Persistence and Smoothness of Invariant Manifolds for Flows

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I. 1. Introduction. We study a differentiable manifold embedded in Euclidean space and invariant under a differentiable flow. It is known that even if the manifold is analytic and asymptotically stable, a C^r-close analytic flow may possess no diffeomorphic invariant manifold [4], [7], [11], [12], [16]. Even when a diffeomorphic invariant manifold persists under perturbation, the new invariant manifold is, in general, less smooth than the original manifold. This loss of smoothness may be thought of as a shock phenomenon associated with attracting limit sets (Moser, [16, pages 305–308]).

Our main results are sufficient conditions for persistence of a diffeomorphic invariant manifold under perturbation of the flow, and a careful study of the smoothness of the perturbed manifold. The conditions admit a simple geometric interpretation. For example, the C^1 perturbation theorem for asymptotically stable invariant manifolds requires that neighborhoods of points on the invariant manifold are flattened out in the direction of the manifold as they are carried forward by the flow. Figure 1 shows the flow near a hyperbolic stationary point of an ordinary differential equation. The unstable manifold M is invariant and asymptotically stable, in an appropriate sense. As the flow carries P to Q it takes the neighborhood U to the flattened neighborhood V. Hence the unstable manifold M satisfies our conditions. If all backward limit sets of the unperturbed flow are stationary points and closed orbits, the conditions may be verified in terms of eigenvalues and Floquet multipliers. It never is necessary to compare eigenvalues or Floquet multipliers of one orbit with those of another orbit. This is an improvement over Sacker's important work on invariant manifolds [20], [21]. See Sacker's example 7.4 [21, pages 748–750].

A compact manifold with boundary is called overflowing invariant under a vector field if the backward orbit through any point in the manifold is contained in the manifold and the field points strictly outward on the boundary. Overflowing invariant manifolds occur as local stable and unstable manifolds. They also may be used to describe regions within an invariant manifold where

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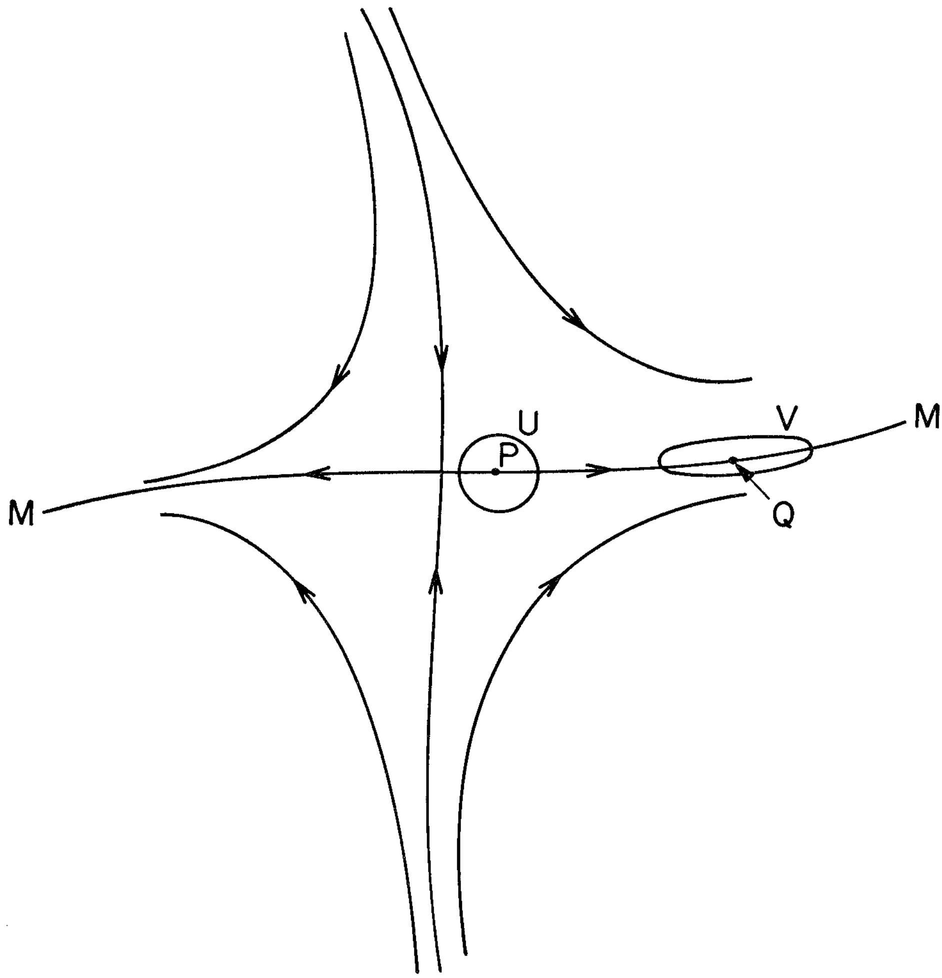

FIGURE 1

loss of smoothness occurs. Our theory is applicable to overflowing invariant manifolds as well as compact invariant manifolds. The overflowing case generally requires more stringent hypotheses than the compact case, so the two cases are treated separately.

Our sufficient conditions are formulated in terms of generalized Lyapunov type numbers. These are real valued functions on an invariant manifold, measuring asymptotic properties of the flow. The type numbers are defined and studied in §II. Strong uniform conditions follow from hypotheses about the asymptotic properties of the flow. These conditions are stated as the Uniformity Lemma, which may be of some independent interest.

Sections III and IV are concerned with the perturbation problem for invariant manifolds. In §III the perturbation theorem for overflowing invariant manifolds is proved in full detail, except for some results of Whitney [22] on smoothing of bundles. The invariant manifold is required to be asymptotically stable. Of course asymptotic stability must be formulated in a way which makes sense for overflowing invariant manifolds. In §IV we study compact manifolds, replacing asymptotic stability with a hyperbolic splitting of the normal bundle. The proof of the perturbation theorem is similar to the proof for overflowing invariant manifolds. Using a trick, we are able to reduce part of the proof to results known from the previous case.

In §V we study stable and unstable manifolds. It makes sense to talk of the unstable manifold of an overflowing invariant manifold, so we express the main theorem in these terms. Then we tie this in with the hyperbolic splittings of §IV. Section VI deals with invariant families of planes transversal to an invariant manifold. These may be useful in studying the flow near the manifold.

A compact manifold with boundary $\bar{M} = M \cup \partial M$ is overflowing invariant under a diffeomorphism F if $\bar{M} \subset F(M)$. Throughout this paper we can replace the continuous variable t by a discrete variable without substantially altering any construction. With this modification all our results hold for a manifold invariant or overflowing invariant under a diffeomorphism.

Our entire study may be extended to invariant submanifolds of an arbitrary differentiable manifold. Only the approximation of sub-bundles of a tangent bundle requires modification.

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I. 2. An example.

Consider a field X given as follows in cylindrical coordinates on R^3,

$$\dot{r} = a(r),$$
$$\dot{\theta} = b(r),$$
$$\dot{z} = c(r)z.$$

Let

$$\begin{aligned} a(r) &= a_1 r \quad \text{for} \quad r < 1/3, \quad \text{where} \quad a_1 < 0 \\ &= a_2(r-1) \quad \text{for} \quad r > 2/3, \quad \text{where} \quad a_2 > 0 \end{aligned}$$

and

$$a(r) = 0 \quad \text{only at} \quad r = 0 \quad \text{and} \quad r = 1.$$

Let

$$\begin{aligned} b(r) &= 2\pi \quad \text{near} \quad r = 1 \\ &= 0 \quad \text{for} \quad r < 1/2 \quad \text{or} \quad r > 2. \end{aligned}$$

This field has a stationary point at $r = 0$ and a closed orbit at $r = 1$. The plane $z = 0$ is invariant. Figure 2 shows the flow in the plane $z = 0$.

The closed disc $\bar{D} = \{(r, \theta, z): 0 \leqq r \leqq 2, z = 0\}$ is overflowing invariant under X. Let k be any integer greater than or equal to 1. It will follow from Theorem 1 that C^k fields C^1 close to X possess overflowing invariant manifolds C^k diffeomorphic to $\bar{D}$ if $c(0) < 0$, $c(1) < 0$ and $c(0) < ka_1$. The first two inequalities are asymptotic stability conditions. $c(1) < 0$ guarantees that

$$\{(r, \theta, z): r = 1, z = 0\}$$

is a hyperbolic closed orbit. It is well known that the unstable manifold of a hyperbolic closed orbit remains smooth under perturbation. Hence D perturbs smoothly except possibly near $r = 0$. The condition $c(0) < ka_1$ means that at $r = 0$ the attraction to D in the normal direction is more than k times stronger

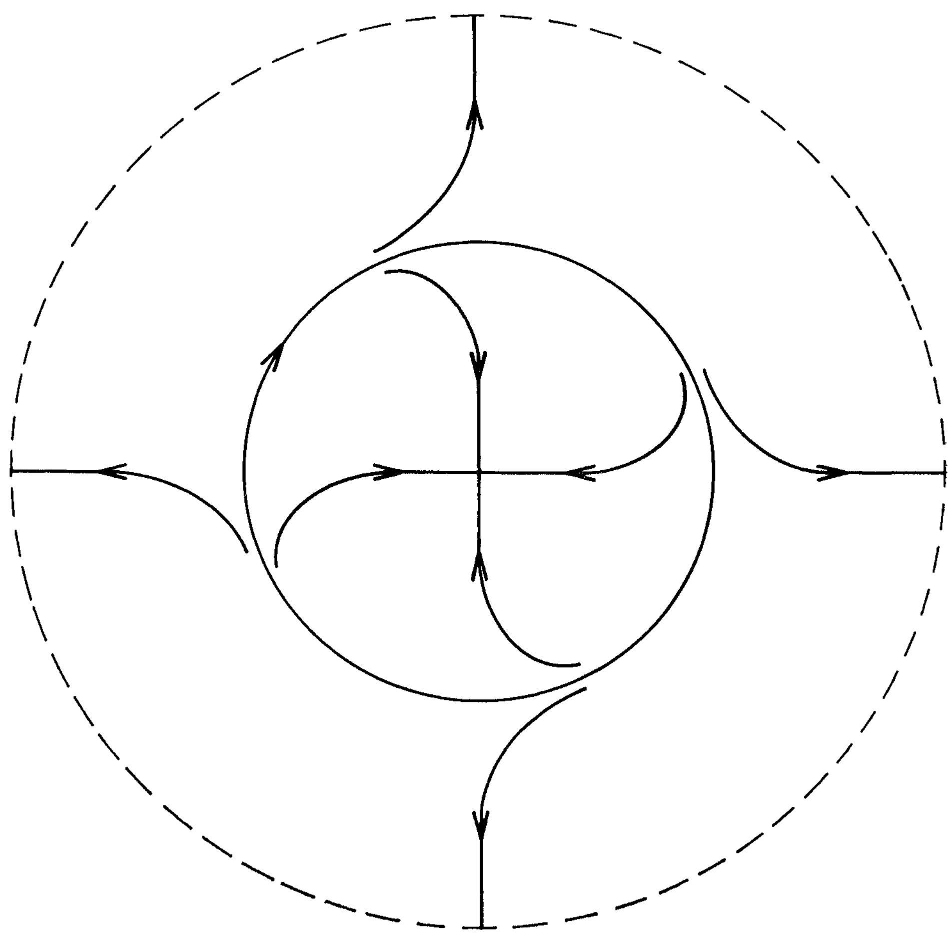

FIGURE 2

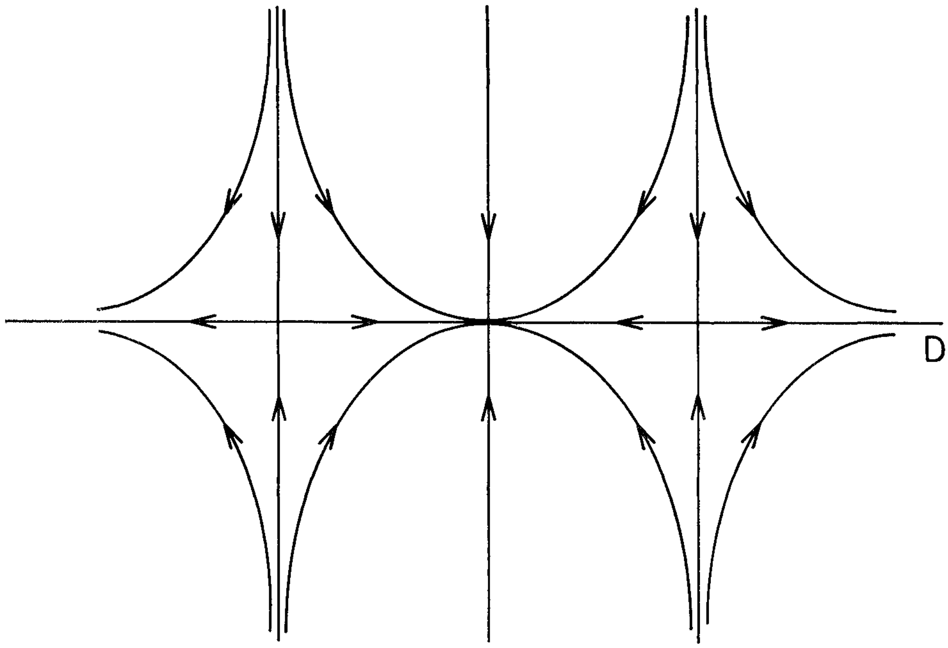

FIGURE 3

than the attraction within D to $r = 0$. As long as $k \geqq 1$ the flow in (r, z) coordinates appears as in Figure 3. All but two orbits approach $r = 0$ tangent to D, and this is why the unstable manifold of the closed orbit remains smooth at $r = 0$. Note that D is asymptotically stable even if $c(r) > 0$ for some values of r other than $r = 0$ and $r = 1$.

I. 3. Background. The geometric idea behind our proof of the persistence of invariant manifolds dates back at least to 1901. In that year Hadamard published a geometric construction for the unstable manifold of a hyperbolic fixed point of a diffeomorphism of the plane [3]. Hadamard constructed the unstable manifold as the limit of a sequence of curves. Some initial curve is chosen, and its successive images under the diffeomorphism form the sequence. These curves converge to the unstable manifold in an appropriate sense.

In 1950 Levinson used a similar approach to construct certain two dimensional tori occurring in the study of weakly coupled oscillators [13]. In this problem the difficulties associated with attracting limit sets are hidden because the unperturbed flow is parallel flow on the torus. Diliberto and his students developed a theory of periodic surfaces extending Levinson's work. See Hufford [6], Kyner [11], and Marcus [14]. See also Diliberto [2].

McCarthy was the first to allow attracting limit sets within the invariant manifold. He showed that if orbits within the invariant manifold come together slower than points near the invariant manifold come toward it, then an invariant manifold persists under perturbation [15]. He also showed why cusps may develop if this condition is violated. We follow Hale's exposition [4, pages 239, 251].

Consider an asymptotically stable invariant circle as in Figure 4. Within the circle there are two stationary points P and Q. P is an attracting point. The rate of approach in the direction tangent to the circle is greater than the

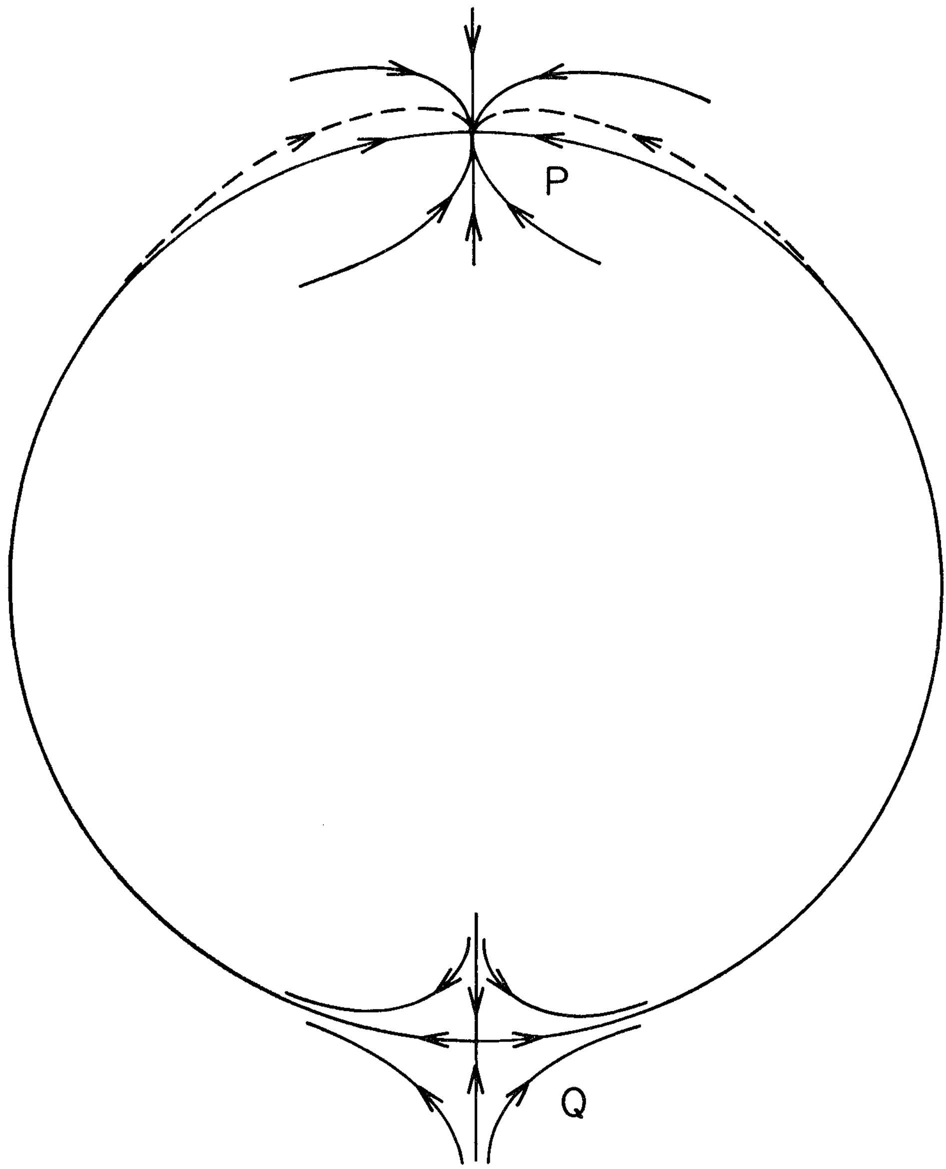

FIGURE 4

rate of approach in the direction normal to the circle. Q is a hyperbolic point. The invariant circle, except for P, is the unstable manifold of Q and hence perturbs smoothly. At P a cusp may develop under perturbation, as indicated by the dotted curve.

Figure 5 shows a similar situation, with the normal rate of approach greater

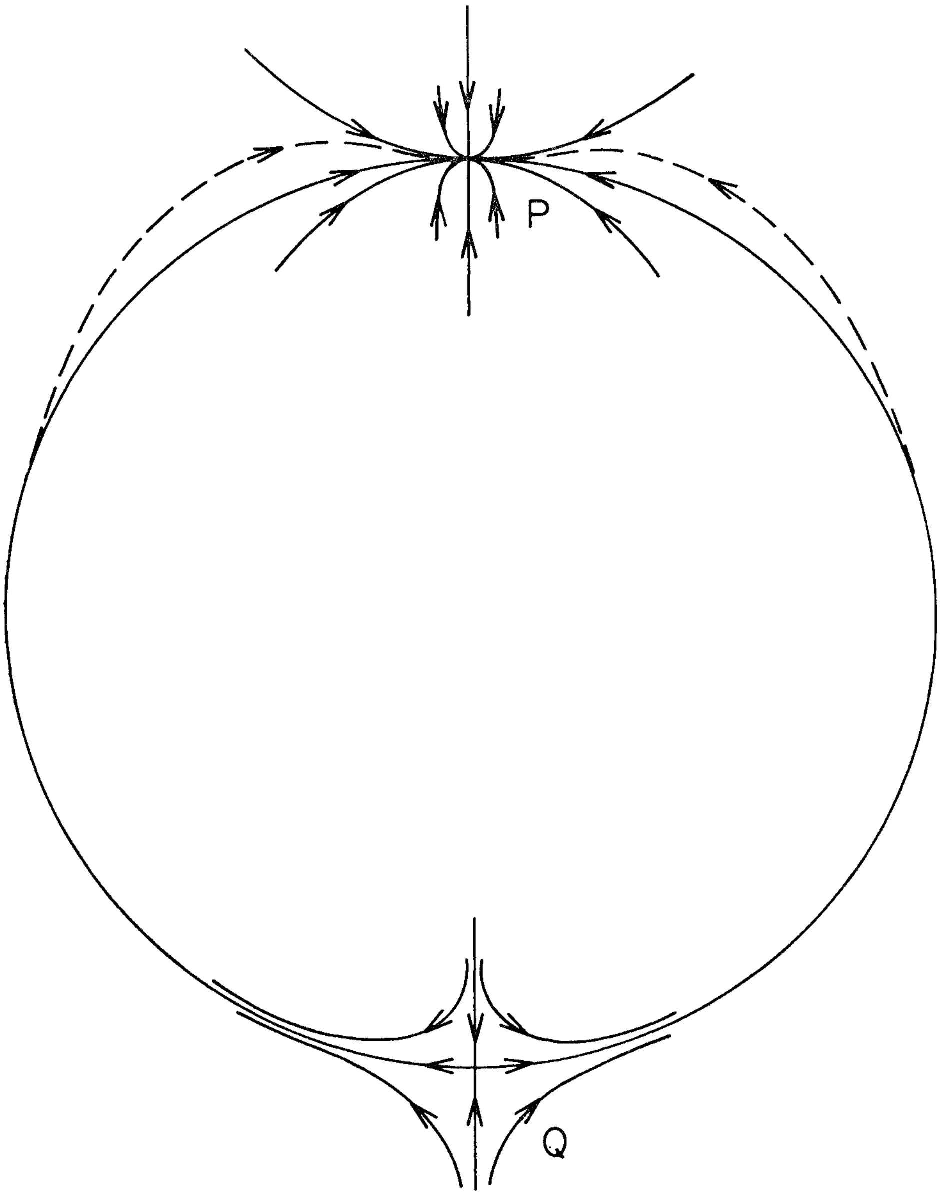

FIGURE 5

than the tangential rate of approach. In this case the flow smooths out any deviations near P, so a differentiable invariant circle persists. In the third case, when the normal and tangential rates of approach are equal, a spiral may develop at P (Sacker [21, pages 747–748]).

Jarnik and Kurzweil [7] have constructed a more pathological example. An asymptotically stable invariant plane changes under perturbation to an invariant set which is not even a topological 2-manifold.

McCarthy's work has been extended by Kyner [12], Kurzweil [10], and Nyemark [18] and [19]. They all require hypotheses uniformly over the invariant manifold, rather than on limit sets. No attempt is made to control higher derivatives. Kupka announced some results dealing with higher derivatives [9].

The higher derivatives of invariant manifolds may be studied using partial differential equations. Manifolds near the unperturbed invariant manifold correspond to normal vector fields on the original manifold. Invariance under the perturbed field is characterized by a first order partial differential equation. Moser [16] studied solutions of such partial differential equations using quadratic convergence techniques. Sacker [20] found better smoothness results using elliptic regularization—adding a small second order operator to form an elliptic second order equation.

The studies mentioned so far are not fully satisfactory because the hypotheses involve a Riemannian metric. Persistence of an invariant manifold does not depend on a Riemannian metric, so the hypotheses of a reasonable perturbation theorem should be invariantly defined. Recently Sacker introduced generalized Lyapunov type members to describe the perturbation theorem [21]. The type numbers are asymptotic rates, and do not depend on the choice of a metric. Sacker uses the flow to construct a preferred metric in which it is easy to estimate finitely many derivatives. This is similar to Moser's introduction of a preferred metric in the study of Anosov systems [17].

Sacker's type numbers are not quite appropriate. To verify his hypotheses even in simple cases it may be necessary to compare eigenvalues or Floquet multipliers of different orbits. Also, the type numbers cannot be defined for overflowing invariant manifolds.

Hirsch, Pugh, and Shub [5] have announced extensions of Sacker's results to certain subsets of a compact invariant manifold. This includes a perturbation theorem for compact invariant manifolds. As in Sacker's work, the hypotheses of their perturbation theorem may require comparison of eigenvalues or Floquet multipliers of different orbits.

I. 4. Notation. We gather here some of the notations used repeatedly below. R^n denotes the n-dimensional Euclidean space. TR^n is the tangent space of R^n. M is a differentiable manifold embedded in R^n; TM is its tangent space. We use a single vertical bar to denote restriction. Thus $TM \subset TR^n|M$. The vertical bar also is used to restrict the domain of definition of functions. All bundles which appear are made up of vectors in TR^n. A manifold $M \subset R^n$ is called properly embedded if each point in M has a neighborhood U and coordinates (x, y) for U, such that $M \cap U = \{y = 0\}$.

The symbol $\dot{}$ denotes differentiation with respect to time; D denotes differentiation with respect to space variables. X is a vector field on R^n. The flow $F^s(x)$ is $x(s)$, where $\dot{x}(t) = X(x(t))$ and $x(0) = x$. $F^s(x)$ may not be defined for all $s \in R$ and all $x \in R^n$.

A manifold with boundary $\bar{M} = M \cup \partial M$ is overflowing invariant under a vector field X on R^n if $X|M$ is tangent to M and X points strictly outward on ∂M. This means that X never is tangent to ∂M.

$L(E, F)$ denotes the space of linear maps from a vector space E to a vector space F. $L^p(E, F) = L(E, L^{p-1}(E, F))$ is the space of p-linear maps from E to F. $C^0(S, T)$ denotes the space of continuous maps between topological spaces S and T.

II. 1. Generalized Lyapunov type numbers, definitions and properties. Our main theorems are phrased in terms of the first order asymptotic behavior of the flow. This means that we are concerned with the action of the flow on tangent and normal vectors of the invariant manifold. The crucial observation in the proof of each theorem is that asymptotic hypotheses about this action lead to uniform conclusions. This is the content of the Uniformity Lemma of the following section.

Let X be a C^1 vector field on R^n with flow F^t. Let $\bar{M} = M \cup \partial M$ be a compact, connected, C^1 manifold with boundary, properly embedded in R^n. Suppose M is invariant under X in the overflowing sense. That is, backward orbits starting in M remain in M, and X points strictly outward on ∂M. From the compactness of $\bar{M}$ it follows that $F^t|\bar{M}$ exists for all t less than zero. By a change of time scale we may guarantee that $F^t|\bar{M}$ exists for all $t \leqq 3$. Let $M_1 = F^1(M)$ and $M_2 = F^2(M)$. $\bar{M}_1$ and $\bar{M}_2$ are properly embedded manifolds, invariant under X in the overflowing sense.

Give R^n its usual metric. This splits $TR^n|M_2$ into $TM_2 \oplus N$, where N is the bundle of vectors normal to TM_2. Let π: $TR^n|M_2 \to N$ be the orthogonal projection. Let $A^t(m) = D(F^{-t}|M_2)(m)$ and $B^t(m) = \pi \cdot DF^t(F^{-t}(m))$. These operators act on vectors of TM_2 and N, respectively, although we may occasionally suppose B^t acts on all of $TR^n|M_2$. Extending B^t to $TR^n|M_2$ does not change its norm because TM_2 lies in the kernel of B^t.

Let m be any point in M_2. Consider any non-zero vectors $v_0 \in T_mM_2$ and $w_0 \in N_m$. Let $v_{-t} = DF^{-t}(m) \cdot v_0$ and $w_{-t} = \pi DF^{-t}(m) \cdot w_0$. We have $v_{-t} = A^t(m)v_0$ and $B^t(m)w_0 = \pi DF^t(F^{-t}(m))\pi DF^{-t}(m)w_0 = \pi w_0 = w_0$. See Figure 6.

For the C^r perturbation theorem we will require that $||w_0||/||w_{-t}|| \to 0$ and $\{||w_0||/||v_0||^r\}/\{||w_{-t}||/||v_{-t}||^r\} \to 0$ as $t \to \infty$. No uniformity is required in the convergence to the limits. The first condition may be interpreted as asymptotic stability along the orbit through m. For $r = 1$ the second condition means that neighborhoods of points on the backward orbit through m are

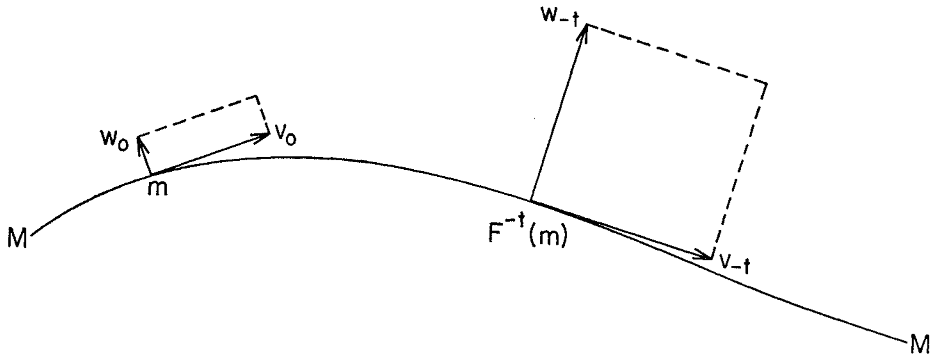

FIGURE 6

flattened as they are carried forward to m by the action of the flow. See the rectangles in Figure 6. For $r > 1$ the second condition requires rapid flattening of neighborhoods.

It is convenient to phrase the limit conditions above quantitatively, in terms of rates of convergence. Define

$$\nu(m) = \inf \{a: (||w_0||/||w_{-t}||)/a^t \to 0 \text{ as } t \to \infty \text{ for all } w_0 \in N_m\}.$$

If $\nu(m) < 1$ for all $m \in M$ define

$$\sigma(m) = \inf \{s: (||w_0||^s/||v_0||)/(||w_{-t}||^s/||v_{-t}||) \to 0 \quad \text{as } t \to \infty$$
$$\text{for all } v_0 \in T_mM \text{ and } w_0 \in N_m\}.$$

ν and σ are called generalized Lyapunov type numbers.

For many purposes it is convenient to phrase the definitions of ν and σ in more analytical terms. It is not hard to see that the following are equivalent definitions.

$$\begin{aligned}\nu(m) &= \inf \{a : ||B^t(m)||/a^t \to 0 \quad \text{as} \quad t \to \infty\} \\ &= \overline{\lim_{t\to\infty}} \, ||B^t(m)||^{1/t} \\ \sigma(m) &= \inf \{s : ||A^t(m)|| \; ||B^t(m)||^s \to 0 \quad \text{as} \quad t \to \infty\} \\ &= \overline{\lim_{t\to\infty}} \frac{\log ||A^t(m)||}{-\log ||B^t(m)||}.\end{aligned}$$

Note that $B^t(F^{-\tau}(m))\cdot B^\tau(m) = B^{t+\tau}(m)$. $B^\tau(m)$ is a bounded invertible operator. From this it follows that $\nu(F^{-t}(m)) = \nu(m)$. That is, ν is constant on orbits. Similarly, σ is constant on orbits. Thus both $\nu(m)$ and $\sigma(m)$ depend only on the flow near the backward limit set of the orbit through m.

DF^t satisfies the variational equation

$$\frac{d}{dt}(DF^t(m)) = DX(F^t(m))\cdot DF^t(m).$$

But DX is bounded on the compact set $\bar{M}$ so DF^t has exponential growth. From this it follows that ν and σ are bounded on $\bar{M}$. In general ν and σ are neither continuous nor semicontinuous.

Proposition 1. *$\nu(m)$ does not depend on the choice of a metric on $TR^n|M$. If $\nu(m) < 1$, $\sigma(m)$ also is independent of the metric.*

Proof. We follow Sacker [21, pages 709–710]. Suppose $(\, , \,)'$ is another metric on TR^n, with norm $|| \; ||'$, orthogonal projection π' and type numbers ν' and σ'. Any two norms are uniformly equivalent over the compact set $\bar{M}_2$. This means that there is a constant c such that $(1/c) \, ||v|| \leqq ||v||' \leqq c \, ||v||$ for any $v \, \varepsilon \, TR^m \mid \bar{M}_2$. Thus for all $m \, \varepsilon \, \bar{M}_2$,

$$\frac{1}{c^2} \, ||A^t(m)|| \leqq ||A^t(m)||' \leqq c^2 \, ||A^t(m)||.$$

We claim that $(1/c) \, ||\pi \cdot v|| \leqq ||\pi \cdot v||' \leqq c \, ||\pi \cdot v||$ for all $v \, \varepsilon \, TR^n|M_2$. For any $v \, \varepsilon \, TR^n \mid M_2$ there is a vector $w \, \varepsilon \, TM_2$ such that $\pi v = v - w$. But

$$||\pi' \, v||' = \inf_{x \varepsilon TM_2} ||v - x||' \leqq ||v - w||'$$
$$\leqq c \, ||v - w|| = c \, ||\pi \cdot v||.$$

The opposite inequality is proved in the same way, giving

$$\frac{1}{c^2} \, ||B^t(m)|| \leqq ||B'^t(m)||' \leqq c^2 \, ||B^t(m)||,$$

where $B'^t(m) = \pi' DF^t(F^{-t}(m))$. Thus $||B^t(m)||/a^t \to 0$ if and only if

$$||B'^t(m)||'/a^t \to 0, \quad \text{so} \quad \nu(m) = \nu'(m).$$

Similarly $||A^t(m)|| \, ||B^t(m)||^s \to 0$ if and only if $||A^t(m)||' \, ||B'^t(m)||'^s \to 0$, so

$$\sigma(m) = \sigma'(m).$$

Proposition 1 shows that the type numbers depend only on X and the differentiable structure of M and R^n. This is important if we want to generalize our theory to invariant manifolds embedded in arbitrary differentiable manifolds.

II. 2. Uniformity Lemma. The Uniformity Lemma is the key to our study of invariant manifolds. According to this lemma, hypotheses about the asymptotic behavior of a flow lead to uniform conclusions about the flow. Sacker used similar results in his construction of a preferred metric. This was hidden in a rather technical construction [21, pages 731–737], and it seems to have received insufficient attention. Our Uniformity Lemma is stronger than Sacker's results, mostly because we do not require the type numbers to be semi-continuous. The proof, however, is essentially the same.

Uniformity Lemma

1) *Suppose $||B^t(m)||/a^t \to 0$ as $t \to \infty$ for all $m \, \varepsilon \, \bar{M}_1$. Then there are constants $\hat{a} < a$ and c such that $||B^t(m)|| < c \, \hat{a}^t$ for all $m \, \varepsilon \, \bar{M}_1$ and $t \geqq 0$.*

2) *Under the hypotheses of* 1), *suppose also that* $a \leqq 1$ *and* $||A^t(m)||\ ||B^t(m)||^s \to 0$ *as* $t \to \infty$ *for all* $m \in \bar{M}_1$. *Then there are constants* $\hat{s} < s$ *and* C *such that* $||A^t(m)||\ ||B^t(m)||^{\hat{s}} < C$ *for all* $m \in \bar{M}_1$ *and* $t \geqq 0$.

3) *If* $\nu(m) < a \leqq 1$ *and* $\sigma(m) < s$ *for all* $m \in M$, *then* $||B^t(m)|| \to 0$ *and* $||A^t(m)||\ ||B^t(m)||^s \to 0$ *as* $t \to \infty$ *uniformly for* $m \in \bar{M}_1$.

4) ν *and* σ *attain their suprema on* M.

Proof. 3) is a direct consequence of 1) and 2). Suppose 1) holds and ν does not attain its supremum on M. Then let $a = \sup \nu$. By 1) there is an $\hat{a} < a$ such that $||B^t(m)|| < c\hat{a}^t$. Then for any a', $\hat{a} < a' < a$, $||B^t(m)||/a'^t \to 0$, so $\nu(m) \leqq a'$, a contradiction. In a similar manner we use 2) to prove that σ attains its maxima.

Assume the hypotheses of 1) and 2). (It will be clear from the proof that 1) does not depend on the hypotheses of 2).) Let a and s be given as in 1) and 2). For each $m \in \bar{M}_1$ there is a number $T(m)$ such that

$$||B^{T(m)}(m)|| < a^{T(m)},$$
$$||A^{T(m)}(m)||\ ||B^{T(m)}(m)||^s < 1.$$

For each $m \in \bar{M}_1$ there is a neighborhood $U(m)$ of m in M_2 such that for all $m' \in \overline{U(m)}$,

$$||B^{T(m)}(m')|| < a^{T(m)},$$
$$||A^{T(m)}(m')||\ ||B^{T(m)}(m')||^s < 1.$$

$\bar{M}_1$ is compact, so we may choose finitely many points $m_1, \cdots, m_N$ such that $\bar{M}_1 \subset U(m_1) \cup \cdots \cup U(m_N)$. Choose $\hat{a} < a$ and $\hat{s} < s$ such that for $m' \in \overline{U(m_i)}$

$$||B^{T(m_i)}(m')|| < \hat{a}^{T(m_i)},$$
$$||A^{T(m_i)}(m')||\ ||B^{T(m_i)}(m')||^{\hat{s}} < 1.$$

Let $m \in \bar{M}_1$ be given. We choose a (non-unique) sequence of integers $i(1)$, $i(2), \cdots$, as follows. Choose $i(1)$ such that $m \in U(m_{i(1)})$. If $i(1), \cdots, i(j)$ have been chosen let $\tau(j) = T(m_{i(1)}) + \cdots + T(m_{i(j)})$. Choose $i(j+1)$ such that $F^{-\tau(j)}(m) \in U(m_{i(j+1)})$.

Let $t > 0$ be given. It is possible to write $t = \tau(j) + r$ for some j, where $0 \leqq r < \max T(m_i)$.

$$\begin{aligned} ||B^t(m)|| &= ||B^r(F^{-\tau(j)}(m)) \cdot B^{T(m_{i(j)})}(F^{-\tau(j-1)}(m)) \cdots B^{T(m_{i(1)})}(m)|| \\ &\leqq ||B^r||\ \hat{a}^{T(m_{i(j)})} \cdots \hat{a}^{T(m_{i(1)})} \\ &\leqq c\hat{a}^t, \end{aligned}$$

where $c = \sup ||B^r(m)||/\hat{a}^r$ and the supremum is taken over all $m \in \bar{M}_1$ and all r, $0 \leqq r \leqq \max T(m_i)$.

In a similar fashion we estimate $||A^t(m)||\ ||B^t(m)||^{\hat{s}}$. In place of the powers of $\hat{a}$ we get 1; the constant is $C = \sup ||A^r(m)||\ ||B^r(m)||^{\hat{s}}$, where the supremum is taken over all $m \in \bar{M}_1$ and $0 \leqq r \leqq \max T(m_i)$.

III. 1. Theorem 1: Construction of the invariant manifold.

Theorem 1. *Let X be a C^r vector field on R^n, $r \geqq 1$. Let $\bar{M} = M \cup \partial M$ be a C^r compact, connected manifold with boundary, properly embedded in R^n and overflowing invariant under X. Suppose $\nu(m) < 1$ and $\sigma(m) < 1/r$ for all $m \in M$. Then for any C^r vector field Y in some C^1 neighborhood of X there is a manifold $\bar{M}_Y$ overflowing invariant under Y and C^r diffeomorphic to $\bar{M}$.*

One might hope to represent M_Y as a section of the normal bundle N introduced in II.1. This is not fine enough, as N is of class C^{r-1}. To control the r-th derivative we jiggle N slightly, retaining transversality to TM. For the proofs of the next two propositions see Whitney [22, Lemma 23]. $k = n - \dim M$ is the fiber dimension of N.

Proposition 2. *There is a C^r k-dimensional bundle $N' \subset TR^n|M_1$ transversal to TM_1.*

Elements of N' are pairs (m, v), where $m \in M_1$ is a point in R^n and v is a tangent vector to R^n at m, transversal to TM_1. Identifying TR^n to R^n in the usual fashion, we define $\varphi: N' \to R^n$, $\varphi(m, v) = m + v$.

Proposition 3. *Let $K \subset M_1$ be any compact subset. φ is a C^r diffeomorphism from a neighborhood of the zero section of N' to a neighborhood of K in R^n.*

We use N' and φ to define local coordinates in R^n near $\bar{M}$. Each point in $\bar{M}$ has a neighborhood in M_1 on which N' possesses a C^r orthonormal basis. Cover $\bar{M}$ by finitely many such neighborhoods, and call them U_i^6, $i = 1, \cdots, s$. Without loss of generality we may assume there are C^r diffeomorphisms $\sigma_i : U_i^6 \to \mathfrak{D}^6$, where $\mathfrak{D}^6$ is the disc about the origin of radius 6 with the same dimension as M. Let $\mathfrak{D}^j$, $j = 1, \cdots, 5$ denote the concentric discs of radius j, and $U_i^j = \sigma_i^{-1}(\mathfrak{D}^j)$. We may assume further that $\bar{M} \subset \bigcup_{i=1}^s U_i^j$ for each $j = 1, \cdots, 6$.

N' inherits the norm of TR^n. Let $N'_\epsilon = \{(m, v) \in N' : ||v|| < \epsilon\}$. It follows from Proposition 3 that there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, φ maps $N'_\epsilon \mid \bigcup_{i=1}^s U_i^5$ diffeomorphically onto a neighborhood of $\bigcup_{i=1}^s \overline{U_i^4}$ in R^n. We generally will not distinguish between $N'_\epsilon \mid \bigcup_{i=1}^s U_i^5$ and this neighborhood.

For each i choose a C^r orthonormal basis for $N' \mid U_i^6$. This is possible by the choice of U_i^6. Define $\tau_i : N' \mid U_i^6 \to R^k$ by $\tau_i(m, v) =$ vector of coordinates of v with respect to the chosen basis at m. Because the basis is orthonormal we have $||\tau_i(m, v)|| = ||v||$. Now define $\sigma_i \times \tau_i : N' \mid U_i^6 \to R^{n-k} \times R^k$ by $\sigma_i \times \tau_i(m, v) = (\sigma_i(m), \tau_i(m, v))$. Clearly $\sigma_i \times \tau_i$ is a C^r diffeomorphism. If ϵ is small enough $N'_\epsilon \mid U_i^5$ is a neighborhood of $\overline{U_i^4}$ in R^n. Thus $\sigma_i \times \tau_i$ specifies a local coordinate system in R^n near part of M. Points in M are characterized by vanishing of the second component. This completes our construction of a local coordinate system near M.

In our construction of the manifold M_Y we have to measure flows and their derivatives in local coordinates. The derivatives along fibers of N' don't give much trouble because we use an orthonormal basis for N' in defining each τ_i.

Derivatives along M bring in factors $D\sigma_i$ and $D\sigma_i^{-1}$. As long as we consider compact sets these factors remain bounded. There is a constant c such that for all i, and all $m \in \overline{U_i^5}$, $||D\sigma_i(m)|| < c$ and $||D\sigma_i^{-1}(\sigma_i(m))|| < c$.

Let π' denote the projection on N' complementary to TM_1. For any $v \in TR^n \mid M_1$, $\pi' v \in N'$ and $v - \pi' v \in TM_1$. Let F^t denote the flow of X. Recall from II.1 that $A^t(m) = D(F^{-t} \mid M_2)(m)$ and $B^t(m) = \pi DF^t(F^{-t}(m))$. Let $B'^t(m) = \pi' DF^t(F^{-t}(m))$. As in the proof that ν does not depend on the metric of TR^n we see that $||B'^t(m)||/||B^t(m)||$ is uniformly bounded for all $t \geqq 0$ and all $m \in \bigcup_{i=1}^s \overline{U_i^5}$. Hence by the Uniformity Lemma, $||B'^t(m)|| \to 0$ and $||A^t(m)||^r\, ||B'^t(m)|| \to 0$, uniformly for $m \in \bigcup_{i=1}^s \overline{U_i^5}$ as $t \to \infty$. For large enough T

$$||B'^T(m)|| < 1/4,$$

$$c^{2r}\, ||A^T(m)||^r\, ||B'^T(m)|| < 1/4$$

for all $m \in \bigcup_{i=1}^s \overline{U_i^5}$. Choose any $T > 1$ such that these inequalities hold, and keep T fixed for the remainder of this construction. Note that if $0 \leqq k \leqq r$,

$$c^{2k}\, ||A^T(m)||^k\, ||B'^T(m)|| < 1/4.$$

The condition $||B'^T|| < 1/4$ means that to first order, F^T decreases lengths by at least a factor 4. Hence if ϵ is chosen small enough,

$$F^T : N'_\epsilon \mid F^{-T}\overline{U_i^4} \to N'_{\epsilon/3} \mid U_i^5 .$$

Denote the flow of any C^1 field Y by F_Y^t. For all Y in some C^1 neighborhood of X,

$$F_Y^T(N'_\epsilon \mid F^{-T}\overline{U_i^4}) \subset N'_\epsilon \mid U_i^5 .$$

Define

$$f_{ij}^0(x, y) = \sigma_j F^T(\sigma_i \times \tau_i)^{-1}(x, y)$$

$$g_{ij}^0(x, y) = \tau_j F^T(\sigma_i \times \tau_i)^{-1}(x, y)$$

$$f_{ij}(x, y) = \sigma_j F_Y^T(\sigma_i \times \tau_i)^{-1}(x, y)$$

$$g_{ij}(x, y) = \tau_j F_Y^T(\sigma_i \times \tau_i)^{-1}(x, y).$$

These are the local expressions for F^T and F_Y^T. They are defined for

$$(x, y) \in (\sigma_i \times \tau_i)N'_\epsilon \mid \overline{U_i^4} \cap F^{-T}\overline{U_j^4} .$$

$g_{ij}^0(x, 0) = 0$ because M is invariant under X. Denote partial differentiation with respect to x and y by D_1 and D_2. $\{D_1 f_{ij}^0(x, 0)\}^{-1}$ is the local representative of A^T at $F^T(\sigma_i^{-1}(x))$. $D_2 g_{ij}^0(x, 0)$ is the local representative of B^T at $F^T\sigma_i^{-1}(x)$. Hence for $(x, y) \in \{\sigma_i \times \tau_i\}N'_\epsilon \mid \overline{U_i^4} \cap F^{-T}\overline{U_j^4}$ and $0 \leqq k \leqq r$,

$$||\{D_1 f_{ij}^0(x, 0)\}^{-1}||^k\, ||D_2 g_{ij}^0(x, 0)|| < 1/4,$$

$$||g_{ij}^0(x, y)|| < \epsilon/3.$$

Let $\eta > 0$ be given. If ϵ is small enough and Y is C^1 close enough to X, we have, for all $(x, y) \in (\sigma_i \times \tau_i)N'_\epsilon \mid \overline{U_i^4} \cap F^{-T}\overline{U_j^4}$ and $0 \leqq k \leqq r$,

$$||\{D_1 f_{ij}(x, y)\}^{-1}||^k \, ||D_2 g_{ij}(x, y)|| < 1/2,$$

$$||g_{ij}(x, y)|| < \epsilon,$$

$$||D_1 g_{ij}(x, y)|| < \eta.$$

The last inequality comes from $g_{ij}^0(x, 0) = 0$. The norms of all first partial derivatives of f_{ij}^0, g_{ij}^0, f_{ij}, g_{ij}, as well as $(D_1 f_{ij})^{-1}$, are bounded on $(\sigma_i \times \tau_i)N'_\epsilon \mid \overline{U_i^4} \cap F^{-T}\overline{U_j^4}$, for all i, j, say by Q.

Let S denote the space of sections of $N'_\epsilon \mid \bigcup_{i=1}^s U_i^3$. Elements of S are maps $u: \bigcup_{i=1}^s U_i^3 \to N'_\epsilon$ taking each point m into the fiber over m. Corresponding to any $u \in S$ there are coordinate functions $u_i : \mathfrak{D}^3 \to R^{n-k}$ defined by $u_i \sigma_i = \tau_i u$. Define

$$\text{Lip} \quad u = \max_i \sup_{x, x' \in \mathfrak{D}^3} \frac{||u_i(x) - u_i(x')||}{||x - x'||},$$

if this exists. Let $S_\delta = \{u \in S\colon \text{Lip } u \leqq \delta\}$.

The image in R^n of any $u \in S_\delta$ is a Lipschitz continuous manifold. In coordinates near U_i^3 the image is the graph of u_i, so we call it graph u. Suppose Y is a vector field near X with flow F_Y^t. A necessary condition for graph u to be overflowing invariant under Y is, for all $t > 0$, graph $u \subset F_Y^t$ (graph u). We define below a graph transform $G\colon S_\delta \to S_\delta$ associated with Y. G has the property that $Gu = u$ if and only if graph $u \subset F_Y^T$ (graph u). We show that G has a unique fixed point u, and that for all $t > 0$, graph $u \subset F_Y^t$ (graph u). Later we show that u is differentiable. Then it is clear that the graph of $u \mid \bar{M}$ is overflowing invariant under Y.

We need the following version of the Implicit Function Theorem, which we use without proof. It may be proved by using the contraction mapping theorem, or by noting that this is one step in the usual proof of the Implicit Function Theorem. (See Dieudonné [1, pages 259–266].)

Implicit Function Theorem. *Let $i(x) = x$ denote the inclusion map from $\mathfrak{D}^4$ into R^{n-k}. There is a neighborhood W of i in the Lipschitz topology for maps from $\mathfrak{D}^4$ into R^{n-k}, such that for all $\varphi \in W$,*

1) *φ is $1 - 1$*
2) $\overline{\mathfrak{D}^3} \subset \varphi(\mathfrak{D}^4) \subset \overline{\varphi(\mathfrak{D}^4)} \subset \mathfrak{D}^5$.

Corollary. Proposition 4. *Let $p\colon N'_\epsilon \to M_1$ denote the fiber projection. Define $\varphi(m) = pF_Y^T uF^{-T}(m)$. There is a C^1 neighborhood $\mathfrak{X}$ of X and a $\delta > 0$ such that for $Y \in \mathfrak{X}$ and $u \in S_\delta$,*

1) *$\varphi(m)$ is defined for all $m \in \bigcup U_i^4$,*
2) $\bigcup \overline{U_i^3} \subset \varphi(\bigcup U_i^4) \subset \overline{\varphi(\bigcup U_i^4)} \subset \bigcup U_i^5$,
3) *Each point in $\bigcup U_i^3$ is the φ image of only one point in $\bigcup U_i^4$.*

Proof. $T > 1$, so if $m \in \bigcup U_i^4 \subset M_1$, $F^{-T}(m) \in M \subset \bigcup U_i^3$, and $F^{-T}(m)$

lies in the domain of definition of u. $||u(F^{-T}(m)|| < \epsilon$ so $||F_Y^T u(F^{-T}(m))|| < \epsilon$ so $pF_Y^T uF^{-T}(m)$ is defined. 2) follows directly from the Implicit Function Theorem. If $\mathfrak{X}$ and δ are small enough and $\varphi(m) \in U_i^3$, then $m \in U_i^4$ and uniqueness also follows from the Implicit Function Theorem.

From Proposition 4 we see that F_Y^T induces a map $G: S_\delta \to S$ defined in coordinates by $(Gu)_j(f_{ij}(x, u_i(x))) = g_{ij}(x, u_i(x))$. We show below that G maps S_δ into S_δ, that G is a contraction in the C^0 norm, and that the graph of the unique fixed point of G is a C^r manifold overflowing invariant under Y.

Proposition 5. $G: S_\delta \to S_\delta$.

Proof. It is sufficient to show that if $u \in S_\delta$, $||(Gu)_j(\xi) - (Gu)_j(\xi')|| \leqq \delta\, ||\xi - \xi'||$ for all $\xi, \xi' \in \mathfrak{D}^3$. It even is sufficient to show, for each ξ that this inequality holds for all ξ' in some neighborhood of ξ. Then for any $\xi, \xi' \in \mathfrak{D}^3$ the segment joining ξ to ξ' lies in $\mathfrak{D}^3$ and is covered by finitely many such neighborhoods.

Take any m, say $m \in U_j^3$. By Proposition 4 there is a point m_- such that $pF_Y^T u(m_-) = m$, say with $m_- \in U_i^3$. Let $x = \sigma_i(m_-)$ and $x' \in \mathfrak{D}^3$, near x. Looking at Gu in local coordinates, we have

$$(Gu)_j(\xi) = g_{ij}(x, u_i(x))$$

and

$$(Gu)_j(\xi') = g_{ij}(x', u_i(x')),$$

where

$$\xi = f_{ij}(x, u_i(x))$$

and

$$\xi' = f_{ij}(x', u_i(x')).$$

We introduce some new notation:

$$A = D_1 g_{ij}(x, u_i(x)),$$
$$B = D_2 g_{ij}(x, u_i(x)),$$
$$C = D_1 f_{ij}(x, u_i(x)),$$
$$E = D_2 f_{ij}(x, u_i(x)).$$

A, B, C, E depend on x, i, j, u and Y. All of $||A||$, $||B||$, $||C||$, $||C^{-1}||$ and $||E||$ are bounded by Q. Furthermore, $||A|| < \eta$, $||B|| < 1/2$, $||B||\, ||C^{-1}|| < 1/2$. We use the "little o" notation; $o(s)$ is any term such that as $s \to 0$, $o(s)/s \to 0$. These terms occur when we use derivatives to approximate functions. Similarly $O(s)$ is any term such that $O(s) \to 0$ as $s \to 0$.

$$\begin{aligned}
||\xi - \xi'|| &= ||f_{ij}(x, u_i(x)) - f_{ij}(x', u_i(x'))|| \\
&\geqq ||f_{ij}(x, u_i(x)) - f_{ij}(x', u_i(x))|| - ||f_{ij}(x', u_i(x)) - f_{ij}(x', u_i(x'))|| \\
&\geqq ||C^{-1}||^{-1}\, ||x - x'|| + o(||x - x'||) - ||E||\, ||u_i(x) - u_i(x')|| \\
&\geqq ||C^{-1}||^{-1} (1 - 2\delta Q^2)\, ||x - x'||.
\end{aligned}$$

We have chosen $||x - x'||$ small enough so that $|o(||x - x'||)| < \delta Q \, ||x - x'||$

$$\begin{aligned} ||(Gu)_j(\xi) - (Gu)_j(\xi')|| &= ||g_{ij}(x, u_i(x)) - g_{ij}(x', u_i(x'))|| \\ &\leqq ||A|| \, ||x - x'|| + ||B|| \, \delta \, ||x - x'|| + o(||x - x'||) \\ &\leqq (2\eta + \delta \, ||B||) \, ||x - x'|| \\ &\leqq (2\eta + \delta \, ||B||) \frac{||C^{-1}||}{(1 - 2\delta Q^2)} \, ||\xi - \xi'||. \end{aligned}$$

We have taken $||x - x'||$ small enough so that $|o(||x - x'||)| < \eta \, ||x - x'||$. δ is chosen so small that $1 - 2\delta Q^2 > 3/4$. η is chosen less than $\delta/8Q$. Then the factor multiplying $||\xi - \xi'||$ is less than δ, so $Gu \in S_\delta$.

Proposition 6. *G is a contraction on S_δ in C^0 norm.*

Proof. Let $u, u' \in S_\delta$. Let $\xi \in \sigma_j U_j^3$ be given. Choose i, x and x' such that $\xi = f_{ij}(x, u_i(x)) = f_{ij}(x', u_i'(x'))$.

$$\begin{aligned} ||(Gu)_j(\xi) - (Gu')_j(\xi)|| &= ||g_{ij}(x, u_i(x)) - g_{ij}(x', u_i(x'))|| \\ &\leqq ||A|| \, ||x - x'|| + ||B|| \, ||u_i(x) - u_i'(x)|| \\ &\quad + ||B|| \, ||u_i(x)' - u_i'(x')|| + o(||x - x'||) \\ &\leqq (2\eta + Q\delta) \, ||x - x'|| + ||B|| \, ||u_i - u_i'||_0. \end{aligned}$$

$||x - x'||$ is small if ϵ is small. We have used this to estimate the $o(||x - x'||)$ term. It remains to estimate $||x - x'||$ in terms of $||u - u'||_0$.

$$\begin{aligned} ||f_{ij}(x, u_i(x)) - f_{ij}(x', u_i(x))|| &\geqq ||C^{-1}||^{-1} \, ||x - x'|| - o(||x - x'||) \\ &\geqq \tfrac{1}{2} \, ||C^{-1}||^{-1} \, ||x - x'||. \end{aligned}$$

$$||f_{ij}(x', u_i'(x')) - f_{ij}(x', u_i(x))|| \leqq Q(\delta \, ||x - x'|| + ||u_i - u_i'||_0).$$

Noting that $f_{ij}(x, u_i(x)) = f_{ij}(x', u_i'(x'))$ we combine these estimates to find $||x - x'|| \leqq \text{constant} \cdot ||u_i - u_i'||_0$. Taking δ and η small enough we find $||Gu - Gu'||_0 \leqq 3/4 \, ||u - u'||_0$. (Note that the norm of a section may be measured in N or in coordinates because the coordinates τ_i are defined in terms of orthonormal bases.)

Corollary. *There is a unique $u \in S$ such that F_Y^t (graph u) $\subset$ graph u for all $t > 0$. Furthermore, $u \in S_\delta$.*

Proof. S_δ is closed under C^0 convergence so G has a unique fixed point in S_δ. Call this u. Uniqueness in S follows from the proof of Proposition 6, noting that only one of u and u' is required to be Lipschitz continuous. Arguing as in Proposition 4, for small $t > 0$, F_Y^t (graph u) $\cap N_\epsilon' \mid \cup U_i^3$ is the graph of an element $u_t \in S_\delta$. Graph $u \subset F_Y^T$ (graph u) so F_Y^t (graph u) $\subset F_Y^t F_Y^T$ (graph u) $= F_Y^T F_Y^t$ (graph u). By Proposition 6, $u_t = u$.

III. 2. Smoothness of the invariant manifold. Now u denotes the invariant section of the preceding corollary. A, B, C, E are defined as in Proposition 5, using this u. In coordinates u is represented by s maps $u_i : \mathfrak{D}^3 \to R^k$. If $u \in C^1$, Du_i assigns to each point in $\mathfrak{D}^3$ a linear map from R^{n-k} to R^k. Thus Du is represented by s maps $v_i \in C^0(\mathfrak{D}^3, L(R^{n-k}, R^k))$. The candidates for Du are of the form $v = (v_1, \cdots, v_s) \in [C^0(\mathfrak{D}^3, L(R^{n-k}, R^k))]^s$. If v is such an s-tuple define

$$||v|| = \max_i \sup_{x \in \mathfrak{D}^3} ||v_i(x)||$$

if this exists, where $||v_i(x)||$ is the operator norm.

In coordinates u satisfies the functional equations $u_j(\xi) = g_{ij}(x, u_i(x))$, where $f_{ij}(x, u_i(x)) = \xi$. Differentiate these formally; v must satisfy $v_j = H_{ij}v_i$, where

$$H_{ij}v_i(\xi) = [A + Bv_i(x)][C + Ev_i(x)]^{-1}.$$

The principal term in $H_{ij}v_i$ is Bv_iC^{-1}, and $||B||\ ||C^{-1}|| < 1/2$, so H_{ij} is suitable for iteration methods. H_{ij} has no coordinate-free significance, however, so we combine all possible H_{ij}'s to form a new functional equation.

Choose C^r functions $\varphi_i : \bigcup U_i^3 \to [0, 1]$ with support of $\varphi_i \subset U_i^2$ and $\sum \varphi_i = 1$ on $\bigcup U_i^1$. Define

$$v_j^0 = 0 \qquad j = 1, \cdots, s$$

and

$$v_j^{n+1}(\xi) = \sum_{i=1}^{s} \varphi_i(m_-)H_{ij}v_i^n(\xi),$$

where $F_Y^T u(m_-) = u(\sigma_j^{-1}(\xi))$. Note that $m_- \in M$, so $\sum \varphi_i(m_-) = 1$.

Proposition 7. *$||v^n|| < \delta$ for each n.*

Proof. Proceeding by induction, it is sufficient to prove $||H_{ij}v_i^n(\xi)|| < \delta$ for each i, j, n and ξ such that $pF_Y^{-T}u(\sigma_j^{-1}(\xi)) \in U_i^3$. This is the sort of estimate used in Proposition 5, so details are omitted.

Proposition 8. $||v_i^{n+1} - v_i^n|| \leqq 3/4\ ||v_i^n - v_i^{n-1}||$.

Proof. It is sufficient to show that $||H_{ij}v_i^n - H_{ij}v_i^{n-1}|| \leqq 3/4\ ||v_i^n - v_i^{n-1}||$. We simply indicate how to break up the differences into terms which readily are estimated.

$$\begin{aligned} H_{ij}v_i^n - H_{ij}v_i^{n-1} &= [A + Bv_i^n][C + Ev_i^n]^{-1} - [A + Bv_i^{n-1}][C + Ev_i^{n-1}]^{-1} \\ &= [A + Bv_i^n][C + Ev_i^n]^{-1}\cdot([C + Ev_i^{n-1}] - [C + Ev_i^n])[C + Ev_i^{n-1}]^{-1} \\ &\quad + ([A + Bv_i^n] - [A + Bv_i^{n-1}])[C + Ev_i^{n-1}]^{-1}. \end{aligned}$$

Corollary. *The sequence v^n converges to a solution v of the equations*

$$v_j = \sum_{i=1}^{s} \varphi_i \cdot H_{ij} v_i \, .$$

Proposition 9. *For all $m \varepsilon U_i^3$, $Du_i(\sigma_i(m))$ exists and equals $v_i(\sigma_i(m))$. Hence $u \varepsilon C^1$ and $v_j = H_{ij} v_i$.*

Proof. We use the definition of derivative, along with the functional equations for u and v. Define an increasing function γ: $(0, 1) \to R$,

$$\gamma(a) = \max_i \sup_{\substack{\xi, \xi' \varepsilon \mathfrak{D}^3 \\ 0 < \|\xi - \xi'\| < a}} \frac{\|u_i(\xi') - u_i(\xi) - v_i(\xi)(\xi' - \xi)\|}{\|\xi' - \xi\|}$$

According to Propositions 5 and 7, γ is bounded by 2δ. To prove Proposition 9 we show that $\gamma(a) \to 0$ as $a \to 0$.

Suppose $\gamma(a)$ satisfies an inequality

$$\gamma(a) \leqq \alpha \cdot \gamma(\beta a) + r(a)$$

for small a, where $r(a)$ decreases to zero as $a \to 0$, and $0 \leqq \alpha < 1$. If $\beta \leqq 1$ we have $\gamma(a) \leqq (1 - \alpha)^{-1} \cdot r(a)$ so we are through. If $\beta > 1$, we replace a successively by $a\beta^{-1}, a\beta^{-2}, \cdots, a\beta^{-n}$, weight the terms with $\alpha^{n-1}, \alpha^{n-2}, \cdots, 1$, and add:

$$\begin{aligned} \gamma(a\beta^{-n}) &\leqq \alpha^n \gamma(a) + r(a\beta^{-n}) + \alpha r(a\beta^{-n+1}) + \cdots + \alpha^{n-1} r(a\beta^{-1}) \\ &\leqq 2\delta\alpha^n + (1 - \alpha)^{-1} r(a\beta^{-1}). \end{aligned}$$

From this it follows that $\gamma(a) \to 0$ as $a \to 0$.

Now let $\xi \varepsilon \mathfrak{D}^3$ be given and suppose for some i, j, $\xi = f_{ij}(x, u_i(x))$, with $x \varepsilon \mathfrak{D}^2$. If d is chosen small enough and $\xi' \varepsilon \mathfrak{D}^3$ with $\|\xi - \xi'\| < d$, there is an $x' \varepsilon \mathfrak{D}^3$ such that $\xi' = f_{ij}(x', u_i(x'))$. d may be chosen independent of ξ, i, j.

To show that γ satisfies the functional inequality above it is sufficient to show that

$$\|u_j(\xi') - u_j(\xi) - H_{ij} v_i(\xi) \cdot (\xi' - \xi)\| \leqq \{\alpha\gamma(\alpha\beta) + r(a)\} \, \|\xi' - \xi\|$$

for all ξ, ξ', i, j as above, for $\|\xi' - \xi\| \leqq a < d$. But

$$\begin{aligned} \xi' - \xi &= f_{ij}(x', u_i(x')) - f_{ij}(x, u_i(x)) \\ &= C(x' - x) + E(u_i(x') - u_i(x)) + o(\|x' - x\|), \\ u_j(\xi') - u_j(\xi) &= g_{ij}(x', u_i(x')) - g_{ij}(x, u_i(x)) \\ &= A(x' - x) + B(u_i(x') - u_i(x)) + o(\|x' - x\|). \end{aligned}$$

From the first equation it follows that $\|x' - x\| \leqq \|C^{-1}\| \, \|\xi' - \xi\|/(1 - 2\delta Q^2)$, and that

$$\begin{aligned} \xi' - \xi = (C + Ev_i(x))(x' - x) \\ + E(u_i(x') - u_i(x) - v_i(x) \cdot (x' - x)) + o(\|x' - x\|). \end{aligned}$$

We use this expression to eliminate $\xi' - \xi$ below.

$$\begin{aligned}
&||u_i(\xi') - u_i(\xi) - H_{ij}v_i(x)\cdot(\xi' - \xi)||\\
&\quad = ||A(x' - x) + B(u_i(x') - u_i(x)) + o(||x' - x||)\\
&\qquad - [A + Bv_i(x)][C + Ev_i(x)]^{-1}\cdot(\xi' - \xi)||\\
&\quad = ||A(x' - x) + B(u_i(x') - u_i(x)) - (A + Bv_i(x))(x' - x)\\
&\qquad - (A + Bv_i(x))(C + Ev_i(x))^{-1}\cdot E(u_i(x') - u_i(x) - v_i(x)\cdot(x' - x))\\
&\qquad + o(||x' - x||)||\\
&\quad = ||[B - (A + Bv_i(x))(C + Ev_i(x))^{-1}\cdot E]\cdot[u_i(x') - u_i(x) - v_i(x)(x' - x)]\\
&\qquad + o(||x' - x||)||\\
&\quad \leqq (||B|| + O(\eta + \delta))\gamma(||x' - x||)\, ||x' - x|| + ||o(||x' - x||)||\\
&\quad \leqq \frac{(||B|| + O(\eta + \delta))\, ||C^{-1}||}{(1 - 2\delta Q^2)}\gamma(||x' - x||)\, ||\xi' - \xi|| + ||o(x' - x)||.
\end{aligned}$$

$||B||\, ||C^{-1}|| < 1/2$, so we take $\alpha = (||B|| + O(\eta + \delta))\, ||C^{-1}||/(1 - 2\delta Q^2)$, and $\alpha < 1$ if η and δ are small enough. Take $\beta = Q/(1 - 2\delta Q^2)$, so that $||x' - x|| \leqq \beta\, ||\xi' - \xi|| < \beta a$, and then $\gamma(||x' - x||) \leqq \gamma(\beta a)$. We have $||\xi' - \xi|| \geqq \beta^{-1}\, ||x' - x||$ so $o(||x' - x||)$ may be bounded by $r(a)\, ||\xi' - \xi||$, where $r(a) \to 0$ as $a \to 0$. This completes the proof of Proposition 9.

Proposition 10. $u \in C^r$.

Proof. We suppose $u \in C^p$, $1 \leqq p < r$, and show that $u \in C^{p+1}$. Note that the C^1 size of the neighborhood of X in Theorem 1 may depend on r.

$D^p u_i(x)$ is a p-linear map from R^{n-k} to R^k. $D^p u$ is represented as an s-tuple $(D^p u_1, \cdots, D^p u_s)$, so

$$D^p u \in [C^0(\mathfrak{D}^3, L^p(R^{n-k}, R^k))]^s.$$

The spaces $L^p(R^{n-k}, R^k)$ have natural operator norms. See Dieudonné [1, sections V.7 and VIII.12]. For $w = (w_1, \cdots, w_s) \in [C^0(\mathfrak{D}^3, L^p(R^{n-k}, R^k))]^s$, define $||w|| =$ max sup $||w_i(x)||$, where the max is taken over i and the sup is taken over $x \in \mathfrak{D}^3$, if the suprema all exist. We use this norm to prove convergence of the p-th derivatives.

The functions v^n used in the construction of v depend on u, so a priori there are not differentiable. Now we known that u is differentiable, however, so it is clear that each v^n is differentiable. We prove, in fact, that the sequence Dv^n converges. By construction,

$$v_j^{n+1}(\xi) = \sum \varphi_i\cdot[A + Bv_i^n(x)][C + Ev_i^n(x)]^{-1}$$

where $\xi = f_{ij}(x, u_i(x))$. Hence

$$\begin{aligned}
Dv_j^{n+1}(\xi) = \sum \varphi_i\cdot[&BDv_i^n(x)(C + Ev_i^n(x))^{-1}\\
&- [A + Bv_i^n(x)][C + Ev_i^n(x)]^{-1}EDv_i^n(x)\cdot[C + Ev_i^n(x)]^{-1}[C + Ev_i(x)]^{-1}
\end{aligned}$$

$+$ (terms not involving derivatives of v's).

The factor $(C + Ev_i(x))^{-1}$ equals $dx/d\xi$. The "terms not involving derivatives of v's" include derivatives of u, φ_i, A, B, C, E. Second derivatives of F_Y^T appear only here. These terms converge as $n \to \infty$ because v^n converges. Note that the operators A, B, C, E depend on i.

It is an easy estimate, using the equation above, to show that for large enough n,

$$\|Dv^{n+1} - Dv^n\| < \alpha \|Dv^n - Dv^{n-1}\| + r_n ,$$

where $r_n <$ constant $\cdot \|v^n - v^{n-1}\| <$ constant $\cdot (3/4)^n$. $\alpha = \|B\| \, \|C^{-1}\|^2 + O(\eta+\delta)$, so $\alpha < 1$ if η and δ are chosen small enough. As in Proposition 9, it follows from this inequality that $\{Dv^n\}$ is a Cauchy sequence. Hence $u \in C^2$.

Suppose $w = D^p u$ exists, $2 \leqq p < r$. Differentiate the functional equation for u p times. w must satisfy

$$w_j(\xi) = \sum \varphi_i \cdot \{Bw_i(x) - (A + Bv_i(x))(C + Ev_i(x))^{-1} Ew_i(x)\} \cdot (C + Ev_i(x))^{-p}$$
$$+ \text{(terms not involving } w).$$

Call the right-hand side of this equation $\mathfrak{I}(w)(\xi)$. $\mathfrak{I}$ is a contraction operator, and so has a unique fixed point. The coefficient governing the contraction properties is $\|B\| \, \|C^{-1}\|^p + O(\eta + \delta)$, and depends only on the C^1 size of $Y - X$.

Let $w^0 = 0$, $w^{n+1} = \mathfrak{I}(w_n)$. w^n converges to the unique fixed point of $\mathfrak{I}$, that is, to $D^p u$. Arguing as for Dv, Dw^n also converges. The coefficient governing the contraction is $\|B\| \, \|C^{-1}\|^{p+1} + O(\eta + \delta)$, and depends only on the C^1 size of $Y - X$.

By induction we have $u \in C^r$, completing the proof of Proposition 10 and Theorem 1.

III. 3. Persistence of the type numbers. For each Y in a C^1 neighborhood of X we have constructed an overflowing invariant manifold M_Y as the graph of a function $u_Y : M \to R^n$. Type numbers ν_Y and $\sigma_Y : M_Y \to R$ are defined. If $\nu_Y < 1$ and $\sigma_Y < 1/r$ we can apply Theorem 1 to Y and M_Y to extend the neighborhood of X in which every field has an invariant manifold diffeomorphic to M. This extension breaks down at any field Y such that $\nu_Y = 1$ or $\sigma_Y = 1/r$ anywhere in M_Y.

The backward limit set of the orbit through any point may change radically under small perturbations of the field, so we don't expect $\nu_Y(u_Y(m))$ and $\sigma_Y(u_Y(m))$ to be continuous in Y even for fixed m. Define

$$\nu(Y) = \sup \nu_Y$$

$$\sigma(Y) = \sup \sigma_Y .$$

Theorem 2. *$\nu(Y)$ and $\sigma(Y)$ are upper-semicontinuous.*

Proof. If Y is close to X, F_Y^T is close to F^T and the tangent planes and normals of M_Y are close to the tangent planes and normals of M. Hence for any $C > 1$, if Y is close enough to X,

$$||D(F_Y^{-T}|M_Y)(u(m))|| < C\ ||D(F^{-T}|M)(m)||$$

and

$$||\pi_Y DF_Y^T(F_Y^{-T}u(m))|| < C\ ||\pi DF^T(F^{-T}(m))||.$$

From the second inequality we have $\nu(Y) \leqq C^{1/T}\nu(X)$. The two inequalities together imply that for any $s > \sigma(x)$ and $t \geqq 0$,

$$||\pi_Y DF_Y^t(F_Y^{-t}(m))||^s\ ||D(F_Y^{-t} \mid M_Y)(u(m))|| < (\text{constant})\ C^{(1+s)t}\ ||\pi DF^t(F^{-t}(m))||^s\ ||D(F^{-t}|M)(m)||.$$

Taking C close enough to 1, the right-hand side goes to zero uniformly in M as $t \to \infty$. This completes the proof of Theorem 2.

$\sigma(x)$ measures the best smoothness we can guarantee throughout M_Y, for Y near X. It may happen, however, that parts of M_Y are quite smooth even though wrinkles develop elsewhere in M_Y. This may be described in terms of overflowing invariant manifolds.

Suppose $\tilde{M} \subset M$ is an overflowing invariant manifold of the same dimension as M, and that X and Y are C^r. If $\sigma(m) < 1/r$ for all $m \in \tilde{M}$ then u is C^r on $\tilde{M}$, even though σ may exceed $1/r$ on M.

IV. Hyperbolic splittings. Let X be a C^1 vector field on R^n, with flow F^t. Let M be a compact, connected C^1 manifold properly embedded in R^n, invariant under X. Let $TR^n \mid M = TM \oplus N^- \oplus N^+$ be a continuous splitting such that $TM \oplus N^-$ and $TM \oplus N^+$ are invariant under DF^t for all t. Let π, π^- and π^+ denote the projections on TM, N^- and N^+, respectively. Define

$$\nu^-(m) = \overline{\lim_{t\to\infty}}\ ||\pi^- DF^t(F^{-t}(m)) \mid N^-||^{1/t},$$

$$\nu^+(m) = \overline{\lim_{t\to-\infty}}\ ||\pi^+ DF^t(F^{-t}(m)) \mid N^+||^{-1/t}.$$

The splitting is called hyperbolic if $\nu^-(m) < 1$ and $\nu^+(m) < 1$ for all $m \in M$.

Invariant manifolds with a hyperbolic splitting possess many properties of asymptotically stable invariant manifolds. Compactness is required because there is no preferred time direction. Only a compact manifold is overflowing invariant under both the forward and the backward flow.

Define

$$\sigma^-(m) = \overline{\lim_{t\to\infty}}\ \frac{\log ||D(F^{-t} \mid M)(m)||}{-\log ||\pi^- DF^t(F^{-t}(m)) \mid N^-||},$$

$$\sigma^+(m) = \overline{\lim_{t\to-\infty}}\ \frac{\log ||D(F^{-t} \mid M)(m)||}{-\log ||\pi^+ DF^t(F^{-t}(m)) \mid N^+||}.$$

If N^+ is absent ν^- and σ^- agree with ν and σ. As in Proposition 1, ν^-, ν^+, σ^- and σ^+ are independent of the metric of R^n. Of course, they generally depend on the splitting of $TR^n|M$.

Theorem 3. *Let X be a C^r vector field on R^n, $r \geqq 1$, M a compact, connected C^r manifold properly embedded in R^n and invariant under X. Suppose a hyperbolic splitting of $TR^n|M$ is given, with $\sigma^-(m) < 1/r$ and $\sigma^+(m) < 1/r$ for all $m \in M$. Then for any C^r vector field Y in some C^1 neighborhood of X there is a C^r manifold M_Y invariant under Y and C^r diffeomorphic to M.*

Proof. For the proof of Theorem 1 we introduced a C^r bundle N' transversal to TM, and (x, y) coordinates in which points with $y = 0$ lie in M and fibers of N' correspond to $x =$ constant. For Theorem 3 we introduce C^r bundles N'^- and N'^+ close to N^- and N^+, and (x, y, z) coordinates in which points $y = 0$, $z = 0$ lie in M, fibers of N'^- correspond to $x =$ constant, $z = 0$, and fibers of N'^+ correspond to $x =$ constant, $y = 0$. x is a variable in $\mathfrak{D}^5$, the disc of radius 5 of the same dimension as M. We use an orthonormal basis to define y and z so that elements of $N'^- \oplus N'^+$ may be measured in coordinates. Define $N'_\epsilon = N'^-_\epsilon \oplus N'^+_\epsilon = \{(n^-, n^+) \in N'^- \oplus N'^+ : ||n^-|| < \epsilon, ||n^+|| < \epsilon\}$. N'_ϵ is diffeomorphic to a neighborhood of M if ϵ is small enough.

Let $\eta > 0$ be given. Choosing T large enough, N'^- and N'^+ close enough to N^- and N^+, ϵ small enough and Y sufficiently C^1 close to X, F_Y^T has local representatives for $||y|| < \epsilon$, $||z|| < \epsilon$,

$$(x, y, z) \to (f(x, y, z), \quad g(x, y, z), \quad h(x, y, z))$$

such that

1) $$||(D_1f)^{-1}||^k \, ||D_2g|| < 1/2, \qquad 0 \leqq k \leqq r,$$
$$||D_1f||^k \, ||(D_3h)^{-1}|| < 1/2, \qquad 0 \leqq k \leqq r.$$

2) $$||D_1g|| < \eta,$$
$$||D_1h|| < \eta,$$
$$||D_3g|| < \eta,$$
$$||D_2h|| < \eta,$$
$$||g(x, 0, 0)|| < \eta,$$
$$||h(x, 0, 0)|| < \eta.$$

3) All first partial derivatives of f, g, h, as well as $(D_1f)^{-1}$ and $(D_2g)^{-1}$ are bounded by Q.

The subscripts i, j have been omitted to reduce the clutter.

The invariant manifold M_Y is constructed as the graph of a pair (u, v) of sections of N'^- and N'^+ satisfying

$$F_Y^T \text{ (graph } (u, v)) = \text{graph } (u, v).$$

Let S denote the set of pairs (u, v) of continuous sections of $N'^- \oplus N'^+$, with $||u, v|| = \max (||u||, ||v||)$. As in Theorem 1, there are local representatives u_i

and v_i. The invariance of graph (u, v) is expressed in coordinates by

$$\xi = f(x, u(x), v(x)),$$
$$u(\xi) = g(x, u(x), v(x)),$$
$$v(\xi) = h(x, u(x), v(x)).$$

Again, the subscripts have been omitted. Let Lip (u, v) be the largest of the Lipschitz constants of u_i and v_i restricted to $\mathfrak{D}^3$, if these exist. Let $S_\delta = \{(u, v) \in S: \|u, v\| < \epsilon^1 = \epsilon/2Q, \text{Lip}\,(u, v) < \delta\}$. We will show that F_Y^T induces a contraction map $G: S_\delta \to S_\delta$. The presence of expansion introduces an asymmetry in the construction of G. Later we use a trick to remove this asymmetry.

Proposition 11. *Let $p: N'_\epsilon \to M$ denote the fiber projection. Let $\varphi(m) = pF_Y^T(u, v)(F^{-T}(m))$. Then there is a $\delta > 0$ and a C^1 neighborhood $\mathfrak{X}$ of X such that if $Y \in \mathfrak{X}$ and $(u, v) \in S_\delta$*

1) *φ is defined,*
2) *φ is $1 - 1$ and onto.*

Proof. See Proposition 4.

Proposition 12. *Let $p^+: N'^- \oplus N'^+ \to N'^+$ be the projection. If δ and $\mathfrak{X}$ in Proposition* 11 *are small enough, there is a section $v^1: M \to N'^+$ such that*

$$v(pF_Y^T(u, v^1)(m)) = p^+F_Y^T(u, v^1)(m)$$

for all $m \in M$.

Proof. In coordinates $v_i^1(x)$ is the unique solution of

$$h_{ij}(x, u_i(x), v_i^1(x)) - v_j(f_{ij}(x, u_i(x), v_i^1(x))) = 0.$$

For $u = 0$, $v = 0$ and $Y = X$, we have $h_{ij}(x, 0, 0) = 0$. D_3h is invertible, and the second term is Lipschitz small, so the existence of a continuous solution v^1 follows from the implicit function theorem for Lipschitz continuous functions. Solutions determined in different coordinate patches are compatible because the solutions are unique.

Define $G: S_\delta \to S$ as follows. Let $(u, v) \in S_\delta$. Find u^1 such that $u_j^1(f_{ij}(x, u_i(x), v_i(x))) = g_{ij}(x, u_i(x), v_i(x))$. u^1 exists by Proposition 13. Find v^1 as in Proposition 14. Let $G(u, v) = (u^1, v^1)$. The next two Propositions and the Corollary are proved as in Theorem 1. The fixed point of G is invariant under the flow of Y and is the only continuous section of $N'^-_\epsilon \oplus N'^+_\epsilon$ whose graph is invariant under the flow of Y.

Proposition 15. $G: S_\delta \to S_\delta$.

Proposition 16. *G is a contraction on S_δ in the C^0 norm.*

Corollary. *G has a unique fixed point (u, v). In coordinates (u, v) satisfies*

$$u(f(x, u(x), v(x))) = g(x, u(x), v(x)),$$

$$v(f(x, u(x), v(x))) = h(x, u(x), v(x)).$$

Now we could study the smoothness of u and v as we did for Theorem 1. This approach is complicated substantially by the asymmetry introduced to handle the expanding directions. We resort to a trick to avoid this problem.

The map F_Y^{-T} has the same properties as F_Y^T, with the roles of N^- and N^+ interchanged. Using the same coordinates, denote the local representatives of F_Y^{-T} by (f', g', h'). We readily show that there is a unique $(u', v') \in S_\delta$ such that graph $(u', v') = F_Y^{-T}$ graph (u', v'). But then F_Y^T graph (u', v') = graph (u', v') so by uniqueness in the Corollary above $(u, v) = (u', v')$. In coordinates (u', v') satisfies

$$u'(f'(x, u')x), v'(x))) = g'(x, u'(x), v'(x)),$$

$$v'(f'(x, u'(x), v'(x))) = h'(x, u'(x), v'(x)).$$

Finally, we take one set of equations for F_Y^T and one set for F_Y^{-T} and look at the functional equations

$$u''(f(x, u''(x), v''(x))) = g(x, u''(x), v''(x)),$$

$$v''(f'(x, u''(x), v''(x))) = h'(x, u''(x), v''(x)).$$

(u, v) is a solution of this system and the solution is unique in S_δ.

Now we have a system of functional equations in which u and v appear symmetrically. This system is of the form studied in III.2, so its solution is of class C^r.

V. Stable and unstable manifolds. Under the hypotheses of Theorem 3 we will construct local stable and unstable manifolds. The compact invariant manifold persists under perturbation if both the stable and the unstable manifold persist. It is as smooth as the rougher of the two.

We offer an example to show that the unstable manifold may persist even though the compact invariant manifold develops a singularity. Define a field in cylindrical coordinates for R^3, deleting the z axis:

$$\dot{r} = -c(r - 1), \qquad c > 1,$$

$$\dot{z} = z,$$

$$\dot{\theta} = \sin \theta.$$

See Figure 7.

The circle γ: $\{r = 1, z = 0\}$ is invariant. It has a hyperbolic structure specified by the r and z directions. The cylinder U: $\{r = 1, -1 \leqq z \leqq 1\}$ is overflowing invariant, a local unstable manifold of γ. The annulus S: $\{\frac{1}{2} \leqq r \leqq 2, z = 0\}$ is overflowing invariant under the flow with time reversed. S is a local stable manifold of γ.

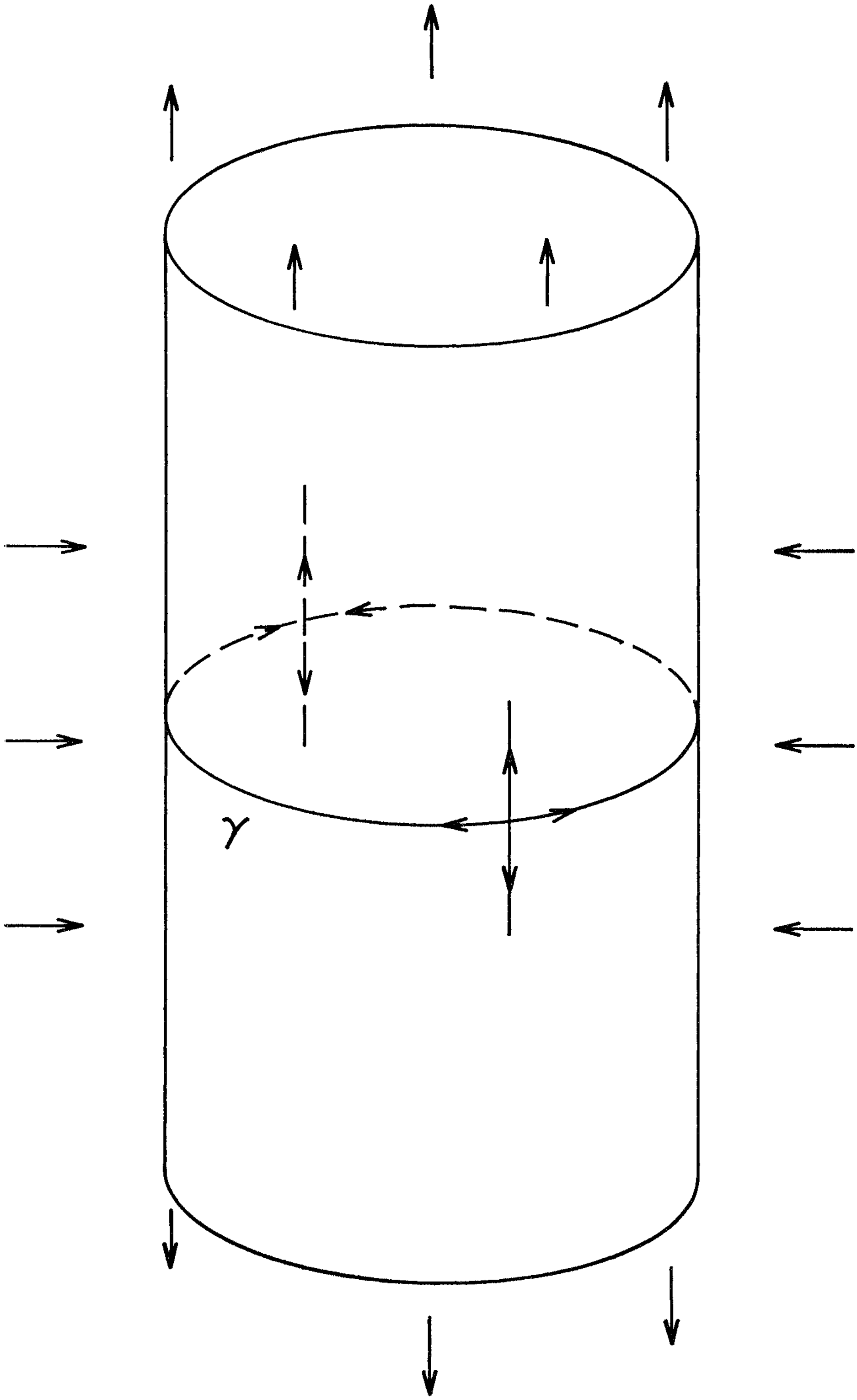

FIGURE 7

All type numbers are computed easily.

On γ:

$$\begin{aligned} \nu^- &= e^{-c}, \\ \nu^+ &= e^{-1}, \\ \sigma^+ &= 1 && \text{at} \quad \theta = 0 \\ &= -1, && \theta \neq 0, \\ \sigma^- &= -1/c, && \theta \neq \pi \\ &= 1/c, && \theta = \pi. \end{aligned}$$

On U:

$$\begin{aligned} \nu &= e^{-c}, \\ \sigma &= 1/c, && \theta \neq \pi \\ &= -1/c, && \theta = \pi. \end{aligned}$$

On S:

$$\begin{aligned} \nu &= e^{-1}, \\ \sigma &= 1, && \theta = 0 \\ &= -1, && \theta \neq 0. \end{aligned}$$

Note that the type numbers for S are computed with time reversed.

Theorem 1 guarantees that U persists under perturbation, and is C^r for any $r < c$. The perturbation theorems do not apply to γ and S because σ^+ and σ equal 1 at $\theta = 0$. Indeed, a spiral can develop in U, as shown in Figure 8.

We formulate the unstable manifold theorem so that U may be studied even though γ becomes singular under perturbation. In dealing with unstable manifolds there is a preferred time direction, so we consider overflowing invariant manifolds again.

Let X be a C^1 vector field on R^n, $\bar{M} = M \cup \partial M$ a compact, connected C^1 manifold, overflowing invariant under X. Suppose a continuous splitting $TR^n \mid M = TM \oplus N^- \oplus N^+$ is given, with $TM \oplus N^-$ and $TM \oplus N^+$ invariant under DF^t for any $t \leqq 0$. Define

$$\lambda^+(m) = \overline{\lim_{t\to\infty}} \, ||\pi^+ DF^{-t}(m) \mid N^+||^{1/t},$$

$$\nu^-(m) = \overline{\lim_{t\to\infty}} \, ||\pi^- DF^t(F^{-t}(m)) \mid N^-||^{1/t},$$

$$\sigma^-(m) = \overline{\lim_{t\to\infty}} \, \frac{\log ||D(F^{-t} \mid M)(m))||}{-\log ||\pi^- DF^t(F^{-t}(m)) \mid N^-||}.$$

The splitting is called hyperbolic if $\lambda^+(m) < 1$ and $\nu^-(m) < 1$ for all $m \in M$.

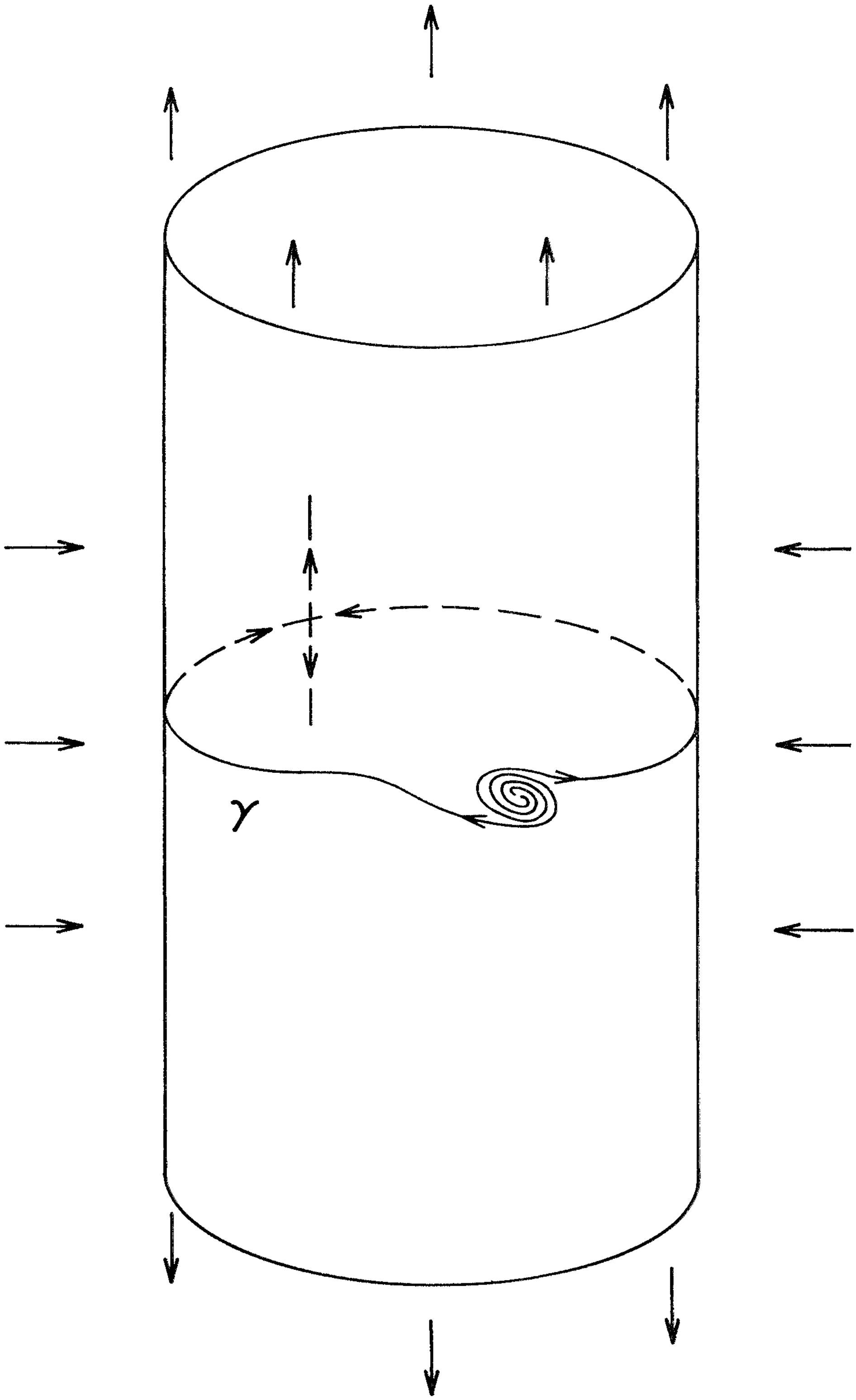

Figure 8

If M is compact we claim this agrees with our previous definition. $\lambda^+ < 1$ and $\nu^+ < 1$ both signify that N^+ consists of vectors diminished at an exponential rate by the flow. λ^+ is measured along backward orbits; ν^+ is measured looking backward from the forward orbit. The Uniformity Lemma lets us tie these together.

Suppose M is compact and $\nu^+(m) < a \leqq 1$ for all $m \in M$. There is a constant C such that

$$||\pi^+ DF^{-t}(F^t(m)) \mid N^+|| < Ca^t$$

for all $m \in M$ and all $t \geqq 0$. But then

$$||\pi^+ DF^{-t}(m) \mid N^+|| < Ca^t$$

for all $m \in M$ and all $t \geqq 0$. Hence $\lambda^+(m) < a$ for all $m \in M$. This argument is extended easily to show that λ^+ and ν^+ have the same supremum on M.

Theorem 4. *Let X be a C^r vector field on R^n, $r \geqq 1$, $\bar{M} = M \cup \partial M$ a C^r compact connected manifold with boundary, overflowing invariant under X. Suppose a hyperbolic splitting of $TR^n|M$ is given, with $\sigma^-(m) < 1/r$ for all $m \in M$. Let N_ϵ^+ denote the set of vector of length less than ϵ in N^+. Then if ϵ is small enough and Y is a C^r vector field C^1 close enough to X, there is a C^r manifold overflowing invariant under Y and homeomorphic to N_ϵ^+.*

Proof. The proof follows the same lines as the proofs of Theorem 1 and Theorem 3, so we omit most details.

N^- and N^+ are perturbed to C^r bundles N'^- and N'^+. These are used to set up (x, y, z) coordinates. Some large value of T is chosen, and F_Y^T has the form

$$x' = f(x, y, z),$$
$$y' = g(x, y, z),$$
$$z' = h(x, y, z).$$

The invariant manifold is constructed as the graph of a section u of N'^- over $N_\epsilon'^+$. In coordinates u satisfies

$$u(f(x, u(x, z), z), h(x, u(x, z), z)) = g(x, u(x, z), z).$$

If ϵ is small and u is Lipschitz small the arguments $f(x, u(x, z), z)$ and $h(x, u(x, z), z)$ cover all of $N_\epsilon'^+$. As in Theorem 1, the functional equation has a unique solution u, and $u \in C^r$.

Theorem 4 also may be proved in two stages. Taking this approach we show first that X has a C^r unstable manifold. Then we show that the unstable manifold satisfies the hypotheses of Theorem 1.

We have seen that under the hypotheses of Theorem 3 the stable and unstable manifolds of M_Y exist and are C^r. It is easy to see that their tangent spaces over M_Y are transversal and intersect in TM_Y. Let Q^- and Q^+ be the orthogonal complements of TM_Y in the tangent space over M_Y of the stable and unstable manifolds. Then

$$TR^n|M_Y = TM_Y \oplus Q^- \oplus Q^+.$$

We compute the type numbers of Y with respect to this splitting, and let $\nu^-(Y)$, $\nu^+(Y)$, $\sigma^-(Y)$, $\sigma^+(Y)$ be the suprema of the type numbers. Arguing as in Proposition 1 and Theorem 2, we see that these are upper semicontinuous functions of Y. Summarizing, we have

Theorem 5. Under the hypotheses of Theorem 3 there is a C^{r-1} hyperbolic splitting $TR^n \mid M_Y = TM_Y \oplus Q^- \oplus Q^+$. The functions $\nu^-(Y)$, $\nu^+(Y)$, $\sigma^-(Y)$, $\sigma^+(Y)$ are upper-semicontinuous.

VI. 1. Invariant transversals. We have not needed to assume the splitting of $TR^n|M$ is invariant under the flow. Invariant splittings should be useful, because they let us set up coordinates in which the flow has an especially simple form. In this section and the next we study existence and smoothness of invariant splittings.

Let X be a C^1 vector field on R^n, with flow F^t. Let M be a compact, connected C^1 manifold, properly embedded in R^n and invariant under X. Let $TR^n|M = TM \oplus N^- \oplus N^+$ be a continuous splitting such that $TM \oplus N^-$ and $TM \oplus N^+$ are invariant under DF^t for all t. Let π, π^- and π^+ be the projections on TM, N^- and N^+. Define

$$\begin{aligned} R_t^-(m) &= \|D(F^{-t} \mid M)(m)\| \, \|\pi^- DF^t(F^{-t}(m)) \mid N^-\|, \\ R_t^+(m) &= \|D(F^{-t} \mid M)(m)\| \, \|\pi^+ DF^t(F^{-t}(m)) \mid N^+\|, \\ \rho^-(m) &= \overline{\lim_{t\to\infty}} \, (R_t^-(m))^{1/t}, \\ \rho^+(m) &= \overline{\lim_{t\to-\infty}} \, (R_t^+(m))^{-1/t}. \end{aligned}$$

Theorem 6. If $\rho^-(m) < 1$ and $\rho^+(m) < 1$ for all $m \in M$, there are bundles I^- and I^+ in $TR^n|M$, homeomorphic to N^- and N^+ and invariant under DF^t for all t. $I^- \oplus I^+$ is transversal to TM.

Proof. We construct I^- as the graph of a continuous family of linear operators $u(m)\colon N_m^- \to T_mM$. The condition for invariance of the graph of u is

$$u(F^t(m))\cdot(\pi^- DF^t(m)(x + u(m)x)) = \pi DF^t(m)(x + u(m)x)$$

for all $m \in N_m^-$. We may omit x and deal with this as a functional equation for the family of operators u.

As in the previous theorems we prove uniqueness of a solution invariant under the flow at some fixed time. This guarantees the invariance under the flow at arbitrary time. Using an extension of the Uniformity Lemma we choose a large T such that

$$R_T^-(m) < 1/2$$

for all $m \varepsilon M$. Noting that $\pi^- DF^t u = 0$ because TM is invariant, we write the condition for invariance at time T:

$$u(m) = D(F^{-T}|M)(F^T(m))u(F^T(m))\pi^- DF^T(m) - D(F^{-T}|M)(F^T(m))\pi DF^T(m).$$

This is in the form of a linear contraction scheme for u, using the supremum over M of the operator norms of $u(m)$ to norm families u. Hence there is a unique solution u, and I^- exists. The construction of I^+ is similar to the construction of I^-. Transversality to TM is clear from the construction, so the theorem is proved.

If M is overflowing invariant and N^+ is absent, the arguments of the scheme for I^- are carried inward, not outward. The construction of continuous splittings gives rise to a boundary value problem which does not, in general, have a unique solution.

Note that Theorem 6 does not assume a hyperbolic splitting. It is applicable to the flow shown in Figure 9, as long as the normal attraction at P is stronger than the tangential attraction, and the normal rejection at Q is weaker than the tangential rejection.

VI. 2. Smooth invariant transversals. Any C^1 transversal bundle of appropriate dimension defines a coordinate system near M, as in Proposition 3. If the invariant transversals are C^1, they define preferred coordinates which may simplify the study of the flow near M. To simplify our construction we assume the given splitting is C^{r-1}. This is true, for example, if N^+ is absent and N^- is the normal bundle of M, or if N^+ and N^- are constructed as in Theorem 5.

Let $A^t(m) = D(F^{-t}|M)(m)$. Define

$$\textstyle\sum^-(m) = \overline{\lim_{t\to\infty}} \frac{\log ||(A^t(m))^{-1}||}{-\log R_t^-(m)},$$

$$\textstyle\sum^+(m) = \overline{\lim_{t\to-\infty}} \frac{\log ||(A^t(m))^{-1}||}{-\log R_t^+(m)}.$$

As with σ, we may define

$$\textstyle\sum^- (m) = \inf \{S: ||(A^t(m))^{-1}||\, (R_t^-(m))^S \to 0 \text{ as } t \to \infty\},$$

with a similar definition for $\sum^+$. If $\sum^- (m) < 1$, we have $||(A^t(m))^{-1}||\, ||A^t(m)|| \cdot ||\pi^- DF^t(F^{-t}(m)) \mid N^-|| \to 0$ as $t \to \infty$. But for any invertible linear map L, $||L||\, ||L^{-1}|| \geqq 1$, so $||\pi^- DF^t(F^{-t}(m)) \mid N^-|| \to 0$. Hence if $\sum^- (m) < 1$ and $\sum^+ (m) < 1$ for all $m \varepsilon M$ the splitting is hyperbolic.

Theorem 7. *Let X be a C^r vector field on R^n, $r \geqq 2$. Let M be a compact, connected C^r manifold invariant under X. Suppose a C^{r-1} splitting of $TR^n|M$ is given, with $\rho^-(m) < 1$, $\rho^+(m) < 1$, $\sum^- (m) < 1/(r - 1)$ for all $m \varepsilon M$. Then the bundle I^- of Theorem 6 is C^{r-1}.*

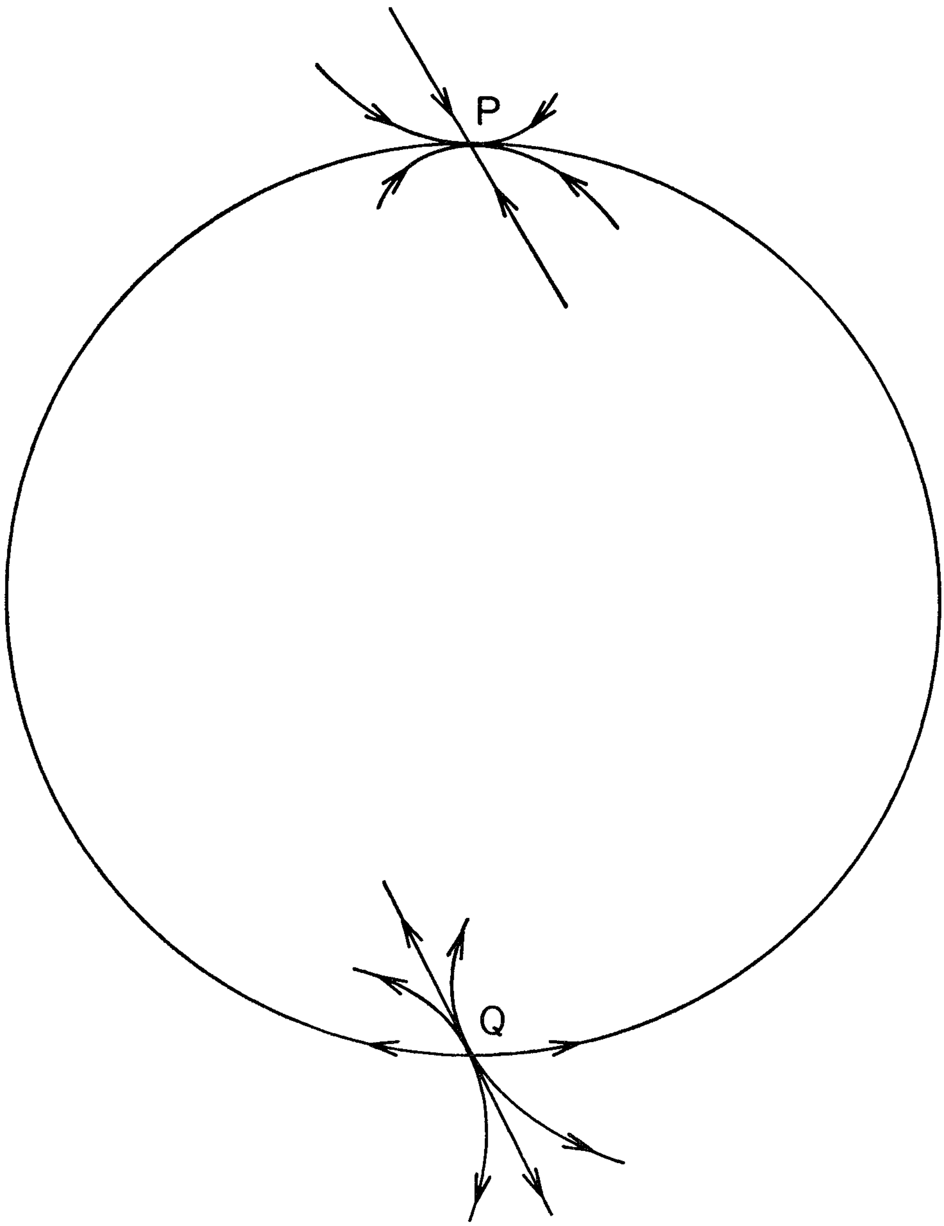

FIGURE 9

Proof. Using N^- and N^+ we set up C^{r-1} coordinates (x, y, z) near M, where $||x|| < 1$, $||y|| < \epsilon$, $||z|| < \epsilon$. x lies in R^l, where l is the dimension of M. y and z lie in R^p and R^q where p and q are the fiber dimensions of N^- and N^+. In coordinates F^T is expressed as

$$\xi = f(x, y, z),$$
$$\eta = g(x, y, z),$$
$$\zeta = h(x, y, z).$$

If T is large enough we have

$$||(D_1 f_{ij})^{-1}|| \, ||D_2 g_{ij}|| \, ||D_1 f_{ij}||^k < 1/2$$

for $0 \leqq k \leqq r - 1$, where all the derivatives are evaluated at $y = 0$, $z = 0$.

The invariant transversal is constructed as the graph of a family of maps u_i, where $u_i(x)$ is a linear map from R^p to R^l. Let

$$T_{ij} u_j(x) = (D_1 f_{ij}(x, 0, 0))^{-1} u_j(f_{ij}(x, 0, 0)) D_2 g_{ij}(x, 0, 0)$$
$$- (D_1 f_{ij}(x, 0, 0))^{-1} D_2 f_{ij}(x, 0, 0).$$

The condition for invariance of the transversals is $u_i = T_{ij} u_j$, wherever this makes sense. As in the smoothness proof in Theorem 1 we combine the T_{ij}'s using a partition of unity to get a contraction scheme for the u_i's. Each time this scheme is differentiated the highest order term gains a factor of $D_1 f_{ij}(x, 0, 0)$. Thus the derivatives up to the $(r - 1)$-st of the u_i's satisfy linear contraction relations, and $u_i \in C^{r-1}$.

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