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Persistent rigid-body motions and study's "Ribaucour" problem

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Abstract. In this work we show that the concept of a one-parameter persistent rigid-body motion is a slight generalisation of a class of motions called Ribaucour motions by Study. This allows a simple description of these motions in terms of their axode surfaces. We then investigate other special rigid-body motions, and ask if these can be persistent. The special motions studied are line-symmetric motions and motions generated by the moving frame adapted to a smooth curve. We are able to find geometric conditions for the special motions to be persistent and, in most cases, we can describe the axode surfaces in some detail. In particular, this work reveals some subtle connections between persistent rigid-body motions and the classical differential geometry of curves and ruled surfaces.

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1. Introduction

Historically kinematics was seen as a branch of geometry. Mathematical interest in the subject however, declined over the last century. Recent advances in the Engineering side of the discipline have sparked renewed interest in the classical geometry underlying the subject.

In a recent series of papers Carricato and co-workers introduced the concept of persistent screw systems [7,11]. A mechanism generates a 'persistent screw system' if the end-effector twist system remains invariant up to a rigid displacement under arbitrary finite displacements, away from singular configurations. In this case, the output screw system preserves its internal pattern and 'shape', but it moves in space like a rigid body. The submanifold of $SE(3)$ 'enveloped' by the output twist system can be referred to, in brief, as a 'persistent manifold'.

Persistent manifolds can be observed in many complex spatial kinematic chains with useful limited mobility of the end-effector [11]. In particular, Carricato [8–10] classified all persistent submanifolds of dimension smaller than 5 that can be generated by *serial* kinematic chains; these submanifolds are products of subgroups. The notion of persistence, however, has a wider applicability and applies to general chains generating submanifolds of $SE(3)$. That is, some sub-manifolds have the property of persistence others, most others, do not. There are persistent submanifolds that cannot be generated by serial chains (*i.e.* by products of subgroups), but still have important applications. A noteworthy example is provided by homokinetic couplings or zero-torsion parallel manipulators, whose persistence property emerges from the in-parallel connections of mirror-symmetric kinematic chains, see [2, 14, 30].

In [28] Study describes what he calls the 'Ribaucour problem' and gives a general solution for one dimensional submanifolds of $SE(3)$. Here, we show that these solutions are a particular type of persistent submanifold and this allows us to generalise Study's Ribaucour problem to arbitrary pitches and hence to characterise all persistent one-dimensional submanifolds of $SE(3)$ in a similar fashion to the one-dimensional Ribaucour manifolds described by Study.

Unfortunately, for higher dimensional submanifolds solutions to Study's Ribaucour problem and the notion of a persistent submanifold diverge, so knowledge of one does not help study of the other.

However, the main focus of the work is to examine some special rigid motions and ask if they can be persistent. In particular, we look at motions defined by a frame attached to a curve and line-symmetric motions generated by ruled surfaces. This leads us to revisit some classical differential geometry of curves and ruled surfaces.

2. Study's Ribaucour problem

Study seeks 1, 2 and 3-dimensional submanifolds of the group of rigid-body displacements such that, the instantaneous twist velocity is always a pure rotation, that is, has pitch 0. In the present work only 1-dimensional submanifolds are considered.

Suppose a rigid-body motion is given by a curve in the group of rigid-displacements, $G(t) \in SE(3)$. The instantaneous twist S_d of the motion $G(t)$ is given by

$$S_d = \frac{dG(t)}{dt} G^{-1}(t). \tag{2.1}$$

This is, of course, the Lie algebra element corresponding to the tangent vector to the curve $G(t)$. It is well known that elements of the Lie algebra $se(3)$ can be described as lines with a pitch. If $G(t)$ is given in the standard 4×4 representation of $SE(3)$, sometimes called the homogeneous representation, then a general Lie algebra element can be written as

$$S_d = \begin{pmatrix} 0 & -P_{03}, & P_{02} & P_{23} + pP_{01} \\ P_{03} & 0 & -P_{01} & P_{31} + pP_{02} \\ -P_{02} & P_{01} & 0 & P_{21} + pP_{03} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the P_{ij}s are the Plücker coordinates of the axis of the twist and p is the pitch of the element. When $p = 0$, the motion is, instantaneously, a pure rotation about the axis. More generally, the motion will be instantaneously a screw motion, that is a rotation about the axis together with a translation along the axis. The ratio of the rotational and translational velocities is given by the pitch of the twist. It is also possible to have motions that are instantaneously pure translations, in which case, the pitch is often said to be infinite.

The Lie algebra elements can also be written in a partitioned form, as

$$S_d = \begin{pmatrix} \Omega & \boldsymbol{v} \\ 0 & 0 \end{pmatrix},$$

where Ω is a 3×3 anti-symmetric matrix corresponding to the angular velocity $\boldsymbol{\omega} = (P_{01}, P_{02}, P_{03})^T$. We will sometimes also write the Lie algebra elements as 6-vectors, in partitioned form as

$$\boldsymbol{s}_d = \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix}.$$

In general, any rigid-body motion can be generated by the motion of a moving axode rolling and slipping on a fixed axode, see [3, Chapter 6, §5]. The fixed axode of a motion $G(t) \in SE(3)$ is given by the axis of S_d as t varies. The instantaneous twist in the moving reference frame is given by $S_b = G^{-1}(t)S_dG(t)$, that is, by the adjoint action on the twist in the fixed frame. The instantaneous twist S_b can also be found from the relation

$$S_b = G^{-1}(t)\frac{dG(t)}{dt}.$$

The moving axode is then the axis of S_b as t varies. Notice that, since the adjoint action preserves the pitch of a twist, the adjoint action which maps the twist velocity in the fixed frame to the twist in the moving frame will also map the fixed axode to the moving axode.

Study's description of the one-dimensional Ribaucour motions consists of trivial cases produced by cones or cylinders rolling without slipping on each other, together with non-trivial cases given by two general ruled surfaces rolling without slipping on each other. Notice that among the trivial cases are rotations about a fixed point, given by general cones rolling on each other and the planar motions given by general cylinders rolling on each other.

This solution is straightforward if we think of the axodes of the motion, as described above. The screw-axis of the instantaneous twist of the motion is given by the generating lines of the two axode surfaces which coincide at the instant under consideration. The pitch of the instantaneous twist will be given by the ratio of the slipping and rolling velocities. Hence, if there is no slipping, the pitch of the instantaneous twist will be zero, that is a pure rotation. So,

it is clear that any motion generated in this way, by rolling a ruled surface on another ruled surface without slipping, will produce a Ribaucour motion. Moreover, since any rigid motion can be realised as a motion given by the moving axode rolling and slipping on the fixed axode, this is the only way to produce such a motion.

Suppose that $G(t)$ is a Ribaucour motion, then the instantaneous twist S_d will have zero pitch. Let L_0 be a fixed twist, that is an element of the Lie algebra $se(3)$, with pitch zero. As mentioned, the pitch of a twist is invariant under the adjoint action of the group. Moreover the adjoint action is transitive on lines in space. Hence, the instantaneous twist velocity of $G(t)$ can be written as $S_d = HL_0H^{-1}$ where $H = H(t)$ is some other smooth path in the group. From Eq. (2.1) we get a differential equation for a general Ribaucour motion,

$$\frac{dG(t)}{dt} = HL_0H^{-1}G(t), \tag{2.2}$$

where H is an arbitrary smooth motion. In this case, the fixed axode of the motion is just HL_0H^{-1} and the moving axode is given by $G^{-1}HL_0H^{-1}G$.

3. Persistent motions

Suppose that M is a submanifold of $SE(3)$ and assume that G is some fixed point in M. The tangent space to the submanifold at G is given by T_GM. Translating the tangent space back to the identity in the group gives a subspace of the Lie algebra, $(T_GM)G^{-1} \subseteq se(3)$. Such a subspace is usually called a screw system. With this notation, the definition of a persistent submanifold can be stated as follows.

Definition 3.1. Let G_1, G_2 be any pair of points in a submanifold $M \subseteq SE(3)$, M is *persistent* if and only if the screw systems determined by the tangent spaces at these points are congruent. That is,

$$(T_{G_1}M)G_1^{-1} = H(T_{G_2}M)G_2^{-1}H^{-1}$$

for some $H \in SE(3)$.

In the rest of this paper only persistent 1-dimensional rigid-body motions will be considered. For such a one-dimensional submanifold persistence just means that the velocity twist at any point must have constant pitch. In other words,

$$\frac{dG}{dt} = HL_pH^{-1}G, \tag{3.1}$$

where L_p is a fixed twist with pitch p and H as before is an arbitrary smooth motion in the group. Clearly, when $p = 0$, this is exactly Study's Ribaucour problem.

The characterisation of these motions in terms of axodes is also similar to that given in the previous section. Finally here, we look at a couple of small results relating the twist velocities in the fixed and moving coordinate frames.

Lemma 3.2. *Suppose that a persistent motion has velocity twist in the global frame given by HL_pH^{-1}, then in the moving frame the twist velocity is given by $U^{-1}L_pU$, where $G = HU$ is the motion of the body.*

Proof. Since the fixed axode is HL_pH^{-1}, the moving axode is,

$$G^{-1}HL_pH^{-1}G = U^{-1}H^{-1}(HL_pH^{-1})HU = U^{-1}L_pU.$$

$\square$

Lemma 3.3. *Consider a persistent motion $G = HU$ with twist velocity HL_pH^{-1} as above, then*

$$L_p = Z_b + Z_d,$$

where $Z_b = H^{-1}\dot{H}$ and $Z_d = \dot{U}U^{-1}$.

Proof. Substitute $G = HU$ into Eq. (3.1) above to get

$$\frac{dG}{dt}G^{-1} = HL_pH^{-1} = (\dot{H}U + H\dot{U})U^{-1}H^{-1}.$$

Hence,

$$L_p = H^{-1}\dot{H} + \dot{U}U^{-1}. \tag{3.2}$$

$\square$

Remark 3.4. If H and L_p are given, then the motion U can be found by integrating,

$$\frac{dU}{dt} = (L_p - Z_b)U.$$

Symmetrically, if U and L_p are given, H can be found by integrating

$$\frac{dH}{dt} = H(L_p - Z_d).$$

3.1. Examples

In general it is difficult to integrate the differential equations for the motions H and U, given in Remark 3.4 above. The exceptional cases are when the Lie algebra elements are constant and the solutions are just exponentials. So we look at these as our first examples.

Assume that $Z_b = H^{-1}\dot{H}$ is constant, say $Z_b = S$. Hence $H = e^{tS}$, if we assume the initial condition $H(0) = I$. Now, to produce a p-persistent motion with pitch p we must solve,

$$\frac{dU}{dt} = (L_p - S)U,$$

where L_p is a twist with pitch p. Since the factor $(L_p - S)$ is constant this is again an exponential,

$$U = e^{t(L_p - S)}U(0)$$

where $U(0)$ is the initial condition which we will again assume to be the identity $U(0) = I$. Hence the persistent motion will be given by the product

$$G(t) = HU = e^{tS} e^{t(L_p - S)}.$$

Substituting $S_1 = S$ and $S_2 = L_p - S$, we see that

$$G(t) = e^{tS_1} e^{tS_2}$$

will be a persistent motion, where the pitch of the motion will be given by the pitch of the twist $S_1 + S_2$. This result, for an arbitrary 2-joint kinematic chain, can be found in [7,11].

In this simple case, it is also possible to describe the fixed and moving axodes of motion. These are given by

$$e^{tS_1} L_0 e^{-tS_1} \qquad \text{and} \qquad e^{-tS_2} L_0 e^{tS_2},$$

respectively, where L_0 is the line with the same axis as the twist L_p. The nature of these surfaces depend on the pitch of the twists in the exponent. If S has pitch 0 then the ruled surface $e^{tS} L_0 e^{-tS}$ will be a regulus of a circular hyperboloid, or the tangent lines to a circle if S and L_0 are perpendicular. If the pitch of S is non-zero (but finite) then the ruled surface will be, in general, a ruled helicoid. It may happen that the perpendicular distance between S and L_0 is δ and the angle between these axes is $\arctan(\delta/p)$ where p is the pitch of S. In such a case, the ruled surface traced out by L_0 will be the tangent developable surface of a helix. The helix is the curve traced out by the foot of the common perpendicular on L_0, between L_0 and the axis of S.

4. Persistent Frenet–Serret motions

4.1. General Frenet–Serret motions

Another way to specify a rigid-body motion uses a curve. Given a curve in space we demand that a specified point on the body follows the curve and the orientation is determined by the Frenet–Serret frame to the curve, see [3, Chapter 9, §2].

We recall here the basic ideas concerning Frenet–Serret motions as a way to fix notation. For a Frenet–Serret motion the motion of the body is fixed with respect to the Frenet frame of a curve $\boldsymbol{\gamma}(\mu)$. The frame equations of the Frenet frame can be written as

$$\frac{d}{d\mu} \boldsymbol{t} = \nu \boldsymbol{\omega} \times \boldsymbol{t},$$

$$\frac{d}{d\mu} \boldsymbol{n} = \nu \boldsymbol{\omega} \times \boldsymbol{n},$$

$$\frac{d}{d\mu} \boldsymbol{b} = \nu \boldsymbol{\omega} \times \boldsymbol{b},$$

where $\boldsymbol{\omega} = \tau \boldsymbol{t} + \kappa \boldsymbol{b}$ is the Darboux vector with $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ the tangent, normal and binormal vectors to the curve and ν, κ and τ the speed, curvature and torsion of the curve. The motion is given by,

$$G(\mu) = \begin{pmatrix} R & \boldsymbol{\gamma} \\ 0 & 1 \end{pmatrix},$$

with,

$$R = (\boldsymbol{t} \mid \boldsymbol{n} \mid \boldsymbol{b}).$$

The conditions for a Frenet–Serret motion to be persistent turn out to be quite stringent. The result follows from classical theorems,

Theorem 4.1. *The instantaneous twist of a Frenet–Serret motion has pitch, $\tau/(\kappa^2 + \tau^2)$, where κ and τ are the curvature and torsion functions of the curve that the motion is based on.*

This result appears to be well known, it appears in [3] as an exercise (Example 2 in section 2 of Chapter 9). Also from Bottema and Roth we have the following results.

Theorem 4.2. *The fixed axode of a Frenet–Serret motion consists of generator lines parallel to the curve's Darboux vector $\boldsymbol{\omega}$ and passing through a point that is displaced from the curve generating the motion along the principle normal to the curve. So the fixed axode is given by*

$$\boldsymbol{a}(\mu, \lambda) = \left(\boldsymbol{\gamma}(\mu) + \frac{\kappa(\mu)}{\kappa^2(\mu) + \tau^2(\mu)} \boldsymbol{n}(\mu) \right) + \lambda \boldsymbol{\omega}(\mu).$$

In general this ruled surface is not developable.

and also

Theorem 4.3. *The moving axode of a Frenet–Serret motion is a conoid.*

We include proofs of these results for completeness.

Proof of Theorems 4.1 and 4.2. The instantaneous velocity twist of a Frenet–Serret motion is given by

$$\frac{dG(\mu)}{d\mu} G^{-1}(\mu) = \nu \begin{pmatrix} \Omega R & \boldsymbol{t} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R^T & -R^T \boldsymbol{\gamma} \\ 0 & 1 \end{pmatrix} = \nu \begin{pmatrix} \Omega & \boldsymbol{t} - \boldsymbol{\omega} \times \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix},$$

where Ω is the 3×3 antisymmetric matrix corresponding to $\boldsymbol{\omega}$. Now, we can write

$$\boldsymbol{t} = \boldsymbol{\omega} \times \left(\frac{-\kappa}{\kappa^2 + \tau^2} \boldsymbol{n} \right) + \frac{\tau}{\kappa^2 + \tau^2} \boldsymbol{\omega},$$

so the instantaneous twist of the motion can be written as

$$\boldsymbol{s}_d = \nu \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix}, \tag{4.1}$$

with

$$\boldsymbol{v} = \left(\boldsymbol{\gamma} + \frac{\kappa}{\kappa^2 + \tau^2} \boldsymbol{n} \right) \times \boldsymbol{\omega} + \frac{\tau}{\kappa^2 + \tau^2} \boldsymbol{\omega} \tag{4.2}$$

The pitch of this twist is given by

$$p = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} = \frac{\tau}{(\kappa^2 + \tau^2)}.$$

This show Theorem 4.1, to see Theorem 4.2 observe that, from Eq. (4.2) the lines of the axode surface pass through the point,

$$\boldsymbol{\gamma} + \frac{\kappa}{\kappa^2 + \tau^2}\boldsymbol{n}$$

with direction $\boldsymbol{\omega}$. The parametric form of the ruled surface then follows. □

Proof of Theorem 4.3. The instantaneous velocity twist of a Frenet–Serret motion in the body frame is given by

$$G^{-1}(\mu)\frac{dG(\mu)}{d\mu} = \nu \begin{pmatrix} R^T & -R^T\boldsymbol{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega R & \boldsymbol{t} \\ 0 & 0 \end{pmatrix} = \nu \begin{pmatrix} R^T\Omega R & R^T\boldsymbol{t} \\ 0 & 0 \end{pmatrix}.$$

By construction of the Frenet–Serret motion the coordinate axes of the body fixed frame correspond to the Frenet frame of the curve at $\mu = 0$. So let us write $\boldsymbol{t}_0 = \boldsymbol{t}(0) = R^T(\mu)\boldsymbol{t}(\mu)$ and so forth. In particular we will denote $\boldsymbol{\omega}_0 = R^T(\mu)\boldsymbol{\omega}(\mu)$. The body-fixed velocity twist will be,

$$\boldsymbol{s}_b = \nu \begin{pmatrix} \boldsymbol{\omega}_0 \\ \boldsymbol{v}_0 \end{pmatrix},$$

with

$$\boldsymbol{v_0} = \frac{\kappa}{\kappa^2 + \tau^2}\boldsymbol{n}_0 \times \boldsymbol{\omega}_0 + \frac{\tau}{\kappa^2 + \tau^2}\boldsymbol{\omega}_0.$$

Notice that $\boldsymbol{\gamma}(0) = \boldsymbol{0}$ in these coordinates. The generator lines of the moving axode are thus given by,

$$\ell(\mu) = \begin{pmatrix} \boldsymbol{\omega}_0 \\ \frac{\kappa}{\kappa^2+\tau^2}\boldsymbol{n}_0 \times \boldsymbol{\omega}_0 \end{pmatrix},$$

where $\boldsymbol{\omega}_0 = \tau\boldsymbol{t}_0 + \kappa\boldsymbol{b}_0$. These lines are all reciprocal and perpendicular to the fixed line,

$$\ell_n = \begin{pmatrix} \boldsymbol{n}_0 \\ \boldsymbol{0} \end{pmatrix},$$

that is, they all intersect the line orthogonally, hence the ruled surface is a conoid. □

4.2. Persistence

Theorem 4.1 has the straightforward corollary.

Corollary 4.4. *The Frenet–Serret motion based on a curve is persistent if and only if $\tau/(\kappa^2 + \tau^2)$ is a constant, where κ and τ are the curvature and torsion functions of the curve.*

Theorem 4.5. *If the curve $\boldsymbol{\gamma}(\mu)$ generates a p-persistent Frenet–Serret motion $(p \neq 0)$, that is if $p = \tau/(\kappa^2 + \tau^2)$ is constant, the curvature and torsion of $\boldsymbol{\gamma}(\mu)$ can be parametrised by*

$$\kappa = \frac{1}{2p} \cos \phi, \qquad \tau = \frac{1}{2p}(\sin \phi + 1),$$

where ϕ is the parameter. The curvature and torsion functions can also be parametrised by the rational functions,

$$\kappa = \frac{1 - t^2}{2p(1 + t^2)}, \qquad \tau = \frac{(1 + t)^2}{2p(1 + t^2)}.$$

Proof of Theorem 4.5. If $p \neq 0$ the relation $p = \tau/(\kappa^2 + \tau^2)$ can be rearranged to give,

$$\kappa^2 + \left(\tau - \frac{1}{2p}\right)^2 = \left(\frac{1}{2p}\right)^2.$$

In a plane with κ and τ as coordinates this relation represents a circle of radius $(1/2p)$ centred at a distance of $(1/2p)$ along the τ axis. The trigonometric parameterisation of this circle gives the result.

The alternative, rational parameterisation is obtained using the tangent-half-angle substitution with parameter $t = \tan(\phi/2)$. $\qquad \square$

Notice that, if ϕ is constant then the curvature κ and torsion τ will also be constant and the curve $\boldsymbol{\gamma}(\mu)$ will be a helix. Also, for a Ribaucour motion, that is a motion with $p = 0$, we must have $\tau = 0$, so that $\boldsymbol{\gamma}(\mu)$ will be a plane curve in this case.

Remark 4.6. Curves with constant $\tau/(\kappa^2 + \tau^2)$ do not seem to have been studied in the classical literature.[1] Standard theory of curves tells us that, given a curvature and a torsion function, there will be a unique curve, up to rigid displacement, with these properties. So, illustrations of curves with the curvature and torsion functions given by the parameterisations above can be produced by numerical integration, see Fig. 1.

4.3. Moving axode of the persistent Frenet–Serret motion

Here we show that,

Theorem 4.7. *The moving axode of a persistent Frenet–Serret motion, with $p \neq 0$, is an equilateral hyperbolic paraboloid (also known as an orthogonal hyperbolic paraboloid).*

Proof. From Theorem 4.3 above, the moving axode of a general Frenet–Serret motion is given in body-fixed coordinates as,

$$\boldsymbol{a}(\mu,\, \lambda) = \frac{\kappa}{\kappa^2 + \tau^2}\boldsymbol{n}_0 + \lambda\boldsymbol{\omega}_0.$$

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[1]The quantity $\tau(\kappa^2 + \tau^2)^{-1}$ does however appear as the distribution parameter of a ruled surface formed from the principle normals to a general curve, see [29].
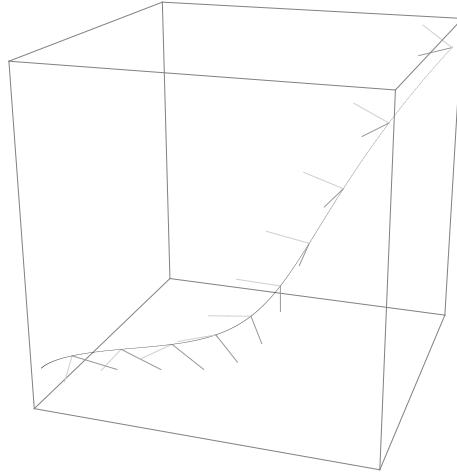
FIGURE 1 A curve with a persistent Frenet–Serret motion, $\kappa = (1 - t^2)/(3(1 + t^2))$ and $\tau = (1 + t)^2/(3(1 + t^2))$, that is, $p = 1.5$. The *grey lines* are normal vectors and *light grey lines* are binormals

Taking x, y and z as coordinates along the $\boldsymbol{t}_0$, $\boldsymbol{n}_0$ and $\boldsymbol{b}_0$ axes respectively, a point on the surface is given by the parametric equations,

$$x(\mu, \lambda) = \lambda\tau,$$
$$y(\mu, \lambda) = \kappa/(\kappa^2 + \tau^2),$$
$$z(\mu, \lambda) = \lambda\kappa.$$

For a persistent Frenet–Serret motion we have that $p = \tau/(\kappa^2 + \tau^2)$, so in the above parameterisation $y(\mu, \lambda) = p\kappa/\tau$. Eliminating κ, τ and λ gives,

$$xy = pz,$$

the equation of an equilateral hyperbolic paraboloid. $\qquad\qquad\qquad\square$

5. Bishop motions

Bishop motions, also called rotation minimising frame motions, are similar to the Frenet–Serret motions studied above. The only difference is that the orientation of the body is required to follow the Bishop frame to the curve. This type of motion has been advocated by several authors over the years for different applications in robotics and computer aided design, see for example [16, 25, 26].

The main result here is that the fixed axode of such a motion is a ruled surface familiar from the classical differential geometry of curves.

5.1. Generalities

In [27] it was shown that Bishop motions are characterised by the fact that their body-frame velocity twist must always lie in a IIB three system of screw with modulus $p = 0$. The fact that this is a II system indicates that (almost) all the twists in the system have the same pitch, the B here means that the system contains a single exceptional twist with infinite pitch. Finally the modulus $p = 0$ means that (almost) all the twists in the screw system have pitch zero. Hence any Bishop motion will be Ribaucour motion.

Given a curve in space $\boldsymbol{\gamma}(\mu)$, there is a moving frame of reference associated with the curve called the Bishop frame. Actually, there are many Bishop frames each determined by a choice of the initial frame. The tangent to the curve is given by the vector $(d\boldsymbol{\gamma}/d\mu) = \nu\boldsymbol{t}$, where ν is the speed $ds/d\mu$, of the curve; that is, the derivative of the arc-length s with respect to the parameter μ. Now, for the Bishop frame there are two normal vectors to the curve, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, and the frame equations are

$$\frac{d}{d\mu}\boldsymbol{t} = \nu(k_1\boldsymbol{n}_1 + k_2\boldsymbol{n}_2),$$

$$\frac{d}{d\mu}\boldsymbol{n}_1 = -\nu k_1\boldsymbol{t},$$

$$\frac{d}{d\mu}\boldsymbol{n}_2 = -\nu k_2\boldsymbol{t}.$$

The functions k_1 and k_2 are curvature-like functions. At every instant the unit vectors $\boldsymbol{t}$, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ form a right-handed ortho-normal frame and satisfy

$$\boldsymbol{t} \times \boldsymbol{n}_1 = \boldsymbol{n}_2, \qquad \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \boldsymbol{t}, \qquad \boldsymbol{n}_2 \times \boldsymbol{t} = \boldsymbol{n}_1.$$

Comparing these vectors with the usual Frenet–Serret vectors, we can see from the derivative of the tangent that

$$\kappa\boldsymbol{n} = k_1\boldsymbol{n}_1 + k_2\boldsymbol{n}_2,$$

where $\boldsymbol{n}$ is the principal normal vector. Since all these vectors have unit length the curvature satisfies $\kappa^2 = k_1^2 + k_2^2$. The binormal vector is defined as $\boldsymbol{b} = \boldsymbol{t}\times\boldsymbol{n}$, so it is easy to see that

$$\kappa\boldsymbol{b} = -k_2\boldsymbol{n}_1 + k_1\boldsymbol{n}_2.$$

Inverting the equations for $\boldsymbol{n}$ and $\boldsymbol{b}$ gives

$$\boldsymbol{n}_1 = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}\boldsymbol{n} - \frac{k_2}{\sqrt{k_1^2 + k_2^2}}\boldsymbol{b}, \qquad \boldsymbol{n}_2 = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}\boldsymbol{n} + \frac{k_1}{\sqrt{k_1^2 + k_2^2}}\boldsymbol{b}.$$

This shows that the Bishop frame rotates about the tangent vector with respect to the Frenet frame. Calling the rotation angle θ, we see that $\cos\theta = k_1/\sqrt{k_1^2 + k_2^2}$ and $\sin\theta = k_2/\sqrt{k_1^2 + k_2^2}$. By differentiating the equation for the binormal it is possible to show that $d\theta/d\mu = \nu\tau$, where τ is the torsion of the curve. Hence the rotation angle is given by the integral $\theta = \int \nu\tau\, d\mu + \theta_0$. The

constant of integration θ_0 represents the choice we have for the initial orientation of the Bishop frame. This freedom does not really affect the motion based on the Bishop frame.

The Bishop motion based on the curve $\boldsymbol{\gamma}(\mu)$ will be given by a curve in the group $SE(3)$

$$G(\mu) = \begin{pmatrix} R & \boldsymbol{\gamma} \\ 0 & 1 \end{pmatrix}$$

where the rotation matrix R has columns given by the tangent and normal vectors of the Bishop frame,

$$R = (\boldsymbol{t} \mid \boldsymbol{n}_1 \mid \boldsymbol{n}_2).$$

5.2. Axodes

The fixed and moving axodes of a Bishop motion are ruled surfaces familiar from classical differential geometry. The Bishop frame was only introduced in 1975 [1], by which time the heyday of classical kinematic geometry was almost over. So it is unlikely that these results appear in the classical literature. The result are expressed as the following pair of theorems.

Theorem 5.1. *The fixed axode of a Bishop motion based on a curve $\boldsymbol{\gamma}(\mu)$ is the polar developable surface of the curve.*

Theorem 5.2. *The moving axode of a Bishop motion consists of lines lying in a fixed plane.*

Proof of Theorem 5.1. At any parameter value μ the point on the curve $\boldsymbol{\gamma}(\mu)$ is instantaneously rotating about an axis. This axis will pass through the centres of the circles of curvature at the current point and will be a generator line of the fixed axode. The line through the centres of the circles of curvature is known to be perpendicular to the osculating plane of the curve at the current point, that is it is parallel to the binormal vector to the curve. The line is also known to pass through the centre of the osculating sphere to the curve. As the parameter varies the centre of the osculating sphere traces out the pole curve of $\boldsymbol{\gamma}(\mu)$ and the line trough the centres of the circles of curvature is tangent to the pole curve. The ruled surface traced by these lines is clearly the fixed axode of the motion, classically it is known as polar developable surface of the curve. See for example, [13, §13] or [12, Chapter 1, §9].   □

Proof of Theorem 5.2. The centre of the osculating sphere is known to lie in the normal plane to the curve and hence so does the instantaneous rotation axis described above as the line through the centre of the osculating sphere in the direction of the binormal vector. As the body moves along the curve the normal plane will be fixed in the body and hence all the lines of the moving axode all lie in this plane.   □

These results can also be shown by direct computations similar to those of Sect. 4.

5.3. Constructing Bishop motions

As an example of the possible use of these results a construction of a Bishop motion is presented here. In Sect. 5.1 above, Bishop motions were described as the composition of a Frenet–Serret motion with a rotation about the tangent line to the curve. Here an alternative construction is given but also based on a Frenet–Serret motion. Let $\boldsymbol{\sigma}(\mu)$ be a regular curve in space, and let $H(\mu)$ be the Frenet–Serret motion based on this curve. The construction uses the tangent developable surface of the $\boldsymbol{\sigma}(\mu)$ as the fixed axode of the Bishop motion. So if T_0 is the tangent line to $\boldsymbol{\sigma}(\mu)$ at $\mu = 0$ then the tangent developable can be written as $T(\mu) = H(\mu)T_0 H^{-1}(\mu)$. Now, Eq. (3.2) gives,

$$T_0 = H^{-1}\dot{H} + \dot{U}U^{-1}, \tag{5.1}$$

where $H^{-1}\dot{H}$ is the Darboux twist of the Frenet–Serret motion in the moving frame,

$$H^{-1}\dot{H} = \nu \begin{pmatrix} 0 & \kappa & 0 & 1 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

see the proof of Theorems 4.1 and 4.2 above. So setting

$$\dot{U}U^{-1} = -\nu \begin{pmatrix} 0 & \kappa & 0 & 1 \\ -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

satisfies Eq. (5.1), up to multiplication by a scalar function. The equation for U is not too difficult to solve since this is only a planar problem. Writing,

$$U(\mu) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & x \\ \sin\theta & \cos\theta & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the equation for $U(\mu)$ can be expanded into the three linear differential equations,

$$\frac{d\theta}{d\mu} = \nu\kappa,$$

$$\frac{dx}{d\mu} + \nu\kappa y = \nu,$$

$$\frac{dy}{d\mu} - \nu\kappa x = 0.$$

As a concrete example the Bishop motion whose fixed axode is the tangent developable to a circular helix can be constructed. The computations are straightforward, but long and not very instructive. The motion is illustrated in Fig. 2. In Sect. 5.1 Bishop motions were defined using the Bishop frame to a regular
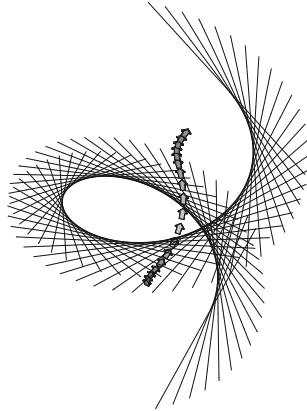
FIGURE 2 A Bishop motion (the *arrows*) whose fixed axode is the tangents to a circular helix

curve, in the present example this curve has not been found. The motion of an arbitrarily placed rigid body, the arrow, is illustrated in the figure.

6. Persistent aeroplane motions

The most general rigid-body motion associated with a curve is a general frame motion which was called an aeroplane motion in [27]. Such a motion can be written as the product $G(\mu) = G_2(\mu)G_1(\mu)$, where $G_2(\mu)$ is a Frenet–Serret motion associated with the curve $\boldsymbol{\gamma}$, and $G_1(\mu)$ is an arbitrary rotation about the tangent vector to the curve. Does this extra freedom allow for more persistent motions? The results can be stated as the following theorems:

Theorem 6.1. *On a smooth curve with minimum radius of curvature ρ, there are two p-persistent frame motions for any $p \neq 0$ and $-(\rho/2) < p < (\rho/2)$.*

Theorem 6.2. *Every regular curve has a unique frame motion which is a Ribaucour motion. This motion is given by any Bishop frame to the curve.*

Proof of Theorems 6.1 *and* 6.2. The instantaneous twist velocity of an aeroplane motion can be written in the fixed frame as

$$\frac{dG(\mu)}{d\mu}G^{-1}(\mu) = \dot{G}_1 G_1^{-1} + G_1 \dot{G}_2 G_2^{-1} G_1^{-1},$$

or, in the 6-vector representation of the Lie algebra $se(3)$, as

$$\boldsymbol{s}_d = \nu \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix} - \lambda \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{\gamma} \times \boldsymbol{t} \end{pmatrix},$$

where $\boldsymbol{\omega}$ and $\boldsymbol{v}$ are as in Sect. 4 and λ represents the rotational velocity about the tangent vector. The negative sign multiplying λ is for consistency with the

Bishop motion studied in Sect. 5. The pitch of $\boldsymbol{s}_d$ is given by

$$p = \frac{\nu(\nu\tau - \lambda)}{\lambda^2 - 2\nu\lambda\tau + \nu^2(\kappa^2 + \tau^2)}.$$

This can be rearranged to give a quadratic equation in λ,

$$p\lambda^2 - (2\nu p\tau - \nu)\lambda + \nu^2(p\kappa^2 + p\tau^2 - \tau) = 0.$$

Now, for the curve to have a p-persistent aeroplane motion, we must be able to solve for λ in terms of the speed, curvature and torsion functions of the curve. The discriminant of this quadratic simplifies to

$$\Delta = \nu^2(1 - 4p^2\kappa^2).$$

Hence, we can find real solutions to the quadratic, and hence p-persistent motions for p satisfying

$$\frac{-1}{2\kappa} < p < \frac{1}{2\kappa}.$$

So, the value of p is limited by half the minimum radius of curvature of the curve, $\pm(\rho/2)$, where $\rho = 1/\kappa$. The two roots of the quadratic will give the two possible p-persistent aeroplane motions when the condition is satisfied. This settles Theorem 6.1.

For Theorem 6.2, we can see that every regular curve has an aeroplane motion which is a Ribaucour motion given by setting $\lambda = \nu\tau$.   □

7. Persistent line-symmetric motions

In this section we explore the problem of finding persistent line-symmetric motions. A line-symmetric motion is given by reflecting a rigid-body in the successive generators of a ruled surface. These motions are of fundamental importance in kinematics, see for example [3]. They were extensively studied in a series of papers by Krames, see [18–24].

The following results are due to Krames [18]:

Theorem 7.1. *The pitch of the instantaneous twist velocity of a line symmetric motion is equal to the distribution parameter of the ruled surface generating the motion.*

Theorem 7.2. *The fixed axode of a general line-symmetric motion is the ruled surface generated by the central tangent lines to the surface generating the motion.*

This gives the straightforward corollary,

Corollary 7.3. *A line-symmetric motion generated by a ruled surface with constant distribution parameter p is a p-persistent motion. In particular, a line-symmetric motion generated by a developable ruled surface is a Ribaucour motion.*

Although the proof of the above theorems given in [18] are elegant and subtle, they rely on a detailed knowledge of synthetic geometry which is not common these days. Hence, in the interests of clarity and completeness modern proofs are offered below. These may not be as elegant as the originals but are straightforward and rely on little extra knowledge.

We begin with a brief description of these motions. Reflection in a line is simply a rotation of π radians about the line, sometimes called a half-turn. The Lie algebra element corresponding to a line can be given as 4×4 matrix,

$$L = \begin{pmatrix} \Omega & \boldsymbol{v} \\ 0 & 0 \end{pmatrix},$$

where Ω is the 3×3 antisymmetric matrix corresponding to the direction of the line $\boldsymbol{\omega}$ and $\boldsymbol{v}$ is moment of the line $\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega}$ for some point $\boldsymbol{r}$ on the line, see Sect. 2 above. These quantities constitute the Plücker coordinates of the line. It is easy to verify that these matrices satisfy the relation $L^3 = -L$, when $|\boldsymbol{\omega}|^2 = 1$. Hence, the exponential of L, corresponding to a rotation of θ about the line is given by the Rodrigues formula,

$$e^{\theta L} = I_4 + \sin\theta L + (1 - \cos\theta)L^2.$$

A half-turn about the line will be represented by the matrix

$$e^{\pi L} = I_4 + 2L^2.$$

The velocity twist of these motions can be calculated and the result is given by the following lemma.

Lemma 7.4. *Let $L(t)$ be the base surface for a line-symmetric motion. The twist velocity of this motion is given by the commutator $S_d = 2[L(t), \dot{L}(t)]$.*

Proof. Consider reflecting a rigid-body in the successive generators of the ruled surface given by $L(t)$. In order that the motion passes through the identity in the group when $t = 0$ we can compose the reflections in the generators with a reflection in the initial generator. The motion can be parametrised as,

$$G(t) = e^{\pi L(t)} e^{\pi L(0)} = (I_4 + 2L(t)^2)(I_4 + 2L(0)^2).$$

The instantaneous twist velocity of this motion is given, in the fixed frame, by

$$\frac{dG(t)}{dt} G^{-1}(t) = S_d = 2(\dot{L}L + L\dot{L})(I_4 + 2L^2)$$

where the explicit dependence on t has been dropped for brevity. Expanding this in terms of the Plücker coordinates of the original lines gives,

$$S_d = 2 \begin{pmatrix} \Omega\dot{\Omega} - \dot{\Omega}\Omega + 2\Omega\dot{\Omega}\Omega^2 & \Omega\dot{\boldsymbol{v}} - \dot{\Omega}\boldsymbol{v} + 2\Omega\dot{\Omega}\Omega\boldsymbol{v} \\ 0 & 0 \end{pmatrix}.$$

Using the fact that $\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1$ and hence $\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}} = 0$, it is possible to show that $\Omega\dot{\Omega}\Omega = 0$. So,

$$S_d = 2 \left\{ \begin{pmatrix} \Omega & \boldsymbol{v} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\Omega} & \dot{\boldsymbol{v}} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \dot{\Omega} & \dot{\boldsymbol{v}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega & \boldsymbol{v} \\ 0 & 0 \end{pmatrix} \right\} = 2[L(t), \dot{L}(t)]$$

$$\square$$

Remark 7.5. In terms of the 6-vector representation of $se(3)$, the result of Lemma 7.4 can be written as

$$\boldsymbol{s}_d = 2 \begin{pmatrix} \boldsymbol{\omega} \times \dot{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \times \dot{\boldsymbol{v}} + \boldsymbol{v} \times \dot{\boldsymbol{\omega}} \end{pmatrix}.$$

This is the vector product of the two twists corresponding to L and $\dot{L}$.

The proof of the main theorem is now straightforward.

Proof of Theorem 7.1. Remembering that $\boldsymbol{\omega} \cdot \boldsymbol{v} = 0$ and $|\boldsymbol{\omega}|^2 = 1$ since these are the Plücker coordinates of a line, the pitch of the velocity twist $\boldsymbol{s}_d$, given in Remark 7.5, can be evaluated as

$$p = \frac{(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{v}} + \boldsymbol{v} \times \dot{\boldsymbol{\omega}})}{|\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}|^2} = \frac{\dot{\boldsymbol{\omega}} \cdot \dot{\boldsymbol{v}}}{|\dot{\boldsymbol{\omega}}|^2}.$$

This can be compared with standard formulas for the distribution parameters of a ruled surface, for example see [6]. After accounting for the fact that we have set $|\boldsymbol{\omega}|^2 = 1$ the formulas agree.   □

Finally, the result for the velocity twist of these motions can be used to prove Theorem 7.2.

Proof of Theorem 7.2. Assume that the ruled surface generating the line-symmetric motion is given by,

$$\boldsymbol{r}(t, \lambda) = \boldsymbol{s}(t) + \lambda \boldsymbol{\omega}(t),$$

where the directions of the generating lines $\boldsymbol{\omega}(t)$ are unit vectors so that $\boldsymbol{\omega} \cdot \dot{\boldsymbol{\omega}} = 0$. Moreover, assume that the directrix curve $\boldsymbol{s}(t)$ is actually the striction curve of the ruled surface, so that $\dot{\boldsymbol{s}} \cdot \boldsymbol{\omega} = 0$. The central normal vector is given by $\boldsymbol{\omega} \times \dot{\boldsymbol{s}}$ and hence the central tangent vector is given by $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{s}})$. The vector can be shown to be parallel to the vector $\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}$: all that is needed is to take the cross product of the two vectors and simplify using the relations above. Hence, we see that the axis of the motion's velocity twist, as given in Remark 7.5, is parallel to the central tangent of the surface. To show that the axis of the twist passes through the striction point, at time t, we look at the translational part of the twist. Substituting $\boldsymbol{v} = \boldsymbol{s} \times \boldsymbol{\omega}$ in the twist velocity gives

$$\boldsymbol{\omega} \times \dot{\boldsymbol{v}} + \boldsymbol{v} \times \dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times (\dot{\boldsymbol{s}} \times \boldsymbol{\omega} + \boldsymbol{s} \times \dot{\boldsymbol{\omega}}) + (\boldsymbol{s} \times \boldsymbol{\omega}) \times \dot{\boldsymbol{\omega}}$$
$$= \boldsymbol{\omega} \times (\dot{\boldsymbol{s}} \times \boldsymbol{\omega}) + \boldsymbol{s} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}).$$

The first term on the right of the equation above is parallel to the direction of the axis $(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})$. The second term is thus the moment of the line and clearly it passes through $\boldsymbol{s}$, the striction point on the generator.   □

7.1. An example persistent line-symmetric motion

To produce a p-persistent line-symmetric motion we need examples of ruled surfaces with constant distribution parameter. Such surfaces have been studied in the context of classical differential geometry by several workers, see for example [5].

The circular hyperboloid and ruled helicoid met in Sect. 3.1 are examples. Note that the motions generated using these surfaces as axodes and those generated by the same surfaces as line-symmetric motions will be different in general.

Another set of known examples consists of the binormal lines to curves with constant torsion. Given a space curve, the set of lines along the binormal vectors to the curve form a ruled surface. It is straightforward to show that the distribution parameter of such a surface is equal to the torsion of the original curve. In fact the striction curve to the surface is the original curve from which the surface was constructed, see [29].

Before working on line-symmetric motions, Krames found a special class of Cayley's cubic ruled surfaces with constant distribution parameter, [17]. Later Brauner showed that these were the only cubic ruled surfaces with constant distribution parameter, [4]. Line-symmetric motions based on Cayley's cubic ruled surface were studied by Husty [15], but this work did not use a surface with constant distribution parameter.

The Cayley cubic ruled surface with constant distribution parameter is given in Plücker coordinates by

$$
\begin{array}{ll}
P_{01} = 2t^3, & P_{23} = -6dt, \\
P_{02} = 3t^2 + 1, & P_{31} = 6dt^2, \\
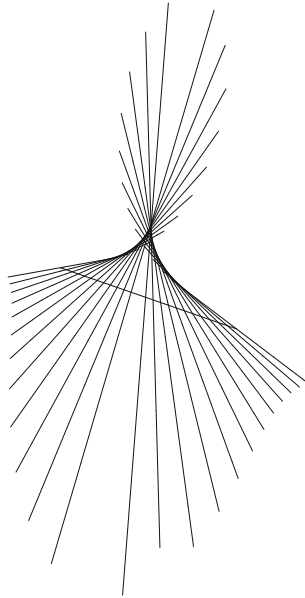P_{03} = \sqrt{3}(t^2 + 1), & P_{12} = -2\sqrt{3}dt^2.
\end{array}
$$

FIGURE 3 A cubic Cayley ruled surface with constant distribution parameter ($d = 1/3$)

This has been adapted very slightly from the result given in [4], the parameter d has been scaled so that in this representation the distribution parameter of the surface is exactly d. An illustration of this surface is given in Fig. 3. This diagram also shows the generator line which is also a directrix of the surface, that is a curve which meets all the generators. The directrix corresponds to $t = 0$ in the parameterisation given above.

As a 4×4 matrix, the motion is then given by

$$G(t) = (I_4 + 2L(t)^2)(I_4 + 2L(0)^2)$$

$$= \begin{pmatrix} \frac{-t^6+3t^4+3t^2+1}{(t^2+1)^3} & \frac{t^3}{(t^2+1)^3} & \frac{\sqrt{3}(2t^5+t^3)}{(t^2+1)^3} & -\frac{2\sqrt{3}dt^2(3t^2+2)}{(t^2+1)^3} \\ -\frac{3t^5+t^3}{(t^2+1)^3} & \frac{t^6+3t^4+6t^2+2}{2(t^2+1)^3} & -\frac{\sqrt{3}t^2(t^4-3t^2-2)}{2(t^2+1)^3} & \frac{\sqrt{3}dt(2t^4-3t^2-3)}{(t^2+1)^3} \\ -\frac{\sqrt{3}t^3}{(t^2+1)^2} & -\frac{\sqrt{3}t^2(t^2+2)}{2(t^2+1)^2} & \frac{-t^4+4t^2+2}{2(t^2+1)^2} & \frac{3d(2t^3+t)}{(t^2+1)^2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Plücker coordinates of the fixed axode of the motion are given by

$$\begin{aligned} F_{01} &= -2\sqrt{3}t, & F_{23} &= -6\sqrt{3}dt, \\ F_{02} &= \sqrt{3}t^2(t^2+3), & F_{31} &= -3\sqrt{3}d(t^4+1), \\ F_{03} &= 3t^2(t^2+1), & F_{12} &= -3d(t^4+2t^2-1), \end{aligned}$$

namely a quartic ruled surface.

Acknowledgments

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