Persistently ϵ-Optimal Strategies — Source link

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PERSISTENTLY ϵ-OPTIMAL STRATEGIES

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Abstract

There are strategies available for nonnegative gambling problems which are not only ϵ-optimal but persist in being conditionally ϵ-optimal along every history.

Key words: gambling, optimal strategies, probability, finite additivity, dynamic programming, stochastic control, decision theory.

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1. Introduction.

Let (F, Γ, u) be a gambling problem where, as in [3], F is the set of fortunes or states, Γ is the gambling house, and u the utility function. The gambling problem is nonnegative if u is a nonnegative, possibly unbounded function.

Let $0 < \epsilon < 1$, $f \in F$, and $V(f)$ be the most a gambler with fortune f can achieve ([3], section 3.3). A strategy σ available at f in Γ is (multiplicatively) ϵ-optimal at f if $u(\sigma) \geq (1-\epsilon)V(f)$. Let $p = (f_1,\ldots,f_n)$. Then σ is conditionally (multiplicatively) ϵ-optimal given p if the conditional strategy $\sigma[p]$ is multiplicatively ϵ-optimal at f_n. If σ is (multiplicatively) ϵ-optimal at f and is conditionally (multiplicatively) ϵ-optimal given p for every partial history p, then σ is said to be persistently (multiplicatively) ϵ-optimal at f. If u is bounded or, more generally, if u is nonnegative and V is everywhere finite, then persistently ϵ-optimal strategies are always available (Theorem 1 below). If, in addition, the gambling problem is sufficiently measurable, there exist measurable strategies which are persistently ϵ-optimal (Theorem 2).

The notion of persistently (or thoroughly) optimal strategies was introduced in [3, section 3.5]. It is related to the concept of stationarity. For suppose $\bar{\sigma}$ is a stationary family of ϵ-optimal strategies. Then, for every f, $\bar{\sigma}(f)$ is persistently ϵ-optimal since $\bar{\sigma}(f)[f_1,\ldots,f_n] = \bar{\sigma}(f_n)$.

Gambles in [3] were taken to be finitely additive probability measures defined on all subsets of F. Here a gamble γ can be regarded as a nonnegative functional with domain the collection $\mathfrak{h}$ of nonnegative, real-valued functions defined on F and satisfying

$$\gamma(u + v) = \gamma u + \gamma v ,$$

$$\gamma(a\, u) = a\, \gamma\, u$$

for $u, v \in \mathfrak{h}$ and $a \geq 0$. A gamble in the present sense is, when restricted to the collection of indicator functions, a gamble in the sense of [3]. Thus the gambling problems considered here are slightly more general than those of [3]. However, much of the theory remains unchanged and appropriate definitions and results from [3] will be used without further comment.

2. Preliminaries.

Let $H = F \times F \times \ldots$ be the set of histories. A mapping t from H to $\{1,2,\ldots\} \cup \{\infty\}$ is called a stopping time if, whenever h and h' are in H, $t(h) = n < \infty$, and h' agrees with h in the first n coordinates, then $t(h') = n$. A stopping time which assumes only finite values is a stop rule. Let n be a positive integer and $h = (f_1,\ldots,f_n,\ldots)$ be a history. Recall that $p_n(h) = (f_1,\ldots,f_n)$. (See [3] for a detailed explanation of the notation and terminology.) If r is a stopping time, the conditional stopping time given $p_n(h)$ is defined by $t[p_n(h)](h') = t(p_n(h)h') - n$ for $h' \in H$. Notice that $t[p_n(h)]$ is a stopping time or identically zero according as $r(p_n(h)h')$ is greater than or equal to n.

Let (σ, t) be a policy and s be a stop rule. Make the convention that $u(\sigma[p_s(h)], t[p_s(h)])$ is $u(f_t(h))$ when $s(h) \geq t(h)$. Then the formula

$$(1) \qquad u(\sigma, t) = \int u(\sigma[p_s], t[p_s])d\sigma$$

is a special case of formula 3.7.1, [3]. The similar formula

$$(2) \qquad u(\sigma) = \int u(\sigma[p_s])d\sigma$$

follows from Theorem 3.2.1, [3] and an argument by induction on the structure of p_s. The first lemma below generalizes (2) and gives a formula which, in a sense, separates the utility of a strategy into that earned before a given time r and that earned afterwards.

For simplicity, assume henceforth that u is nonnegative and all strategies have finite utility.

Lemma 1. Let σ be a strategy and r be a stopping time. Then

$$(3) \qquad u(\sigma) = \limsup_{t\to\infty} u_r(\sigma,t)$$

where $u_r(\sigma,t) = \int_{[t\le r]} u(f_t)d\sigma + \int_{[t>r]} u(\sigma[p_r])d\sigma$

and the lim sup is taken over the directed set of stop rules.

Proof: By definition ([3], section 3.2)

$$(4) \qquad u(\sigma) = \limsup_{t\to\infty} u(\sigma,t) \ .$$

For t a stop rule, apply (1) with $s = t\wedge r$ to get

$$(5) \qquad u(\sigma,t) = \int u(\sigma[p_{t\wedge r}], \ t[p_{t\wedge r}])d\sigma$$

$$= \int_{[t\le r]} u(f_t)d\sigma + \int_{[t>r]} u(\sigma[p_r], \ t[p_r])d\sigma \ .$$

Let $\epsilon > 0$.

Claim: There is a stop rule t_0 such that, for every stop rule t,

$$\int_{[t>r,t_0\le r]} u(\sigma[p_r], \ t[p_r])d\sigma < \epsilon \ .$$

Suppose the claim is false. Then, for every stop rule t, there is a stop rule s such that

$$\int_{[s>r,t\le r]} u(\sigma[p_r], \ s[p_r])d\sigma \ge \epsilon \ .$$

Define $t_1 = t$ if $t > r$ or $s \le t$,

$\qquad\qquad = s$ if $t \le r$ and $s > t$.

Then t_1 is a stop rule, $t_1 \ge t$ and

$$\int_{[t_1>r]} u(\sigma[p_r],\ t_1[p_r])d\sigma = \int_{[t>r]} u(\sigma[p_r],\ t[p_r])d\sigma + \int_{[t\le r,s>r]} u(\sigma[p_r],\ s[p_r])d\sigma$$

$$\ge \int_{[t>r]} u(\sigma[p_r],\ t[p_r])d\sigma + \epsilon\ .$$

Similarly, there exist stop rules $t_2, t_3, \ldots$ such that $t_n \le t_{n+1}$ and

$$\int_{[t_{n+1}>r]} u(\sigma[p_r],\ t_{n+1}[p_r])d\sigma \ge \int_{[t_n>r]} u(\sigma[p_r],\ t_n[p_r])d\sigma + \epsilon\ .$$

Thus, for $n = 1,2,\ldots,$ $t_n \ge t$ and

$$u(\sigma,t_n) \ge \int_{[t_n>r]} u(\sigma[p_r],\ t_n[p_r])d\sigma \ge n\ \epsilon\ .$$

Hence, $u(\sigma) = \infty$, a contradiction which establishes the claim.

Next, define a stop rule t' thus: If $t_0(h) \le r(h)$, set $t'(h) = t_0(h)$. If $t_0(h) > r(h)$, choose a stop rule t_h such that $t_h \ge t_0[p_r(h)]$ and, for every stop rule $s \ge t_h$,

$$(6) \qquad u(\sigma[p_r(h)],s) \le u(\sigma[p_r(h)] + \epsilon\ .$$

Then set $t'(h) = r(h) + t_h(f_{r(h)+1}, f_{r(h)+2},\ldots)$ so that $t'[p_r(h)] = t_h$. Notice that $t' \ge t_0$.

Suppose t is a stop rule and $t \ge t'$. Then

$$u(\sigma,t) = \int_{[t\le r]} u(f_t)d\sigma + \int_{[t_0>r]} u(\sigma[p_r],\ t[p_r])d\sigma + \int_{[t>r,t_0\le r]} u(\sigma[p_r],\ t[p_r])d\sigma$$

(by (5))

$$\le \int_{[t\le r]} u(f_t)d\sigma + \int_{[t_0>r]} u(\sigma[p_r])d\sigma + 2\ \epsilon$$

(by (6) and the Claim)

$$\leq \int_{[t\leq r]} u(f_t)d\sigma + \int_{[t>r]} u(\sigma[p_r])d\sigma + 2\,\epsilon$$

$$= u_r(\sigma,t) + 2\,\epsilon\ .$$

Hence, $u(\sigma) \leq \limsup_{t\to\infty} u_r(\sigma,t)$.

To prove the opposite inequality, again let $\epsilon > 0$ and s be a stop rule. Define a stop rule t as follows: If $s(h) \leq r(h)$, let $t(h) = s(h)$. If $s(h) > r(h)$, choose t_h to be a stop rule at least as large as $s[p_r(h)]$ and such that

$$(7)\qquad u(\sigma[p_r(h)],\ t_h) \geq u(\sigma[p_r(h)] - \epsilon\ .$$

Set $t(h) = r(h) + t_h(f_{r(h)+1}, f_{r(h)+2}, \ldots)$ so that $t[p_r(h)] = t_h$. Then $t \geq s$ and

$$u(\sigma,t) \geq u_r(\sigma,t) - \epsilon \qquad \text{(by (5) and (7))}$$

$$= u_r(\sigma,s) - \epsilon\ .$$

Hence, $u(\sigma) \geq \limsup_{t\to\infty} u_r(\sigma,t)$. □

Let r be a stopping time. The strategies σ and σ' <u>agree up to time $r-$</u> if $\sigma_0 = \sigma'_0$ and, for every $h \in H$ and $0 < n < r(h)$, $\sigma_n(p_n(h)) = \sigma'_n(p_n(h))$. Here is a formulation of the nearly obvious fact that if σ and σ' agree up to some time and if, given the past up to that time, σ' does conditionally nearly as well as σ, then unconditionally σ' does nearly as well as σ.

<u>Lemma 2.</u> If r is a stopping time, σ and σ' are strategies which agree up to time $r-$ and $\epsilon > 0$, then each of the following conditions

implies its successor:

(i) σ' is available in Γ and $u(\sigma[p_r(h)]) \geq (1-\epsilon)V(f_r(h))$ whenever $r(h) < \infty$;

(ii) $u(\sigma[p_r(h)] \geq (1-\epsilon)u(\sigma'[p_r(h)])$ whenever $r(h) < \infty$;

(iii) $u(\sigma) \geq (1-\epsilon)u(\sigma')$.

Proof: To see that (i) implies (ii), observe that $u(\sigma'[p_r(h)]) \leq V(f_r(h))$ when $r(h) < \infty$. Use Lemma 1 for an easy proof that (ii) implies (iii). □

The next lemma states that a good strategy earns relatively little income after it becomes conditionally bad. Moreover, any strategy which agrees with a good strategy until it becomes conditionally bad is itself a fairly good strategy. A related fact (Lemma 3.1 in [6]) is that a sufficiently good strategy is unlikely to ever become conditionally bad. First some notation: If σ is a strategy and $0 < \alpha < 1$, let $r(\sigma,\alpha)$ be the first time (if any) when σ is not conditionally α-optimal. That is,

$$r(\sigma,\alpha)(h) = \inf\{n : u(\sigma[p_n(h)]) < (1-\alpha)V(f_n)\} \ . \tag{8}$$

The infimum of the empty set is taken to be infinite.

Lemma 3. Let α and β be numbers in (0,1). Suppose σ is available at f in Γ and $u(\sigma) \geq (1-\beta)V(f)$. If $r = r(\sigma,\alpha)$, then, for every stop rule t,

$$\int_{[t>r]} V(f_r)d\sigma \leq \beta\, \alpha^{-1}V(f) \ . \tag{9}$$

Furthermore, if σ' is available at f in Γ and if σ' and σ agree up to time $r-$, then

$$(10) \qquad u(\sigma') \ge (1-\epsilon)V(f)$$

where $\epsilon = \beta(1 + \alpha^{-1})$.

Proof: Let t be a stop rule and set $v_t = \int_{[t>r]} V(f_r)d\sigma$. Then

$$\begin{aligned}(1-\beta)V(f) &\le u(\sigma) \\ &= \int u(\sigma[p_{t\wedge r}])d\sigma \quad \text{(by (2))} \\ &= \int_{[t\le r]} u(\sigma[p_t])d\sigma + \int_{[t>r]} u(\sigma[p_r]d\sigma \\ &\le \int_{[t\le r]} V(f_t)d\sigma + (1-\alpha)\int_{[t>r]} V(f_r)d\sigma \\ &= \int V(f_{t\wedge r})d\sigma - \alpha\, v_t \\ &\le V(f) - \alpha\, v_t \qquad \text{(Corollary 3.3.4, [3])}\ .\end{aligned}$$

Inequality (9) is now clear.

Now let s be a stop rule and $\epsilon' > 0$. By Lemma 1 there is a stop rule t such that $t \ge s$ and

$$(11) \qquad \begin{aligned}(1-\beta)V(f) - \epsilon' &\le \int_{[t\le r]} u(f_t)d\sigma + \int_{[t>r]} u(\sigma[p_r])d\sigma \\ &\le \int_{[t\le r]} u(f_t)d\sigma + v_t\ .\end{aligned}$$

Hence,

$$\begin{aligned}\int_{[t\le r]} u(f_t)d\sigma' + \int_{[t>r]} u(\sigma'[p_r])d\sigma' &\ge \int_{[t\le r]} u(f_t)d\sigma' \\ &= \int_{[t\le r]} u(f_t)d\sigma\end{aligned}$$

(since σ and σ' agree up to r-)

$$\geq (1-\beta)V(f) - v_t - \varepsilon'$$

(by (11))

$$\geq (1-\varepsilon)V(f) - \varepsilon' .$$

(by (9))

Inequality (10) now follows from another application of Lemma 1. □

3. <u>The existence proof.</u>

Let (F, Γ, u) be a nonnegative gambling problem. A *family of strategies* is a mapping from F to the set of strategies. Let $\bar{\rho}$ be a family of strategies and $0 < \alpha < 1$. Define another family of strategies $\bar{\sigma} = \psi(\bar{\rho}, \alpha)$ as follows: For every $f' \in F$, let

$$(12) \qquad r(f') = r(\bar{\rho}(f'), \alpha) \quad \text{(see formula (8))} .$$

Fix $f \in F$. Set $s(1) = r(f)$ and, for $n = 1,2,\ldots$, let $s(n+1)$ be the composition of $s(n)$ with the family $r(\cdot)$ (p. 22, [3]); that is, for $h = (f_1, f_2, \ldots) \in H$, let

$$(13) \qquad \begin{aligned} s(n+1)(h) &= s(n)(h) + r(f_{s(n)(h)})(f_{s(n)(h)+1}, f_{s(n)(h)+2}, \ldots) \\ &\qquad\qquad \text{if } s(n)(h) < \infty , \\ &= \infty \qquad\quad \text{if } s(n)(h) = \infty . \end{aligned}$$

Define $\bar{\sigma}(f)$ to be that strategy which agrees with $\bar{\rho}(f)$ up to time $r(f)-$ and is such that, for $n \geq 1$ and $h \in H$, $\sigma[p_{s(n)}(h)]$ agrees with $\bar{\rho}(f_{s(n)}(h))$ up to time $r(f_{s(n)}(h))-$. The family $\bar{\sigma} = \psi(\bar{\rho}, \alpha)$ is now defined.

Let $0 < \beta < 1$. Let $S(\beta)$ be the collection of families of strategies $\bar{\rho}$ such that, for every f, $\bar{\rho}(f)$ is available at f in Γ and $u(\bar{\rho}(f)) \geq (1-\beta)V(f)$. Define $P(\alpha, \beta)$ to be the set of families of strategies $\bar{\sigma}$ such that $\bar{\sigma} = \psi(\bar{\rho}, \alpha)$ for some $\bar{\rho}$ in $S(\beta)$. Intuitively, to construct an element of $P(\alpha, \beta)$, start with a family $\bar{\rho}$ of β-optimal strategies; use the strategy which $\bar{\rho}$ specifies at the initial fortune until it becomes conditionally less than α-optimal; then switch to the strategy specified by $\bar{\rho}$ at the current fortune and use it until it becomes conditionally less than α-optimal and so forth.

Notice that if V is everywhere finite then $S(\beta)$ and $P(\alpha, \beta)$ are not empty.

<u>Theorem 1.</u> Let (F, Γ, u) be a nonnegative gambling problem with V everywhere finite. For every $f \in F$ and $0 < \epsilon < 1$, there is available at f in Γ a strategy σ which is persistently (multiplicatively) ϵ-optimal. Indeed if

$$(14)\qquad 0 < \alpha < 1,\ 0 < \beta < 1,\ (1-\beta(1 + \alpha^{-1}))(1-\alpha) \geq 1 - \epsilon ,$$

and $\bar{\sigma} \in P(\alpha, \beta)$, then $\bar{\sigma}(f)$ is persistently (multiplicatively) ϵ-optimal at f.

<u>Proof:</u> Choose α, β to satisfy (14). Let $\bar{\rho} \in S(\beta)$ and $\bar{\sigma} = \psi(\bar{\rho}, \alpha) \in P(\alpha, \beta)$. Set $\sigma = \bar{\sigma}(f)$. Let $r(f)$ and $s(n)$ be as in the definition of $\bar{\sigma}$. Set $\epsilon' = \beta(1 + \alpha^{-1})$. Since $u(\bar{\rho}(f)) \geq (1-\beta)V(f)$ and σ agrees with $\bar{\rho}(f)$ up to time $r(f)-$, Lemma 3 applies to show $u(\sigma) \geq (1 - \epsilon')V(f) \geq (1-\epsilon)V(f)$. Similarly, for each positive integer m and $h \in H$, it follows from Lemma 3 that, whenever $s(m)(h) < \infty$,

$$(15)\qquad u(\sigma[p_{s(m)}(h)]) \geq (1 - \epsilon')V(f_{s(m)}(h)) \geq (1 - \epsilon)V(f_{s(m)}(h)) .$$

Consider next a partial history $p = (f_1,\ldots,f_n)$ which, for every $k = 1,2,\ldots,$ is not of the form $p_{s(k)}(h)$. Let m be the least positive integer such that $n < s(m)(ph')$ for some h' (and, hence, every h'). If $m \geq 2$, let $k = s(m-1)(f_1,\ldots,f_n,\ldots)$ so that $f_k = f_{s(m-1)}(f_1,\ldots,f_n,\ldots)$. If $m = 1$, let $k = 0$ and $f_k = f$. Set $\sigma' = \bar{\rho}(f_k)[(f_{k+1},\ldots,f_n)]$ and

$r_0 = r(f_k)[(f_{k+1},\ldots,f_n)]$. Use the definitions of $r(f_k)$ and $s(m)$ (formulas (12) and (13)) to see that

$$(16) \qquad u(\sigma') \geq (1 - \alpha)V(f_n) \; .$$

Check that $\sigma[p]$ and σ' agree up to time r_0-. Also, for $h' \in H$,

$$\begin{aligned} u(\sigma[p][p_{r_0}(h')]) &= u(\sigma[p_{s(m)}(ph')]) \\ &\geq (1 - \epsilon')V(f_{s(m)}(ph')) \qquad \text{(by (15))} \\ &= (1 - \epsilon')V(f_{r_0}(h')) \; , \end{aligned}$$

whenever $r_0(h') < \infty$. Apply Lemma 2 to the strategies $\sigma[p]$ and σ' to conclude

$$(17) \qquad u(\sigma[p]) \geq (1 - \epsilon')u(\sigma') \; .$$

By (16) and (17),

$$\begin{aligned} u(\sigma[p]) &\geq (1 - \epsilon')(1 - \alpha)V(f_n) \\ &\geq (1 - \epsilon)V(f_n) \; . \end{aligned}$$

The proof of the theorem is now complete. □

Even if optimal strategies are available at every fortune, persistently 0-optimal strategies need not be as can be seen from the following example.

Example. Let $F = \{0, 1,\ldots\}$; $u(1) = 1$, $u(n) = 0$ if $n \neq 1$; $\Gamma(n) = \{\delta(n)\}$ for $n \leq 1$, $\Gamma(n) = \{\delta(n), (1-n^{-1})\delta(1) + n^{-1}\delta(0), \gamma\}$ for $n \geq 2$ where γ is a diffuse gamble on F.

Corollary 1. Let (F, Γ, u) be a gambling problem with u bounded. For each $f \in F$ and $\epsilon > 0$, there is a strategy σ available at f in Γ such that $u(\sigma) \geq V(f) - \epsilon$ and $u(\sigma[p]) \geq V(f_n) - \epsilon$ for every $p = (f_1,\ldots,f_n)$.

Proof: There is no real loss of generality in assuming u is nonnegative. Since V is bounded, the conclusion follows easily from the theorem. □

An example of Blackwell in [1] can easily be modified to show that for unbounded, even nonnegative u, there need not exist strategies which are persistently (additively) ϵ-optimal, that is, strategies which satisfy the conclusion of Corollary 1.

4. Measurable gambling problems.

Let X be a separable metric space. Call X analytic if there is a continuous function from the set of irrationals in the unit interval onto X. (See Kuratowski [4] or Blackwell, Freedman, and Orkin [2] for a discussion of analytic sets.) Let $\mathcal{B}(X)$ denote the sigma-field of Borel subsets of X. A nonnegative, real-valued function g defined on X is semi-analytic if $\{x:g(x) > a\}$ is analytic for all nonnegative a. (See [2] and [4].) Denote by $\mathcal{P}(X)$ the set of countably additive probability measures defined on $\mathcal{B}(X)$. Equip $\mathcal{P}(X)$ with the weak-star topology. Then $\mathcal{P}(X)$ is an analytic set if X is. (Lemma (25) in [2]). A function g from an analytic set X into an analytic set Y is universally measurable or measurable for short if, for every $S \in \mathcal{B}(Y)$ and $p \in \mathcal{P}(X)$, $g^{-1}(S)$ is in the completion of $\mathcal{B}(X)$ under p.

A gambling problem (F, Γ, u) is nonnegative analytic if F is an analytic set, u is semi-analytic, and the set $\{(f,\gamma):\gamma \in \Gamma(f)\}$ is an analytic subset of $F \times \mathcal{P}(F)$. (Here each gamble γ is identified with its restriction to $\mathcal{B}(F)$ and is assumed to be countably additive on $\mathcal{B}(F)$.) This definition of analytic gambling problems was inspired by Blackwell, Freedman, and Orkin [2]. Analytic gambling problems include the Borel measurable gambling problems defined by Strauch [5].

A strategy σ is measurable if, for $n = 1,2,\ldots,$ the mappings $(f_1,\ldots,f_n) \to \sigma_n(f_1,\ldots,f_n)$ are measurable from F^n to $\mathcal{P}(F)$. A measurable strategy σ determines a probability measure $p(\sigma)$ on the Borel subsets of the product space $H = F \times F \times \cdots$ as follows: the $p(\sigma)$-marginal distribution of f_1 is σ_0 and, for every $(f_1,\ldots,f_n)$

the $p(\sigma)$-distribution of f_{n+1} given $(f_1,\ldots,f_n)$ is $\sigma_n(f_1,\ldots,f_n)$. For simplicity, denote $p(\sigma)$ by σ below. A <u>measurable family of strategies</u> is a mapping $\bar{\sigma}$ which assigns to each $f \in F$ a measurable strategy $\bar{\sigma}(f)$ in such a way that, for every $n = 0, 1,\ldots,$ the function $(f,f_1,\ldots,f_n) \to \bar{\sigma}(f)_n(f_1,\ldots,f_n)$ is measurable from F^{n+1} to $\mathcal{P}(F)$. The family is <u>available</u> if $\bar{\sigma}(f)$ is available at f for every f.

<u>Lemma 4.</u> If $\bar{\sigma}$ is a measurable family of strategies and u is a bounded Borel function from F to the reals, then the mapping $\varphi: f \to u(\bar{\sigma}(f))$ is measurable from F to the reals and the mappings $\varphi_n:(f, h) \to u(\bar{\sigma}(f)[p_n(h)])$ are measurable from $F \times H$ to the reals for $n = 1,2,\ldots$.

<u>Proof:</u> If g is a nonnegative Borel function from H to the real line and if g depends only on a finite number of coordinates, then, by Corollary (41a) of [2], the mapping

$$\varphi_g: f \to \int g \, d\, \bar{\sigma}(f)$$

is measurable. The collection $\mathcal{G}$ of functions g such that φ_g is measurable includes, in particular, the indicator functions of Borel cylinder sets. Since $\mathcal{G}$ is closed under linear combinations and increasing limits of nonnegative functions, conclude that $\mathcal{G}$ contains all nonnegative Borel functions on H. By Theorem 3.2 of [8], $u(\bar{\sigma}(f)) = \int u^* \, d\bar{\sigma}(f)$ where $u^*(f_1,f_2,\ldots) = \limsup_{n\to\infty} u(f_n)$. Thus $\varphi = \varphi_{u^*}$ is measurable. The proof that the φ_n are measurable is similar. □

<u>Theorem 2.</u> Let (F, Γ, u) be a nonnegative analytic gambling problem and assume u is a bounded, Borel measurable function. Suppose that, for every $\beta \in (0, 1)$, there is a measurable family of (multiplicatively)

β-optimal strategies available. Then, for every $\epsilon \in (0,1)$, there is available a measurable family of strategies which are persistently (multiplicatively) ϵ-optimal.

Proof: First notice that V is measurable. To see this, let $\bar{\sigma}_n$ be a measurable family of n^{-1}-optimal strategies for $n = 1,2,\ldots$. Then $V(f) = \sup u(\bar{\sigma}_n(f))$ and, by Lemma 4, V is measurable.

To prove the assertion of the theorem, it suffices, by Theorem 1, to show that, given $\alpha, \beta \in (0, 1)$, there is a measurable family $\bar{\sigma}$ in $P(\alpha, \beta)$. By assumption, there is a measurable family $\bar{\rho} \in S(\beta)$. Let $\bar{\sigma} = \psi(\bar{\rho}, \alpha)$. By definition, $\bar{\sigma} \in P(\alpha, \beta)$. In checking that $\bar{\sigma}$ is measurable, the only real difficulty lies in showing that the map $(f,h) \to r(f)(h)$ is measurable from $F \times H$ to the real line. (See formulas (12) and (8).) This in turn follows easily from the measurability of V together with Lemma 4. □

Perhaps, the conclusion of Theorem 2 holds for every nonnegative analytic gambling problem which has V everywhere finite. However, up to the present only two special cases have been treated. It was shown in [4] that good measurable strategies are available for leavable, Borel measurable problems with a bounded utility function. In fact, good measurable families of strategies are always available for leavable, nonnegative analytic problems and so Theorem 2 applies. The same is true for nonleavable problems in which u is the indicator of a single fortune (see [7] for the Borel case). Of course, if F is countable, every strategy is measurable and Theorem 1 yields the existence of persistently good measurable strategies.

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