

# PERSISTENCE PROPERTIES AND UNIQUE CONTINUATION OF SOLUTIONS OF THE CAMASSA-HOLM EQUATION

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ABSTRACT. It is shown that a strong solution of the Camassa-Holm equation, initially decaying exponentially together with its spacial derivative, must be identically equal to zero if it also decays exponentially at a later time. In particular, a strong solution of the Cauchy problem with compact initial profile can not be compactly supported at any later time unless it is the zero solution.

## 1. INTRODUCTION

This work is concerned with the nonperiodic Camassa-Holm equation

$$(1.1) \quad \partial_t u - \partial_t \partial_x^2 u + 3u \partial_x u - 2\partial_x u \partial_x^2 u - u \partial_x^3 u = 0, \quad t, x \in \mathbb{R}.$$

This equation appears in the context of hereditary symmetries studied by Fuchssteiner and Fokas [FF]. It was first written explicitly, and derived physically as a water wave equation by Camassa and Holm [CH], who also studied its solutions. Equation (1.1) is remarkable for its properties such as infinitely many conserved integrals, bi-hamiltonian structure or its non-smooth travelling wave solutions known as “peakons” (see formula (1.9)). It was also derived as an equation for geodesics of the  $H^1$ -metric on the diffeomorphism group, see [Mi]. For a discussion of how it relates to the theory of hereditary symmetries see [F]. The inverse scattering approach to the Camassa-Holm equation has also been developed in several works, for example see [CH], [CoMc], [Mc1], [BSS], and references therein.

A considerable amount of work has been devoted to the study of the corresponding Cauchy problem in both nonperiodic and periodic cases. Among these results, of relevance to the present paper will be the fact that (1.1) is locally well-posed (in Hadamard’s sense) in  $H^s(\mathbb{R})$  for any  $s > 3/2$ , see for example [LO], [R], [D]. The long time behaviour of solutions has been studied and conditions which guarantee their global existence and their finite blow up have been deduced. In particular, in [Mc1] a necessary and sufficient condition was established on the initial datum to guarantee finite time singularity formation for the corresponding strong solution. For further results in this direction we refer to [Mc1], [CoE] and the survey article [Mo] and references therein. For well-posedness results in the periodic case we refer to [HM1], [Mi], and [DKT], where the equation is studied in its integral-differential form (see (1.2) below) as an ODE on the space of diffeomorphisms of the circle. A recent result demonstrating that the solution map  $u_0 \rightarrow u$

for the Camassa-Holm equation is not locally uniformly continuous in Sobolev spaces can be found in [HM2].

Also the Camassa-Holm equation has been studied as an integrable infinite-dimensional Hamiltonian system, and several works have been devoted to several aspect of its scattering setting, see [CH], [CoMc], [Mc1], [BSS] and references therein.

It is convenient to rewrite the equation in its formally equivalent integral-differential form

$$(1.2) \quad \partial_t u + u \partial_x u + \partial_x G * \left( u^2 + \frac{1}{2} (\partial_x u)^2 \right) = 0,$$

where  $G(x) = \frac{1}{2} e^{-|x|}$ .

Our first objective here is to formulate decay conditions on a solution, at two distinct times, which guarantee that  $u \equiv 0$  is the unique solution of equation (1.1). The idea of proving unique continuation results for nonlinear dispersive equations under decay assumptions of the solution at two different times was motivated by the recent works [EKPV1], [EKPV2] on the nonlinear Schrödinger and the  $k$ -generalized Korteweg-de Vries equations respectively.

In the recent works [Co], [He] and [Z] it was shown that  $u$  cannot preserve compact support in a non-trivial time interval (i.e. for  $t \in [0, \epsilon]$ ,  $\epsilon > 0$ ) except if  $u \equiv 0$ . However, this result does not preclude the possibility of the solution having compact support at a later time. In fact, in [Z] the question concerning the possibility of a smooth solution of (1.1) having compact support at two different times was explicitly stated. In particular, our first result, Theorem 1.1, gives a negative answer to this question.

**Theorem 1.1.** *Assume that for some  $T > 0$  and  $s > 3/2$*

$$(1.3) \quad u \in C([0, T] : H^s(\mathbb{R}))$$

*is a strong solution of the IVP associated to the equation (1.2). If  $u_0(x) = u(x, 0)$  satisfies that for some  $\alpha \in (1/2, 1)$*

$$(1.4) \quad |u_0(x)| \sim o(e^{-x}), \quad \text{and} \quad |\partial_x u_0(x)| \sim O(e^{-\alpha x}) \quad \text{as } x \uparrow \infty,$$

*and there exists  $t_1 \in (0, T]$  such that*

$$(1.5) \quad |u(x, t_1)| \sim o(e^{-x}) \quad \text{as } x \uparrow \infty,$$

*then  $u \equiv 0$ .*

Notation We shall say that

$$|f(x)| \sim O(e^{ax}) \quad \text{as } x \uparrow \infty \quad \text{if} \quad \lim_{x \rightarrow \infty} \frac{|f(x)|}{e^{ax}} = L,$$

and

$$|f(x)| \sim o(e^{ax}) \quad \text{as } x \uparrow \infty \quad \text{if} \quad \lim_{x \rightarrow \infty} \frac{|f(x)|}{e^{ax}} = 0.$$

Remarks (a) Theorem 1.1 holds with the corresponding decay hypothesis in (1.4)-(1.5) stated for  $x < 0$ .

(b) The time interval  $[0, T]$  is the maximal existence time interval of the strong solution. This guarantees that the solution is uniformly bounded in the  $H^s$ -norm in this interval (see (2.12)), and that our solution is the strong limit of smooth ones such that the integration by parts in the proof (see (2.21), (2.29)) can be justified.

The proof of Theorem 1.1 will be a consequence of the following result concerning some persistence properties of the solution of the equation (1.2) in  $L^\infty$ -spaces with exponential weights.

**Theorem 1.2.** *Assume that for some  $T > 0$  and  $s > 3/2$*

$$(1.6) \quad u \in C([0, T] : H^s(\mathbb{R}))$$

*is a strong solution of the IVP associated to the equation (1.2) and that  $u_0(x) = u(x, 0)$  satisfies that for some  $\theta \in (0, 1)$*

$$(1.7) \quad |u_0(x)|, \quad |\partial_x u_0(x)| \sim O(e^{-\theta x}) \quad \text{as } x \uparrow \infty.$$

*Then*

$$(1.8) \quad |u(x, t)|, \quad |\partial_x u(x, t)| \sim O(e^{-\theta x}) \quad \text{as } x \uparrow \infty,$$

*uniformly in the time interval  $[0, T]$ .*

The following result establishes the optimality of Theorem 1.1 and tells us that a strong non-trivial solution of (1.2) corresponding to data with fast decay at infinity will immediately behave asymptotically, in the  $x$ -variable at infinity, as the ‘‘peakon’’ solution

$$(1.9) \quad v_c(x, t) = c e^{-|x-ct|}, \quad t > 0.$$

**Theorem 1.3.** *Assume that for some  $T > 0$  and  $s > 3/2$*

$$(1.10) \quad u \in C([0, T] : H^s(\mathbb{R}))$$

*is a strong solution of the IVP associated to the equation (1.2) and that  $u_0(x) = u(x, 0)$  satisfies that for some  $\alpha \in (1/2, 1)$*

$$(1.11) \quad |u_0(x)| \sim O(e^{-x}), \quad |\partial_x u_0(x)| \sim O(e^{-\alpha x}) \quad \text{as } x \uparrow \infty$$

*for some  $\alpha \in (1/2, 1)$ . Then*

$$(1.12) \quad |u(x, t)| \sim O(e^{-x}) \quad \text{as } x \uparrow \infty,$$

*uniformly in the time interval  $[0, T]$ .*

In the case when the solution  $u(x, t)$  possesses further regularity and its data  $u_0$  has stronger decay properties we shall give a more precise description of its behavior at infinity in the space variable.

As it was noted in both [Co] and [Z] in the case of compactly supported initial data  $u_0$  the difference  $h(x, t)$  of the solution and its second derivative, i.e.

$$(1.13) \quad h(x, t) = (1 - \partial_x^2)u(x, t),$$

remains compactly supported. Thus, if  $u_0$  is supported in the interval  $[a, b]$  in its lifespan, one has that  $h(x, t)$  has compact support in the time interval  $[\eta(a, t), \eta(b, t)]$ , (for the definition of  $\eta(\cdot, \cdot)$  see (2.38)).

**Theorem 1.4.** *Assume that for some  $T > 0$  and  $s > 5/2$*

$$(1.14) \quad u \in C([0, T] : H^s(\mathbb{R}))$$

*is a strong solution of the IVP associated to the equation (1.2).*

(a) *If  $u_0(x) = u(x, 0)$  has compact support, then for any  $t \in (0, T]$*

$$(1.15) \quad u(x, t) = \begin{cases} c_+(t) e^{-x}, & \text{for } x > \eta(b, t), \\ c_-(t) e^x, & \text{for } x < \eta(a, t). \end{cases}$$

(b) *If for some  $\mu > 0$*

$$(1.16) \quad \partial_x^j u_0 \sim O(e^{-(1+\mu)|x|}) \quad \text{as } |x| \uparrow \infty \quad j = 0, 1, 2,$$

*then for any  $t \in (0, T]$*

$$(1.17) \quad h(x, t) = (1 - \partial_x^2)u(x, t) \sim O(e^{-(1+\mu)|x|}) \quad \text{as } |x| \uparrow \infty,$$

*and*

$$(1.18) \quad \lim_{x \rightarrow \pm\infty} e^{\pm x} u(x, t) = c_{\pm}(t),$$

*where in (1.15), (1.18)  $c_+(\cdot)$ ,  $c_-(\cdot)$  denote continuous non-vanishing functions, with  $c_+(t) > 0$  and  $c_-(t) < 0$  for  $t \in (0, T]$ . Furthermore,  $c_+(\cdot)$  is strictly increasing function, while  $c_-(\cdot)$  is strictly decreasing.*

Theorem 1.4 tells us that, as long as it exists, the solution  $u(x, t)$  is positive at infinity and negative at minus infinity regardless of the profile of a fast-decaying data  $u_0 \neq 0$ .

Finally, as a consequence of some of the estimates obtained in the proofs of the previous results we shall show that any strong solution corresponding to a data with compact support blows up in finite time. For details on the structure of the blow-up we refer to [Mc1], [Mc2] and references therein. Other blow up results are discussed in [Mo].

**Corollary 1.1.** *Assume that for some  $T > 0$  and  $s > 5/2$*

$$(1.19) \quad u \in C([0, T] : H^s(\mathbb{R}))$$

is a strong solution associated to the equation (1.2) with initial data  $u(x, 0) = u_0(x)$  having compact support. Then the solution  $u(x, t)$  blows up in finite time, i.e. there exists a time interval  $[0, T^*)$  such that  $u \in C([0, T^*) : H^s(\mathbb{R}))$  and

$$(1.20) \quad \int_0^{T^*} \|\partial_x u(t)\|_{L^\infty} dt = \infty.$$

## 2. PROOF OF THE RESULTS

First, assuming the result in Theorem 1.2 we shall prove Theorem 1.1.

*Proof of Theorem 1.1.* Integrating equation (1.2) over the time interval  $[0, t_1]$  we get

$$(2.1) \quad u(x, t_1) - u(x, 0) + \int_0^{t_1} u \partial_x u(x, \tau) d\tau + \int_0^{t_1} \partial_x G * (u^2 + \frac{1}{2}(\partial_x u)^2)(x, \tau) d\tau = 0.$$

By hypothesis (1.4) and (1.5) we have

$$(2.2) \quad u(x, t_1) - u(x, 0) \sim o(e^{-x}) \quad \text{as } x \uparrow \infty.$$

From (1.4) and Theorem 1.2 it follows that

$$(2.3) \quad \int_0^{t_1} u \partial_x u(x, \tau) d\tau \sim O(e^{-2\alpha x}) \quad \text{as } x \uparrow \infty,$$

and so

$$(2.4) \quad \int_0^{t_1} u \partial_x u(x, \tau) d\tau \sim o(e^{-x}) \quad \text{as } x \uparrow \infty.$$

We shall show that if  $u \neq 0$ , then the last term in (2.1) is  $O(e^{-x})$  but not  $o(e^{-x})$ . Thus, we have

$$(2.5) \quad \int_0^{t_1} \partial_x G * (u^2 + \frac{1}{2}(\partial_x u)^2)(x, \tau) d\tau = \partial_x G * \int_0^{t_1} (u^2 + \frac{1}{2}(\partial_x u)^2)(x, \tau) d\tau \\ = \partial_x G * \rho(x),$$

where by (1.4) and Theorem 1.2

$$(2.6) \quad 0 \leq \rho(x) \sim O(e^{-2\alpha x}), \quad \text{so that } \rho(x) \sim o(e^{-x}) \quad \text{as } x \uparrow \infty.$$

Therefore

$$(2.7) \quad \partial_x G * \rho(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(x-y) e^{-|x-y|} \rho(y) dy \\ = -\frac{1}{2} e^{-x} \int_{-\infty}^x e^y \rho(y) dy + \frac{1}{2} e^x \int_x^{\infty} e^{-y} \rho(y) dy.$$

From (2.6) it follows that

$$(2.8) \quad e^x \int_x^\infty e^{-y} \rho(y) dy = o(1)e^x \int_x^\infty e^{-2y} dy \sim o(1)e^{-x} \sim o(e^{-x}),$$

and if  $\rho \neq 0$  one has that

$$(2.9) \quad \int_{-\infty}^x e^y \rho(y) dy \geq c_0, \quad \text{for } x \gg 1.$$

Hence, the last term in (2.5) and (2.7) satisfies

$$(2.10) \quad -\partial_x G * \rho(x) \geq \frac{c_0}{2} e^{-x}, \quad \text{for } x \gg 1$$

which combined with (2.1)-(2.3) yields a contradiction. Thus,  $\rho(x) \equiv 0$  and consequently  $u \equiv 0$ , see (2.5).  $\square$

*Proof of Theorem 1.3.* This proof is similar to that given for Theorem 1.1 and therefore it will be omitted.  $\square$

We proceed to prove Theorem 1.2.

*Proof of Theorem 1.2.* We introduce the following notations

$$(2.11) \quad F(u) = u^2 + \frac{(\partial_x u)^2}{2},$$

and

$$(2.12) \quad M = \sup_{t \in [0, T]} \|u(t)\|_{H^s}.$$

Multiplying the equation (1.2) by  $u^{2p-1}$  with  $p \in \mathbb{Z}^+$  and integrating the result in the  $x$ -variable one gets

$$(2.13) \quad \int_{-\infty}^\infty u^{2p-1} \partial_t u dx + \int_{-\infty}^\infty u^{2p-1} u \partial_x u dx + \int_{-\infty}^\infty u^{2p-1} \partial_x G * F(u) dx = 0.$$

The estimates

$$(2.14) \quad \int_{-\infty}^\infty u^{2p-1} \partial_t u dx = \frac{1}{2p} \frac{d}{dt} \|u(t)\|_{2p}^{2p} = \|u(t)\|_{2p}^{2p-1} \frac{d}{dt} \|u(t)\|_{2p}$$

and

$$(2.15) \quad \left| \int_{-\infty}^\infty u^{2p-1} u \partial_x u dx \right| \leq \|\partial_x u(t)\|_\infty \|u(t)\|_{2p}^{2p}$$

and Hölder's inequality in (2.13) yield

$$(2.16) \quad \frac{d}{dt} \|u(t)\|_{2p} \leq \|\partial_x u(t)\|_\infty \|u(t)\|_{2p} + \|\partial_x G * F(u)(t)\|_{2p}$$

and therefore, by Gronwall's inequality

$$(2.17) \quad \|u(t)\|_{2p} \leq \left( \|u(0)\|_{2p} + \int_0^t \|\partial_x G * F(u)(\tau)\|_{2p} d\tau \right) e^{Mt}.$$

Since  $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  implies

$$(2.18) \quad \lim_{q \uparrow \infty} \|f\|_q = \|f\|_\infty,$$

taking the limits in (2.17) (notice that  $\partial_x G \in L^1$  and  $F(u) \in L^1 \cap L^\infty$ ) from (2.18) we get

$$(2.19) \quad \|u(t)\|_\infty \leq \left( \|u(0)\|_\infty + \int_0^t \|\partial_x G * F(u)(\tau)\|_\infty d\tau \right) e^{Mt}.$$

Next, differentiating (1.2) in the  $x$ -variable produces the equation

$$(2.20) \quad \partial_t \partial_x u + u \partial_x^2 u + (\partial_x u)^2 + \partial_x^2 G * \left( u^2 + \frac{1}{2} (\partial_x u)^2 \right) = 0.$$

Again, multiplying the equation (2.20) by  $\partial_x u^{2p-1}$  ( $p \in \mathbb{Z}^+$ ) integrating the result in the  $x$ -variable and using integration by parts

$$(2.21) \quad \int_{-\infty}^{\infty} u \partial_x^2 u (\partial_x u)^{2p-1} dx = \int_{-\infty}^{\infty} u \partial_x \left( \frac{(\partial_x u)^{2p}}{2p} \right) dx = -\frac{1}{2p} \int_{-\infty}^{\infty} \partial_x u (\partial_x u)^{2p} dx$$

one gets the inequality

$$(2.22) \quad \frac{d}{dt} \|\partial_x u(t)\|_{2p} \leq 2 \|\partial_x u(t)\|_\infty \|\partial_x u(t)\|_{2p} + \|\partial_x^2 G * F(u)(t)\|_{2p}$$

and therefore as before

$$(2.23) \quad \|\partial_x u(t)\|_{2p} \leq \left( \|\partial_x u(0)\|_{2p} + \int_0^t \|\partial_x^2 G * F(u)(\tau)\|_{2p} d\tau \right) e^{2Mt}.$$

Since  $\partial_x^2 G = G - \delta$ , we can use (2.18) and pass to the limit in (2.23) to obtain

$$(2.24) \quad \|\partial_x u(t)\|_\infty \leq \left( \|\partial_x u(0)\|_\infty + \int_0^t \|\partial_x^2 G * F(u)(\tau)\|_\infty d\tau \right) e^{2Mt}.$$

We shall now repeat the above arguments using the weight

$$(2.25) \quad \varphi_N(x) = \begin{cases} 1, & x \leq 0, \\ e^{\theta x}, & x \in (0, N), \\ e^{\theta N}, & x \geq N \end{cases}$$

where  $N \in \mathbb{Z}^+$ . Observe that for all  $N$  we have

$$(2.26) \quad 0 \leq \varphi'_N(x) \leq \varphi_N(x) \quad \text{a.e. } x \in \mathbb{R}.$$

Using notation in (2.11), from equation (1.2) we obtain

$$(2.27) \quad \partial_t(u \varphi_N) + (u \varphi_N) \partial_x u + \varphi_N \partial_x G * F(u) = 0,$$

while from (2.20) we get

$$(2.28) \quad \partial_t(\partial_x u \varphi_N) + u \partial_x^2 u \varphi_N + (\partial_x u \varphi_N) \partial_x u + \varphi_N \partial_x^2 G * F(u) = 0,$$

We need to eliminate the second derivatives in the second term in (2.28). Thus, combining integration by parts and (2.26) we find

$$(2.29) \quad \begin{aligned} & \left| \int_{-\infty}^{\infty} u \partial_x^2 u \varphi_N (\partial_x u \varphi_N)^{2p-1} dx \right| \\ &= \left| \int_{-\infty}^{\infty} u (\partial_x u \varphi_N)^{2p-1} (\partial_x (\partial_x u \varphi_N) - \partial_x u \varphi_N') dx \right| \\ &= \left| \int_{-\infty}^{\infty} u \partial_x \left( \frac{(\partial_x u \varphi_N)^{2p}}{2p} \right) dx - \int_{-\infty}^{\infty} u \partial_x u \varphi_N' (\partial_x u \varphi_N)^{2p-1} dx \right| \\ &\leq 2(\|u(t)\|_{\infty} + \|\partial_x u(t)\|_{\infty}) \|\partial_x u \varphi_N\|_{2p}^{2p} \end{aligned}$$

Hence, as in the weightless case (2.19) and (2.24), we get

$$(2.30) \quad \begin{aligned} & \|u(t)\varphi_N\|_{\infty} + \|\partial_x u(t)\varphi_N\|_{\infty} \leq e^{2Mt} (\|u(0)\varphi_N\|_{\infty} + \|\partial_x u(0)\varphi_N\|_{\infty}) \\ &+ e^{2Mt} \int_0^t (\|\varphi_N \partial_x G * F(u)(\tau)\|_{\infty} + \|\varphi_N \partial_x^2 G * F(u)(\tau)\|_{\infty}) d\tau. \end{aligned}$$

A simple calculation shows that there exists  $c_0 > 0$ , depending only on  $\theta \in (0, 1)$  (see (1.7) and (2.25)) such that for any  $N \in \mathbb{Z}^+$

$$(2.31) \quad \varphi_N(x) \int_{-\infty}^{\infty} e^{-|x-y|} \frac{1}{\varphi_N(y)} dy \leq c_0.$$

Thus, for any appropriate function  $f$  one sees that

$$(2.32) \quad \begin{aligned} |\varphi_N \partial_x G * f^2(x)| &= \left| \frac{1}{2} \varphi_N(x) \int_{-\infty}^{\infty} \operatorname{sgn}(x-y) e^{-|x-y|} f^2(y) dy \right| \\ &\leq \frac{1}{2} \varphi_N(x) \int_{-\infty}^{\infty} e^{-|x-y|} \frac{1}{\varphi_N(y)} \varphi_N(y) f(y) f(y) dy \\ &\leq \frac{1}{2} \left( \varphi_N(x) \int_{-\infty}^{\infty} e^{-|x-y|} \frac{1}{\varphi_N(y)} dy \right) \|\varphi_N f\|_{\infty} \|f\|_{\infty} \\ &\leq c_0 \|\varphi_N f\|_{\infty} \|f\|_{\infty}. \end{aligned}$$

Since  $\partial_x^2 G = G - \delta$  the argument in (2.32) also shows that

$$(2.33) \quad |\varphi_N \partial_x^2 G * f^2(x)| \leq c_0 \|\varphi_N f\|_{\infty} \|f\|_{\infty}.$$



Thus, inserting (2.32)-(2.33) into (2.30) and using (2.11)-(2.12) it follows that there exists a constant  $\tilde{c} = \tilde{c}(M; T) > 0$  such that

$$(2.34) \quad \begin{aligned} & \|u(t)\varphi_N\|_\infty + \|\partial_x u(t)\varphi_N\|_\infty \leq \tilde{c}(\|u(0)\varphi_N\|_\infty + \|\partial_x u(0)\varphi_N\|_\infty) \\ & + \tilde{c} \int_0^t (\|u(\tau)\|_\infty + \|\partial_x u(\tau)\|_\infty) (\|u(\tau)\varphi_N\|_\infty + \|\partial_x u(\tau)\varphi_N\|_\infty) d\tau \\ & \leq \tilde{c} \left( \|u(0)\varphi_N\|_\infty + \|\partial_x u(0)\varphi_N\|_\infty + \int_0^t (\|u(\tau)\varphi_N\|_\infty + \|\partial_x u(\tau)\varphi_N\|_\infty) d\tau \right). \end{aligned}$$

Hence, for any  $N \in \mathbb{Z}^+$  and any  $t \in [0, T]$  we have

$$(2.35) \quad \begin{aligned} \|u(t)\varphi_N\|_\infty + \|\partial_x u(t)\varphi_N\|_\infty & \leq \tilde{c}(\|u(0)\varphi_N\|_\infty + \|\partial_x u(0)\varphi_N\|_\infty) \\ & \leq \tilde{c}(\|u(0)e^{\theta x}\|_\infty + \|\partial_x u(0)e^{\theta x}\|_\infty). \end{aligned}$$

Finally, taking the limit as  $N$  goes to infinity in (2.35) we find that for any  $N \in \mathbb{Z}^+$  and any  $t \in [0, T]$

$$(2.36) \quad \sup_{t \in [0, T]} (\|u(t)e^{\theta x}\|_\infty + \|\partial_x u(t)e^{\theta x}\|_\infty) \leq \tilde{c} (\|u(0)e^{\theta x}\|_\infty + \|\partial_x u(0)e^{\theta x}\|_\infty)$$

which completes the proof of Theorem 2.  $\square$

It remains to prove Theorem 1.4.

*Proof of Theorem 1.4.* A simple calculation shows that the solution  $u$  of equation (1.1) satisfies the identity

$$(2.37) \quad (1 - \partial_x^2)u \circ \eta (\partial_x \eta)^2 = (1 - \partial_x^2)u_0,$$

(it has a mechanical interpretation as conservation of spacial angular momentum). Here  $\eta = \eta(x, t)$  is the flow of  $u$ , that is

$$(2.38) \quad \begin{cases} \frac{d\eta(x, t)}{dt} = u(\eta(x, t), t), \\ \eta(x, 0) = x, \end{cases}$$

so that by the assumption and the standard ODE theory  $t \rightarrow \eta(t)$  is a smooth curve of  $C^1$ -diffeomorphisms of the line, close to the identity map and defined on the same time interval as  $u$  (see [Mi] for details in the periodic case). From (2.37) we then have

$$(2.39) \quad u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} h(y, t) dy = \frac{1}{2} e^{-x} \int_{-\infty}^x e^y h(y, t) dy + \frac{1}{2} e^x \int_x^{\infty} e^{-y} h(y, t) dy,$$

where

$$(2.40) \quad h(x, t) = (1 - \partial_x^2)u(x, t) = \frac{(1 - \partial_x^2)u_0(\eta^{-1}(x, t))}{(\partial_x \eta(\eta^{-1}(x, t), t))^2}.$$

Let us first prove part (a). Thus, from (2.40) it follows that if  $u_0$  has compact support in  $x$  in the interval  $[a, b]$ , then so does  $h(\cdot, t)$  in the interval  $[\eta(a, t), \eta(b, t)]$ , for any  $t \in [0, T]$ . Moreover, defining

$$(2.41) \quad E_+(t) = \int_{\eta(a,t)}^{\eta(b,t)} e^y h(y, t) dy \quad \text{and} \quad E_-(t) = \int_{\eta(a,t)}^{\eta(b,t)} e^{-y} h(y, t) dy,$$

one has from (2.40) that

$$(2.42) \quad u(x, t) = \frac{1}{2} e^{-|x|} * h(x, t) = \frac{1}{2} e^{-x} E_+(t), \quad x > \eta(b, t),$$

and

$$(2.43) \quad u(x, t) = \frac{1}{2} e^{-|x|} * h(x, t) = \frac{1}{2} e^x E_-(t), \quad x < \eta(a, t).$$

Hence, it follows that for  $x > \eta(b, t)$

$$(2.44) \quad u(x, t) = -\partial_x u(x, t) = \partial_x^2 u(x, t) = \frac{1}{2} e^{-x} E_+(t),$$

and for  $x < \eta(a, t)$

$$(2.45) \quad u(x, t) = \partial_x u(x, t) = \partial_x^2 u(x, t) = \frac{1}{2} e^x E_-(t).$$

Next, integration by parts, (2.44), (2.45), and the equation in (1.1) yield the identities

$$(2.46) \quad \begin{aligned} E_+(0) &= \int_{-\infty}^{\infty} e^y h(y, 0) dy = \int_{-\infty}^{\infty} e^y u_0(y) dy - \int_{-\infty}^{\infty} e^y \partial_x^2 u_0(y) dy \\ &= \int_{-\infty}^{\infty} e^y u_0(y) dy + \int_{-\infty}^{\infty} e^y \partial_x u_0(y) dy = 0 \end{aligned}$$

and

$$(2.47) \quad \begin{aligned} &\frac{dE_+(t)}{dt} \\ &= - \int_{-\infty}^{\infty} e^y u \partial_x u dy + \int_{-\infty}^{\infty} e^y \partial_x^2 (u \partial_x u) dy - \int_{-\infty}^{\infty} e^y \partial_x F(u) dy \\ &= e^y (\partial_x (u \partial_x u) - u \partial_x u) |_{-\infty}^{\infty} - e^y \left( u^2 + \frac{(\partial_x u)^2}{2} \right) |_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^y F(u) dy \\ &= \int_{-\infty}^{\infty} e^y \left( u^2 + \frac{(\partial_x u)^2}{2} \right) dy > 0. \end{aligned}$$

Therefore, in the life-span of the solution  $u(x, t)$ ,  $E_+(t)$  is an increasing function. Thus, from (2.46) it follows that  $E_+(t) > 0$  for  $t \in (0, T]$ .

Similarly, it is easy to see that  $E_-(t)$  is decreasing with  $E_-(0) = 0$ , therefore  $E_-(t) < 0$  for  $t \in (0, T]$ .

Taking  $c_{\pm}(t) = \frac{1}{2} E_{\pm}(t)$  we obtain (1.15).

Next, let us consider part (b). Since  $h(x, t) = (1 - \partial_x^2)u(x, t)$  satisfies the equation

$$(2.48) \quad \partial_t h(x, t) + u(x, t) \partial_x h(x, t) = -2 \partial_x u(x, t) h(x, t), \quad (x, t) \in \mathbb{R} \times [0, T],$$

an argument similar to that given in the proof of Theorem 1.2 shows that

$$(2.49) \quad \sup_{t \in [0, T]} \|h(t) e^{(1+\mu)|x|}\|_\infty \leq \tilde{c} \|h(0) e^{(1+\mu)|x|}\|_\infty,$$

with  $\tilde{c}$  depending only on  $M$  in (2.12) and  $T$ , and that for any  $\theta \in (0, 1)$

$$(2.50) \quad \partial_x^j u(t) \sim O(e^{-\theta|x|}) \quad \text{as } |x| \uparrow \infty \quad \text{for } j = 0, 1, 2.$$

Thus, the definitions in (2.41) make sense with the integrals extended to the whole real line and the computations in (2.46)-(2.47) can be carried out in the same fashion. Finally, using (2.49) in (2.39) we obtain (1.18).  $\square$

*Proof of Corollary. 1.1* Suppose that the support of  $u_0$  is contained in some interval  $[a, b]$ . According to a result of McKean [Mc2] the corresponding solution  $u(x, t)$  persists for all time only if either  $h_0(x) = (1 - \partial_x^2)u_0(x)$  is of one sign or

$$0 \neq h_0(x) \leq 0 \quad \text{when } a \leq x < x_o \quad \text{and} \quad 0 \neq h_0(x) \geq 0 \quad \text{when } x_o < x \leq b$$

for some  $x_o$  in  $(a, b)$ . Otherwise there exists  $T^* < \infty$  such that  $\|\partial_x u(t)\|_{L^\infty} \rightarrow \infty$  as  $t \uparrow T^*$ .

Assuming that the solution is global in time and computing as in (2.46) we find that

$$\int_a^b e^x h_0(x) dx = \int_a^b e^{-x} h_0(x) dx = 0.$$

This implies that  $h_0(x)$  must change sign. Furthermore, since  $e^x$  is strictly increasing we have the inequalities

$$-e^{x_o} \int_a^{x_o} h_0(x) dx > - \int_a^{x_o} e^x h_0(x) dx = \int_{x_o}^b e^x h_0(x) dx > e^{x_o} \int_{x_o}^b h_0(x) dx$$

and similarly

$$-e^{-x_o} \int_a^{x_o} h_0(x) dx < - \int_a^{x_o} e^{-x} h_0(x) dx = \int_{x_o}^b e^{-x} h_0(x) dx < e^{-x_o} \int_{x_o}^b h_0(x) dx.$$

These two sets of inequalities cannot both hold and so  $T^* < \infty$  is the maximal existence time of the solution  $u$ .

In order to prove (1.20), we can combine (2.16), (2.18) and (2.22) to get

$$\begin{aligned} e^{-2 \int_0^t \|\partial_x u(\tau)\|_\infty d\tau} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty) &\leq \|u(0)\|_\infty + \|\partial_x u(0)\|_\infty + \\ &+ \int_0^t e^{-\int_0^\tau \|\partial_x u(r)\|_\infty dr} (\|\partial_x G * F(u)(\tau)\|_\infty + \|p_x^2 G * F(u)(\tau)\|_\infty) d\tau \end{aligned}$$

for any  $t < T^*$ . Since

$$\|\partial_x G * F(u)\|_\infty \leq c \|\partial_x u\|_\infty (\|u\|_\infty + \|\partial_x u\|_\infty)$$

and

$$\|\partial_x^2 G * F(u)\|_\infty \leq c \|\partial_x u\|_\infty (\|u\|_\infty + \|\partial_x u\|_\infty)$$

we have

$$\begin{aligned} e^{-2 \int_0^t \|\partial_x u(\tau)\|_\infty d\tau} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty) &\leq \|u(0)\|_\infty + \|\partial_x u(0)\|_\infty + \\ &+ c \int_0^t \|\partial_x u(\tau)\|_\infty e^{-2 \int_0^\tau \|\partial_x u(r)\|_\infty dr} (\|u(\tau)\|_\infty + \|\partial_x u(\tau)\|_\infty) d\tau. \end{aligned}$$

Applying Gronwall's inequality again we obtain

$$\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty \leq (\|u(0)\|_\infty + \|\partial_x u(0)\|_\infty) \exp\left(c \int_0^t \|\partial_x u(\tau)\|_\infty d\tau\right)$$

valid for any  $t < T^*$ . This estimate implies that if  $\int_0^{T^*} \|\partial_x u(t)\|_\infty dt$  were finite then the solution  $u$  could be extended beyond time  $T^*$ . The corollary follows.  $\square$

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