

PERSISTENCE PROPERTY IN WEIGHTED SOBOLEV SPACES FOR NONLINEAR DISPERSIVE EQUATIONS

BY

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Abstract. We generalize the Abstract Interpolation Lemma proved by the authors in Carvajal and Neves (2010). Using this extension, we show in a more general context the persistence property for the generalized Korteweg-de Vries equation in the weighted Sobolev space with low regularity in the weight. The method used can be applied for other nonlinear dispersive models, for instance the multidimensional nonlinear Schrödinger equation.

1. Introduction. We are mainly concerned with the question of the persistence property in weighted Sobolev spaces for dispersive partial differential equations. Thus, the aim of this study is to generalize the Abstract Interpolation Lemma proved by the authors in [2] and to apply this new result to show, in a more general context, the persistence property of the initial-value problem for the nonlinear dispersive equations. To be more precise, let us recall the persistence result we established in [2] for the Cauchy Problem for higher order nonlinear Schrödinger equation, that is,

$$\begin{cases} \partial_t u + i a \partial_x^2 u + b \partial_x^3 u + i c |u|^2 u + d |u|^2 \partial_x u + e u^2 \partial_x \bar{u} = 0, & (t, x) \in \mathbb{R}^2, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where u is a complex valued function, a, b, c, d and e are real parameters and u_0 is a given initial data. We restate the main theorem of [2]:

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THEOREM 1.1. The IVP (1.1) is globally well-posed in $X^{2,\theta}$ for any $0 \leq \theta \leq 1$ fixed. Moreover, the solution u of (1.2) satisfies, for each $t \in [-T, T]$,

$$\|u(t)\|_{L^2(d\mu_\theta)}^2 \leq C \left(\|u_0\|_{L^2}^2 + \|u_0\|_{L^2(d\mu_\theta)}^2 + 1 \right),$$

where $C = C(\theta, \|u(t)\|_{H^s}, \|u(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u_{xx}(0)\|_{L^2}, T), s > 1/2$.

The notion of well-posedness for dispersive equations is given below, and the particular notation used throughout this paper is given in Section 1.1. Therefore, one of the main issues of this article is to extend the persistence property proved before for $\theta \in [0, 1]$ to more general values of the exponent θ . In particular, we explore our strategy on the generalized KdV equation (see (1.2) below); i.e., we consider the 1-dimensional case. However, the extension of the Abstract Interpolation Lemma proved in this paper to show the persistence property for more general exponents θ also allows us to demonstrate the persistence property for multi-dimensional equations as presented in this paper.

Consider the initial value problem (IVP)

$$\begin{cases} \partial_t u + a(u)\partial_x u + \partial_x^3 u = 0, & (t, x) \in \mathbb{R}^2, \\ u(0, x) = u_0(x), \end{cases} \tag{1.2}$$

where u is the real valued function we are seeking, u_0 is the initial data given in some convenient space, and $a(u)$ is a given C^∞ (weaker differentiability is sufficient for most results) real value function. Moreover, we may assume that $a(u)$ satisfies, as in Kato [4], the following condition:

$$\limsup_{|\lambda| \rightarrow \infty} \frac{2}{|\lambda|^6} \int_0^\lambda (\lambda - s) a(s) ds \leq 0. \tag{1.3}$$

Now, we introduce the typical notion of well-posedness that we are going to use throughout this paper. First, we consider the integral equation associated with (1.2),

$$u(t) = U(t) u_0 + \int_0^t U(t - \tau) a(u(\tau)) \partial_x u(\tau) d\tau, \tag{1.4}$$

where $U(t)$ is the unitary group solution of the linear KdV equation. It is not difficult to show that if u is a solution for the Cauchy Problem (1.2), then it satisfies (1.4). Then, we have the following.

DEFINITION 1.2. Let X, Y be two Banach spaces such that X is continuously embedded in Y . Suppose that, for each $u_0 \in X$, there exists $T > 0$ and a unique function

$$u \in C([0, T]; X) \tag{1.5}$$

satisfying (1.4) for all $t \in [0, T]$, and also $\partial_t u \in C((0, T]; Y)$. The Cauchy Problem (1.2) is said to be locally well-posed in X when the map $u_0 \mapsto u$ is continuous from X to $C([0, T]; X)$. If T can be taken arbitrarily large, then (1.2) is said to be globally rather than locally well-posed in X . Moreover, (1.5) implies the persistence property of the initial data.

If we consider the initial data in Sobolev spaces with sufficient regularity, for example in $H^s(\mathbb{R}), s \geq 2$, it is not difficult to prove the unique existence of the solution of the IVP (1.2) in the weighted Sobolev spaces. However, proving the persistence property,

also continuous dependence, is not so easy and it is quite involved when we are working in weighted Sobolev spaces. Our main focus in this paper is to show the persistence property with respect to more general exponents, as explained below. To accomplish this, in the present paper we establish an extension of the Abstract Interpolation Lemma proved in [2]. In fact, the interpolation extension proved here is quite general and applies to several dispersive equations provided they satisfy certain *a priori* estimates. These *a priori* estimates are related to the conserved quantities and are as follows:

$$\|u(t)\|_{L^2} \leq C\|u_0\|_{L^2}, \tag{1.6}$$

$$\|u(t)\|_{\dot{H}^{a(r)}} \leq A_1(\|u_0\|_{H^{a(r)}}), \tag{1.7}$$

and

$$\|u(t)\|_{L^2(d\hat{\mu}_r)} \leq C\|u_0\|_{L^2(d\hat{\mu}_r)} + A_2(\|u_0\|_{H^{a(r)}}), \tag{1.8}$$

where $a(r) \geq 1, r \in \mathbb{Z}^+, A_j$ are nonnegative continuous functions with $A_1(0) = 0, A_2(0) = 0$. Here, we consider that the IVP (1.2) satisfies (1.6)–(1.8) (for that we refer the reader to Kato [4]) as we are going to explain below. A typical equation that satisfies the properties (1.6)–(1.8) is the IVP associated to the generalized Korteweg-de Vries (gKdV) equation,

$$\begin{cases} \partial_t u + u^k \partial_x u + \partial_x^3 u = 0, & (t, x) \in \mathbb{R}^2, k = 1, 2, 3, \dots, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.9}$$

Before stating the main result of this work, we discuss some similar results previously obtained in the same direction of the main issue of this paper. The IVP associated to the nonlinear Schrödinger (NLS) equation

$$\begin{cases} i\partial_t u + \Delta u = \mu|u|^{\alpha-1}u, & \mu = \pm 1, \alpha > 1, x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(x, 0) = u_0(x) \end{cases} \tag{1.10}$$

has been studied in [3] for given data in the weighted Sobolev spaces. More precisely, the following theorem which deals with the persistence property has been proved in [3]:

THEOREM 1.3. Suppose that $u_0 \in H^s(\mathbb{R}^n) \cap L^2(|x|^{2m} dx), m \in \mathbb{Z}^+,$ with $m \leq \alpha - 1$ if α is not an odd integer.

A. If $s \geq m,$ then there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x, t)$ of the IVP (1.10) with

$$u \in C([-T, T]; H^s \cap L^2(|x|^{2m} dx)) \cap L^q([-T, T]; L^p_s \cap L^p(|x|^{2m} dx)). \tag{1.11}$$

B. If $1 \leq s < m,$ then (1.11) holds with $[s]$ instead of $m,$ and

$$\Gamma^\beta u = (x_j + 2it\partial_{x_j})^\beta u \in C([-T, T]; L^2) \cap L^q([-T, T]; L^p), \tag{1.12}$$

for any $\beta \in (\mathbb{Z}^+)^n$ with $|\beta| \leq m.$

The power m of the weight in Theorem 1.3 is assumed to be a positive integer. In the recent study of Nahas and Ponce [9], this restriction in m is relaxed by proving that the persistence property holds for positive real $m.$ To be more precise, the result in [9] is the following.

THEOREM 1.4. Suppose that $u_0 \in H^s(\mathbb{R}^n) \cap L^2(|x|^{2m} dx)$, $m > 0$, with $m \leq \alpha - 1$ if α is not an odd integer.

A. If $s \geq m$, then there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x, t)$ of the IVP (1.10) with

$$u \in C([-T, T]; H^s \cap L^2(|x|^{2m} dx)) \cap L^q([-T, T]; L^p_s \cap L^p(|x|^{2m} dx)). \tag{1.13}$$

B. If $1 \leq s < m$, then (1.13) holds with $[s]$ instead of m , and

$$\Gamma^b \Gamma^\beta u \in C([-T, T]; L^2) \cap L^q([-T, T]; L^p), \tag{1.14}$$

where $\Gamma^b = e^{i|x|^2/4t} 2^b t^b D^b (e^{-i|x|^2/4t})$ with $|\beta| = [m]$ and $b = m - [m]$.

In the next section (see the IVP (1.20), Theorem 1.11 and Remark 1.12) we establish the conditions to apply our technique, and hence we obtain similar results for the above NLS equation.

Now, we recall that Kato [4] studied the IVP (1.2) for the given initial data in the weighted Sobolev spaces and proved the following result.

THEOREM 1.5. Let r be a positive integer. Then, the IVP (1.2) is locally well-posed in weighted Sobolev spaces $\mathcal{X}^{2r,r}$ and globally well-posed in $\mathcal{X}^{2r,r}$ if the initial data satisfies $\|u_0\|_{L^2} < \gamma$, for some positive γ .

The proof of Theorem 1.5 is given in Kato’s Theorem 8.1 and Theorem 8.2; see [4].

In fact, it seems that the persistence property for dispersive equations has been widely discussed recently, as in Nahas [7] and Nahas and Ponce [8]. Moreover, the results in [7] were extended recently by Nahas to the generalized KdV equation; see [6]. In this paper we are interested in removing the requirement that the power of the weight in Theorem 1.5 be an integer by proving that a similar result is obtained for noninteger values of r . The main result of this article is the following.

THEOREM 1.6. Assume $r \geq 1$. If the IVP (1.2) is locally well-posed in H^s for $s \geq 2r$ and satisfies the *a priori* estimates (1.6)–(1.8), then the IVP (1.2) has the properties of the unique existence and persistence in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, for $s \geq 2r$ and $\theta \in [0, r]$.

One observes that in the above theorem r is a real number. Moreover, from the proof of Theorem 1.6, it can be inferred that if one has the local well-posedness result for given data in H^s and if the model under consideration satisfies *a priori* estimates (1.6)–(1.8), then with the help of the Abstract Interpolation Lemma, it is easy to prove the persistence property in weighted Sobolev spaces.

As an application of Theorem 1.6 we have the following result.

THEOREM 1.7. Let $r \geq 1$ be a real number. Then, the IVP for the gKdV equation (1.9) is locally well-posed in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, for $s \geq 2r$ and $0 \leq \theta \leq r$. Moreover, it is globally well-posed in $\mathcal{X}^{s,\theta}$, for $0 \leq \theta \leq r$ and $s \geq 2r$, when the initial data satisfies $\|u_0\|_{L^2} < \gamma$ for some positive γ .

The paper is organized as follows: In the rest of this section we fix the notation and some background used throughout the paper. The Abstract Interpolation Lemma is

given in Section 2. In Section 3, we first show some conserved quantities and prove a nonlinear estimate. Then, we formulate the approximate problems associated to the IVP (1.2) from them; then we gain continuous dependence in H^s norms, which is used to show mainly Theorem 1.7 at the end of this section.

1.1. *Notation and background.* We follow the notation introduced in our earlier paper [2]. For the sake of clarity we recall most of it here, clearly adapted for the multi-dimensional setting and for the more general case of $\theta \in [0, r]$, $r \geq 1$. Moreover, we present some results used through the paper.

We use dx to denote the Lebesgue measure on \mathbb{R}^n and

$$\begin{aligned} d\mu_\theta(x) &:= (1 + \|x\|^2)^\theta dx, \\ d\dot{\mu}_\theta(x) &:= \|x\|^{2\theta} dx \end{aligned}$$

to denote the Lebesgue-Stieltjes measures on \mathbb{R}^n . Hence, given a set X , a measurable function $f \in L^2(X; d\mu_\theta)$ means that

$$\|f\|_{L^2(X; d\mu_\theta)}^2 = \int_X |f(x)|^2 d\mu_\theta(x) < \infty.$$

When $X = \mathbb{R}^n$, we write: $L^2(d\mu_\theta) \equiv L^2(\mathbb{R}^n; d\mu_\theta)$, and for simplicity

$$L^2 \equiv L^2(d\mu_0), \quad L^2(d\mu) \equiv L^2(d\mu_1)$$

and similarly for the measure $d\dot{\mu}_\theta$. We will use the Lebesgue space-time $L_x^p \mathcal{L}_\tau^q$ endowed with the norm

$$\|f\|_{L_x^p \mathcal{L}_\tau^q} = \| \|f\|_{\mathcal{L}_\tau^q} \|_{L_x^p} = \left(\int_{\mathbb{R}} \left(\int_0^\tau |f(x, t)|^q dt \right)^{p/q} dx \right)^{1/p} \quad (1 \leq p, q < \infty).$$

When the integration in the time variable is on the whole real line, we use the notation $\|f\|_{L_x^p L_t^q}$. The notation $\|u\|_{L^p}$ is used when there is no doubt about the variable of integration. We adopt similar notation as above when p or q is ∞ . As usual, $H^s \equiv H^s(\mathbb{R}^n)$, $\dot{H}^s \equiv \dot{H}^s(\mathbb{R}^n)$ are the classic Sobolev spaces in \mathbb{R}^n , endowed respectively with the norms

$$\|f\|_{H^s} := \|\widehat{f}\|_{L^2(d\mu_s)}, \quad \|f\|_{\dot{H}^s} := \|\widehat{f}\|_{L^2(d\dot{\mu}_s)}.$$

We study in this work the solutions of dispersive equations in the weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, defined as

$$\mathcal{X}^{s,\theta} := H^s \cap L^2(d\mu_\theta), \tag{1.15}$$

with the norm

$$\|f\|_{\mathcal{X}^{s,\theta}} := \|f\|_{H^s} + \|f\|_{L^2(d\mu_\theta)}.$$

We remark that $\mathcal{X}^{s,r} \subseteq \mathcal{X}^{s,\theta}$, for all $s \in \mathbb{R}$ and $\theta \in [0, r]$. Indeed, using Hölder's inequality, we have

$$\|f\|_{L^2(d\dot{\mu}_\theta)} \leq \|f\|_{L^2}^{1-\theta/r} \|f\|_{L^2(d\dot{\mu}_r)}^{\theta/r}. \tag{1.16}$$

Moreover, we recall the classical notation of pseudo-differential operators. For any real number m , we define the set

$$S^m := \{a \in C^\infty(\mathbb{R}^{2n}; \mathbb{C}) : |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|}, \forall \alpha, \beta \in (\mathbb{Z}^+)^n\}.$$

For $a \in \mathcal{S}^m$, we consider the differential operator $a(x, D)$, defined for any $f \in \mathcal{S}(\mathbb{R}^n)$ in the following sense:

$$(a(\widehat{x, D})f)(\xi) = a(x, \xi)\widehat{f}(\xi).$$

The proof of the next two lemmas can be found in [9].

LEMMA 1.8. If $a \in \mathcal{S}^0$, then for each $b > 0$,

$$a(x, D) : L^2(\mathbb{R}^n; d\mu_b) \rightarrow L^2(\mathbb{R}^n; d\mu_b)$$

is a bounded differential operator.

LEMMA 1.9. Let $a, b > 0$. If $D^a f \in L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n; d\mu_b)$, then for each $\theta \in [0, 1]$,

$$\|D^{(1-\theta)a} f\|_{L^2(d\mu_{\theta b})} \leq C \|f\|_{L^2(d\mu_b)}^\theta \|D^a f\|_{L^2}^{1-\theta}. \tag{1.17}$$

Now, applying Lemma 1.8 we have the following:

LEMMA 1.10. Let $\beta \in (\mathbb{Z}^+)^n$ be a multi-index and $b > 0$ fixed. If $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\|\partial^\beta f\|_{L^2(d\dot{\mu}_b)} \leq C \|D^{|\beta|} f\|_{L^2(d\dot{\mu}_b)} + C \|f\|_{L^2(d\dot{\mu}_b)}. \tag{1.18}$$

Proof. Let us consider $a(x, \xi) = \frac{\xi^\beta}{(1 + |\xi|^2)^{|\beta|/2}}$; we can see that $a \in \mathcal{S}^0$. Then, applying Lemma 1.8, the associated operator $a(x, D)$ is bounded in $L^2(\mathbb{R}^n; d\mu_b)$. Therefore, it follows that

$$\|a(\cdot, D)g\|_{L^2(d\dot{\mu}_b)} \leq C \|g\|_{L^2(d\dot{\mu}_b)}. \tag{1.19}$$

If $\widehat{J^\beta f}(\xi) = (1 + |\xi|^2)^{|\beta|/2} \widehat{f}(\xi)$, considering $g = J^\beta f$, then $a(D)g = (1/i^{|\beta|})\partial^\beta f$, and the lemma is proved. □

Now, we consider the following evolution equation:

$$\begin{cases} \partial_t u + Lu + F(u, \nabla_x u) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \end{cases} \tag{1.20}$$

where the linear part of the equation Lu is defined by

$$\widehat{Lu}(\xi) = i h(\xi) \widehat{u}(\xi),$$

for some polynomial symbol $h(\xi)$ real valued, and $F(x, y)$ is a function with $F(0, 0) = 0$ (for the KdV equation $h(\xi) = -\xi^3$, $\xi \in \mathbb{R}$, $F(x, y) = a(x)y$, and for the nonlinear Schrödinger equation $h(\xi) = \sum_{k=1}^n \xi^{2e_k} = |\xi|^2$ where e_k is the k -th unit vector, $\xi \in \mathbb{R}^n$, $F(x, y) = |x|^{\alpha-1}x$, $\alpha > 1$).

THEOREM 1.11. Let $r \geq 1$ and $u \in C([-T, T]; \mathcal{X}^{s,r})$ be a smooth solution of the linear IVP

$$\begin{cases} \partial_t u + Lu = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \end{cases} \tag{1.21}$$

where the linear operator L is defined with symbol $h(\xi) = \sum_{j=1}^p C_j \xi^{\beta_j}$, $\xi \in \mathbb{R}^n$, $\beta_j \in (\mathbb{Z}^+)^n$, $|\beta_j| > 1$, $j = 1, \dots, p$. Then, u satisfies the inequality (1.8) with

$$a(r) = \left(\max_{j=1, \dots, p} |\beta_j| - 1 \right) r. \tag{1.22}$$

Proof. By the Bona-Smith approximation argument, we can suppose that $u(t) \in S(\mathbb{R}^n)$ or in some $\mathcal{X}^{s_0, r}$ with $s \ll s_0$. Moreover, without loss of generality, we can suppose that $h(\xi) = \xi^\beta$, for some multi-index β , $|\beta| > 1$. Multiplying (1.21) by $|x|^{2r} \bar{u}$, taking the real part and integrating, we have

$$0 = \partial_t \int |x|^{2r} |u|^2 dx + 2\text{Re} \int (x \cdot \bar{x})^r \bar{u} Lu dx. \tag{1.23}$$

Using the notation of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, \dots, n$, we have respectively the multi-nomial and Leibniz formula

$$\left(\sum_{j=1}^n x_j^2 \right)^r = \sum_{|\alpha|=r} \binom{r}{\alpha} x^{2\alpha}, \quad \partial^\alpha (f(\xi)g(\xi)) = \sum_{\eta \leq \alpha} \binom{\alpha}{\eta} \partial^\eta f(\xi) \partial^{\alpha-\eta} g(\xi). \tag{1.24}$$

Applying the definition of the Fourier transform, we obtain

$$\partial_\xi^\alpha \widehat{u}(\xi) = (-i)^{|\alpha|} \widehat{x^\alpha u}(\xi), \tag{1.25}$$

and by the multi-nomial formula, Plancherel equality and (1.25), we can write

$$\int |x|^{2r} |u(x)|^2 dx = (-1)^{|\alpha|} \int \sum_{|\alpha|=r} \binom{r}{\alpha} |\partial_\xi^\alpha \widehat{u}(\xi)|^2 d\xi. \tag{1.26}$$

Now, considering the second term in (1.23), we have

$$\begin{aligned} \int (x \cdot \bar{x})^r \bar{u} Lu dx &= \int \left(\sum_{j=1}^n x_j^2 \right)^r \bar{u} Lu dx \\ &= \int \sum_{|\alpha|=r} \binom{r}{\alpha} x^{2\alpha} \bar{u} Lu dx \\ &= \sum_{|\alpha|=r} \binom{r}{\alpha} \int x^\alpha \bar{u} x^\alpha Lu dx \\ &= \sum_{|\alpha|=r} \binom{r}{\alpha} \int \overline{x^\alpha u} \widehat{x^\alpha Lu} d\xi, \end{aligned} \tag{1.27}$$

where in the last equality we used Plancherel equality. By the Leibniz formula, identity (1.25) and definition of L with $h(\xi) = \xi^\beta$, we have

$$\begin{aligned} \int \overline{x^\alpha u} \widehat{x^\alpha Lu} d\xi &= (-1)^{|\alpha|} \int \overline{\partial_\xi^\alpha \widehat{u}(\xi)} \partial_\xi^\alpha \widehat{Lu}(\xi) d\xi \\ &= i (-1)^{|\alpha|} \int \overline{\partial_\xi^\alpha \widehat{u}(\xi)} \partial_\xi^\alpha (h(\xi) \widehat{u}(\xi)) d\xi \\ &= i (-1)^{|\alpha|} \int \overline{\partial_\xi^\alpha \widehat{u}(\xi)} \sum_{\eta \leq \alpha} \binom{\alpha}{\eta} (\partial^\eta \xi^\beta) (\partial_\xi^{\alpha-\eta} \widehat{u}(\xi)) d\xi. \end{aligned} \tag{1.28}$$

One observes that when $\eta = (0, \dots, 0) := \mathbf{0}$ in (1.28), we obtain

$$i (-1)^{|\alpha|} \int |\partial_\xi^\alpha \widehat{u}(\xi)|^2 d\xi,$$

and thus this term in (1.23) is equal to zero. We conclude from (1.26), (1.27) and (1.28) that

$$\begin{aligned}
 & 2\text{Re} \int |x|^{2r} \bar{u} Lu \, dx \\
 &= 2\text{Re} i (-1)^{|\alpha|} \sum_{|\alpha|=r} \binom{r}{\alpha} \int \overline{\partial_\xi^\alpha \widehat{u}(\xi)} \sum_{\substack{\eta \leq \alpha \\ \eta \neq \mathbf{0}}} \binom{\alpha}{\eta} (\partial^\eta \xi^\beta) (\partial_\xi^{\alpha-\eta} \widehat{u}(\xi)) \, d\xi \\
 &\leq \sum_{|\alpha|=r} \binom{r}{\alpha} \int \left(|\partial_\xi^\alpha \widehat{u}(\xi)|^2 + C_r \sum_{\substack{\eta \leq \alpha \\ \eta \neq \mathbf{0}}} \binom{\alpha}{\eta}^2 \left| \partial^\eta \xi^\beta \partial_\xi^{\alpha-\eta} \widehat{u}(\xi) \right|^2 \right) \, d\xi \\
 &\leq \int |x|^{2r} |u(x)|^2 \, dx + C_r \sum_{|\alpha|=r} \binom{r}{\alpha} \sum_{\substack{\eta \leq \alpha \\ \eta \neq \mathbf{0}}} \binom{\alpha}{\eta}^2 \int \left| \partial^\eta \xi^\beta \partial_\xi^{\alpha-\eta} \widehat{u}(\xi) \right|^2 \, d\xi. \tag{1.29}
 \end{aligned}$$

In order to estimate the second term in (1.29), we consider a multi-index $\eta \leq \alpha$, $\eta \neq \mathbf{0}$, and the expression $J(\alpha, \beta, \eta) = \|\partial^\eta \xi^\beta \partial_\xi^{\alpha-\eta} \widehat{u}(\xi)\|_{L_\xi^2}$. Then, for $\mathbf{0} \neq \eta \leq \alpha$, $\eta \leq \beta$, using (1.25), Plancherel equality, and the Leibniz formula, we obtain

$$\begin{aligned}
 J(\alpha, \beta, \eta) &= \|(\partial^\eta \xi^\beta) \partial_\xi^{\alpha-\eta} \widehat{u}(\xi)\|_{L_\xi^2} = \frac{\beta!}{(\beta-\eta)!} \|\xi^{\beta-\eta} \partial_\xi^{\alpha-\eta} \widehat{u}(\xi)\|_{L_\xi^2} \\
 &= \frac{\beta!}{(\beta-\eta)!} \|\xi^{\beta-\eta} \widehat{(x^{\alpha-\eta} u)}(\xi)\|_{L_\xi^2} \\
 &= \frac{\beta!}{(\beta-\eta)!} \|\partial_x^{\beta-\eta} (x^{\alpha-\eta} u)\|_{L_x^2} \\
 &\leq \frac{\beta!}{(\beta-\eta)!} \sum_{\nu \leq \beta-\eta} \binom{\beta-\eta}{\nu} \|(\partial^\nu x^{\alpha-\eta}) (\partial_x^{\beta-\eta-\nu} u)\|_{L_x^2}. \tag{1.30}
 \end{aligned}$$

Now, we proceed to estimate $\|(\partial^\nu x^{\alpha-\eta}) (\partial_x^{\beta-\eta-\nu} u)\|_{L_x^2}$. We know that the function $\partial^\nu x^{\alpha-\eta} \neq 0$ if $\nu \leq \alpha-\eta$ and zero otherwise. Thus we suppose that $\nu \leq \alpha-\eta$, $\nu \leq \beta-\eta$, and since $\eta \neq \mathbf{0}$, we have

$$r_0 = |\alpha - \eta - \nu| = |\alpha| - |\eta| - |\nu| = r - |\eta| - |\nu| < r$$

and

$$r_1 = |\beta - \eta - \nu| = |\beta| - |\eta| - |\nu| < |\beta|.$$

Therefore, applying Lemma 1.10 we obtain

$$\begin{aligned}
 \|(\partial^\nu x^{\alpha-\eta}) (\partial_x^{\beta-\eta-\nu} u)\|_{L_x^2} &= \frac{(\alpha-\eta)!}{(\alpha-\eta-\nu)!} \|x^{\alpha-\eta-\nu} \partial_x^{\beta-\eta-\nu} u\|_{L_x^2} \\
 &\leq \frac{(\alpha-\eta)!}{(\alpha-\eta-\nu)!} \| |x|^{r_0} \partial_x^{\beta-\eta-\nu} u \|_{L_x^2} \\
 &\leq C_{\alpha,\eta,\nu} (\| |x|^{r_0} D_x^{r_1} u \|_{L_x^2} + \| |x|^{r_0} u \|_{L_x^2}). \tag{1.31}
 \end{aligned}$$

We observe that $\eta \neq \mathbf{0}$ implies $|\eta| + |\nu| \geq 1$, and this inequality implies

$$1 - \frac{|\eta| + |\nu|}{r} \leq 1 - \frac{|\beta| - |\eta| - |\nu|}{(|\beta| - 1)r}.$$

Now we choose θ such that $1 - \frac{|\eta| + |\nu|}{r} \leq \theta \leq 1 - \frac{|\beta| - |\eta| - |\nu|}{(|\beta| - 1)r}$; it follows that $\theta \in [0, 1]$. Thus applying the Intermediate Value Theorem, there exist $b \in [0, r]$ and $a \in [0, (|\beta| - 1)r]$ such that $r_0 = \theta b$ and $r_1 = (1 - \theta)a$. Using Lemma 1.9 and the interpolation (1.16), we obtain

$$\|(\partial^\nu x^{\alpha-\eta})(\partial_x^{\beta-\eta-\nu} u)\|_{L_x^2} \leq C \|u\|_{L^2(d\mu_r)}^\theta \|u\|_{H^{(|\beta|-1)r}}^{1-\theta} + C \|x^r u\|_{L_x^2}^{(1-\kappa_0)} \|u\|_{L^2}^{\kappa_0},$$

where $\kappa_0 = r_0/r$, and this concludes the proof of the theorem. □

REMARK 1.12. i) One observes that for the generalized KdV equation, we have $a(r) = 2r$ and for the nonlinear Schrödinger equation $a(r) = r$.

ii) In order to obtain the estimate (1.8) for the Cauchy problem (1.20), we multiply (1.20) by $|x|^{2r}\bar{u}$, take the real part and integrate to obtain

$$0 = \partial_t \int |x|^{2r} |u|^2 dx + 2\text{Re} \int (x \cdot \bar{x})^r \bar{u} L u dx + 2\text{Re} \int (|x|^r \bar{u}) |x|^r F(u, \nabla_x u) dx. \tag{1.32}$$

Then, by Theorem 1.11 we only need to estimate the third term in (1.32) (for the nonlinear Schrödinger this term is zero). Using the Cauchy-Schwartz inequality,

$$2\text{Re} \int (|x|^r \bar{u}) |x|^r F(u, \nabla_x u) dx \leq 2 \| |x|^r \bar{u} \|_{L_x^2} \| |x|^r F(u, \nabla_x u) \|_{L_x^2}. \tag{1.33}$$

Thus we need an estimate of the following form:

$$\| |x|^r F(u, \nabla_x u) \|_{L_x^2} \leq C \| |x|^r \bar{u} \|_{L_x^2} A(\|u\|_{H^{a(r)}}), \tag{1.34}$$

and it is possible if for example $F(x, y) = x G(x, y)$, where G is a polynomial function and $a(r) > n/2 + 1$, in order to use immersion of u and $\nabla_x u$ in L_x^∞ and therefore $|G(u, \nabla_x u)| \leq A(\|u\|_{H^{a(r)}})$.

2. The Generalized Interpolation Lemma. In this section we generalize the Abstract Interpolation Lemma established by the authors in [2]. In fact, we extend in two directions: First, we generalize to the multi-dimensional setting. The second extension is concerned with the exponent θ of the weight.

Let $s > n/2$, $r \geq 1$ be fixed. For each $T > 0$, we consider a family \mathcal{A} of functions f from $[-T, T]$ in $H^s(\mathbb{R}^n)$, satisfying the following conditions:

(C1) The measure $\mathcal{L}^n(\{\xi \in \mathbb{R}^n; f(t, \xi) \neq 0\})$ is positive, where $\mathcal{L}^n(E)$ is the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$.

(C2) There exist positive constants C_0, \tilde{C}_0 and a function $A_0 \geq 0$ which do not depend on f and t such that

$$\|f(t)\|_{L^2}^2 \leq C_0 \|f(0)\|_{L^2}^2, \tag{2.35}$$

$$\|f(t)\|_{L^2(d\dot{\mu}_r)}^2 \leq \tilde{C}_0 \|f(0)\|_{L^2(d\dot{\mu}_r)}^2 + A_0(\|f(0)\|_{H^{a(r)}}). \tag{2.36}$$

(C3) For all $\theta \in [0, r]$, there exist $\Theta > 0$, which does not depend on f and t , and $\gamma_1 \in (0, 1/2)$ such that

$$\int_{\{|f(t)|^2 < \Theta\}} |f(t)|^2 d\dot{\mu}_\theta \leq \gamma_1 \int_{\mathbb{R}^n} |f(t)|^2 d\dot{\mu}_\theta. \tag{2.37}$$

(C4) There exist $R > 0$ and $\gamma_2 \in (0, 1)$ (both independent of f) such that

$$\int_{\mathbb{R}^n \setminus B(0,R)} |f(0)|^2 d\mu_r \leq \gamma_2 \int_{\mathbb{R}^n} |f(0)|^2 d\mu_r. \tag{2.38}$$

Clearly the set \mathcal{A} depends on the constants $C_0, \tilde{C}_0, R, \gamma_2$, and also the functions $A_0, \Theta(\theta)$. In the following, we present two different families which satisfy the conditions (C1)–(C4). The former example is a nonenumerable set of functions which are not necessarily solutions of a partial differential equation. On the other hand, the elements of the family in the second example are solutions of the dispersive equation (1.20).

EXAMPLE 2.1. Let $R_0, T > 0, r \geq 1$ be constants and $b > 0$ such that, for each $\theta \in [0, r]$,

$$\int_{\{R_0 \leq |\xi| \leq R_0 + b\}} |\xi|^{2\theta} d\xi \leq \frac{1}{3(T+1)^2} \int_{\{|\xi| \leq R_0\}} |\xi|^{2\theta} d\xi. \tag{2.39}$$

Let \mathcal{B}_0 be the set of continuous functions in \mathbb{R}^n such that

$$g(\xi) = \begin{cases} 0, & \text{if } |\xi| > R_0 + b, \\ L, & \text{if } |\xi| \leq R_0, \end{cases}$$

and $0 \leq g(\xi) \leq L$, where L is any positive real number, fixed. Now, we set

$$\mathcal{B}_1 = \{f(t, \xi) = g(\xi)(1 + |t|); t \in [-T, T], g \in \mathcal{B}_0\}.$$

Then, the family \mathcal{B}_1 satisfies the (C1)–(C4) conditions. Indeed, condition (C1) is clearly satisfied. The condition (C2) is satisfied with $C_0 = \tilde{C}_0 = 1 + T$. The condition (C4) is satisfied with $R = R_0 + b$ for all $\gamma_2 \in (0, 1)$, since the first integral in (2.38) is null. And the condition (C3) is satisfied with $\Theta = L^2$ and $\gamma_1 = 1/3$, since (2.39) implies

$$\begin{aligned} \int_{\{|f(t)|^2 < L^2\}} |\xi|^{2\theta} |f(t, \xi)|^2 d\xi &\leq (1 + T)^2 L^2 \int_{\{R_0 \leq |\xi| \leq R_0 + b\}} |\xi|^{2\theta} d\xi \\ &\leq \frac{(1 + T)^2 L^2}{3(1 + T)^2} \int_{\{|\xi| \leq R_0\}} |\xi|^{2\theta} d\xi \\ &= \frac{1}{3} \int_{\{|\xi| \leq R_0\}} |\xi|^{2\theta} |g(\xi)|^2 d\xi \\ &\leq \frac{1}{3} \int_{\{|\xi| \leq R_0\}} |\xi|^{2\theta} |f(t, \xi)|^2 d\xi \\ &\leq \frac{1}{3} \int_{\mathbb{R}^n} |\xi|^{2\theta} |f(t, \xi)|^2 d\xi. \end{aligned}$$

We remark that the following example will be used in the proofs of Theorems 1.6 and 1.7.

EXAMPLE 2.2. We consider the evolution equation (1.20) under the conditions in Remark 1.12. We assume that

$$u_0(x) \in \mathcal{X}^{s,r}, \quad u_0 \neq 0. \tag{2.40}$$

Now, let (u_0^k) be a sequence of regular functions (in $S(\mathbb{R}^n)$ or in some $\mathcal{X}^{s_0,\theta}$, with $s \ll s_0$) such that

$$u_0^k \rightarrow u_0 \quad \text{in } \mathcal{X}^{s,\theta} \quad \text{when } k \rightarrow \infty. \tag{2.41}$$

If the IVP (1.20) satisfies the conditions (1.6)–(1.8), and it is well-posed in $C([-T, T]; H^s)$, then the set of solutions

$$\mathcal{C} = (u^k(t)) \quad (k > N_0, \text{ for some } N_0 > 0) \tag{2.42}$$

of the IVP (1.20) with initial data u_0^k satisfies the conditions (C1)–(C4). Indeed, we have the following:

Condition (C1)

We prove this by contradiction. First, we suppose that

$$\forall N_0, \exists k \geq N_0, \exists t \in [-T, T]; \mathcal{L}^n(\{x \in \mathbb{R}^n; u^k(t, x) \neq 0\}) = 0.$$

Then, there exist $k_m \geq m, m = 1, 2, \dots$, and $t_m \in [-T, T]$ such that

$$u^{k_m}(t_m, x) = 0, \quad x - a.e. \tag{2.43}$$

By (2.41), (2.43) and the continuous dependence of the initial data, the sequence of solutions $u^k(t)$, associated to IVP (1.20) and initial data u_0^k , satisfies

$$\|u(t_m)\|_{H^s} = \|u^{k_m}(t_m) - u(t_m)\|_{H^s} \leq \sup_{t \in [-T, T]} \|u^{k_m}(t) - u(t)\|_{H^s} \xrightarrow{m \rightarrow \infty} 0. \tag{2.44}$$

As $u \in C([-T, T]; H^s), t_m \in [-T, T]$, by compactness we can assume that $t_m \rightarrow t_0 \in [-T, T]$. Thus $\|u(t_m)\|_{H^s} \rightarrow \|u(t_0)\|_{H^s}$ and by (2.44), it follows that $\|u(t_0)\|_{H^s} = 0$, which implies

$$u(t_0, x) = 0, \quad x - a.e.$$

By uniqueness of solutions, we have for any $t \in [-T, T], u(t, x) = 0$ almost everywhere. In particular, $u(0, x) = u_0 = 0$, which is to say a contradiction with (2.40).

Condition (C2)

It is a direct consequence given from the fact that the solution u of the IVP (1.20) satisfies the conditions (1.6) and (1.8).

Condition (C3)

We must prove that

$$\forall \theta \in [0, r], \exists \Theta > 0, \text{ s.t. } \forall k > N_0, \forall t \in [-T, T] \text{ and for some } \gamma_1 \in (0, 1/2),$$

we have

$$\int_{\{|u^k(t)|^2 < \Theta\}} |u^k(t)|^2 d\dot{\mu}_\theta \leq \gamma_1 \int_{\mathbb{R}^n} |u^k(t)|^2 d\dot{\mu}_\theta. \tag{2.45}$$

Again, we prove this condition by contradiction. We suppose that

$$\exists \theta \in [0, r], \forall \Theta > 0, \exists k > N_0, \exists t \in [-T, T] \text{ and } \forall \gamma_1 \in (0, 1/2)$$

and we have

$$\int_{\{|u^k(t)|^2 < \Theta\}} |u^k(t)|^2 d\dot{\mu}_\theta > \gamma_1 \int_{\mathbb{R}^n} |u^k(t)|^2 d\dot{\mu}_\theta.$$

Then, there exist $k_j > N_0, t_j \in [-T, T], j \in \mathbb{Z}^+$ and $\gamma_0 \in (0, 1/2)$ such that

$$\int_{\{|u^{k_j}(t_j)|^2 < 1/j\}} |u^{k_j}(t_j)|^2 d\dot{\mu}_\theta > \gamma_0 \int_{\mathbb{R}^n} |u^{k_j}(t_j)|^2 d\dot{\mu}_\theta. \tag{2.46}$$

Now, without loss of generality, we can suppose that

$$t_j \rightarrow t_0 \in [-T, T] \quad \text{when } j \rightarrow \infty. \tag{2.47}$$

Further, we consider the map

$$\Phi : \mathbb{Z}^+ \rightarrow \mathcal{V} = \{k_j; j \in \mathbb{Z}^+\}, \quad \Phi(j) = k_j, \tag{2.48}$$

and the following cases:

Case I, the set \mathcal{V} is not finite: In this case we can suppose that $k_j \rightarrow \infty$ when $j \rightarrow \infty$. By the immersion ($s > n/2$), (2.41) and continuous dependence of the initial data, the subsequence of solutions $u^{k_j}(t_j)$, associated to IVP (1.20) and initial data $u_0^{k_j}$, satisfies

$$\begin{aligned} |u^{k_j}(t_j, x) - u(t_0, x)| &\leq C\|u^{k_j}(t_j) - u(t_j)\|_{H^s} + C\|u(t_j) - u(t_0)\|_{H^s} \\ &\leq \sup_{t \in [-T, T]} \|u^{k_j}(t) - u(t)\|_{H^s} + C\|u(t_j) - u(t_0)\|_{H^s} \xrightarrow{j \rightarrow \infty} 0, \end{aligned} \tag{2.49}$$

where we have used that $u \in C([-T, T]; H^s)$.

Then, using (2.49), the Dominated Convergence Theorem and (2.46), we obtain a contradiction.

Case II, the set \mathcal{V} is finite: In this case, concerning the application Φ , there exists $k_q \in \mathcal{V}$ such that $\mathcal{V}_0 := \Phi^{-1}\{k_q\} = \{q_1, q_2, \dots\} \subseteq \mathbb{Z}^+$ must not be finite, with $q_j < q_{j+1}$, for each $j \in \mathbb{Z}^+$. Therefore, by (2.46) we get

$$\int_{\{|u^{k_q}(t_{q_j})|^2 < 1/q_j\}} |u^{k_q}(t_{q_j})|^2 d\mu_\theta > \gamma_0 \int_{\mathbb{R}^n} |u^{k_q}(t_{q_j})|^2 d\mu_\theta. \tag{2.50}$$

If $j \rightarrow \infty$, then $q_j \rightarrow \infty$ and by (2.47) $t_{q_j} \rightarrow t_0$. As $u^{k_q} \in C([-T, T]; H^s)$, by immersion, we have for any $x \in \mathbb{R}^n$,

$$|u^{k_q}(t_{q_j}, x) - u^{k_q}(t_0, x)| \leq C\|u^{k_q}(t_{q_j}) - u^{k_q}(t_0)\|_{H^s} \xrightarrow{j \rightarrow \infty} 0. \tag{2.51}$$

Therefore, arguing as previously in Case I and taking the limit in (2.50), we obtain a contradiction.

Condition (C4)

We prove: There exist $R > 0$ and $\gamma_2 \in (0, 1)$ such that for any $k > N_0$,

$$\int_{\mathbb{R}^n \setminus B(0, R)} |u^k(0, x)|^2 |x|^{2r} dx \leq \gamma_2 \int_{\mathbb{R}^n} |u^k(0, x)|^2 |x|^{2r} dx.$$

Again by contradiction, we suppose that

$$\forall R > 0, \forall \gamma_2 \in (0, 1), \exists k > N_0, \text{ such that}$$

$$\int_{\{|x| > R\}} |u^k(0, x)|^2 |x|^{2r} dx > \gamma_2 \int_{\mathbb{R}^n} |u^k(0, x)|^2 |x|^{2r} dx. \tag{2.52}$$

In particular, this proposition implies for any $m \in \mathbb{Z}^+$ that there exists $k_m > N_0$ such that

$$\int_{\{|x| > m\}} |u_0^{k_m}(x)|^2 |x|^{2r} dx > \left(1 - \frac{1}{m}\right) \int_{\mathbb{R}^n} |u_0^{k_m}(x)|^2 |x|^{2r} dx. \tag{2.53}$$

Let us consider the map

$$\Gamma : \mathbb{Z}^+ \rightarrow \mathcal{W} = \{k_m; m \in \mathbb{Z}^+\}, \quad \Gamma(m) = k_m, \tag{2.54}$$

and the following cases:

Case I, the set \mathcal{W} is not finite: In this case we can suppose that $k_m \rightarrow \infty$ when $m \rightarrow \infty$, and thus by (2.41), we obtain

$$\int_{\{|x|>m\}} |u_0^{k_m}(x) - u_0(x)|^2 |x|^{2r} dx \leq \int_{\mathbb{R}^n} |u_0^{k_m}(x) - u_0(x)|^2 |x|^{2r} dx \xrightarrow{m \rightarrow \infty} 0 \quad (2.55)$$

and

$$\int_{\mathbb{R}^n} |u_0^{k_m}(x)|^2 |x|^{2r} dx \xrightarrow{m \rightarrow \infty} \int_{\mathbb{R}^n} |u_0(x)|^2 |x|^{2r} dx. \quad (2.56)$$

Moreover, as

$$\begin{aligned} \int_{\{|x|>m\}} |u_0^{k_m}(x)|^2 |x|^{2r} dx &\leq \int_{\{|x|>m\}} |u_0^{k_m}(x) - u_0(x)|^2 |x|^{2r} dx \\ &\quad + \int_{\{|x|>m\}} |u_0(x)|^2 |x|^{2r} dx \xrightarrow{m \rightarrow \infty} 0, \end{aligned} \quad (2.57)$$

from (2.53)–(2.57), we arrive at a contradiction.

Case II, the set \mathcal{W} is finite: In this case, again concerning the application Γ , there exists $k_p \in \mathcal{W}$ such that $\mathcal{W}_0 := \Gamma^{-1}\{k_p\} = \{p_1, p_2, \dots\} \subseteq \mathbb{Z}^+$ is not finite, with $p_i < p_{i+1}$, $i \in \mathbb{Z}^+$. Therefore, by (2.53) we get

$$\int_{\{|x|>p_m\}} |u_0^{k_p}(x)|^2 |x|^{2r} dx > \left(1 - \frac{1}{p_m}\right) \int_{\mathbb{R}^n} |u_0^{k_p}(x)|^2 |x|^{2r} dx. \quad (2.58)$$

Similarly to Case I before, taking the limit in (2.58) when $m \rightarrow \infty$ ($p_m \rightarrow \infty$), we obtain a contradiction.

Now we pass to the Generalized Abstract Interpolation Lemma.

LEMMA 2.3. Let $r \geq 1$ be a real number, and \mathcal{A} a family satisfying the conditions (C1)–(C4). Then, for each $\theta \in (0, r)$, there exists a positive constant $\rho(\theta, r)$ such that, for each $t \in [-T, T]$,

$$\|f(t)\|_{L^2(d\mu_\theta)}^2 \leq \|f(t)\|_{H^s}^{2\rho} \left(K_0 \|f(0)\|_{L^2}^2 + K_1 \|f(0)\|_{L^2(d\mu_\theta)}^2 + K_2 \right) \quad (2.59)$$

for all $f \in \mathcal{A}$, where

$$K_0 = C_0 R^{2\theta} \left(\frac{4}{\Theta}\right)^{\rho+1}, \quad K_1 = \frac{\tilde{C}_0}{\rho(1-\gamma_2)} \left(\frac{4}{\Theta}\right)^\rho, \quad K_2 = \frac{A_0(\|f(0)\|_{H^{\alpha(r)}})}{\rho R^{2\theta\rho}}.$$

Proof. The technique is similar to the proof of Lemma 2.2 in [2], but we give a proof for the sake of completeness. For simplicity, we write $f(t, \xi) \equiv f(\xi)$ and $f(0, \xi) \equiv f_0(\xi)$. Let $\kappa_j > 0$ ($j = 0, 1$) be constants independent of t , and for $\theta \in [0, r]$, we set

$$\begin{aligned} I_1^{\kappa_1} &:= \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 \chi_{\{|f(\xi)|^2 > \kappa_1\}} d\xi, \\ I_2^{\kappa_1} &:= \kappa_1 \int_{\mathbb{R}^n} \|\xi\|^{2\theta} \chi_{\{|f(\xi)|^2 > \kappa_1\}} d\xi, \\ I_3^{\kappa_1} &:= \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 \chi_{\{|f(\xi)|^2 \leq \kappa_1\}} d\xi, \end{aligned}$$

where χ_E is the characteristic function of the set E . Then, we have

$$I := \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 d\xi = I_1^{\kappa_1} + I_3^{\kappa_1} = I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1} + I_3^{\kappa_1} + \kappa_0 I_2^{\kappa_1}.$$

It is not difficult to show that $I_2^{\kappa_1} < I_1^{\kappa_1}$, hence $\kappa_0 I_2^{\kappa_1} < \kappa_0 (I_1^{\kappa_1} + I_3^{\kappa_1}) = \kappa_0 I$. Consequently, we have

$$(1 - \kappa_0) I < I - \kappa_0 I_2^{\kappa_1} = I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1} + I_3^{\kappa_1}. \tag{2.60}$$

Now, we show that there exist $\theta_1 > 0$ independent of $f, t \in [-T, T]$, and a positive constant $\beta < 1$ such that $I_3^{\kappa_1} < \beta I_1^{\kappa_1}$. Indeed, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 \chi_{\{|f|^2 \leq \kappa_1\}} d\xi &\leq \beta \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 \chi_{\{|f|^2 > \kappa_1\}} d\xi \\ &= \beta \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 d\xi \\ &\quad - \beta \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(t, \xi)|^2 \chi_{\{|f|^2 \leq \kappa_1\}} d\xi, \end{aligned}$$

and hence, we must have

$$\int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 \chi_{\{|f|^2 \leq \kappa_1\}} d\xi \leq \frac{\beta}{1 + \beta} \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f(\xi)|^2 d\xi,$$

which is satisfied since $f \in \mathcal{A}$. Consequently, we take $\kappa_1 = \Theta$ in inequality (2.37). One observes that since $\beta < 1$, it follows that $\beta/(1 + \beta) < 1/2$. It follows that there exists a positive constant $\alpha < 1/2$ such that

$$I_3^{\kappa_1} < \alpha (I_1^{\kappa_1} + I_3^{\kappa_1}) = \alpha I. \tag{2.61}$$

Hence we fix $\kappa_0 = (3/4 - \alpha) > 1/4$ and, from (2.60), (2.61), we obtain

$$I < \frac{I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1}}{1 - (\kappa_0 + \alpha)} = 4 (I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1}). \tag{2.62}$$

At this point, we claim that, there exist $N_1 \in \mathbb{N}$ and a constant $C_1 > 0$ both independent of f and t , such that, for all $\eta \geq N_1$

$$\int_{\{\|\xi\| < \eta\}} |f(\xi)|^2 \|\xi\|^{2r} d\xi \leq C_1 \int_{\{\|\xi\| < \eta\}} |f_0(\xi)|^2 \|\xi\|^{2r} d\xi + \tilde{C}_1.$$

In order to prove the claim, we show that

$$\begin{aligned} \int_{\mathbb{R}^n} |f(\xi)|^2 \|\xi\|^{2r} d\xi - \int_{\{\|\xi\| \geq \eta\}} |f(\xi)|^2 \|\xi\|^{2r} d\xi \\ \leq C_1 \int_{\mathbb{R}^n} |f_0(\xi)|^2 \|\xi\|^{2r} d\xi - C_1 \int_{\{\|\xi\| \geq \eta\}} |f_0(\xi)|^2 \|\xi\|^{2r} d\xi + \tilde{C}_1, \end{aligned}$$

for each $\eta \geq N_1$. Therefore, from (2.36) and supposing $C_1 > \tilde{C}_0$, it is enough to show that

$$\begin{aligned} \tilde{C}_1 + \tilde{C}_0 \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^{2r} d\xi - \int_{\{|\xi| \geq \eta\}} |f(\xi)|^2 |\xi|^{2r} d\xi \\ \leq C_1 \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^{2r} d\xi - C_1 \int_{\{|\xi| \geq \eta\}} |f_0(\xi)|^2 |\xi|^{2r} d\xi + \tilde{C}_1. \end{aligned}$$

By a simple algebraic manipulation, it is sufficient to show that

$$\int_{\{\|\xi\|\geq\eta\}} |f_0(\xi)|^2 \|\xi\|^{2r} d\xi \leq \frac{C_1 - \tilde{C}_0}{C_1} \int_{\mathbb{R}^n} |f_0(\xi)|^2 \|\xi\|^{2r} d\xi,$$

which is true for each $f \in \mathcal{A}$, and we take $N_1 = R$ of inequality (2.38).

Now, we proceed to estimate $I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1}$. If $\theta \in \{0, r\}$, then by (2.35) and (2.36), it is obvious that

$$I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1} \leq C_0 \int_{\mathbb{R}^n} \|\xi\|^{2\theta} |f_0(\xi)|^2 d\xi.$$

Then, we consider in the following $\theta \in (0, r)$. Denoting $\kappa = (\kappa_0 \kappa_1)^{1/2\theta}$, it follows that

$$\begin{aligned} I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1} &= \int_{\mathbb{R}^n} (\|\xi\|^{2\theta} |f(\xi)|^2 - \kappa^{2\theta} \|\xi\|^{2\theta}) \chi_{\{|f(\xi)|^2 > \kappa_1\}} d\xi \\ &= \int_{\mathbb{R}^n} ((\|\xi\| |f(\xi)|^{1/\theta})^{2\theta} - \kappa \|\xi\|^{2\theta}) \chi_{\{|f(\xi)|^2 > \kappa_1\}} d\xi \\ &= \int_{\mathbb{R}^n} \int_{\kappa \|\xi\|}^{\|\xi\| |f(\xi)|^{1/\theta}} \varphi'(\eta) d\eta d\xi \\ &= 2\theta \int_0^\infty \eta^{2\theta-1} \mathcal{L}^n(E(\eta)) d\eta, \end{aligned}$$

where for each $\eta > 0$, $\varphi(\eta) = \eta^{2\theta}$ and

$$E(\eta) := \{ \xi \in \mathbb{R}^n / |f(\xi)|^{1/\theta} \|\xi\| > \eta \} \cap \{ \xi \in \mathbb{R}^n / \kappa \|\xi\| < \eta \}.$$

One observes that for each $\eta > 0$, $E(\eta) \neq \emptyset$ (in the geometric measure sense). Indeed, assume to the contrary that $E(\eta) = \emptyset$; then $\mathcal{L}^n(E(\eta)) = 0$ and thus $I < 0$ from (2.62), which is a contradiction by condition (C1) and the definition of I . Moreover, we observe that since

$$1 < \frac{|f(\xi)|^{2r/\theta} \|\xi\|^{2r}}{\eta^{2r}},$$

we could write

$$\mathcal{L}^n(E(\eta)) \leq \int_{\{\|\xi\| < \eta/\kappa\}} \frac{|f(\xi)|^{2r/\theta} \|\xi\|^{2r}}{\eta^{2r}} d\xi.$$

Therefore, we have

$$\begin{aligned} I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1} &\leq 2\theta \int_0^\infty \eta^{2\theta-1} \int_{\{\|\xi\| < \eta/\kappa\}} \frac{|f(\xi)|^{2r/\theta} \|\xi\|^{2r}}{\eta^{2r}} d\xi d\eta \\ &\leq 2\theta \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_0^\infty \eta^{2\theta-2r-1} \int_{\{\|\xi\| < \eta/\kappa\}} |f(\xi)|^2 \|\xi\|^{2r} d\xi d\eta, \end{aligned}$$

where we have used the Sobolev Embedding Theorem. Hence applying (2.35), we obtain

$$\begin{aligned}
 I_1^{\kappa_1} - \kappa_0 I_2^{\kappa_1} &\leq 2\theta \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_0^{N_1} \eta^{2\theta-2r-1} \int_{\{\|\xi\| < \eta/\kappa\}} |f(\xi)|^2 \frac{\eta^{2r}}{\kappa^{2r}} d\xi d\eta \\
 &+ 2\theta \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_{N_1}^\infty \eta^{2\theta-2r-1} \int_{\{\|\xi\| < \eta/\kappa\}} |f(\xi)|^2 \|\xi\|^{2r} d\xi d\eta \\
 &\leq \frac{C_0 N_1^{2\theta}}{\kappa^{2r}} \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_{\mathbb{R}^n} |f_0(\xi)|^2 d\xi \\
 &+ 2\theta C_1 \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_{\mathbb{R}^n} |f_0(\xi)|^2 \|\xi\|^{2r} \int_{\{\eta > \kappa \|\xi\|\}} \eta^{2\theta-2r-1} d\eta d\xi + \Xi \\
 &\leq C_0 \left(\frac{4}{\kappa_1}\right)^{r/\theta} N_1^{2\theta} \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_{\mathbb{R}^n} |f_0(\xi)|^2 d\xi \\
 &+ C_1 \left(\frac{4}{\kappa_1}\right)^{(r-\theta)/\theta} \frac{\theta}{r-\theta} \|f(t)\|_{H^s}^{(2r/\theta)-2} \int_{\mathbb{R}^n} |f_0(\xi)|^2 \|\xi\|^{2\theta} d\xi + \Xi,
 \end{aligned}$$

where

$$\Xi = \tilde{C}_1 \frac{\theta}{r-\theta} \frac{\|f(t)\|_{H^s}^{(2r/\theta)-2}}{N_1^{2(r-\theta)}}.$$

□

3. Statement of the well-posedness result. This is the section where we prove the well-posedness of the Cauchy problem (1.2) in weighted Sobolev space $\mathcal{X}^{s,\theta}$, for $s \geq 2r$ and $\theta \in [0, r]$.

3.1. *Proof of Theorems 1.6 and 1.7.*

Proof of Theorem 1.6. Consider $r \geq 1$, $u_0 \in \mathcal{X}^{s,\theta}$, $s \geq 2r$, $\theta \in [0, r]$, with $u_0 \neq 0$. We know that there exists a function $u \in C([-T, T], H^s)$ such that the IVP (1.2) is globally well-posed in H^s . It is well known that $\mathbf{S}(\mathbb{R})$ is dense in $\mathcal{X}^{s,\theta}$. Therefore, for $u_0 \in \mathcal{X}^{s,\theta}$ there exists a sequence (u_0^λ) in $\mathbf{S}(\mathbb{R})$ such that

$$u_0^\lambda \rightarrow u_0 \quad \text{in } \mathcal{X}^{s,\theta}. \tag{3.63}$$

By continuous dependence, the sequence of solutions $u^\lambda(t)$ associated to IVP (1.2) and with initial data u_0^λ satisfies

$$\sup_{t \in [-T, T]} \|u^\lambda(t) - u(t)\|_{H^s} \xrightarrow{\lambda \rightarrow \infty} 0. \tag{3.64}$$

Now, assuming conditions (1.3), suppose temporarily that the solutions u^λ of the IVP

$$\begin{cases} \partial_t u^\lambda + a(u^\lambda) \partial_x u^\lambda + \partial_x^3 u^\lambda = 0, & (t, x) \in \mathbb{R}^2, \\ u^\lambda(x, 0) = u_0^\lambda(x) \end{cases} \tag{3.65}$$

satisfy the conditions (C1)–(C4) of Section 2. Therefore Lemma 2.3 gives

$$\int_{\mathbb{R}} |\xi|^{2\theta} |u^\lambda(t, \xi)|^2 d\xi \leq C \left(\int_{\mathbb{R}} |u^\lambda(0, \xi)|^2 d\xi + \int_{\mathbb{R}} |\xi|^{2\theta} |u^\lambda(0, \xi)|^2 d\xi + 1 \right),$$

where $C = C(\theta, \|u^\lambda(t)\|_{H^s}, \|u^\lambda(0)\|_{L^2}, \|u_x^\lambda(0)\|_{L^2}, \|u_{xx}^\lambda(0)\|_{L^2}, T)$, and taking the limit when $\lambda \rightarrow \infty$, (3.64) implies

$$\int_{\mathbb{R}} |\xi|^{2\theta} |u(t, \xi)|^2 d\xi \leq C \left(\int_{\mathbb{R}} |u(0, \xi)|^2 d\xi + \int_{\mathbb{R}} |\xi|^{2\theta} |u(0, \xi)|^2 d\xi + 1 \right),$$

where $C = C(\theta, \|u(t)\|_{H^s}, \|u(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u_{xx}(0)\|_{L^2}, T)$. Thus $u(t) \in \mathcal{X}^{s,\theta}$, $t \in [-T, T]$, which proves the persistence. The local well-posedness theory in H^s implies the uniqueness, and thanks to that we obtain uniqueness in $\mathcal{X}^{s,\theta}$.

Finally, following the proof in Example 2.2, we prove that the sequence of solutions $(u^{\lambda_n}(t))$ satisfies the conditions (C1)–(C4). □

Proof of Theorem 1.7. By Theorem 1.6 is sufficient to prove continuous dependence in the norm $\|\cdot\|_{L^2(d\hat{\mu}_\theta)}$. Let $u(t)$ and $v(t)$ be two solutions in $\mathcal{X}^{s,\theta}$ of the IVP (1.2) with initial data u_0 and v_0 respectively, let $u^\lambda(t)$, $v^\lambda(t)$ be the solutions of the IVP (1.9) with initial data u_0^λ and v_0^λ respectively such that $u_0^\lambda, v_0^\lambda \in \mathbf{S}(\mathbb{R})$, $u_0^\lambda \rightarrow u_0$, $v_0^\lambda \rightarrow v_0$ in $\mathcal{X}^{s,\theta}$ and with $\lambda \gg 1$, so we have

$$\begin{aligned} \|u(t) - v(t)\|_{L^2(d\hat{\mu}_\theta)} &\leq \|u(t) - u^\lambda(t)\|_{L^2(d\hat{\mu}_\theta)} + \|u^\lambda(t) - v^\lambda(t)\|_{L^2(d\hat{\mu}_\theta)} \\ &\quad + \|v^\lambda(t) - v(t)\|_{L^2(d\hat{\mu}_\theta)}. \end{aligned}$$

Convergence in (3.64) implies for $\lambda \gg 1$ that

$$|u(x, t) - u^\lambda(x, t)| \leq 2|u(x, t)| \quad \text{and} \quad |v(x, t) - v^\lambda(x, t)| \leq 2|v(x, t)|,$$

and Lebesgue’s Dominated Convergence Theorem gives

$$\|u(t) - u^\lambda(t)\|_{L^2(d\hat{\mu}_\theta)} \rightarrow 0 \quad \text{and} \quad \|v^\lambda(t) - v(t)\|_{L^2(d\hat{\mu}_\theta)} \rightarrow 0.$$

Let $w^\lambda := u^\lambda - v^\lambda$; then w^λ satisfies the equation

$$w_t^\lambda + w_{xxx}^\lambda + (u^\lambda)^k w_x^\lambda + v_x^\lambda A(u^\lambda, u^\lambda) w_x^\lambda = 0,$$

where $A(x, y) = x^{k-1} + x^{k-2}y + \dots + xy^{k-1} + y^{k-1}$.

Then, we multiply the above equation by \bar{w}^λ , integrate on \mathbb{R} and take two times the real part to obtain

$$\partial_t \int_{\mathbb{R}} |w^\lambda(t, x)|^2 dx \leq h(\|u_0\|_{H^2}, \|v_0\|_{H^2}) \int_{\mathbb{R}} |w^\lambda(t, x)|^2 dx,$$

where h is a polynomial function with $h(0, 0) = 0$ and we have used (1.6)–(1.8) and convergence (3.63). Therefore, by Gronwall’s Lemma, we have

$$\|w^\lambda(t)\|_{L^2} \leq \exp(T h(\|u_0\|_{H^2}, \|v_0\|_{H^2})) \|w_0^\lambda\|_{L^2},$$

which gives the continuous dependence in case $\theta = 0$. Moreover, when $\theta = r$ with an analogous argument as used in the proof of Theorem 1.5 in Section 1,

$$\begin{aligned} \|w^\lambda(t)\|_{L^2(d\hat{\mu}_r)} &\leq \exp(T h_1(\|u_0\|_{H^{2r}}, \|v_0\|_{H^{2r}})) \\ &\quad \times \left(\|w_0^\lambda\|_{L^2(d\hat{\mu}_r)} + h_1(\|u_0\|_{H^{2r}}, \|v_0\|_{H^{2r}}) \right), \end{aligned}$$

where h_1 is a continuous function with $h_1(0, 0) = 0$.

Consequently, applying the Abstract Interpolation Lemma, we obtain the continuous dependence for $\theta \in (0, r)$, where we have assumed that the family (w^λ) satisfies the

hypothesis of the Abstract Interpolation Lemma. Indeed, these properties for the family (w^λ) are demonstrated in a similar way as in the proof of Theorem 1.6. \square

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REFERENCES

- [1] Jöran Bergh and Jörgen Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin, 1976. MR0482275 (58 #2349)
- [2] X. Carvajal and W. Neves, *Persistence of solutions to higher order nonlinear Schrödinger equation*, J. Differential Equations **249** (2010), no. 9, 2214–2236, DOI 10.1016/j.jde.2010.05.013. MR2718656 (2011g:35346)
- [3] Nakao Hayashi, Kuniaki Nakamitsu, and Masayoshi Tsutsumi, *Nonlinear Schrödinger equations in weighted Sobolev spaces*, Funkcial. Ekvac. **31** (1988), no. 3, 363–381. MR987792 (90d:35265)
- [4] Tosio Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128. MR759907 (86f:35160)
- [5] Tosio Kato and Kyūya Masuda, *Nonlinear evolution equations and analyticity. I* (English, with French summary), Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), no. 6, 455–467. MR870865 (88h:34041)
- [6] Joules Nahas, *A decay property of solutions to the k -generalized KdV equation*, Adv. Differential Equations **17** (2012), no. 9-10, 833–858. MR2985676
- [7] J. Nahas, *A decay property of solutions to the mKdV equation*, PhD thesis, University of California-Santa Barbara, June 2010.
- [8] J. Nahas and G. Ponce, *On the persistence properties of solutions of nonlinear dispersive equations in weighted Sobolev spaces*, Harmonic analysis and nonlinear partial differential equations, RIMS Kōkyūroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011, pp. 23–36. MR2883844 (2012k:35516)
- [9] J. Nahas and G. Ponce, *On the persistent properties of solutions to semi-linear Schrödinger equation*, Comm. Partial Differential Equations **34** (2009), no. 10-12, 1208–1227, DOI 10.1080/03605300903129044. MR2581970 (2011a:35498)
- [10] T. Ozawa and Y. Tsutsumi, *Space-time estimates for null gauge forms and nonlinear Schrödinger equations*, Differential Integral Equations **11** (1998), no. 2, 201–222. MR1741843 (2000m:35167)
- [11] K. Porsezian, P. Shanmugha Sundaram, and A. Mahalingam, *Coupled higher-order nonlinear Schrödinger equations in nonlinear optics: Painlevé analysis and integrability*, Phys. Rev. E (3) **50** (1994), no. 2, 1543–1547, DOI 10.1103/PhysRevE.50.1543. MR1381874 (96k:35171)
- [12] Catherine Sulem and Pierre-Louis Sulem, *The nonlinear Schrödinger equation*, Self-focusing and wave collapse. Applied Mathematical Sciences, vol. 139, Springer-Verlag, New York, 1999. MR1696311 (2000f:35139)
- [13] Hideo Takaoka, *Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity*, Adv. Differential Equations **4** (1999), no. 4, 561–580. MR1693278 (2000e:35221)
- [14] Yoshio Tsutsumi, *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. **30** (1987), no. 1, 115–125. MR915266 (89c:35143)