Persistent clusters in lattices of coupled nonidentical chaotic systems

I. Belykh[a)]
Laboratory of Nonlinear Systems, Swiss Federal Institute of Technology Lausanne (EPFL), CH-1015 Lausanne, Switzerland

V. Belykh and K. Nevidin
Mathematics Department, Volga State Academy, 5, Nesterov st., Nizhny Novgorod, 603 600 Russia

M. Hasler
Laboratory of Nonlinear Systems, Swiss Federal Institute of Technology Lausanne (EPFL), CH-1015 Lausanne, Switzerland

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Two-dimensional (2D) lattices of diffusively coupled chaotic oscillators are studied. In previous work, it was shown that various cluster synchronization regimes exist when the oscillators are identical. Here, analytical and numerical studies allow us to conclude that these cluster synchronization regimes persist when the chaotic oscillators have slightly different parameters. In the analytical approach, the stability of almost-perfect synchronization regimes is proved via the Lyapunov function method for a wide class of systems, and the synchronization error is estimated. Examples include a 2D lattice of nonidentical Lorenz systems with scalar diffusive coupling. In the numerical study, it is shown that in lattices of Lorenz and Rössler systems the cluster synchronization regimes are stable and robust against up to 10%–15% parameter mismatch and against small noise. [DOI: 10.1063/1.1514202]

Lattices of coupled chaotic oscillators model many systems of interest in physics, biology, and engineering. In particular, the phenomenon of cluster synchronization, i.e., the synchronization of groups of oscillators, has received much attention. This phenomenon depends heavily on the coupling configuration between the oscillators. While in a network of globally coupled identical oscillators in principle any subset of the oscillators may synchronize, in diffusively coupled oscillators only very few of the decompositions into subsets of synchronized oscillators are possible. They have been characterized in recent papers in detail (see Refs. 14–16). In that work, the oscillators were assumed to be identical and the symmetries of the resulting system of coupled oscillators were exploited. In more realistic models of physical systems, however, the individual oscillators have slightly different parameters and therefore the perfect symmetries in the coupled systems no longer exist. Similarly, perfect cluster synchronization cannot exist anymore, but approximate synchronization is still possible. The question then arises whether the cluster synchronization regimes that are observed in systems with identical oscillators persist approximatively under small parameter mismatch and the addition of small noise. This paper gives a positive answer and explores the limit of approximate synchronization both analytically and numerically.

I. INTRODUCTION

Chaotic synchronization was first discovered by Fujisaka and Yamada[1] for identical coupled oscillators and by Afraimovich *et al.*[2] for nonidentical subsystems. Since then this phenomenon has received a great deal of attention in the mathematical and physical literature (see, e.g., Ref. 7, and references therein). In the subsequent years, new synchronization phenomena were found including the most interesting cases of *full*[1–9] and *cluster* synchronization,[10–17] *generalized*,[18] *phase*,[19] and *lag*[20] synchronization, *riddled basins of attraction*,[21] *attractor bubbling*,[22] *on–off intermittency*,[23] etc.

In *full* chaotic synchronization, all oscillators acquire identical chaotic behavior when a threshold value of the coupling parameter is reached. Full synchronization in two-dimensional (2D) and 3D lattices of locally coupled limit-cycle[26,27] systems and chaotic oscillators[33–36,16] has recently been studied analytically.

Cluster synchronization (or clustering) is observed when the network of oscillators splits into subgroups, called clusters, such that all oscillators within one cluster move in perfect synchrony but the motion of different clusters is not synchronized at all. Cluster synchronization was mainly studied in networks of coupled identical maps but the interest is now shifting towards the analysis of coupled continuous time systems that have a more direct relation to the properties of real physical systems. Clustering is considered to be particularly significant in biology where one often encounters coupled cells or functional units which have complicated nonlinear behavior.[10,11,24,25] Recently cluster synchronization in an array of three chaotic lasers was reported[13] as well.

Considerable attention has been devoted to the problem of the persistence of full chaotic synchronization in networks of identical oscillators when a parameter mismatch between the systems is introduced.[2,28–36] In particular, a simple system of two coupled nonidentical skew tent maps was consid-

[a)]Electronic mail: igor.belykh@epfl.ch

ered and the synchronization error was estimated.[28] It was also shown that it is possible to maintain excellent locally stable chaotic synchronization between a drive and a response system even when there is a large parameter mismatch between them and they are coupled only through a scalar signal.[29] Mismatching effects on locally stable synchronization of two coupled Rössler systems were studied.[31] Conditions of asymptotic stability of full synchronization in 2D lattices of nonidentical chaotic oscillators were obtained in a series of papers.[33–36] In particular, the stability conditions of synchronization in a square lattice of forced Duffing type nonidentical oscillators with a dissipative nearest-neighbor coupling and Dirichlet boundary conditions were presented.[33] In another paper,[35] these results were extended to the case of Neumann or periodic boundary conditions. Using the same approach, sufficient conditions of global stability of full synchronization in a 2D lattice of nonidentical Lorenz systems with vector diffusive coupling were also obtained.[36] However, the question of the persistence of cluster synchronization modes in lattices of chaotic oscillators has not yet been systematically investigated.

We have recently studied the phenomena of clustering in lattices of locally coupled identical oscillators.[14–16] This phenomenon is directly related to the existence of stable linear invariant manifolds to which the trajectories of the synchronous modes of clusters of oscillators are restrained. These invariant hyperplanes define the strict set of all possible clusters of synchronized oscillators that can occur in the lattice. Their existence is imposed by the symmetries of the diffusive coupling and boundary conditions and strongly depends on the number of oscillators composing the lattice. Our previous analysis was limited to arrays of oscillators that are strictly identical. This idealization is convenient mathematically, but it ignores a small diversity that is always present in reality. The question that naturally arises is whether the discovered cluster synchronization regimes are robust against small parameter mismatch or whether they are only a fragile product of the symmetries from the idealized case.

This paper revisits the existence and stability of cluster synchronization modes in lattices of diffusively coupled oscillators with chaotic behavior, this time in the context of their persistence under parameter mismatch. For small parameter mismatch, we prove the existence of stable clusters of almost-perfectly synchronized oscillators. These cluster regimes, defined by the cluster synchronization manifolds existing in the identical oscillators case, are preserved even up to a fairly large mismatch between the oscillators.

The layout of the paper is as follows. First, in Sec. II, we recall our results on the existence of chaotic clusters in a 1D chain of coupled identical oscillators. Then, in Sec. III, we study the existence and stability of cluster synchronization manifolds in a 2D lattice of identical oscillators. We describe the set of possible cluster modes, prove global stability of one cluster synchronization manifold, and discuss the question of cluster appearance with increasing coupling. In Sec. IV, we prove first the general results on the persistence and stability of corresponding δ-synchronization regimes in the 2D lattice of nonidentical oscillators. Next, we apply these results for a 2D lattice of nonidentical Lorenz systems with *scalar* diffusive coupling. We prove asymptotic stability of the synchronization between lines of nonsynchronized oscillators in the 2D lattice. We obtain estimates for the corresponding coupling constant threshold and for the synchronization error δ which are explicitly expressed in terms of the parameters of the coupled Lorenz systems. In Sec. V, we confirm our theoretical results with numerical simulation of lattices of nonidentical Lorenz and Rössler systems. Finally, a brief discussion of the obtained results is given.

II. 1D CHAIN OF IDENTICAL OSCILLATORS

We start off with a 1D array of diffusively coupled identical oscillators,

$$\dot{X}_i = F(X_i) + \varepsilon C(X_{i+1} - 2X_i + X_{i-1}),\quad i = 1,2,\ldots,N, \tag{1}$$

with zero-flux ($X_0 \equiv X_1$, $X_N \equiv X_{N+1}$) or periodic ($X_0 \equiv X_N$, $X_{N+1} \equiv X_1$) boundary conditions (BC).

Here, X_i is the m-dimensional vector of the ith oscillator variables, $F(X_i): R^m \to R^m$ is a vector function. $K = N \cdot m$ is the dimension of the whole system (1). $\varepsilon > 0$ defines the coupling strength between the oscillators. The $m \times m$ coupling matrix C is diagonal, $C = \mathrm{diag}(c_1, c_2, \ldots, c_m)$, where $0 \leqslant c_k \leqslant 1$, $k = 1,2,\ldots,l$ and $c_k = 0$ for $k = l+1,\ldots,m$. Nonzero elements of the matrix C determine by which variables the oscillators are coupled.

Cluster synchronization regimes in the array are defined by linear invariant manifolds of the system (1). To proceed with the study of cluster synchronization, we need first to introduce some notions. A manifold M^* is said to be *invariant* with respect to a dynamical system $\dot{x} = F(x,t)$ if for $\forall x \in M^*$, the trajectory $\varphi(t,x)$ lies in the manifold M^*. Let us now specify this definition for linear invariant manifolds of the system (1). Let the set of vertices of the 1D chain be decomposed into the disjoint subsets $V = V_1 \cup \cdots \cup V_d$, $V_\gamma \cap V_\mu = \varnothing$ given by the equalities of groups of the coordinates of oscillators. If the decomposition of the vectors is compatible with the system (1) then the manifold $M(d) \equiv M(V_1, \ldots, V_d)$ is invariant under the dynamics given by Eq. (1) and is said to be a *cluster synchronization manifold*. The coordinates in the manifold $M(d)$ are $\chi_r = X_{i_r}$, $r = 1,2,\ldots,d$. In this section we provide a detailed description of cluster synchronization manifolds existing in the system (1). When studying cluster synchronization in lattices of slightly nonidentical oscillators (Sec. IV), we will take into account only diagonal-like invariant manifolds defining almost perfect cluster synchronization. All other situations concerned with generalized and phase synchronization will be disregarded.

In contrast to networks of globally coupled oscillators[10] where all cluster decompositions are possible, the array (1) may exhibit only a few of them. Main questions of interest here are the following. Which clusters can arise in the coupled system (1) with increasing coupling and how do these clusters depend on the number of oscillators N and boundary conditions?

To tackle this problem, we consider first the existence of possible cluster synchronization manifolds $M(d)$ for $1 < d$

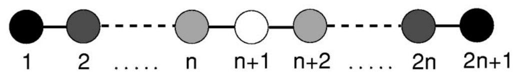

FIG. 1. Cluster regime defined by the cluster synchronization manifold $M^{c}(n+1)$. Oscillators with the same gray shading belong to one cluster.

$<N$. Obviously, for $d=1$ the system (1) has a well-known synchronization manifold $M(1)=\{X_1=X_2=\cdots=X_N=\chi_1\}$. Here, the dynamics of the coupled system is restricted to the manifold $M(1)$ such that the oscillators are all doing the same thing at the same moment, even though it is chaotic motion. The manifold $M(1)$, often called the diagonal, exists for any N and boundary conditions, and it is embedded in every cluster synchronization manifold.

Statement 1 (zero-flux or periodic BC):[14]

(a) If the chain (1) is composed of an odd number of oscillators $N=2n+1$, then there exists a cluster synchronization manifold $M^{c}(n+1)$ which is given by the equalities $\{X_1=X_{2n+1}, X_2=X_{2n}, \ldots, X_{n-1}=X_{n+3}, X_n=X_{n+2}\}$ defining $n+1$ clusters of synchronized oscillators.

(b) For even $N=2n$, there exists a cluster synchronization manifold $M^{c}(n)$ given by the equalities $\{X_1=X_{2n}, X_2=X_{2n-1}, \ldots, X_n=X_{n+1}\}$.

The hyperplanes $M^{c}(n+1)$ and $M^{c}(n)$ define a central symmetry of synchronized oscillators with respect to the middle of the chain. In the case of $N=2n+1$, the oscillators are synchronized in pairs around the middle element $(n+1)$ (see Fig. 1).

Statement 2:[14]

For a factorizable number of oscillators $N=p\cdot n$, where p and n are any arbitrary integers, the system (1) with zero-flux BC has an invariant hyperplane $M^{\mathrm{alt}}(n)$ defined by the equalities $\{X_i=X_{i+2nk}, k=1,2,\ldots,\mathrm{int}((p-1)/2), X_i=X_{-i+1+2nk}, k=1,2,\ldots,\mathrm{int}(p/2), i=1,2,\ldots,n\}$. For periodic BC and even n, the system (1) has a similar manifold.

The manifold $M^{\mathrm{alt}}(n)$ defines a cluster synchronization regime under which the chain of oscillators is decomposed into p equal palindromic subchains of n nonsynchronized oscillators (see Fig. 2).

It follows from Statements 1 and 2 that in the case of periodic BC, each cell of the array may be considered as a first element, and the system (1) has $N-1$ additional manifolds M^{c} and $N-1$ additional manifolds M^{alt}. Some of them may be identical.

By means of Statements 1 and 2, one can study the set of possible cluster decompositions in the chain (1) with the concrete number of oscillators N.

Figure 3 shows the set of all possible modes of cluster synchronization in the chain (1) that is composed of $N=6$ and $N=7$ with zero-flux BC. For $N=6$, there exist three

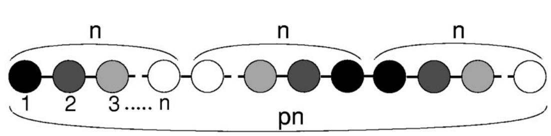

FIG. 2. Palindromic cluster regime for the factorizable number of oscillators $N=p\cdot n$ in the chain.

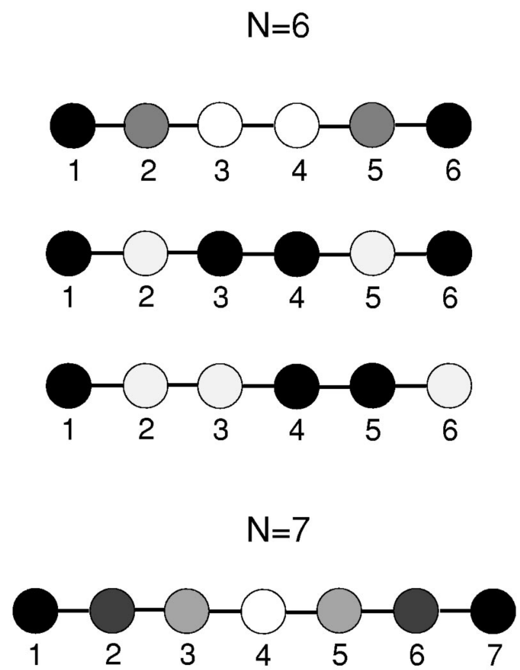

FIG. 3. Collection of all possible clusters in the chain for $N=6$ and $N=7$ and zero-flux BC.

cluster synchronization hyperplanes $M^{c}(3)=\{X_1=X_6, X_2=X_5, X_3=X_4\}$, $M^{c}(2)=\{X_1=X_3=X_4=X_6, X_2=X_5\}$, and $M^{\mathrm{alt}}(2)=\{X_1=X_4=X_5, X_2=X_3=X_6\}$ defining corresponding clusters. For $N=7$, the only cluster synchronization hyperplane is $M^{c}(4)=\{X_1=X_7, X_2=X_6, X_3=X_5\}$ defining a four-cluster regime (Fig. 3).

We note that these two similar arrays that are composed of close numbers of oscillators $N_1=6$ and $N_2=7$ may exhibit completely different regimes of cluster synchronization. This means that one can completely change possible modes of cluster synchronization in a large array by adding only one oscillator to the network. In a broader context, it may be related to a challenging problem of the control of a given number synchronous motions in many physical systems. Obviously, the main problem for such a control and selection of a particular synchronous mode is to ensure its stability.[15]

We proceed now with the existence of possible modes of cluster synchronization in a 2D lattice of locally coupled oscillators. Then we discuss the stability of the corresponding synchronization regimes and their persistence in the presence of a parameter mismatch between the systems.

III. 2D LATTICE OF IDENTICAL OSCILLATORS

We first study a square 2D lattice of chaotic oscillators that are coupled with four nearest-neighbor elements with equal coupling strength,

$$\dot{X}_{i,j}=F(X_{i,j})+\varepsilon C\cdot(\Delta X)_{i,j}, \tag{2}$$

where $(\Delta X)_{i,j}=X_{i+1,j}+X_{i-1,j}+X_{i,j+1}+X_{i,j-1}-4X_{i,j}$, $i,j=\overline{1,N}$. We assume zero-flux or periodic BC, all other notations are similar to those of the system (1).

Note that the system (2) represents a discrete version of two-dimensional spatially extended reaction-diffusion system and plays a significant role as a model of simple forms of turbulence and spatiotemporal chaos. On the other hand, such a model is of interest in connection with the description of coupled biological cells. The individual cells often display complicated forms of bursting and spiking behavior, and it is an obvious challenge to describe how the units function together and which stable synchronization patterns are possible.

A. Existence of cluster synchronization manifolds

To study the existence of possible modes of cluster synchronization in the lattice (2), we apply the results on the cluster synchronization manifolds from the 1D chain case in a straightforward manner.[16]

Statement 3:

(a) The system (2) has a family of cluster synchronization manifolds $M(d_1,d_2)$ that are an intersection of invariant manifolds $M(d_1)$ and $M(d_2)$ existing in the case of the 1D system (1). The corresponding cluster regimes are a topological product of synchronization regimes in the two directions of the 2D lattice.

(b) The lattice (2) has simple symmetries with respect to the principal and secondary diagonals, i.e., there exist invariant manifolds $M^{\mathrm{pr}}(d):\{X_{j,i}=X_{i,j}=\chi_{i,j}\,,\ i=\overline{1,N},\ N\geqslant j\geqslant i\}$ and $M^{\mathrm{sc}}(d):\{X_{N-j+1,N-i+1}=X_{i,j}=\chi_{i,j}\,,\ i=\overline{1,N},\ 1\leqslant j\leqslant N-i+1\}$. Obviously, there also exists the intersection symmetrical invariant manifold $M^{\mathrm{pr-sc}}(d_s)=M^{\mathrm{pr}}(n_1)\cap M^{\mathrm{sc}}(n_2)$, where

$$d_s=\begin{cases}(n+1)^2, & \text{for odd } N=2n+1\\ n(n+1), & \text{for even } N=2n.\end{cases}$$

(c) There exists a cluster synchronization manifold $M^{\mathrm{star}}(d)=M^{\mathrm{c}}(d_1,d_1)\cap M^{\mathrm{pr-sc}}(d_s)$ which defines simultaneously the symmetries of synchronized oscillators with respect to the two diagonals and to the middles of the rows and columns of the lattice. Number of clusters is

$$d=\begin{cases}(n+1)(n+2)/2, & \text{for } N=2n+1\\ n(n+1)/2, & \text{for } N=2n\end{cases}.$$

Figure 4 presents an example of two cluster regimes which exist in the 5×5 lattice (2) and which are defined by the manifolds $M^{\mathrm{c}}(3,3)$ and $M^{\mathrm{star}}(6)$, respectively.

The invariant manifold $M^{\mathrm{c}}(3,3)$ defines synchronization between oscillators with respect to the middles of the rows and columns of the lattice [see Fig. 4(a)]. The invariant manifold $M^{\mathrm{star}}(6)=M^{\mathrm{c}}(3,3)\cap M^{\mathrm{pr-sc}}(15)$ defines simultaneously the symmetries of synchronized oscillators with respect to the two diagonals and to the middles of the rows and columns of the lattice [see Fig. 4(b)].

Obviously, for this example of the prime number $N=5$ there also exist the following product manifolds: the manifold $M^{\mathrm{c}}(3,1)$ ($M^{\mathrm{c}}(1,3)$) defining symmetrical 3-cluster synchronization between the rows (columns) of synchronized oscillators, the manifold $M(1,5)$ ($M(5,1)$) defining full synchronization of the rows (columns) and out of synchronization between the columns (rows).

For the factorizable number of oscillators N, the collection of possible modes of cluster synchronization in the 2D lattice is even richer and additional intersection invariant manifolds may be obtained as the topological product of the cluster patterns.

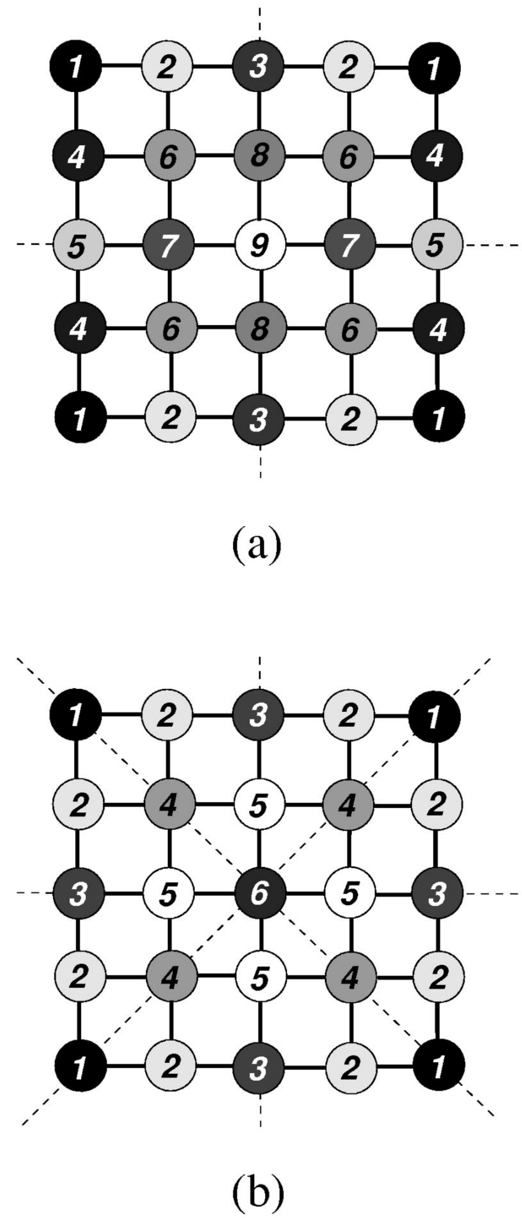

FIG. 4. (a) Product cluster defined by the manifold $M^{\mathrm{c}}(3,3)$. Oscillators synchronize with respect to the middle row and column (depicted by dashed lines). Oscillators labeled by the same digit belong to the same cluster. (b) Symmetrical cluster defined by the manifold $M^{\mathrm{star}}(6)$. Oscillators synchronize with respect to the dashed lines.

B. Eventual dissipativeness of the coupled system

To go further with the stability of cluster synchronization manifolds, we need to show first the eventual dissipativeness of the coupled system (2).

Assume that the individual system $\dot{X}_{i,j}=F(X_{i,j})$ is eventually dissipative, i.e., there exists the Lyapunov directing function $V_{i,j}=X_{i,j}^T\cdot Q\cdot X_{i,j}/2$, where $Q=\mathrm{diag}(q_1,q_2,\ldots,q_m)$, $q_k>0$ for $k=1,\ldots,m$ such that the time derivative with respect to the individual subsystem

$$\dot{V}_{i,j}=X_{i,j}^T\cdot Q\cdot F(X_{i,j})<0 \tag{3}$$

outside of the compact set $\tilde{B}_{i,j}=\{||X_{i,j})||<b_1\}$. The compact set $\tilde{B}_{i,j}$ belongs to the absorbing domain $B_{i,j}=\{V_{i,j}(x_{i,j})<b_2\}$. Since $\dot{V}<0$ outside of the absorbing domain $B_{i,j}$, this compact set $B_{i,j}$ attracts all trajectories of the system $\dot{X}_{i,j}=F(X_{i,j})$ from the outside. Note that many chaotic dynamical systems satisfy this natural assumption.

Statement 4: Under the above conditions, the 2D lattice system (2) is eventually dissipative. The absorbing domain B of the system (2), such that every trajectory of the system reaches B and remains there forever, is a topological product of the absorbing domains $B_{i,j}$.

Proof: Consider the directing Lyapunov function $W=\Sigma_{i,j=1}^N V_{i,j}$. Its derivative with respect to the system (2) has the form,

$$\dot{W}=\sum_{i,j=1}^N \dot{V}_{i,j}=\sum_{i,j=1}^N (X_{i,j}^T\cdot Q\cdot F(X_{i,j}) + X_{i,j}^T\cdot Q\cdot \varepsilon C(\Delta X)_{i,j}). \tag{4}$$

The first sum in the expression (4) is negative outside of the compact set B due to the assumption (3). The second sum takes the form,

$$\sum_{i,j=1}^N (X_{i,j}^T\cdot Q\cdot \varepsilon C(\Delta X)_{i,j})=-\varepsilon(S_1+S_2),$$

where

$$S_1=\sum_{i=1}^N\sum_{j=1}^{N_0}(X_{i,j}-X_{i,j+1})^T\cdot Q\cdot C(X_{i,j}-X_{i,j+1}),$$
$$S_2=\sum_{j=1}^N\sum_{i=1}^{N_0}(X_{i,j}-X_{i+1,j})^T\cdot Q\cdot C(X_{i,j}-X_{i+1,j}), \tag{5}$$

and the superscript N_0 stands for N (for $N-1$) for periodic (zero-flux) BC. The product matrix QC is diagonal and positive definite, therefore the quadratic forms S_1 and S_2 are positive definite for $X_{i,j}\neq 0$. Hence the derivative $\dot{W}$ is negative outside of the compact set B, and the system (2) is eventually dissipative. □

C. Stability of the invariant manifolds

Our first objective is to obtain conditions of global asymptotic stability of full synchronization in the system (2). We study the conditions of global stability of the cluster synchronization manifold $M(1,N)$ which defines complete synchronization between rows of the lattice (2). Having obtained these conditions, we can apply them directly to the stability of the cluster manifold $M(N,1)$ defining complete synchronization between the columns. The intersection of these conditions gives the conditions for full global synchronization. Stability of other cluster synchronization manifolds can be also obtained in a similar way.

Using the approach developed in the previous papers,[5,14,16] we proceed now with the study of global stability of the cluster synchronization manifold $M(1,N)=\{X_{i,j}=X_j,\ i,j=1,\dots,N\}$.

Introducing the notation for the differences

$$U_{i,j}=X_{i,j}-X_{i+1,j}, \tag{6}$$

we derive the finite difference equations

$$\dot{U}_{i,j}=\overline{DF}U_{i,j}+\varepsilon C(\Delta U)_{i,j}, \tag{7}$$

with $i=1,\dots,N-1$, $j=1,\dots,N$, and $U_{0,j}=U_{N,j}=0$ for the case of zero-flux BC, and $U_{0,j}=U_{N,j}$ for periodic BC. In the following we will consider only zero-flux BC. $\overline{DF}$ is an $m\times m$ Jacobi matrix of $F(X^*(t))$, where $X^*(X_{i,j},X_{i+1,j})\in[X_{i,j},X_{i+1,j}]$ is driven by the system (2) and comes from Mean Value Theorem.

Note that the Jacobian $\overline{DF}$ can be calculated explicitly via the parameters of the individual subsystem and for infinitesimal $U_{i,j}$ it becomes the Jacobian of the variational system.

Adding and subtracting an additional term $AU_{i,j}$ to the system (7), we obtain the system

$$\dot{U}_{i,j}=(-A+\overline{DF})U_{i,j}+AU_{i,j}+\varepsilon C(\Delta U)_{i,j}, \tag{8}$$

where the $m\times m$ matrix A is diagonal and, similarly to the coupling matrix C, satisfies the conditions $A=\mathrm{diag}(a_1,a_2,\dots,a_m)$, $a_k\geqslant 0$ for $k=1,2,\dots,l$ and $a_k=0$ for $k=l+1,\dots,m$.

The matrix $-A$ is added to damp instabilities caused by eigenvalues with nonnegative real parts of the Jacobian $\overline{DF}$. At the same time, the instability introduced by the positive definite matrix $+A$ in Eq. (8) can be damped by the coupling terms. The positive coefficients a_k are put in the matrix A only at the places corresponding to the variables by which oscillators are coupled, and therefore they can be compensated by the negative coupling terms.

We develop now this approach as follows. Let us introduce the auxiliary system,

$$\dot{U}_{i,j}=(-A+\overline{DF})U_{i,j}. \tag{9}$$

We assume that there exists the Lyapunov function,

$$\tilde{V}_{i,j}=U_{i,j}^T\cdot H\cdot U_{i,j}/2, \tag{10}$$

where $H=\mathrm{diag}(h_1,h_2,\dots,h_l,H_1)$, the numbers $h_k>0$ for $k=1,\dots,l$, and the $(m-l)(m-l)$ matrix H_1 is positive definite.

We require its derivative with respect to the system (9) to be negative

$$\dot{\tilde{V}}_{i,j}=U_{i,j}^T H(-A+\overline{DF})U_{i,j}<0,\quad U_{i,j}\neq 0. \tag{11}$$

Note that this assumption is realizable. For example, it is valid for some $a_k>a^*>0$, $k=1,...,l$ and for the real spectrum $\lambda_{l+1}(t),\dots,\lambda_m(t)<\lambda^*<0$ of the $(m-l)(m-l)$ block matrix of $\overline{DF}$ corresponding to the last $m-l$ variables of $U_{i,j}^{(k)}$. When the spectrum is not real, the conditions for the Lyapunov exponents to be negative essentially depend on the imaginary part of the eigenvalues. Moreover, these conditions may fail while the imaginary part of the eigenvalues is increasing.

To obtain the conditions for global stability of the manifold $M(1,N)$, we consider the Lyapunov function

$$\tilde{W}=\sum_{j=1}^{N}\sum_{i=1}^{N-1}\tilde{V}_{i,j} \tag{12}$$

for the system (8). The corresponding time derivative has the form

$$\dot{\tilde{W}}=\sum_{j=1}^{N}\sum_{i=1}^{N-1}\{U_{i,j}^{T}H(-A+\overline{DF})U_{i,j}\}$$
$$+\sum_{j=1}^{N}\sum_{i=1}^{N-1}\{U_{i,j}^{T}H(AU_{i,j}+\varepsilon C(\Delta U)_{i,j})\}. \tag{13}$$

The first sum in Eq. (13) is negative definite due to Eq. (11). The second one falls into two terms S_1+S_2. The term S_1,

$$S_1=-\varepsilon\sum_{j=1}^{N}\sum_{i=1}^{N-1}(U_{i,j}-U_{i,j+1})^{T}HC(U_{i,j}-U_{i,j+1})$$

is negative definite as $HC>0$. The term S_2 takes the form,

$$S_2=\varepsilon\sum_{j=1}^{N}\sum_{k=1}^{l}h_k c_k\sum_{i=1}^{N-1}\{U_{i,j}^{(k)}U_{i+1,j}^{(k)}+U_{i,j}^{(k)}U_{i-1,j}^{(k)}$$
$$-2(1-a_k/(2\varepsilon c_k))(U_{i,j}^{(k)})^2\}. \tag{14}$$

The conditions for the quadratic form S_2 to be negative definite can be presented as follows:

$$\varepsilon c_k>a_k/|\lambda_{\max}|,\quad k=1,2,\ldots,l, \tag{15}$$

where $\lambda_{\max}=-4\sin^2(\pi/2N)$ is the well-known maximal nonzero eigenvalue of the nearest neighbor coupling matrix with Neumann (zero-flux) BC.

Thus, we arrive at the following conclusion:

Statement 5: Under the assumption (11) and the condition (15), the manifold $M(1,N)$ is globally stable.

Obviously, the synchronization threshold coupling value ε^* can be estimated by the expression

$$\varepsilon^*=\max_{k\in[1,l]}(a_k/c_k)/|\lambda_{\max}|. \tag{16}$$

Remark: It is easy to verify that the conditions for global synchronization in the rectangular lattice $N_1\times N_2$ are two inequalities (15) written for N_1 and N_2 separately. One can also obtain synchronization conditions for periodic BC similar to those of Eq. (15).

To prove global stability of cluster synchronization manifolds that are described in previous sections, one can study the corresponding finite difference equations that are similar to Eq. (7) with a reduced number N and changed boundary conditions. For example, for the cluster manifold $M^c(\mathrm{int}(N/2),N)$ providing symmetrical cluster synchronization between the rows of the 2D lattice, the number N in Eq. (7) is defined by the number of clusters in one lattice direction and becomes $\mathrm{int}(N/2)$. The corresponding sufficient conditions can be written similarly to Eq. (15).

The sufficient conditions (11)–(15) for the necessary coupling strength may give large overestimates but they are useful for a rough estimation of the range of coupling strength required for synchronization. They guarantee the stability of the synchronization regime and solve rigorously the problem of whether synchronization occurs in a concrete lattice with increasing coupling or not. In fact, for a large number of examples of coupled continuous time systems, synchronization arises with increasing coupling and remains up to infinite coupling strength. However, a few examples of coupled systems for which this is not the case were reported.[6,14] Among them is a lattice of coupled Rössler systems in which the stability of the synchronization regime was lost with an increase of coupling. These desynchronization bifurcations were called short-wavelength bifurcations.[6] In a recent paper,[14] we have linked this, at first sight surprising, phenomenon with the equilibria disappearance bifurcations. In fact, usually for a fairy small coupling the coupled system (with a fixed number of oscillators N) has a finite number of equilibria n_{coupl} which is usually less than n_{sngl}^{N}, where n_{sngl} is the number of equilibria of the individual oscillator. Most of these equilibria lie outside the synchronization manifold. With increasing coupling strength, this part of the equilibria disappears via saddle-node bifurcations such that when the synchronization regime becomes globally stable there are no equilibria outside the manifold. We proposed the conditions on the individual oscillator and the place of coupling for which this sequence of the equilibria disappearance is broken and some equilibria are always present outside the manifold. In this case, the synchronization behavior depends dramatically on whether these equilibria are a unique limiting set or if they have some neighboring attractor outside the diagonal manifold. In the last case, the existence of these equilibria has direct relation to the mechanism of desynchronization. The lattice of coupled Rössler systems satisfies our desynchronization conditions and will be described in Sec. V.

Let us also comment on the order of cluster stabilization with increasing coupling. There are two main scenarios to complete synchronization. In the most widespread case, with increasing coupling, the diagonal manifold becomes globally stable simultaneously with all the other cluster manifolds in which it is embedded, and full synchronization arises right away. The cluster manifolds are globally stable and attract trajectories from the outside but the cluster regimes are unstable since the trajectories within the manifolds are then being attracted by the diagonal. In other cases, when the coupling is increased, the dynamics is restricted to stable cluster manifolds of lower and lower dimension while the diagonal remains unstable. This decreasing sequence of dimensions of the cluster synchronization (number of clusters) is determined by the order of the embedding of the manifolds. However, this sequence may be interrupted at any place when the diagonal manifold becomes globally stable. In fact, the appearance of clusters depends on the vector field of the single system and on the corresponding variational stability equation. One drawback of the sufficient conditions of the stability is the inability to predict, in the general case, which cluster mode will be stable, and the numerical study of the transversal Lyapunov exponents is often the only method available for predicting the stability.

Thus after having considered and discussed the existence and stability of synchronization manifolds in the identical oscillators case, the main problem is to show that the corresponding clusters are indeed robust against small perturbations introducing by a small parameter mismatch, i.e., they can be realized in real physical systems.

IV. PARAMETER MISMATCH

A. General results

We consider now the 2D lattice (2) with an additional mismatch term

$$\dot{X}_{i,j}=F(X_{i,j})+\mu f_{i,j}(X_{i,j})+\varepsilon C(\Delta X)_{i,j}, \tag{17}$$

where μ is a positive scalar parameter and $f_{i,j}:R^m\rightarrow R^m$ is a smooth mismatch function. Boundary conditions are assumed to be zero-flux.

We assume that all our assumptions related to the unperturbed system (2) are valid for the system (17). In particular, we assume that each individual subsystem of the lattices (17) (for $\varepsilon=0$) has an absorbing domain $B_{i,j}(\mu)$ for some region of the parameter μ. Therefore, due to Statement 4, the lattice system (17) is eventually dissipative and has the absorbing domain $B(\mu)$.

Consider a cluster synchronization manifold $M(d)$ of the unperturbed system (2). Recall that the index $r=1,2,\ldots,d$ indicates the index of a cluster from the d clusters. Let the index $s=1,2,\ldots,s^r$ indicate the place of the oscillator within the cluster. Two oscillators from the same cluster are denoted by the indexes (r,s_1) and (r,s_2), respectively.

Thus due to the chosen identification $(r,s)\rightarrow(i,j)$ we can rename the coordinates of the manifold $M(d)$ as X_{rs}, $r=1,2,\ldots,d$ and $s=1,2,\ldots,s^{(r)}$.

Let $X_{r,s}(t,X^0,\mu)$, $X^0=\{X^0_{i,j},\ i,j=1,\ldots,N\}$ be the coordinates of the oscillators defining the dynamics of a given cluster (r,s) and satisfying the initial conditions $X_{r,s}(0,X^0,\mu)=X^0_{r,s}$.

Definition: Clusters of the nonperturbed system (2) are said to be clusters of *δ-synchronized* oscillators of the perturbed system (17) if the following property of global asymptotical synchronization is fulfilled. For any initial state X^0 of the lattice there exists

$$T\ \text{such that}\ \|X_{r,s_1}(t,X^0,\mu)-X_{r,s_2}(t,X^0,\mu)\|<\delta(\mu)\quad \text{for}\ >T \tag{18}$$

for any $r=1,...,d$, $s_{1,2}\in[1,s^r]$, and $\lim_{\mu\rightarrow 0}\delta(\mu)=0$.

In other words, this means that the δ-neighborhood of the manifold $M(d)|_{B(\mu)}$ is globally stable and attracts all trajectories of the system (17).

We consider now global stability of a cluster δ-synchronization regime defined by the generating (pristine) manifold $M(1,N)$ of the nonperturbed system (2). Global stability of this manifold determining synchronization between the rows of the 2D lattice (2) was considered in the previous section.

Using the differences (6) and similar to Eq. (7), we obtain the finite difference equations,

$$\dot{U}_{i,j}=\overline{DF_{i,j}}U_{i,j}+\mu\overline{e_{i,j}}+\varepsilon C(\Delta U)_{i,j}, \tag{19}$$

where $\overline{e_{i,j}}=[f_{i,j}(X_{i,j})-f_{i+1,j}(X_{i+1,j})]|_{B(\mu)}$ is a mismatch difference calculated within the absorbing domain $B(\mu)$, and both $\overline{DF_{i,j}}$ and $\overline{e_{i,j}}$ are driven by the system (17).

We use here the same technique as we utilized in the previous section, except that we add and subtract now two additional terms $AU_{i,j}$ and $H^{-1}PU_{i,j}$ to the system (19),

$$\dot{U}_{i,j}=(-A+H^{-1}P+\overline{DF}_{i,j})U_{i,j}-H^{-1}PU_{i,j}+AU_{i,j}+\mu\overline{e_{i,j}}+\varepsilon C(\Delta U)_{i,j}. \tag{20}$$

Matrices A,H are identical to those of the systems (9) and (10), and the matrix $P=\mathrm{diag}(p_1,p_2,\ldots,p_m)$, $p_k>0$ for $k=1,2,\ldots,m$.

Our purpose is to obtain the conditions under which all attractors of the system (20) lie in the vicinity of $U_{i,j}=0$, and thus they correspond to the cluster δ-synchronization. To compensate the instability defined by the mismatch term in Eq. (20), the matrix $-H^{-1}P$ is added. Instabilities introduced by the positive definite matrix $H^{-1}P$ that in turn arose in the system (20) can be damped now by the appropriate choice of the values of the matrix A.

We introduce the new auxiliary system

$$\dot{U}_{i,j}=(-A+H^{-1}P+\overline{DF}_{i,j})U_{i,j}. \tag{21}$$

Assume that the derivative of the Lyapunov function (10) along the trajectories of the auxiliary system (21) is negative,

$$\dot{V}_{i,j}=U^T_{i,j}H(-A+\overline{DF})U_{i,j}+U^T_{i,j}PU_{i,j}<0,\quad U_{i,j}\neq 0. \tag{22}$$

In the system (21), we "spoiled" the Jacobian $\overline{DF_{i,j}}$ by inserting the matrix $H^{-1}P$ and we choose the maximal values of p_k, $k=1,\ldots,m$ in such a way that it would be still possible to compensate the increased instability, defined by the terms $(H^{-1}P+\overline{DF}_{i,j})U_{i,j}$, by the term $-AU_{i,j}$. Obviously, the matrix $-A$ must be more stable than that used in the previous section. In the simplest (from the stability viewpoint) case where the oscillators are coupled by all variables, i.e., all coefficients a_k,c_k,p_k, $k=1,\ldots,m$ are positive, the values of p_k must be proportional to a_k, $a_k=a^0+\alpha p_k$. Hence, the maximal instability terms p_k in Eq. (21) are proportional to the coupling strength εc_k via Eq. (15).

Applying the Lyapunov function (12) for the system (19), we obtain

$$\dot{\tilde{W}}=\sum_{j=1}^{N}\sum_{i=1}^{N-1}\dot{V}_{i,j}+S_1+S_2-S_\mu, \tag{23}$$

where

$$S_\mu=\sum_{j=1}^{N}\sum_{i=1}^{N-1}[U^T_{i,j}PU_{i,j}-\mu U^T_{i,j}H\overline{e_{i,j}}]. \tag{24}$$

The first three terms in Eq. (23) are similar to those of Eqs. (13)–(14), and they are negative definite due to Eq. (22) under the conditions (15)–(16) which we assume to be true.

To obtain the conditions on the region of negative definiteness of the quadratic form $\dot{\tilde{W}}$ it remains now to attack the quadratic form S_μ.

The values $\overline{e^{(k)}_{i,j}}$, $k=1,\ldots,m$ are bounded in the absorbing domain $B(\mu)$ for each cluster mode, i.e., $|\overline{e^{(k)}_{i,j}}|<\overline{e^{(k)}}$. Denote $M^{(k)}=|\Sigma^m_{n=1}h_{kn}\overline{e^{(n)}}|$. Then the sum (24) satisfies the inequality,

$$S_\mu > \sum_{j=1}^{N} \sum_{i=1}^{N-1} \sum_{k=1}^{m} \{(p_k |U_{i,j}^{(k)}| - \mu M^{(k)}) |U_{i,j}^{(k)}|\}.$$

Therefore $S_\mu > 0$, and hence $\dot{\tilde{W}} < 0$, for $|U_{i,j}^{(k)}| > \mu M^{(k)}/p_k$, $k = 1, \ldots, m$.

To estimate the domain where the quadratic form $\dot{\tilde{W}}$ is positive definite and thus to obtain an estimate of the maximal values of the transversal deviations $U_{i,j}^{(k)}$ from the pristine manifold $M(1,N)$ one should enclose the domain $\{\dot{\tilde{W}} < 0\}$ into some region bounded by a certain level W_0 of the Lyapunov function (12).

The enclosure $\{|U_{i,j}^{(k)}| < \mu M^{(k)}/p_k, \quad k=1,...,m\} \subset \{\tilde{W} < W_0\}$ determines that $\dot{\tilde{W}}$ is negative outside of the region

$$|U_{i,j}^{(k)}| < w_k M^{(k)} \mu / p_k, k = 1, \ldots, m, \tag{25}$$

where the constants w_k are defined by the level W_0.

Concluding the proof of the stability of the δ-synchronization regime we come to the following assertion.

Statement 6: Under the conditions (15), (16) and the assumptions related to the auxiliary system (21), a d-cluster synchronization regime of the system (2) defines d clusters of δ-synchronized oscillators of the system (17), where

$$\delta = \left[\max_{k \in [1,m]} w_k M^{(k)}/p_k \right] \mu, \quad k = 1, \ldots, m. \tag{26}$$

While the auxiliary parameters p_k are increasing, the estimated synchronization threshold $\varepsilon^*(p)$ increases whereas the synchronization error $\delta(p)$ decreases. Since we deal with the sufficient conditions, it is often possible to put some optimal value $p = \text{const}$.

The law of the (δ,ε) dependence is implicitly expressed via the dependence on p.

Answering the question of the persistence of the invariant manifolds under small perturbations, we remark on the following. The system (17) in a neighborhood of a cluster synchronization manifold $M(d)$ of the nonperturbed system (2) may be cast into the general form,

$$\begin{aligned} \dot{\mathbf{U}} &= \tilde{G}(\mathbf{X})\mathbf{U} + \mu \tilde{e}(\mathbf{U},\mathbf{X}), \\ \dot{\mathbf{X}} &= \tilde{F}(\mathbf{X}) + \mu \tilde{f}(\mathbf{U},\mathbf{X}), \end{aligned} \tag{27}$$

having the invariant manifold $\{\mathbf{U} = 0\}$ for $\mu = 0$.

It follows from Theory of Invariant Manifolds[37] and Central Manifold Theorem[38] that if the matrix $\tilde{G}(\mathbf{X})|_C$, where C is a compact, has the eigenvalues bounded from the left to zero, then the system (27) has a stable invariant manifold $\mathbf{U} = \tilde{\mathbf{U}}(\mathbf{X},\mu)$, $\mathbf{X} \in C$, $\tilde{\mathbf{U}}(\mathbf{X},0) = 0$. Hence, if the linear invariant manifold $M(d)$ satisfies the above conditions, then it is preserved under small perturbations such that the perturbed manifold $\tilde{M}_d$ defines the persistent clusters. However this approach seems to be less effective since it does not allow us to estimate the synchronization error and may not work for not infinitesimal values of the mismatch parameter.

To conclude, in this section we have proven the attracting property of the δ-neighborhood of generating cluster manifolds and obtained the estimate on the synchronization error δ. Our approach allows to investigate the persistence of synchronization *regimes*. When δ is no longer small, the invariant manifolds may be no longer preserved but their stable neighborhood provides stable δ-synchronization clusters.

We shall make now the general ideas of our approach concrete by investigating a 2D lattice of coupled Lorenz oscillators.

B. Example: 2D lattice of nonidentical Lorenz systems

Stability of asymptotic full synchronization in a 2D lattice of nonidentical Lorenz oscillators was recently studied[36] for the case of vector diffusive coupling and mismatch introduced in all the individual variables. The use of the all-variables coupling configuration and a large coupling strength allowed the authors to compensate the mismatch effect and provide asymptotic synchronization.

In contrast to this work, we apply our general results to a more difficult case of a scalar coupling and mismatch parameters that are present in all three equations of the individual Lorenz system.

We consider the 2D lattice (2)–(17) with the Lorenz system as an individual oscillator,

$$\begin{aligned} \dot{x}_{i,j} &= (\sigma + \sigma_{i,j})(y_{i,j} - x_{i,j}) + \varepsilon(\Delta x)_{i,j}, \\ \dot{y}_{i,j} &= (\gamma + \gamma_{i,j})x_{i,j} - y_{i,j} - x_{i,j}z_{i,j}, \\ \dot{z}_{i,j} &= -(b + b_{i,j})z_{i,j} + x_{i,j}y_{i,j}, \end{aligned} \tag{28}$$

for which $X_{i,j} = \text{column}(x_{i,j}, y_{i,j}, z_{i,j})$, and all other notations are similar to those of the system (2)–(17). We assume the perturbations of the parameters to be uniformly bounded $|\sigma_{i,j}| < \mu$, $|\gamma_{i,j}| < \mu$, and $|b_{i,j}| < \mu$.

Let us study global stability of the cluster δ-synchronization regime defined by the generating manifold $M(1,N) = \{X_{i,j} = X_j,\ X_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j}),\ i,j = 1, \ldots, N\}$ of the system (17) with $\mu = 0$. To do so, we shall follow the steps of the above study.

(1) The individual nonperturbed Lorenz system ($\mu = 0$, $\varepsilon = 0$) is eventually dissipative[39] and has an absorbing domain,

$$B = \{x^2 + y^2 + (z - \alpha)^2 < b^2 \alpha^2 / 4(b-1)\}, \quad \alpha = \gamma + \sigma.$$

Hence, the coordinates of the attractor of the individual Lorenz system are estimated to be bounded by

$$|\psi| < b\alpha / 2\sqrt{b-1}, \quad \psi = x, y, (z - \alpha). \tag{29}$$

Due to Statement 4, the estimates (29) are valid for coordinates of each oscillator of the coupled system (2)–(28).

(2) The finite difference equations (19) for $U_{i,j}^{(x)} = x_{i,j} - x_{i+1,j}$, $U_{i,j}^{(y)} = y_{i,j} - y_{i+1,j}$, $U_{i,j}^{(z)} = z_{i,j} - z_{i+1,j}$ of the coupled Lorenz systems (28) without mismatch ($\sigma_{i,j} = 0$, $\gamma_{i,j} = 0$, and $b_{i,j} = 0$) have the matrix

$$\overline{DF_{i,j}} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \gamma - \Theta^{(z)} & -1 & -\Theta^{(x)} \\ \Theta^{(y)} & -\Theta^{(x)} & -b \end{pmatrix}, \tag{30}$$

where $\Theta^{(\xi)} = (\xi_{i,j} + \xi_{i+1,j})/2$ for $\xi = x, y, z$. In the matrix (30), we have succeeded to get rid of the crossing terms with the help of the formula,

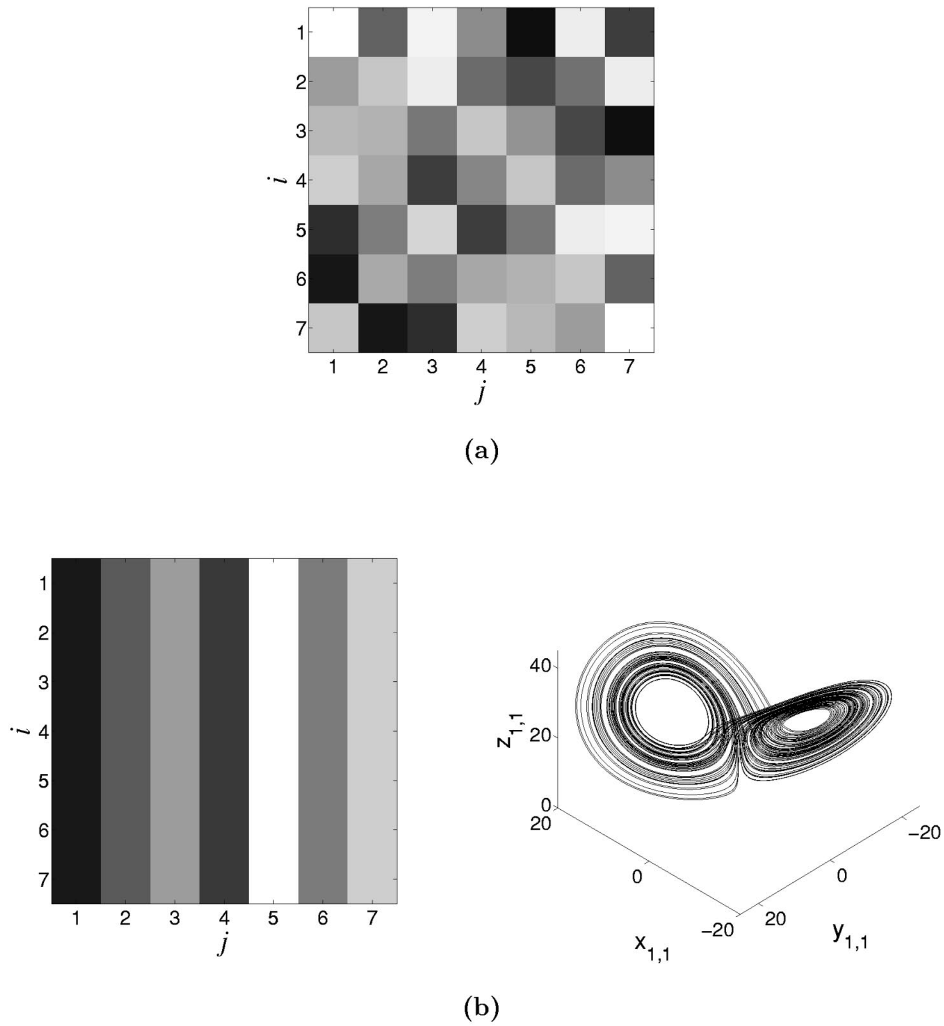

FIG. 5. Snapshots of clusters in the 7×7 lattice of nonidentical Lorenz systems. Different shades of gray are proportional to the amplitudes $x_{i,j}(t)$. Oscillators with identical gray shading belong to the same cluster. (a) Oscillators synchronize with respect to the principal diagonal of the lattice ($\varepsilon = 52$). (b) Synchronized rows of oscillators ($\varepsilon = 70$) (left). Temporal behavior of the (1,1) oscillator in the regime of synchronization of the rows (right).

$$\xi_{i,j}\eta_{i,j} - \xi_{i+1,j}\eta_{i+1,j} = \Theta^{(\eta)}(\xi_{i,j} - \xi_{i+1,j}) + \Theta^{(\xi)}(\eta_{i,j} - \eta_{i+1,j}).$$

To study the stability of the system (19)–(28) with the matrix (30), we use the simple quadratic form (10) with the unit matrix $H = I$. The auxiliary matrices are $A = \mathrm{diag}(a,0,0)$ and $P = \mathrm{diag}(p,p,p)$.

Then the condition (22) for the auxiliary system (21) to be stable is the condition for the symmetrized matrix $S = -[H(-A + \overline{DF} + P)]_s$,

$$S = \begin{pmatrix} a + \sigma - p & (-\alpha + \Theta^{(z)})/2 & -\Theta^{(y)}/2 \\ (-\alpha + \Theta^{(z)})/2 & 1 - p & 0 \\ -\Theta^{(y)}/2 & 0 & b - p \end{pmatrix} \quad (31)$$

to have positive eigenvalues. From Eq. (31) it follows that the auxiliary parameter p must be chosen from the interval (0,1) since the parameter b is assumed to be greater than 1 (in the original Lorenz system $b = 8/3$).

Taking into account the estimate (29) for the coordinates $\Theta^{(y)}$ and $\Theta^{(z)}$, we obtain the following sufficient condition for the matrix (31) to be positive definite:

$$a > a^*(p) = b^2\alpha^2/16(b-1)(1-p) + p - \sigma. \quad (32)$$

Hence, the sufficient conditions of the stability of the systems (19)–(28) may be written similar to Eq. (16) and take the form,

$$\varepsilon > \varepsilon^* = a^*(p)/4\sin^2(\pi/N). \quad (33)$$

(3) Synchronization error δ depends on the mismatch functions,

$$\mu f^{(1)}_{i,j}(X_{i,j}) = \sigma_{i,j}(y_{i,j} - x_{i,j}),$$

$$\mu f^{(2)}_{i,j}(X_{i,j}) = \gamma_{i,j}x_{i,j},$$

$$\text{and} \quad \mu f^{(3)}_{i,j}(X_{i,j}) = -b_{i,j}z_{i,j}. \quad (34)$$

The differences $\overline{e}^{(k)}_{i,j} = f^{(k)}_{i,j}(X_{i,j}) - f^{(k)}_{i+1,j}(X_{i+1,j})$ are linear functions of the coordinates of the system (28). Hence, they can be estimated via the absorbing domains $B_{i,j}$ that are in turn estimated by Eq. (29).

Taking into account Eq. (34) and using $H = I$, we obtain the estimate on the maximal mismatch functions difference,

$$M = \max_k \{\overline{e}^{(k)}_{i,j} \quad k = 1,2,3\}$$

$$= 2(b+\mu)(\alpha + 2\mu)/\sqrt{b + \mu - 1}. \quad (35)$$

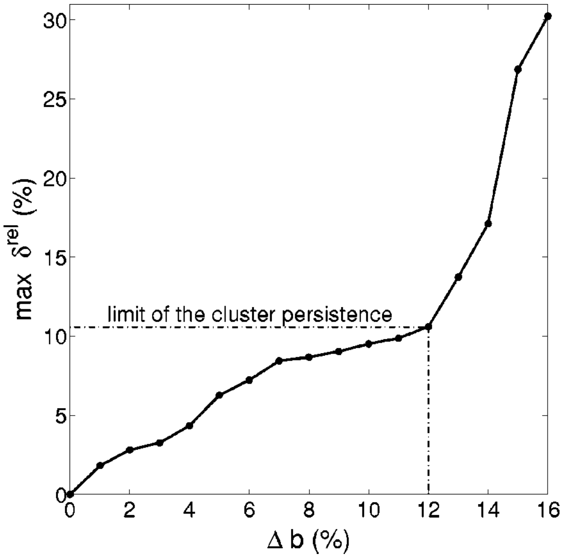

FIG. 6. 2D lattice of nonidentical Lorenz systems ($\varepsilon=70$). Maximum relative cluster synchronization error δ^{rel} versus parameter mismatch Δb.

Thus the synchronization error δ is estimated as follows:

$$\delta=4[(b+\mu)(\alpha+2\mu)/\sqrt{b+\mu-1}]\mu \tag{36}$$

for the chosen auxiliary parameter $p=1/2$. The constants w_k, $k=1,2,3$ from Eq. (25) are equal here to $\sqrt{2}$. While the mismatch parameter μ is small, the estimate (36) presents a quasilinear law of the dependence of the synchronization error on the mismatch coefficient.

Thus, we finally arrive at the conclusion that the cluster δ-synchronization regime, defined by the generating manifold $M(1,N)$ of the system (17) with $\mu=0$, is stable when the coupling strength ε reaches the threshold value ε^*. The value ε^* is a sufficient condition and gives an overestimate, therefore the δ-synchronization regime may become stable early under weaker coupling.

We estimate the relative synchronization error, expressed as the ratio of the maximal amplitude $A=\max\{X_{i,j}, j=1,N\}$ of the attractor, as follows:

$$\delta^{rel}=2\sqrt{2}M\mu/A. \tag{37}$$

Since for the coupled Lorenz systems (28), the maximal value M of the mismatch function difference is estimated by the maximal value A of the coordinates of the attractor, therefore the estimate (37) takes the form,

$$\delta^{rel}=2\sqrt{2}\mu. \tag{38}$$

In the general case, the synchronization error δ^{rel} may be not small. To be of physical relevance, it must though be essentially less than the difference between the corresponding coordinates of oscillators from two different clusters.

To validate the results on the existence and persistence of the described clusters with respect to parameter mismatch perturbations as well as the real observability of the predicted cluster synchronization modes, let us consider several numerical examples.

V. NUMERICAL EXAMPLES

A. 2D lattice of nonidentical Lorenz systems

To check the theoretical results we consider the 2D lattice (2)–(28) with $N=7$ and zero-flux BC. The parameters are $\sigma=10$, $\gamma=51$, $b=2.67$, $\sigma_{i,j}=0$, $\gamma_{i,j}=0$, and $b_{i,j}=\Delta b \cdot q$. The mismatch parameter Δb is expressed as a percentage of b, and values of the parameter q are chosen randomly from the interval $(-1,1)$.

We study numerically the order of appearance of persistent cluster synchronization modes when increasing the coupling parameter ε from zero ($\Delta b=5\%$). With increasing coupling ($\varepsilon=51.5$), a stable cluster, defining synchronization in pairs of oscillators with respect to the principal diagonal of the lattice, arises from a spatial disorder [Fig. 5(a)]. For $\varepsilon=69.5$, this cluster loses its stability, and chaotic synchronization between the lines of the lattice, defined by the generating manifold $M(1,7)$, arises [Fig. 5(b)]. For $\varepsilon=84.2$, complete synchronization becomes finally stable.

We study now the persistence of the cluster defining the synchronization between the lines. Numerical simulation shows that the limit of the persistence is being reached for $\Delta b=12\%$ (see Fig. 6). Here, the maximum relative synchronization error $\delta^{rel}=\max(x_{1,1}-x_{2,1})/\max(x_{1,1},x_{2,1})$ between the oscillators from one cluster is relatively small (up to $\Delta b=12\%$), whereas the amplitudes of different clusters are essentially different. For $\Delta b=13\%$ the transversal fluctuations from the generating manifold are no longer small with respect to the differences between the amplitudes of clusters, therefore the cluster regime is no longer recognizable.

The maximum synchronization error corresponding to the limit of the persistence is $\delta^{rel}=10.6\%$, and the theoretical δ^{rel} calculated from Eq. (38) for $\mu=\Delta b=12\%$ equals $\delta^{rel}=33.9\%$. Obtained from sufficient conditions, it can be considered as a good estimate of the synchronization error in the region of small μ.

B. Chain of nonidentical Rössler oscillators

The individual Rössler system does not satisfy our stability conditions (9)–(11) and our theory, strictly speaking, cannot be applied. However, we knowingly chose this difficult example to show that even here the δ-synchronization modes, while staying chaotic, are robust and stable.

We consider the system (1) of 9 x-coupled Rössler oscillators with zero-flux BC. The individual system reads

$$\begin{aligned}\dot{x}&=-(y+z),\\ \dot{y}&=x+a(1+\Delta a\cdot q)y,\\ \dot{z}&=b+(x-c)z.\end{aligned} \tag{39}$$

The parameters are $a=0.2$, $b=0.2$, $c=5.7$. Values of the parameter q are chosen randomly from the interval $(-1,1)$. The parameter Δa, expressed as a percentage of a, introduces the parameter mismatch.

Numerical simulation shows that a cluster of δ-synchronized oscillators defined by the generating symmetrical manifold $M^c(5)=\{X_1=X_9, X_2=X_8, X_3=X_7, X_4=X_6\}$ is stable and observed in a fairly wide region of the

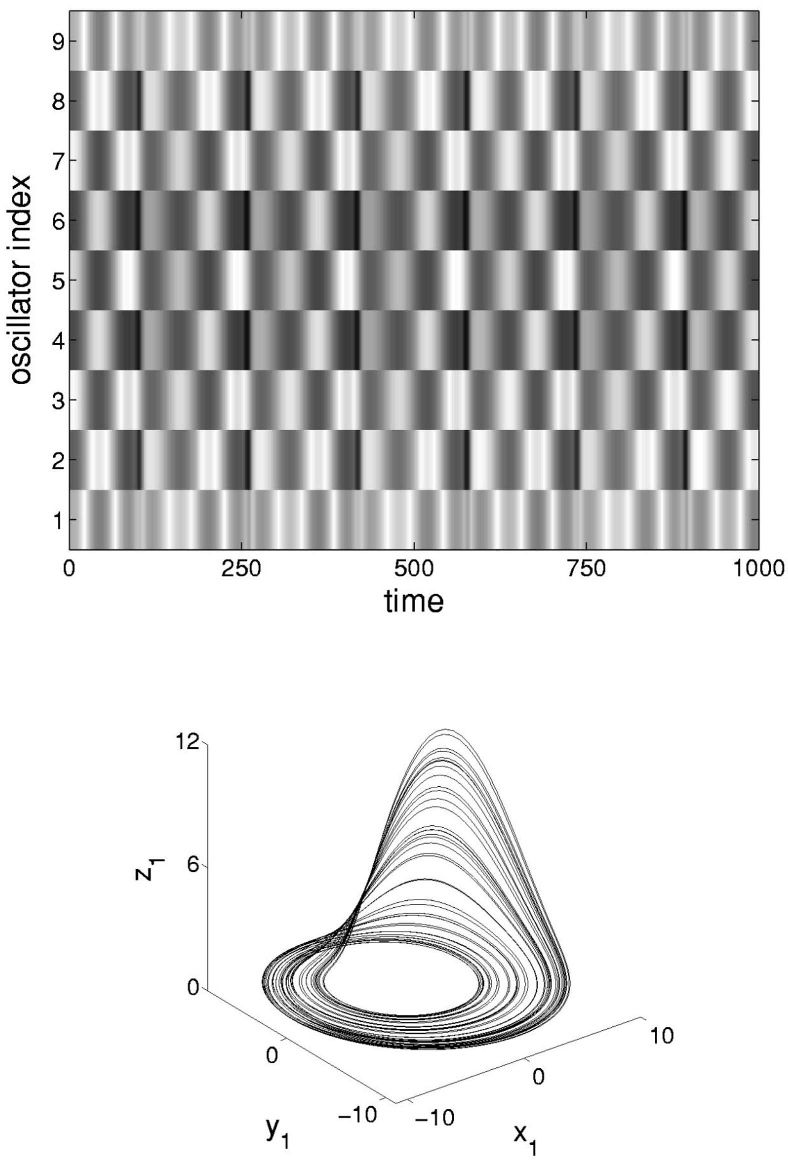

FIG. 7. Cluster synchronization in a chain of 9 Rössler systems. Different shades of gray are proportional to the amplitudes $x_i(t)$. The mismatch parameter $\Delta a = 5\%$. Oscillators synchronize in pairs around the fifth element (top). Chaotic attractor defining the dynamics of the first and the ninth oscillators in the regime of cluster synchronization (bottom).

parameter ε (see Fig. 7). The two other possible cluster regimes, defined by the manifolds $M^{\mathrm{alt}}(3)$ and $M^{\mathrm{c}}(2)$, are not observable in this particular case.

This stable chaotic cluster is persistent up to parameter mismatch $\Delta a = 11\%$ (see Fig. 8). Similar to the case of the Lorenz system, the maximum relative synchronization error δ^{rel} is calculated as follows: $\delta^{\mathrm{rel}} = \max(x_1 - x_9)/\max(x_1, x_9)$. Up to $\Delta a = 11\%$, the difference between the oscillators from one cluster is relatively small whereas the amplitudes of different clusters are essentially different. For $\Delta a = 12\%$ the transversal fluctuations from the generating manifold are no longer small with respect to the differences between the amplitudes of clusters, therefore the limit of the persistence is being reached. However, while the amplitudes of oscillators, that are supposed to form one cluster, develop in different manners, their phases seem to be close. In this case one can expect the phenomenon of cluster phase synchronization.[40] Here, the generating cluster manifold may still define the rule of the existence of phase synchronized clusters.

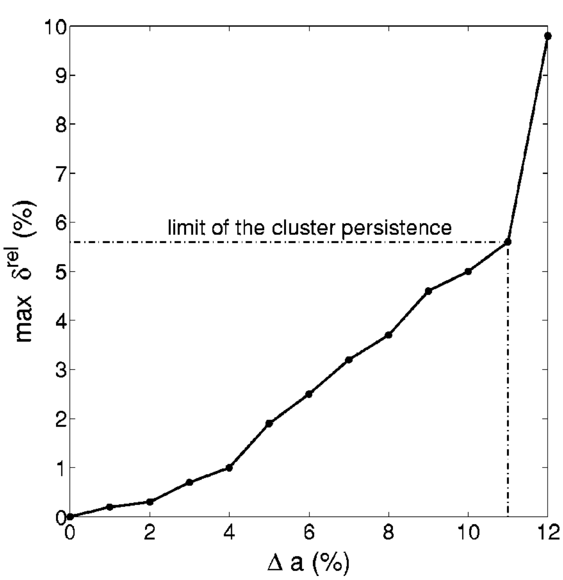

FIG. 8. Chain of nonidentical Rössler systems ($\varepsilon = 1.14$). Maximum relative cluster synchronization error δ^{rel} versus parameter mismatch Δa.

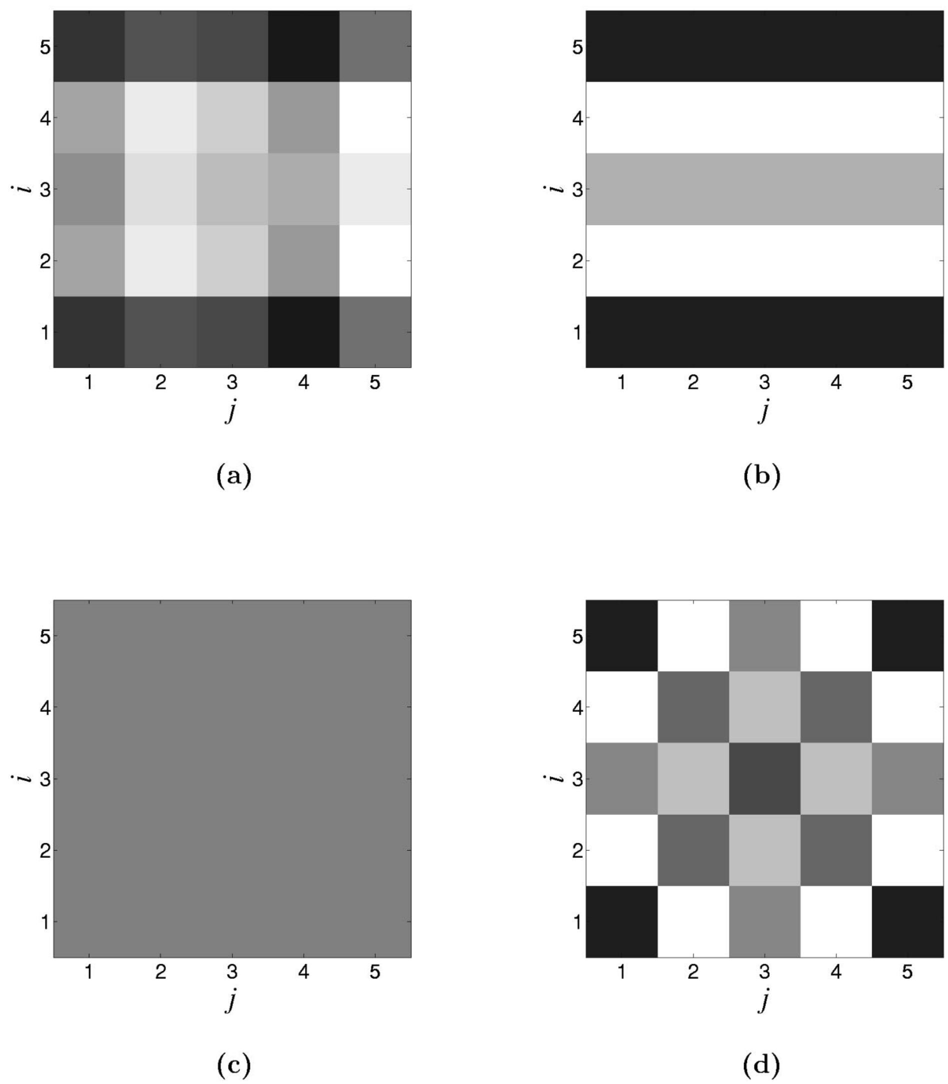

FIG. 9. Snapshots of clusters in the 5×5 lattice of chaotic Rössler oscillators with mismatch noise. (a) Rows of nonsynchronized oscillators synchronize in pairs around the middle row ($\varepsilon = 0.12$). (b) Oscillators synchronize within the rows ($\varepsilon = 0.49$). (c) Full synchronization ($\varepsilon = 0.58$). (d) Oscillators are synchronized with respect to the diagonals of the lattice and with respect to the middles of the rows and columns ($\varepsilon = 0.65$).

C. 2D lattice of nonidentical Rössler oscillators

As a second example of 2D lattices we consider the lattices (2)–(39) of N x-coupled Rössler oscillators with zero-flux BC. In the system (39), we introduce uniformly distributed mismatch noise at the interval $(-1,1)$ defined by the function $\psi(t)$ which stands for the parameter q. In contrast to the previous cases where mismatch was introduced by constant parameters, here we perturb the generating cluster synchronization manifolds by small mismatch noise.

Once again, the lattice of Rössler systems belongs to the class of coupled systems for which the synchronization regime is losing its stability as the coupling is increased. As it was discussed before, these desynchronization bifurcations can be directly related to the presence of saddle-foci (in the case of the Rössler system) which lie outside of the diagonal manifold and are preserved for any coupling strength. Their existence is imposed by a singularity of the individual Rössler system and the use of x-coupling.[14] Thus the cluster appearance may have the reverse order: as the coupling is increased, the number of clusters is also increased.

1. 5×5 lattice

Figure 9 presents the sequence of appearance of the persistent cluster δ-synchronization regimes with increasing coupling for a fairly large mismatch $\Delta a = 10\%$.

With increasing coupling from zero ($\varepsilon = 0.12$), a stable cluster defined by the generating manifold $M^{c}(3,5)$ arises [Fig. 9(a)]. This cluster defines a symmetrical spatiotemporal regime under which rows of the lattice synchronize in pairs around the middle (third) row. Oscillator within the rows are not synchronized. With further increased coupling ($\varepsilon = 0.49$), this regime gradually develops into a cluster defined by the manifold $M^{c}(3,1)$. Oscillators within the synchronized rows start to synchronize [Fig. 9(b)]. For $\varepsilon = 0.58$, full synchronization becomes locally stable and the corresponding homogeneous cluster arises [Fig. 9(c)]. For $\varepsilon = 0.65$, the spatiohomogeneous pattern decays due to the desynchronization bifurcations and a regime defined by the generating manifold $M^{\text{star}}(6)$ becomes stable [Fig. 9(d)]. Here, the oscillators are synchronized with respect to the diagonals of the square lattice and with respect to the middles of the rows and

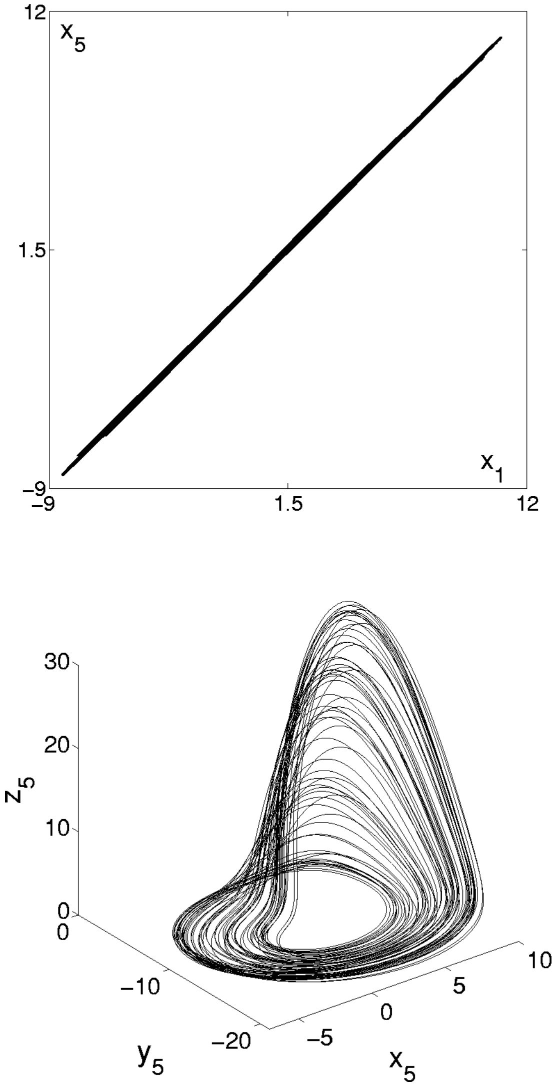

FIG. 10. (Top) δ-synchronized motion of oscillators forming the symmetrical cluster shown in Fig. 9(d). Temporal behavior of the (5,5) oscillator in the regime of cluster synchronization (bottom).

columns [see also Fig. 4(b)]. Projection of the attracting δ-neighborhood onto the plane (x_1,x_5) is shown in Fig. 10 (top). Figure 10 (bottom) shows a chaotic attractor defining the temporal behavior of the (5,5) oscillator in the regime of the symmetrical cluster synchronization.

Finally, with gradually increasing coupling this cluster synchronization regime becomes unstable and develops into a completely unsynchronized pattern.

2. 33×33 lattice

The numerical study of this fairly large network of oscillators is intended to show two things. First, it shows that the chaotic clusters predicted in the theoretical study are indeed stable in lattices composed of a large number of oscillators. Second, these modes are robust against small mismatch perturbations.

Figure 11 shows the sequence of the stabilization of the main symmetrical clusters in the lattice (2)–(39) with mismatch noise $\psi(t)$. For $\varepsilon=0.57$, a cluster δ-synchronization mode defined by the manifold $M(17,33)$ becomes stable and defines synchronization in pairs of oscillators of the lattice around the middle row [Fig. 11(a)]. Oscillators with identical gray shading belong to the same cluster. Unfortunately, there are not enough distinguishable gray shades (the number of clusters $d=17\times33=561$) to differentiate between all distinct elements. For $\varepsilon=0.6$, there arises a symmetrical cluster

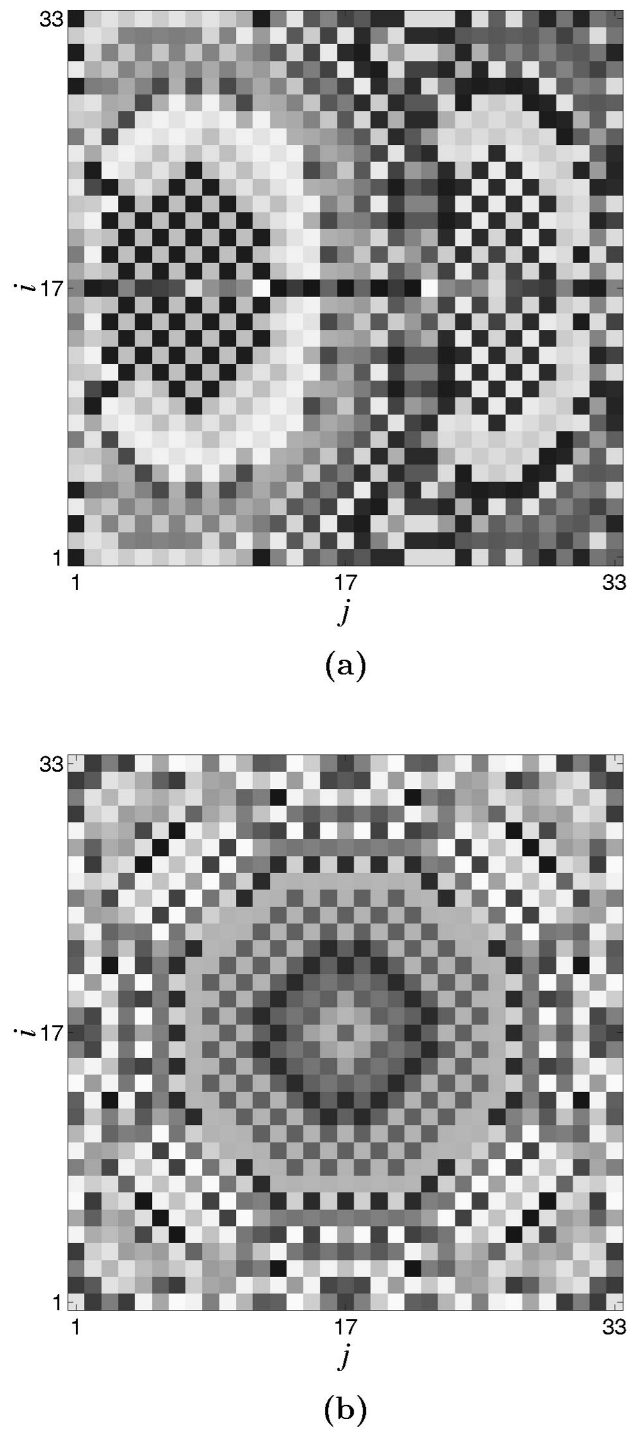

FIG. 11. Snapshots of stable clusters with chaotic dynamics in the 33×33 lattice of Rössler oscillators for $\Delta a=10\%$ (zero-flux BC and random initial conditions). (a) $\varepsilon=0.57$. Cluster similar to that of Fig. 9(a). (b) $\varepsilon=0.6$. Cluster similar to that of Fig. 9(d). Different shades of gray are proportional to the amplitudes $x_{i,j}(t)$. Unfortunately, there are not enough distinguishable gray shades to differentiate between all distinct elements.

defined by the generating manifold $M^{\text{star}}((n+1)(n+2)/2)$ for $N=2n+1=33$ with the number of clusters $d=153$ [Fig. 11(b)]. Here, oscillators synchronize with respect to the principal and secondary diagonals of the lattice and with respect to the middle rows and columns [pattern similar to that of Fig. 9(d)]. The (17,17) oscillator remains unsynchronized and defines one separate cluster.

These regimes provide good-quality cluster δ-synchronization up to the mismatch parameter $\Delta a \approx 15\%$. Similar to the 1D chain case, one can recognize persistent clusters of approximatively synchronized oscillators even up to $\Delta a = 100\%$. The amplitudes of ''synchronized'' oscillators differ essentially but the phases are close and probably locked.

VI. CONCLUSIONS

We have learned from the study, that apart from the fully synchronized and completely desynchronized solutions, the lattices of diffusively coupled oscillators may exhibit certain (strictly defined) kinds of clustering. These modes are defined by the invariant manifolds which exist regardless of the individual dynamics of the oscillators. Their stability and persistence essentially depends in turn on the vector field of the individual cells and on the variables by which the oscillators are coupled (scalar or vector diffusive coupling).

The main contribution of this paper is a systematic analysis of the persistence of cluster synchronization regimes in lattices of nonidentical chaotic oscillators. We have proven the global stability of cluster δ-synchronization regimes and obtained a good estimate for the synchronization error δ^{rel}. We have also shown numerically that these clusters are well preserved up to 10%–15% of parameter mismatch. Even in the case of larger parameter mismatch, the knowledge of the cluster manifold existence from the identical case is still useful since these manifolds may serve as a frame for possible regimes of lag and phase cluster synchronization in large lattices of diffusively coupled systems. Obviously these phenomena are subjects for future study.

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