $C_n = \frac{1}{n+1}\binom{2n}{n}$ such sequences $(c_1, \dots, c_n)$.*
(8) $\sum A_{c_1,\dots,c_n} = n!\, C_n$.

Property (3) follows from the well-know fact that $A_{kn} = n! \operatorname{Vol} \Delta_{kn}$; and property (4) follows from the result of [ERS] about the mixed volume of two adjacent hypersimplices. Property (8) was numerically noticed by Richard Stanley. Moreover, he conjectured the following claim.

Theorem 6.2. *Let us write $(c_1, \dots, c_n) \sim (c'_1, \dots, c'_n)$ whenever $(c_1, \dots, c_n, 0)$ is a cyclic shift of $(c'_1, \dots, c'_n, 0)$. Then, for fixed $(c_1, \dots, c_n)$, we have*

$$\sum_{(c'_1,\dots,c'_n)\sim(c_1,\dots,c_n)} A_{c'_1,\dots,c'_n} = n!$$

In other words, the sum of mixed Eulerian numbers in each equivalence class is $n!$.
There are exactly the Catalan number $\frac{1}{n+1}\binom{2n}{n}$ equivalence classes of sequences.

For example, we have $A_{1,\dots,1} = n!$ and $A_{n,0,\dots,0} + A_{0,n,0,\dots,0} + A_{0,0,n,\dots,0} + \cdots + A_{0,\dots,0,n} = n!$, because the sum of usual Eulerian numbers $\sum_k A_{kn}$ is $n!$.

Remark 6.3. Every equivalence class contains exactly one sequence $(c_1, \dots, c_n)$ such that $c_1 + \cdots + c_i \geq i$, for $i = 1, \dots, n$. For this special sequence, the mixed Eulerian number is given by the simple product $A_{c_1,\dots,c_n} = 1^{c_1} \cdots n^{c_n}$; see Theorem 6.1.(7).

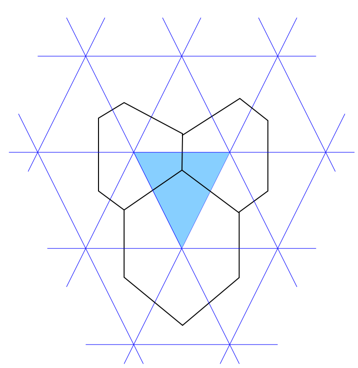
Theorem 6.2 follows from the following claim.

Theorem 6.4. *Let $\hat{U}_n(z_1, \dots, z_{n+1}) = \operatorname{Vol} P_n$. (It does not depend on z_{n+1}.)*

$$\hat{U}_n(z_1, \dots, z_{n+1}) + \hat{U}_n(z_{n+1}, z_1, \dots, z_n) + \cdots + \hat{U}_n(z_2, \dots, z_{n+1}, z_1) = \\ = (z_1 + \cdots + z_{n+1})^n$$

This claim extends to any Weyl group W. It has a simple geometric proof using alcoves of the associated affine Weyl group. Cyclic shifts come from symmetries of type A extended Dynkin diagram.

Idea of Proof. The volume of the fundamental alcove times $|W|$ equals the sum of volumes of $n+1$ adjacent permutohedra. For example, the 6 areas of the blue triangle on the following picture is the sum of the areas of three hexagons. □



Corollary 6.5. *Fix* $z_1, \dots, z_{n+1}$, $\lambda_1, \dots, \lambda_{n+1}$ *such that* $\lambda_1 + \cdots + \lambda_{n+1} = 0$. *Symmetrizing the expression*

$$\frac{1}{n!}\,\frac{(\lambda_1 z_1 + (\lambda_1+\lambda_2)z_2 + \cdots (\lambda_1+\cdots+\lambda_{n+1})z_{n+1})^n}{(\lambda_1-\lambda_2)\cdots(\lambda_n-\lambda_{n+1})}$$

with respect to $(n+1)!$ *permutations of* $\lambda_1, \dots, \lambda_{n+1}$ *and* $(n+1)$ *cyclic permutations of* $z_1, \dots z_{n+1}$, *we obtain*

$$(z_1 + \cdots + z_{n+1})^n.$$

It would be interesting to find a direct proof of this claim.

7. Third formula

Let us give a combinatorial interpretation for the mixed Eulerian numbers based on plane binary trees.

Let T be an *increasing plane binary tree* with n nodes labelled $1, \dots, n$. It is well-known that the number of such trees is $n!$. Let v_i be the node of T labelled i, for $i = 1, \dots, n$. In particular, v_1 is the top node of T. Let us define a different labelling of the nodes $v_1, \dots, v_n$ of T by numbers $d_1, \dots, d_n \in [n]$ based on the *depth-first search algorithm.* This labelling is uniquely characterized by the following condition: For any node v_i in T and any v_j in the left (respectively, right) branch of v_i, we have $d_j < d_i$ (respectively, $d_i < d_j$). In particular, for the left-most node v_l in T, we have $d_l = 1$ and, for the right-most node v_r, we have $d_r = n$. Then, for any node v_i, the numbers d_j, for all descendants v_j of v_i (including v_i), form a *consecutive interval* $[l_i, r_i]$ of integers. (In particular, $l_i \le d_i \le r_i$.)

Remark 7.1. For a plane binary tree T, the collection N of intervals $[l_i, r_i]$, $i = 1, \dots, n$, is a maximal nested family for the building set B formed by all consecutive intervals in $[n]$, i.e., the building set for the usual associahedron, see Section 5. The map $T \mapsto N$, is a bijection between plane binary trees and vertices of the associahedron in Loday's realization [L].

Let $\mathbf{i} = (i_1, \dots, i_n) \in [n]^n$ be a sequence of integers. Let us say that an increasing plane binary tree T is $\mathbf{i}$-*compatible* if, $i_k \in [l_k, r_k]$, for $k = 1, \dots, n$. For a node v_k in such a tree, define its weight as

$$wt(v_k) = \begin{cases} \frac{i_k - l_k + 1}{d_k - l_k + 1} & \text{if } i_k \le d_k \\ \frac{r_k - i_k + 1}{r_k - d_k + 1} & \text{if } i_k \ge d_k \end{cases}$$

Define the $\mathbf{i}$-*weight* of an $\mathbf{i}$-compatible tree T as

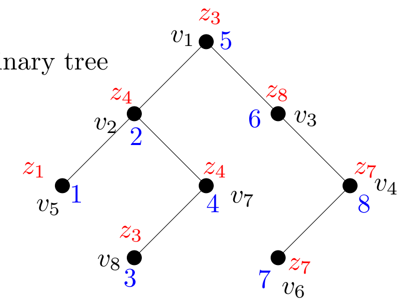
$$wt(\mathbf{i}, T) = \prod_{k=1}^{n} wt(v_k).$$

It is not hard to see that $n!\, wt(\mathbf{i}, T)$ is always a positive integer.

Here is an example of an $\mathbf{i}$-compatible tree T, for $\mathbf{i} = (3, 4, 8, 7, 1, 7, 4, 3)$. The labels d_k of the nodes v_k are $5, 2, 6, 8, 1, 7, 4, 3$. (They are shown on picture in blue color.) The intervals $[l_k, r_k]$ are $[1, 8]$, $[1, 4]$, $[6, 8]$, $[7, 8]$, $[1, 1]$, $[7, 7]$, $[3, 4]$, $[3, 3]$. We also marked each node v_k by the variable z_{i_k}. The $\mathbf{i}$-weight of this tree is $wt(\mathbf{i}, T) = \frac{3}{5}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{2}{2}\cdot\frac{1}{1}$.

an **i**-compartible increasing plane binary tree

$T =$



$\mathbf{i} = (3, 4, 8, 7, 1, 7, 4, 3)$

Theorem 7.2. *The volume of the permutohedron is equal to*

$$V_n = \sum_{\mathbf{i}\in[n]^n} z_{i_1}\cdots z_{i_n} \sum_{T \text{ is } \mathbf{i}\text{-compatible}} wt(\mathbf{i}, T).$$

where the first sum is over n^n sequences $\mathbf{i} = (i_1, \dots, i_n)$ and second sum is over **i**-*compatible increasing plane binary trees with n nodes.*

Let us give a combinatorial interpretation for the mixed Eulerian numbers.

Theorem 7.3. *Let $(i_1, \dots, i_n)$ be any sequence such that $z_{i_1}\cdots z_{i_n} = z_1^{c_1}\cdots z_n^{c_n}$. Then*

$$A_{c_1,\dots,c_n} = \sum_{T \text{ is } \mathbf{i}\text{-compatible}} n!\, wt(\mathbf{i}, T),$$

where the sum is over **i**-*compatible increasing plane binary trees with n nodes.*

Note that all terms $n!\, wt(T)$ in this formula are positive integers. Actually, this theorem gives not just one but $\binom{n}{c_1,\dots,c_n}$ different combinatorial interpretations of the mixed Eulerian numbers $A_{c_1,\dots,c_n}$ for each way to write $z_1^{c_1}\cdots z_n^{c_n}$ as $z_{i_1}\cdots z_{i_n}$.

The proof of this theorem is based on the following recurrence relation for the volume of the permutohedron. Let us write $\operatorname{Vol} P_n$ as a polynomial $U_n(z_1, \dots, z_n)$.

Proposition 7.4. *For any $i = 1, \dots, n$, we have*

$$\frac{\partial}{\partial z_i} U_n(z_1, \dots, z_n) = \sum_{k=1}^{n} \binom{n+1}{k} wt_{i,k,n}\, U_{k-1}(z_1, \dots, z_{k-1})\, U_{n-k-1}(z_{k+1}, \dots, z_n),$$

where

$$wt_{i,k,n} = \frac{1}{n}\begin{cases} i(n-k) & \text{if } 1 \le i \le k; \\ k(n-k) & \text{if } k \le i \le n. \end{cases}$$

Idea of Proof. The derivative $\partial U_n/\partial z_i$ is the rate of change of the volume $\operatorname{Vol} P_n$ of the permutohedron as we move its generating vertex in the direction of the coordinate z_k. (More generally, in the direction of the k-th fundamental weight.) It can be written as a sum of areas of facets of P_n scaled by some factors. Each facet of P_n is a product $P_{k-1} \times P_{n-k-1}$ of two smaller permutohedra. There are exactly $\binom{n+1}{k}$ facets like this. The corresponding factor $wt_{i,k,n}$ tells how fast the facet moves as we shift the generating vertex of P_n.

This formula can be extended to the weight polytope for an arbitrary Weyl group W. In general, the coefficient $wt_{i,k,n}$ equals the inner product of two fundamental weights (ω_i, ω_k). □

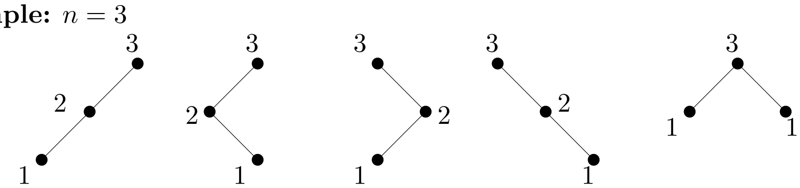
Comparing different formulas for V_n, we obtain many interesting combinatorial identities. For example, we have the following claim.

Corollary 7.5. *We have*

$$(n+1)^{n-1} = \sum_T \frac{n!}{2^n} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right),$$

where is sum is over unlabeled plane binary trees T on n nodes, and $h(v)$ denotes the "hook-length" of a node v in T, i.e., the number of descendants of the node v (including v).

Example: $n = 3$



hook-lengths of binary trees

The identity says that

$$(3+1)^2 = 3 + 3 + 3 + 3 + 4.$$

This identity was combinatorially proved by Seo [S].

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