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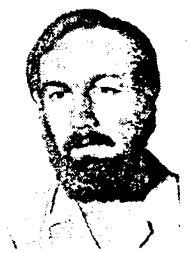

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Persistence of Excitation in Extended Least Squares

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Abstract—**In least-squares parameter estimation schemes, "persistency of excitation" conditions on the plant states are required for consistent estimation. In the case of extended least squares, the persistency conditions are on the state estimates. Here, these "persistency of excitation" conditions are translated into "sufficiently rich" conditions on the plant noise and inputs. In the case of adaptive minimum variance control schemes, the "sufficiently rich" conditions are on the noise and specified output trajectory. With sufficiently rich input signals, guaranteed convergence rates of prediction errors improve, and it is conjectured that the algorithms are consequently more robust.**

I. Introduction

IN LEAST-SQUARES identification schemes, when the underlying assumption is that the measurements y_k are linear in both the unknown parameters θ and the states of the plant x_k, it is not surprising that consistent estimation of θ from the measurements demands that the states be persistently exciting in some sense [1]. Since the plant states x_k cannot be manipulated directly, but only via the plant inputs u_k, it is important to translate "persistency of excitation" conditions on the states of the plant, to "persistence of excitations" or "sufficiently rich" conditions on the plant inputs and noise.

In extended least squares, when state estimates are employed in lieu of the states, then the "persistency of excitation" conditions for consistent estimations are given in terms of the state estimates [2]. Again, it is important to translate these into conditions on the plant inputs and noise.

Consistent parameter estimation is obviously important in the case of plant identification. It is less obviously important in the case of N-step-ahead prediction or in closed-loop least-squares tracking error control. In [2], [3], the convergence of prediction of prediction errors/tracking

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$$\hat{P}_k = \hat{P}_{k-1} - \hat{P}_{k-1}\hat{x}_k\hat{x}_k'\hat{P}_{k-1}\left(\hat{\gamma}_k^{-1} + \hat{x}_k'\hat{P}_{k-1}\hat{x}_k\right)^{-1} = \left(\sum_0^k \hat{\gamma}_i\hat{x}_i\hat{x}_i'\right)^{-1} \tag{2.4b}$$

where $\hat{x}_k$ is given from the inverse system of (2.3) as

$$\hat{x}_{k+1} = \left(F + G_1\hat{\theta}_k'\right)\hat{x}_k + G_2\hat{w}_k + G_3u_k + G_4z_k \tag{2.5a}$$

$$\hat{w}_k = z_k - \hat{z}_{k/k}, \qquad \hat{z}_{k/k} = \hat{\theta}_k'\hat{x}_k. \tag{2.5b}$$

For the ARMAX model (2.1) we have $x_k' = [z_{k-1}z_{k-2}\cdots z_{k-\bar{a}}u_{k-1}u_{k-2}\cdots u_{k-\bar{b}}w_{k-1}w_{1-2}\cdots w_{k-\bar{c}}]$.

iii) Stability Assumption: For open-loop estimation, the plant is assumed for convenience to be asymptotically stable with inputs bounded in L_∞.

iv) Positive Real Condition: For the convergence theory of [2], we require that $\{\frac{1}{2}I + \theta'[zI - (F + G\theta')^{-1}G]\}^{-1} \equiv \{[I - \theta'(zI - F)^{-1}G]^{-1} - \frac{1}{2}I\}$ is strictly positive real where $G = G_1 - G_2$ and z is the z-transform variable. For the ARMAX specialization, this is precisely equivalent to $[C^{-1}(z) - \frac{1}{2}]$ is strictly positive real where $C(z) = 1 + c_1z^{-1} + c_2z^{-2} + \cdots c_{\bar{c}}z^{-\bar{c}}$.

v) Convergence Results. The relevant results which can be extracted from the theoretical approach of [2] are now summarized.

Theorem 2.1: Consider the models (2.1)–(2.3) and estimation schemes (2.4), (2.5) under the stability and positive real condition, taking $\hat{\gamma}_k = 1$. Then with $\tilde{x}_k = x_k - \hat{x}_k$, $\tilde{z}_{k/k-1} = z_k - \hat{z}_{k/k-1}$,

$$\lim_{k\to\infty} \frac{1}{k}\sum_o^k \|\hat{x}_i\|^2 < \infty,$$

$$\limsup_{k\to\infty} \sum_0^k i^{-1}\|\tilde{x}_i\|^2 < \infty \quad \text{a.s.} \tag{2.6}$$

$$\sum_0^\infty k^{-1}\|\tilde{z}_{k/k-1} - w_k\|^2 < \infty \quad \text{a.s.} \tag{2.7}$$

Moreover, with the persistently exciting condition

$$\lim_{k\to\infty}\inf \frac{1}{k}\sum_0^k \hat{x}_i\hat{x}_i' > 0. \tag{2.8}$$

Then, denoting $\tilde{\theta}_k = \theta - \hat{\theta}_k$,

$$\lim_{k\to\infty} \tilde{\theta}_k = 0 \quad \text{a.s.} \tag{2.9}$$

Also, with the related persistently exciting condition bound, for some $\epsilon > 0$

$$\sum_0^\infty k^{-\epsilon}\hat{x}_k'\hat{P}_k\hat{x}_k < \infty, \tag{2.10}$$

together with (2.8); then (2.7), (2.9) are strengthened as

$$\sum_0^\infty k^{-\epsilon}\|\tilde{z}_{k/k-1} - w_k\|^2 < \infty,$$

$$\lim_{k\to\infty} k^{(1-\epsilon)}\|\tilde{\theta}_k\|^2 < \infty \quad \text{a.s..} \tag{2.11}$$

Proof: Follows from Theorem 3.1 of [2]. The result (2.11) requires, in the notation of [2], the selection $\gamma_k = \delta_k = k^{-\epsilon}$ as suggested from the work of [9]. Note that with this γ_k selection, (2.10) ensures that the conditions for the Theorem 3.1 of [2] are satisfied. ▽▽▽

Remarks:

1) The above results extend, as in [2], to the N-step-ahead prediction case. Details are not included here.

2) The result (2.7) can be strengthened in the absence of persistence of excitation using a more sophisticated $\hat{\gamma}_k$ selection—as for adaptive control in Section IV.

3) The result (2.6a) and Assumption (2.8) in effect bounds the condition member of $\hat{P}_k$.

4) The specific γ_k, δ_k selection in the above proof and persistence condition (2.10) is not studied in [2], but is motivated here by the desire to exploit the advantages of sufficiently rich inputs. Thus, the specific results of the above theorem are novel, although they are derived using essentially the same techniques as in [2].

vi) Sufficiently Rich Plant States: Here, we translate the persistence conditions (2.8), (2.10) on $\hat{x}_k$ to persistence conditions on the plant states x_k as an intermediate step for translation to sufficiently rich conditions on the inputs u_k and w_k.

Lemma 2.1: Under the condition of the first part of Theorem 2.1, the plant state persistence implies estimator state persistence as

$$\liminf_{k\to\infty} \frac{1}{k}\sum_0^k x_ix_i' > 0 \Rightarrow \liminf_{k\to\infty} \frac{1}{k}\sum_0^k \hat{x}_i\hat{x}_i' > 0 \quad \text{a.s.} \tag{2.12}$$

$$\left.\begin{aligned} &\sum_0^\infty k^{-(1+\epsilon)}\|x_k\|^2 < \infty \\ &\liminf_{k\to\infty} \frac{1}{k}\sum_0^k x_ix_i' > 0 \end{aligned}\right\} \Rightarrow \sum_0^\infty k^{-\epsilon}\hat{x}_k'\hat{P}_k\hat{x}_k < \infty \quad \text{a.s.} \tag{2.13}$$

Proof: From (2.6), application of the Kronecker lemma [12] gives $\lim 1/k\sum_0^k\|\tilde{x}_i\|^2 = 0$ a.s. Also, adding $(\hat{x} + \tilde{x})(\hat{x} + \tilde{x})' = xx'$ to $(\hat{x} - \tilde{x})(\hat{x} - \tilde{x})' > 0$ gives $2\hat{x}\hat{x}' + 2\tilde{x}\tilde{x}' \geqslant xx'$ from which $2/k\sum_0^k\hat{x}_i\hat{x}_i' \geqslant 1/k\sum_0^k x_ix_i' - 2/k\sum_0^k\|\tilde{x}_i\|^2$. Taking limits, the results (2.12) follows. Likewise, since $\hat{x} = x - \tilde{x}$, then $\|\hat{x}\|^2 \leqslant 2\|x\|^2 + 2\|\tilde{x}\|^2$ and thus,

$$\sum_0^\infty k^{-(1+\epsilon)}\|\hat{x}_k\|^2 \leqslant 2\sum_0^\infty k^{-(1+\epsilon)}\|x_k\|^2 + 2\sum_0^\infty k^{-(1+\epsilon)}\|\tilde{x}_k\|^2 < \infty \quad \text{a.s.} \tag{2.14}$$

The last inequality follows from (2.6) and the lemma assumption. Now applying (2.8), then for some $\kappa > 0$

$$\sum_0^\infty k^{-\epsilon}\hat{x}_k'\hat{P}_k\hat{x}_k \leqslant \kappa\sum_0^\infty k^{-(1+\epsilon)}\|\hat{x}_k\|^2.$$

From (2.14), the result (2.13b) follows. ▽▽▽

theory and indeed appears more concise than the scalar variable theory of [5], [6], [8].

2) The following lemma is useful when there are stochastic inputs such as w_k in model (2.3).

Lemma 3.3: The "sufficiently rich" conditions on the plant inputs and noise are implied as follows:

$$\lim_{m\to\infty}\left(\sum_0^m E[\bar{v}_k\bar{v}_k'|F_{k-1}]\right)^{-1}=0 \Rightarrow \lim_{m\to\infty}\left(\sum_0^m \bar{v}_k\bar{v}_k'\right)^{-1}=0 \quad \text{a.s.} \quad (3.9)$$

$$\liminf_{m\to\infty}\frac{1}{m}\sum_0^m E[\bar{v}_k\bar{v}_k'|F_{k-1}]>0 \Rightarrow \liminf_{m\to\infty}\frac{1}{m}\sum_0^m \bar{v}_k\bar{v}_k'>0 \quad \text{a.s.} \quad (3.10)$$

The implication (3.10) requires also that $E[\|\bar{v}_k\|^4]$ is bounded in terms of $E[\|\bar{v}_k\|^2]$.

Proof: Follows from standard arguments (see Appendix).

Remarks:

1) For the signal model (2.3) when $v_k'=[u_k'w_k']$ and u_k is known and w_k satisfies (2.2), then, since $E[w_k|F_{k-j}]=0$ for $j\geq 1$,

$$E\left[v_{k+j}v_{k+l}'|F_{k-1}\right]=\operatorname{diag}\left\{u_{k+j}u_{k+l}',\ E\left[w_{k+j}w_{k+l}'|F_{k-1}\right]\right\}.$$

Thus, the contributions of u_k and w_k to $E[\bar{v}_k\bar{v}_k'|F_{k-1}]$ are disjoint so that the contribution of w_k and u_k can be studied separately. Thus, consider the case when all the states are controllable through the input w_k and we can apply the theory above with $v_k=w_k$. Then a sufficient condition for persistence is that w_k be "white" zero-mean noise with a nonsingular covariance, or, more precisely, that

$$E\left[w_{k+j}w_{k+l}'|F_{k-1}\right]\geq\sigma^2 I \qquad \text{for } j=l\geq 0 \text{ and zero otherwise.} \quad (3.11)$$

For the case of periodic or almost periodic inputs, u_k, then it is well known [7] that if there are a sufficient number[2] of frequencies present, the inputs u_k are persistent in the required sense.

2) The above lemmas and remarks, and those in Section II, allow the persistence conditions on the state estimates $\hat{x}_k$ for the schemes of sections II, namely, (2.8), to be translated to conditions on the plant inputs u_k and noise w_k. For the related condition (2.10), then Lemma 2.2 suggests that we seek bounds on the control u_k and noise w_k to achieve the state bound as in the following lemma.

3) Although the derivations of the above lemma apply standard arguments, there are, to our knowledge, no such results in the literature and yet the need for such is clear as noted in the above remarks.

[2]The number is the smallest integer bounded below by $n/2$ where n is the dimension of x_k.

Lemma 3.4: For the asymptotically stable plant (2.3), with (2.2a) satisfied, then

$$\sum_0^\infty k^{-(1+\epsilon)}\|u_k\|^2<\infty \Rightarrow \sum_0^\infty k^{-(1+\epsilon)}\|x_k\|^2<\infty. \quad (3.12)$$

Proof: First, we claim that (2.2a) implies that $\sum_0^\infty k^{-(1+\epsilon)}\|w_k\|^2<\infty$. To see this, note that (2.2a) gives

$$\sum_0^\infty k^{-(1+\epsilon)}E\left[\|w_k\|^2|F_{k-1}\right]\leq\sigma_w^2\sum_0^\infty k^{-(1+\epsilon)}<\infty.$$

Also, since

$$\sum_0^\infty k^{-(1+\epsilon)}\left(\|w_k\|^2-E\left[\|w_k\|^2|F_{k-1}\right]\right)<\infty,$$

by virtue of a result in the Appendix, the claim is established.

Next, perceive that (3.12), under the condition $\sum_0^\infty k^{-(1+\epsilon)}\|w_k\|^2<\infty$, is merely a bounded-input, bounded-output (state) property of asymptotically stable linear systems. The property is a variation on the standard one [10], and is proved using the same approach as the standard one, but, in addition, exploiting the fact that $k^{-(1+\epsilon)}$ is monotonically decreasing. A critical intermediate step is that with $A=F+(G_1+G_4)\theta'$,

$$\sum_{i=0}^k \|(k+1)^{-1/2(1+\epsilon)}A^{k-i}i^{1/2(1+\epsilon)}\|\leq\sum_{i=0}^k\|A\|^{k-i}<\kappa<\infty$$

for some κ and all k. ▽▽▽

Extended Least Squares: Summarizing the above results for the extended least-squares schemes of Section II gives an extension to Theorem 2.1 as follows.

Theorem 3.1: Consider the scheme of Theorem 2.1, namely, models (2.1)–(2.3) and estimation scheme (2.4), (2.5) with $\hat{\gamma}_k=1$. Then:

i) sufficiently rich inputs u_k and noise w_k to achieve persistance of excitation of the state estimate as in (2.8), and consequently the consistency result (219) via Theorem 2.1, satisfy

$$\liminf_{m\to\infty}\frac{1}{m}\sum_0^m E[\bar{v}_k\bar{v}_k'|F_{k-1}]>0 \quad (3.13)$$

where $\bar{v}_k'=[v_k'v_{k+1}'\cdots v_{k+n-1}']$ and v_k is given in terms of u_k and w_k such that the states x_k are reachable from v_k as illustrated in (3.1);

ii) with persistence conditions (3.12) and (3.13) on the plant inputs and noise satisfied, the persistence condition (2.10) on the state estimates and consequent results (2.11) are satisfied.

Remarks:

1) To our knowledge, these are the first results translating extended least-squares persistence conditions on the state estimates to sufficiently rich conditions on the noise and plant inputs.

2) It is important to stress that persistence of excitation gives stronger guaranteed convergence rates of prediction

as before, $\bar{v}'_k = [v'_k v'_{k+1} \cdots v'_{k+n-1}]$, but now with v_k in terms of z_k, w_k rather than in terms of u_k, w_k. This gives sufficiently rich conditions on z_k, w_k to assure persistence of the states x_k and thereby those on the state estimates $\hat{x}_k$ of (2.8).

The next step is to translate the sufficiently rich conditions on z_k, w_k to sufficiently rich conditions of z^*_k, w_k. There is no corresponding step required for the open-loop case. The key observation is that the adaptive scheme of [2] guarantees from (2.7), the Kronecker lemma, and the fact that $z^*_k = \hat{z}_{k/k-1}$.

$$\lim_{m\to\infty} \frac{1}{m}\sum_0^m \|z_k - z^*_k - w_k\|^2 = 0 \quad \text{a.s.} \tag{4.10}$$

Consequently, z^*_k can be used in lieu of $(z_k - w_k)$ in the sufficiently rich conditions guaranteeing the persistence condition (2.8). To see this, and to clarify the above approach, let us consider special cases.

Special Case: Consider the signal model (2.3). If now $u_k \equiv 0$, then x_k can be generated from z_k via an inverse system

$$\begin{aligned} x_{k+1} &= [F + (G_1 + G_4)\theta']x_k + (G_2 + G_4)[z_k - \theta' x_k] \\ &= [F + (G_1 - G_2)\theta']x_k + (G_2 + G_4)z_k. \end{aligned}$$

This system has the structure of (3.1) with $v_k = z_k$ so that applying the theory of Section III, if z_k is "sufficiently rich," then x_k is "persistently exciting." More generally, when $u_k \neq 0$, a recursion on (2.3) gives

$$\begin{aligned} z_{k+1} = \theta'[F + (G_1 + G_4)\theta']x_k &+ \theta' G_3 u_k \\ &+ \theta'(G_2 + G_4)w_k + w_{k+1}. \end{aligned}$$

To keep the ideas simple, let $\theta' G_3$ be full rank; otherwise further recursions will be involved. Now observe that an inverse system can be designed to generate $\{x_k\}$ driven by $\{z_k\}$ rather than $\{u_k\}$ as follows:

$$\begin{aligned} x_{k+1} &= [F + (G_1 + G_4)\theta']x_k + G_3 u_k \\ u_k &= (\theta' G_3)^{-1}\{z_{k+1} - w_{k+1} - \theta'(G_2 + G_4)w_k \\ &\quad - \theta'[F + (G_1 + G_4)\theta']x_k\}. \end{aligned}$$

This system has the structure of (3.1) with definitions for v_k in terms of $(z_{k+1} - w_{k+1})$ and w_k, rather than in terms of u_k and w_k as in Section III. Let us define

$$\begin{aligned} \bar{v}'_k &= [(z_k - w_k)'(z_{k-1} - w_{k-1})' \cdots (z_{k-n} - w_{k-n})' \\ &\quad \cdot w'_{k-1} w'_{k-2} \cdots w'_{k-n-1}] \\ \bar{v}^{*\prime}_k &= [z^{*\prime}_k z^{*\prime}_{k-1} \cdots z^{*\prime}_{k-n} w'_{k-1} w'_{k-2} \cdots w'_{k-n-1}]. \end{aligned}$$

Then for some $\kappa > 0$,

$$\begin{aligned} \frac{1}{m}\sum_0^m \bar{v}^*_k \bar{v}^{*\prime}_k &\leq \frac{2}{m}\sum_0^m \bar{v}_k \bar{v}'_k + \frac{2}{m}\sum_0^m (\bar{v}_k - \bar{v}^*_k)(\bar{v}_k - \bar{v}^*_k)' \\ &\leq \frac{2}{m}\sum_0^m \bar{v}_k \bar{v}'_k + \frac{2\kappa}{m}\sum_0^m \|z_k - w_k - z^*_k\|^2. \end{aligned}$$

Taking limits as $m \to \infty$ and applying (4.10) gives that

$$\liminf_{m\to\infty} \frac{1}{m}\sum_0^m \bar{v}^*_k \bar{v}^{*\prime}_k > 0 \Rightarrow \liminf_{m\to\infty} \frac{1}{m}\sum_0^m \bar{v}_k \bar{v}'_k > 0;$$

likewise with conditioned expectations. For this example then, the claims made earlier are substantiated. More general results can be derived via the linear system inverse theory of [10], for example.

The results for adaptive control are now summarized.

Theorem 4.2: For the adaptive control scheme of Theorem 4.1, the persistently exciting condition bound (4.3) is satisfied almost surely if

$$\sum_0^\infty k^{-(1+\epsilon)}\|z^*_k\|^2 < \infty$$

and

$$\liminf_{m\to\infty} \frac{1}{m}\sum_0^m E[\bar{v}_k \bar{v}'_k | F_{k-1}] > 0 \tag{4.11}$$

where $\bar{v}'_k = [v'_k v'_{k+1} \cdots v'_{k+n-1}]$, and v_k is in terms of $(z_k - w_k)$ and w_k and such that x_k are the states of an inverse system driven by v_k. Moreover, z^*_k can be used in lieu of $(z_k - w_k)$.

Remarks:

1) The results of this section can be extended via the theory of [1] to adaptive control of nonminimum phase plants.

2) The above adaptive control results are dual in some sense to the open-loop estimation results of Section III. The interpretation of the condition (4.11) for the control case is more difficult than for the open-loop case.

3) It might be argued that z^*_k is a desired trajectory and is thus not available for manipulation so as to improve convergence rates (or robustness) as suggested is possible in the results of this section. This may be so, but then the possibility of adding to the specified z^*_k a dither signal which is not objectional in its effects could be explored. A tradeoff may be necessary between tracking error for the nominal model structure (perhaps partly due to an added dither signal) and its robustness (enhanced by a dither signal addition).

4) For the control of known linear plants, the separation theorem is a powerful tool. When the plant parameters must be identified on line as here, this theorem does not apply and there is inevitably a tradeoff between quality of state and parameter estimation, and the achievement of the control objectives. Also included in the tradeoffs can be robustness of the algorithm to cope with real plants having possibly higher dimensionality than the model, and/or time delays and nonlinearities. There has been built up over the years some theory, much empirical wisdom, and certain notions supported by simulation experience to assist in such tradeoffs. The results of this paper serve only to give aspects of this accumulated "knowledge" a firmer theoretical base.

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Simultaneous Identification and Adaptive Control of Unknown Systems Over Finite Parameter Sets

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***Abstract*—The problem considered is one of simultaneously identifying an unknown system while adequately controlling it. The system can be any fairly general discrete-time system and the cost criterion can be either of a discounted type or of a long-term average type, the chief restriction being that the unknown parameter lies in a finite parameter set. For a previously introduced scheme of identification and control based on "biased" maximum likelihood estimates, it is shown that 1) every Cesaro-limit point of the parameter estimates is "closed-loop equivalent" to the unknown parameter; 2) for both the discounted and long-term average cost criteria, the adaptive control law Cesaro-converges to the set of optimal control laws; and 3) in the case of the long-term average cost criterion, the actual cost incurred by the use of the adaptive controller is optimal and cannot be bettered even if one knew the value of the unknown parameter at the start.**

I. Introduction

We CONSIDER the problem of simultaneously identifying an unknown system while controlling it adequately. The significant features are as follows.

i) There is a finite set of possible values that the unknown parameter in the system may assume;

ii) the state and control spaces are Polish spaces; and

iii) the underlying cost criterion can be either a discounted cost criterion or a long-term average cost criterion.

The systems studied here are thus fairly general and subject mainly to restriction [i]. As shown in the section on applications, two examples of systems covered are:

a) *Nonlinear stochastic systems* of the type

$$x_{t+1}=f(x_t,u_t,\alpha)+\sigma(x_t,u_t,\alpha)w_t$$

with observations of the complete state x_t.

b) *Markov chain models* with countable or finite state spaces.

We focus our attention on a particular simultaneous identification and control scheme of the "certainty-equivalence" type where, periodically, an "estimate" of the unknown parameter is made, and a control corresponding to the "estimate" is then applied to the unknown system. The criterion used in selecting the estimate is, as in [1], [2], a particular modification of the likelihood function which is well suited to situations where interest centers not only on

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