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# PERTURBATION ANALYSIS FOR THE MOORE–PENROSE METRIC GENERALIZED INVERSE OF CLOSED LINEAR OPERATORS IN BANACH SPACES

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Abstract. In this paper, we characterize the perturbations of the Moore–Penrose metric generalized inverse of closed operator in Banach spaces. Under the condition $R(\delta T) \subset R(T)$, $N(T) \subset N(\delta T)$, respectively, we get some new results about upper-bound estimates of $\|\bar{T}^M\|$ and $\|\bar{T}^M - T^M\|$.

## 1. Introduction

Throughout the present article, let $X$, $Y$ be reflexive strictly convex Banach spaces on real field $\mathbb{R}$. Let $L(X,Y)$, $B(X,Y)$, and $C(X,Y)$ be the set of linear operators, the set of all bounded linear operators, and the set of all densely defined closed linear operators from $X$ to $Y$, respectively. For any $T \in L(X,Y)$, $D(T)$, $R(T)$, and $N(T)$ denote the domain, the range and the kernel of $T$, respectively.

A closed subset $M \subset X$ is called *topological-complemented* if there is a closed subset $N \subset X$ such that $X = M + N$ and $M \cap N = \{0\}$. In this case, we set $X = M \oplus f N$. As we know, for any $T \in C(X,Y)$, if $\overline{R(T)}$ (the closure of $R(T)$) and $N(T)$ are topological-complemented, then there is a linearly generalized inverse $T^+$ such that

$$TT^+T = T \quad \text{on } D(T), \qquad T^+TT^+ = T^+ \quad \text{on } D(T^+).$$


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Furthermore, if $R(T)$ is closed, then $T^+ \in B(Y,X)$ is a bounded linear operator (see [4], [15]).

The linear generalized inverse has been widely studied over the last decades and has many important applications in numerical approximation, statistics, optimization, and more (see [2], [12]). However, linearly generalized inverse cannot deal with the extremal solutions, the minimal norm solutions, and the best approximation solutions of an ill-posed linear operator equation in Banach spaces. In order to solve the best approximation problems for an ill-posed linear operator equation in Banach spaces, Nashed and Votruba introduced the concept of the (set-valued) metric generalized inverse of a linear operator in Banach spaces (see [8]). In 2003, H. Wang and Y. Wang introduced the Moore–Penrose metric generalized inverse for linear operator on Banach space in [14], which is a homogeneous operator.

In recent years, some papers on the perturbation of Moore–Penrose metric generalized inverse have appeared (see [3], [10], [13]). In [9], Ni characterized the Moore–Penrose metric generalized inverse for an arbitrary linear operator in Banach space. In [1], J. Cao and Y. Xue considered the simple expressions of the Moore–Penrose metric generalized inverse and investigated the perturbations for the Moore–Penrose metric generalized inverse of bounded linear operators. Some results on the perturbation of the Moore–Penrose metric generalized inverse similar to the linearly generalized inverse are obtained in [7] by H. Ma et al., under the assumption that $T^M$ is quasiadditive and that metric projection $\pi_{N(T)}$ is linear and that $R(\delta T) \subseteq R(T)$, $N(T) \subseteq N(\delta T)$.

It is well known the metric projection is a homogeneous operator, and then the Moore–Penrose metric generalized inverse is different from the linearly generalized inverse. In the present article, we characterize the Moore–Penrose metric generalized inverse of closed operator with closed range in Banach spaces. Under some conditions, we present the upper bounds of $\|\bar{T}^M\|$ and $\|\bar{T}^M - T^M\|$, respectively.

## 2. Preliminaries

Let $M$ be a subset in $X$. If $\lambda x \in M$ whenever $x \in M$ and $\lambda \in \mathbb{R}$, then we call $M$ a *homogeneous subset.* A nonlinear operator $T : X \to Y$ is called a *bounded homogeneous operator* if $T$ maps every bounded set in $X$ into a bounded set in $Y$ and $T(\lambda x) = \lambda Tx$ for all $\lambda \in \mathbb{R}$. Let $H(X,Y)$ denote the set of all bounded homogeneous operators from $X$ to $Y$. Equipped with the usual linear operations on $H(X,Y)$ and norm on $T \in H(X,Y)$ defined as $\|T\| = \sup_{\|x\|=1} \|Tx\|$, $H(X,Y)$ become a Banach space (see [12]). Obviously, $B(X,Y) \subseteq H(X,Y)$.

Recall that a nonlinear operator $T$ is called *quasi-additive* on subspace $M \subset X$ if

$$T(x+z) = T(x) + T(z), \quad \forall x \in X, \forall z \in M.$$

If a homogeneous operator $T \in H(X,X)$ is quasiadditive on $R(T)$, then we call $T$ a quasilinear operator.

Let $M \subset X$. Then the (set-valued) metric projection $P_M$ defined on $X$ is a mapping from $X$ to $M$,

$$P_M(x) = \{z \in M \mid \|x - z\| = d(x, M), \forall x \in X\},$$

where $d(x, M) = \inf_{\forall y \in M} \|x - y\|$.

If $P_M \neq \emptyset$, then $M$ is called a *proximinal set*. If $P_M$ is a singleton, then $M$ is said to be a *Chebyshev set*. In this case, we denote $P_M$ by $\pi_M$. Moreover, $\pi_M$ satisfies the following properties.

**Proposition 2.1** ([12]). *Let $M \subset X$ be a subspace of $X$. Then*

(1) $\pi_M^2(x) = \pi_M(x)$, $\forall x \in X$ *(i.e., $\pi_M$ is idempotent);*
(2) $\|x - \pi_M(x)\| \leq \|x\|$ *and so* $\|\pi_M(x)\| \leq 2\|x\|$, $\forall x \in X$*;*
(3) $\pi_M(\lambda x) = \lambda \pi(x)$, $\forall x \in X$, $\forall \lambda \in \mathbb{R}$ *(i.e., $\pi_M$ is homogenous);*
(4) $\pi_M(x + z) = \pi_M(x) + \pi_M(z) = \pi_M(x) + z$ *for any* $z \in M$ *(i.e., $\pi_M$ is quasiadditive on $M$);*
(5) $\pi_M$ *is a closed operator if $M$ is a Chebyshev subspace.*

**Lemma 2.2** ([12]). *Let $M \subset X$ be a Chebyshev subspace. Then $\pi_M^{-1}(0)$ is a linear subspace if and only if $\pi_M$ is a linear operator.*

**Lemma 2.3** ([6]). *Let $X$ be a reflexive Banach space. Then $X$ is strictly convex if and only if every nonempty closed convex subset $M \subset X$ is a Chebyshev set.*

Let $X^*$ be the dual space of $X$ and let $M^\perp = \{x^* \in X^* \mid \langle x, x^* \rangle = 0, x \in M\}$. Now, we recall the notation from "dual-mapping."

*Definition* 2.4. The set-valued mapping $F_X : X \to X^*$ defined as

$$F_X(x) = \{x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

is called the *dual-mapping* of $X$, where $\langle x, x^* \rangle = x^*(x)$.

**Lemma 2.5** (Generalized Orthogonal Decomposition Theorem; see [12]). *Let $X$ be a Banach space and let $M \subset X$ be a proximinal subspace. Then for any $x \in X$, we have*

(1) $x = x_1 + x_2$ *with* $x_1 \in M$ *and* $x_2 \in F_X^{-1}(M^\perp)$*;*
(2) *if $M \subset X$ is a Chebyshev subspace, then the decomposition in (1) is unique such that $x = \pi_M(x) + x_2$, and in this case, we write $X = M \dotplus F_X^{-1}(M^\perp)$.*

*Where $F_X^{-1}(M^\perp) = \{x \in X \mid F_X(x) \cap M^\perp \neq \emptyset\}$.*

*Definition* 2.6 ([12], [14]). Let $T \in L(X, Y)$. Assume that $R(T)$ and $N(T)$ are Chebyshev subspaces. If there is a homogeneous operator $T^M : D(T^M) \to D(T)$ such that

(1) $TT^MT = T$, on $D(T)$; (2) $T^MTT^M = T^M$, on $D(T^M)$;

(3) $TT^M = \pi_{R(T)}$, on $D(T^M)$; (4) $T^MT = I - \pi_{N(T)}$, on $D(T)$,

then $T^M$ is called the *Moore–Penrose metric generalized inverse* of $T$. Here, $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp)$.

By Definition 2.6 and Lemma 2.5, if $T^M$ exists, then the spaces $X$, $Y$ have the following unique decompositions:

$$X = N(T) \dotplus F_X^{-1}(N(T)^\perp), \qquad Y = R(T) \dotplus F_Y^{-1}(R(T)^\perp).$$

**Lemma 2.7** ([12]). *Let $T \in L(X,Y)$ be a linear operator with $R(T)$, $N(T)$ as Chebyshev subspaces. Then there exists unique Moore–Penrose metric generalized inverse $T^M$ of $T$ such that*

$$T^M(y) = (T|_{C(T)})^{-1}\pi_{R(T)}(y)$$

*for any $y \in D(T^M)$. Here, $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$.*

**Lemma 2.8.** *Let $T \in L(X,Y)$ be a linear operator with $R(T)$, $N(T)$ as Chebyshev subspaces. Then $T^M = (I - \pi_{N(T)})T^- \pi_{R(T)}$ is independence of the choice of $T^- \in L(Y,X)$ which satisfied $TT^-T = T$.*

*Proof.* Since $N(T)$, $R(T)$ are Chebyshev subspaces, we have that $T^M$ exists and $TT^M = \pi_{R(T)}$, $T^M T = I - \pi_{N(T)}$. Let $B = (I - \pi_{N(T)})T^- \pi_{R(T)}$. Then

$$B = (I - \pi_{N(T)})T^- \pi_{R(T)} = T^M TT^- TT^M = \pi_{R(T)} = T^M. \qquad \square$$

From Lemma 2.8, we know that, if $T^-$ is a bounded linear operator, then $T^M$ is a bounded homogeneous operator and $\|T^M\| \leq 2\|T^-\|$.

**Lemma 2.9** ([11, IV.5, Theorem 5.8]). *Let $X$, $Y$ be Banach spaces and let $T$ be a closed linear operator with $D(T) \subseteq X$, $R(T) \subseteq Y$. Suppose that $T^{-1}$ exists. Then $T^{-1}$ is continuous if and only if $R(T)$ is closed in $Y$.*

**Lemma 2.10.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed. If $\pi_{N(T)}$ is a linear metric projection, then $T^M$ is a closed homogeneous operator that is quasiadditive on $R(T)$.*

*Proof.* Since $X$, $Y$ are reflexive strictly convex Banach spaces and $R(T)$, $N(T)$ are Chebyshev subspaces, $T^M$ exists. By Lemma 2.7, we have

$$T^M(y) = (T|_{C(T)})^{-1}\pi_{R(T)}(y), \quad \forall y \in D(T^M).$$

Noting that $\pi_{N(T)}$ is a linear metric projection, by Lemma 2.2, $\pi^{-1}_{N(T)}(0) = F_X^{-1}(N(T)^{\perp})$ is a linear subspace, and so is $C(T)$. Thus, $T|_{C(T)}$ is a linear operator from $C(T)$ onto $R(T)$ and consequently $(T|_{C(T)})^{-1}$ is a linear operator from $R(T)$ onto $C(T)$.

For any $x_n \in D(T^M)$, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} T^M(x_n) = y$, we have

$$y = \lim_{n\to\infty} T^M(x_n) = \lim_{n\to\infty} (T|_{C(T)})^{-1}\pi_{R(T)}(x_n).$$

Since $R(T)$ is closed, by Lemma 2.9, $(T|_{C(T)})^{-1}$ is a continuous linear operator. Thus, $\lim_{n\to\infty} \pi_{R(T)}(x_n) = T|_{C(T)}y$. By Proposition 2.1(5), $\pi_{R(T)}$ is a closed operator and so $\pi_{R(T)}(x) = T|_{C(T)}y$. Thus, $y = (T|_{C(T)})^{-1}\pi_{R(T)}(x)$. Consequently, $T^M$ is a closed homogeneous operator.

Noting that $\pi_{R(T)}$ is quasiadditive on $R(T)$ and that $(T|_{C(T)})^{-1}$ is a linear operator, it is easy to verify that $T^M$ is quasiadditive on $R(T)$. $\square$

**Proposition 2.11.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed. Then $\pi_{N(T)}$ is a linear metric projection if and only if $T^M$ is quasiadditive on $R(T)$.*

*Proof.* From the proof of Lemma 2.10, we know that if $\pi_{N(T)}$ is a linear metric projection, then $T^M$ is quasiadditive on $R(T)$. On the contrary, suppose that $T^M$ is quasiadditive on $R(T)$. Then, for any $x, y \in D(T)$,

$$\begin{aligned}\pi_{N(T)}(x+y) &= (I - T^M T)(x+y)\\ &= (x+y) - T^M(Tx+Ty)\\ &= x + y - T^M Tx - T^M Ty\\ &= (I - T^M T)x + (I - T^M T)y\\ &= \pi_{N(T)}x + \pi_{N(T)}y.\end{aligned}$$

This shows that $\pi_{N(T)}$ is a linear metric projection. □

## 3. The perturbation analysis of the Moore–Penrose metric generalized inverse

Let $T \in C(X, Y)$ and $\delta T$ be a linear operator with $D(T) \subseteq D(\delta T)$. Recall that $\delta T$ is relatively bounded with respect to $T$ or simply $T$-bounded if there are constants $a, b > 0$ such that

$$\|\delta T x\| \le a\|x\| + b\|Tx\|, \quad \forall x \in D(T).$$

The constant $b$ is called the *$T$-bounded* of $\delta T$. From [5, Theorem 1.1], we know that if $b < 1$, then $\bar{T} = T + \delta T$ is closed if and only if $T$ is closed.

Let $T$ be a linear operator. The reduced modulus $\gamma(T)$ of $T$ is defined as

$$\gamma(T) = \inf\big\{\|Tx\| \bigm| \operatorname{dist}\big(x, N(T)\big) = 1, \forall x \in D(T)\big\}.$$

Here, $\operatorname{dist}(x, N(T)) = \inf_{\forall y \in N(T)} \|x - y\|$. Obviously, $\gamma(T)\operatorname{dist}(x, N(T)) \le \|Tx\|$.

Let $M$, $N$ be the homogeneous subsets of Banach space $X$. Put

$$\eta(M, N) = \begin{cases}\sup\{\operatorname{dist}(x, N) \mid x \in M, \|x\| = 1\}, & M \ne \{0\},\\ 0, & M = \{0\}.\end{cases}$$

We define $\hat{\eta} = \max\{\eta(M, N), \eta(N, M)\}$ the gap between $M$ and $N$ (see [3]). Clearly, $\operatorname{dist}(x, N) \le \|x\|\eta(M, N)$. If $M$ and $N$ are subspaces, then the gap between $M$ and $N$ is denoted by $\hat{\delta} = \max\{\delta(M, N), \delta(N, M)\}$ (see [5]).

**Lemma 3.1.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X, Y)$ with $R(T)$ closed. If $T^M$ is a bounded homogenous operator, then*

$$\frac{1}{\|T^M\|} \le \gamma(T) \le \frac{\|TT^M\|}{\|T^M\|}.$$

*Proof.* Since $T^M$ is a bounded homogenous operator, we have

$$\operatorname{dist}\big(x, N(T)\big) = \|x - \pi_{N(T)}x\| = \|T^M Tx\| \le \|T^M\|\|Tx\|,$$

and so $\gamma(T) \ge \frac{1}{\|T^M\|}$.

Noting that $\operatorname{dist}(x, N(T)) = \|T^M Tx\|$, we have

$$\gamma(T)\|T^M Tx\| = \gamma(T)\operatorname{dist}\big(x, N(T)\big) \le \|Tx\|, \quad \forall x \in D(T).$$

For any $y \in Y$, $T^M y \in D(T)$. Hence,

$$\gamma(T)\|T^M T T^M y\| \leq \|TT^M y\|,$$

and consequently, $\gamma(T) \leq \frac{\|TT^M\|}{\|T^M\|}$. □

**Lemma 3.2** ([15, Lemma 1.3.6])**.** *Let $T, \bar{T} \in C(X,Y)$ with $D(T) = D(\bar{T})$. Assume that there are constants $\lambda > 0$ and $\mu \in \mathbb{R}$ such that*

$$\|\bar{T}x\| \geq \lambda\|Tx\| + \mu\|x\|, \quad \forall x \in D(T).$$

*Then*

$$\gamma(\bar{T}) \geq \lambda\gamma(T)\big(1 - 2\delta\big(N(T), N(\bar{T})\big)\big) + \mu.$$

**Lemma 3.3.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T$ be a $T$-bounded linear operator with $D(T) \subseteq D(\delta T)$, $\bar{T} = T + \delta T$. Assume that $b < 1$ and that*

$$\delta\big(N(T), N(\bar{T})\big) < \frac{(1-b)\gamma(T) - a}{2(1-b)\gamma(T)}.$$

*Then $\bar{T} \in C(X,Y)$ with $R(\bar{T})$ closed.*

*Proof.* $\bar{T} \in C(X,Y)$ is evident since $b < 1$.

Noting that $\delta T$ is a $T$-bounded linear operator, we have

$$\begin{aligned} \|\bar{T}x\| &\geq \|Tx\| - \|\delta T x\| \\ &\geq \|Tx\| - \big[a\|x\| + b\|Tx\|\big] \\ &= (1-b)\|Tx\| - a\|x\|. \end{aligned}$$

By Lemma 3.2, we have

$$\gamma(\bar{T}) \geq (1-b)\gamma(T)\big(1 - 2\delta\big(N(T), N(\bar{T})\big)\big) - a.$$

Thus, if

$$\delta\big(N(T), N(\bar{T})\big) < \frac{(1-b)\gamma(T) - a}{2(1-b)\gamma(T)},$$

then $\gamma(\bar{T}) > 0$ (i.e., $R(\bar{T})$ is closed). □

**Proposition 3.4.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T$ be a $T$-bounded linear operator with $b < 1$ and $D(T) \subseteq D(\delta T)$, $\bar{T} = T + \delta T$. Then*

(1) $\gamma(T)\|(I - \pi_{N(T)})\pi_{N(\bar{T})}\| \leq \frac{a}{1-b}\|\pi_{N(\bar{T})}\|$,
(2) $|\gamma(\bar{T}) - \gamma(T)| \leq \|\pi_{N(\bar{T})} - \pi_{N(T)}\|\gamma(\bar{T}) + b\gamma(T) + a$,
(3) $\gamma(\bar{T})\gamma(T)\|(\pi_{N(\bar{T})} - \pi_{N(T)})x\| \leq \{\gamma(\bar{T}) + (1+b)\gamma(T)\}\|Tx\| + a\gamma(T)\|x\|$ *for any* $x \in D(T)$,
(4) $\|(\pi_{N(\bar{T})} - \pi_{N(T)})\| \leq 2\{\frac{a}{(1-b)\gamma(T)} + \eta(F_X^{-1}(N(\bar{T})^\perp), F_X^{-1}(N(T)^\perp))\}$.

*Proof.* Obviously, $\bar{T} \in C(X, Y)$ and $\pi_{N(\bar{T})}$, $\pi_{N(T)}$ exist by the assumption.

(1) Since $R(T)$ is closed, we have $\gamma(T) > 0$. Thus, for any $x \in X$, we have

$$\begin{aligned}\gamma(T)\big\|(I - \pi_{N(T)})\pi_{N(\bar{T})}x\big\| &= \gamma(T)\operatorname{dist}\big(\pi_{N(\bar{T})}x, N(T)\big) \\ &\leq \|T\pi_{N(\bar{T})}x\| = \big\|(\bar{T} - \delta T)\pi_{N(\bar{T})}x\big\| = \|\delta T\pi_{N(\bar{T})}x\|.\end{aligned}$$

Noting that $\delta T$ is a $T$-bounded linear operator with $b < 1$, we have

$$\begin{aligned}\|\delta T\pi_{N(\bar{T})}x\| &\leq a\|\pi_{N(\bar{T})}x\| + b\|T\pi_{N(\bar{T})}x\| \\ &= a\|\pi_{N(\bar{T})}x\| + b\big\|(\bar{T} - \delta T)\pi_{N(\bar{T})}x\big\| \\ &= a\|\pi_{N(\bar{T})}x\| + b\|\delta T\pi_{N(\bar{T})}x\|.\end{aligned}$$

So, we have $\|\delta T\pi_{N(\bar{T})}x\| \leq \frac{a}{1-b}\|\pi_{N(\bar{T})}x\|$. Consequently, we get

$$\gamma(T)\big\|(I - \pi_{N(T)})\pi_{N(\bar{T})}x\big\| \leq \frac{a}{1-b}\|\pi_{N(\bar{T})}x\|.$$

(2) Since $N(T)$ is a Chebyshev subspace, by Lemma 2.5 (generalized orthogonal decomposition theorem), we have

$$X = N(T) \dotplus F_X^{-1}\big(N(T)^{\perp}\big).$$

For any $x \in F_X^{-1}(N(T)^{\perp})$, $\|x\| = 1$, we have

$$\operatorname{dist}\big(x, N(T)\big) = \|x - \pi_{N(T)}x\| = \|x\| = 1$$

and

$$\begin{aligned}a + b\|Tx\| &\geq \|\delta Tx\| \geq \|\bar{T}x\| - \|Tx\| \\ &\geq \gamma(\bar{T})\operatorname{dist}\big(x, N(\bar{T})\big) - \|Tx\| \\ &= \gamma(\bar{T})\|x - \pi_{N(\bar{T})}x\| - \|Tx\| \\ &\geq \gamma(\bar{T})\big\{\|x - \pi_{N(T)}x\| - \|\pi_{N(\bar{T})}x - \pi_{N(T)}x\|\big\} - \|Tx\| \\ &= \gamma(\bar{T}) - \gamma(\bar{T})\|\pi_{N(\bar{T})} - \pi_{N(T)}\| - \|Tx\|.\end{aligned}$$

This easily implies that

$$a + b\gamma(T) \geq \gamma(\bar{T}) - \gamma(T) - \|\pi_{N(\bar{T})} - \pi_{N(T)}\|\gamma(\bar{T}).$$

Interchanging $\bar{T}$ and $T$, we get

$$\pm\big(\gamma(\bar{T}) - \gamma(T)\big) \leq \|\pi_{N(\bar{T})} - \pi_{N(T)}\|\gamma(\bar{T}) + b\gamma(T) + a.$$

(3) For any $x \in D(T)$, we have

$$\begin{aligned}\|Tx\| + \|\delta Tx\| &\geq \|\bar{T}x\| \geq \gamma(\bar{T})\operatorname{dist}\big(x, N(\bar{T})\big) \\ &= \gamma(\bar{T})\|x - \pi_{N(\bar{T})}x\| \\ &\geq \gamma(\bar{T})\|\pi_{N(T)}x - \pi_{N(\bar{T})}x\| - \gamma(\bar{T})\|x - \pi_{N(T)}x\| \\ &= \gamma(\bar{T})\|\pi_{N(T)}x - \pi_{N(\bar{T})}x\| - \gamma(\bar{T})\operatorname{dist}\big(x, N(T)\big) \\ &\geq \gamma(\bar{T})\|\pi_{N(T)}x - \pi_{N(\bar{T})}x\| - \gamma(\bar{T})\frac{\|Tx\|}{\gamma(T)}.\end{aligned}$$

Thus,

$$\begin{aligned}\gamma(T)\gamma(\bar{T})\|\pi_{N(T)}x - \pi_{N(\bar{T})}x\| &\leq \big(\gamma(\bar{T}) + \gamma(T)\big)\|Tx\| + \gamma(T)\|\delta Tx\| \\ &\leq \big\{\gamma(\bar{T}) + (1+b)\gamma(T)\big\}\|Tx\| + a\gamma(T)\|x\|.\end{aligned}$$

(4) By Lemma 2.5, $\pi_{N(T)}^{-1}(0) = F_X^{-1}(N(T)^{\perp})$. Thus, for any $y \in F_X^{-1}(N(T)^{\perp}) \cap D(T)$, $\pi_{N(T)}y = 0$ and for any $x \in D(T)$, we have

$$\begin{aligned}\big\|\pi_{N(T)}(\pi_{N(\bar{T})} - I)x\big\| &= \big\|\pi_{N(T)}\big[(\pi_{N(\bar{T})} - I)x - y\big]\big\| \\ &\leq \|\pi_{N(T)}\|\big\|(\pi_{N(\bar{T})} - I)x - y\big\| \\ &\leq \|\pi_{N(T)}\| \inf_{y \in F_X^{-1}(N(T)^{\perp})} \big\|(\pi_{N(\bar{T})} - I)x - y\big\| \\ &\leq \|\pi_{N(T)}\| \operatorname{dist}\big((\pi_{N(\bar{T})} - I)x, F_X^{-1}\big(N(T)^{\perp}\big)\big) \\ &\leq \|\pi_{N(T)}\|\big\|(\pi_{N(\bar{T})} - I)x\big\|\eta\big(F_X^{-1}\big(N(\bar{T})^{\perp}\big), F_X^{-1}\big(N(T)^{\perp}\big)\big) \\ &\leq \|\pi_{N(T)}\|\|x\|\eta\big(F_X^{-1}\big(N(\bar{T})^{\perp}\big), F_X^{-1}\big(N(T)^{\perp}\big)\big).\end{aligned}$$

Associated with item (1), we have

$$\begin{aligned}&\big\|(\pi_{N(\bar{T})} - \pi_{N(T)})x\big\| \\ &\quad = \|\pi_{N(\bar{T})}x - \pi_{N(T)}\pi_{N(\bar{T})}x + \pi_{N(T)}\pi_{N(\bar{T})}x - \pi_{N(T)}x\| \\ &\quad = \big\|(I - \pi_{N(T)})\pi_{N(\bar{T})}x + \pi_{N(T)}(\pi_{N(\bar{T})} - I)x\big\| \\ &\quad \leq \big\|(I - \pi_{N(T)})\pi_{N(\bar{T})}x\big\| + \big\|\pi_{N(T)}(\pi_{N(\bar{T})} - I)x\big\| \\ &\quad \leq \gamma(T)^{-1}\|\delta T\pi_{N(\bar{T})}x\| + \eta\big(F_X^{-1}\big(N(\bar{T})^{\perp}\big), F_X^{-1}\big(N(T)^{\perp}\big)\big)\|\pi_{N(T)}\|\|x\| \\ &\quad \leq \frac{a}{(1-b)\gamma(T)}\|\pi_{N(\bar{T})}x\| + \eta\big(F_X^{-1}\big(N(\bar{T})^{\perp}\big), F_X^{-1}\big(N(T)^{\perp}\big)\big)\|\pi_{N(T)}\|\|x\| \\ &\quad \leq 2\Big\{\frac{a}{(1-b)\gamma(T)} + \eta\big(F_X^{-1}\big(N(\bar{T})^{\perp}\big), F_X^{-1}\big(N(T)^{\perp}\big)\big)\Big\}\|x\|.\end{aligned}$$

Hence,

$$\big\|(\pi_{N(\bar{T})} - \pi_{N(T)})\big\| \leq 2\Big\{\frac{a}{(1-b)\gamma(T)} + \eta\big(F_X^{-1}\big(N(\bar{T})^{\perp}\big), F_X^{-1}\big(N(T)^{\perp}\big)\big)\Big\}. \qquad \square$$

**Corollary 3.5.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T \in B(X,Y)$, $\bar{T} = T + \delta T$. Then*

1. $\gamma(T)\|(I - \pi_{N(T)})\pi_{N(\bar{T})}\| \leq \|\delta T\|\|\pi_{N(\bar{T})}\|$,
2. $|\gamma(\bar{T}) - \gamma(T)| \leq \|\pi_{N(\bar{T})} - \pi_{N(T)}\|\gamma(\bar{T}) + \|\delta T\|$,
3. $\gamma(\bar{T})\gamma(T)\|(\pi_{N(\bar{T})} - \pi_{N(T)})x\| \leq \gamma(\bar{T})\|Tx\| + \gamma(T)\|\bar{T}x\|$ *for any* $x \in D(T)$,
4. $\|(\pi_{N(\bar{T})} - \pi_{N(T)})\| \leq 2\{\gamma(T)^{-1}\|\delta T\| + \eta(F_X^{-1}(N(\bar{T})^{\perp}), F_X^{-1}(N(T)^{\perp}))\}$.

**Theorem 3.6.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T$ be a $T$-bounded linear operator with $D(T) \subseteq D(\delta T)$.*

1. *If $\pi_{N(T)}$ is a linear metric projection and $R(\delta T) \subseteq R(T)$, then $T^M\delta T$ is a linear operator. Furthermore, if $R(T)$ is topological-complemented in $Y$, then $T^M\delta T$ is a $T$-bounded linear operator.*

(2) *If $T^M$ is a bounded homogenous operator, then $\delta T T^M$ is bounded homogeneous operator and $\|\delta T T^M\| \leq (a\|T^M\| + 2b)$.*

*Proof.* (1) Since $R(\delta T) \subseteq R(T)$, by Lemma 2.7, we have

$$T^M \delta T = (T|_{C(T)})^{-1}\pi_{R(T)}\delta T = (T|_{C(T)})^{-1}\delta T.$$

Noting that $\pi_{N(T)}$is a linear metric projection, we have that $(T|_{C(T)})^{-1}$ is a linear operator form $R(T)$ onto $C(T)$ by Lemma 2.2. Thus, $T^M\delta T$ is a linear operator.

Since $\pi_{N(T)}$ is a linear metric projection, by Lemma 2.2, $\pi^{-1}_{N(T)}(0) = F_X^{-1}(N(T)^{\perp})$ is a closed linear subspace. This shows that $N(T)$ is topological-complemented in $X$. Note that $R(T)$ is topological-complemented in $Y$, too. Therefore, there is a bounded linear operator $T^- \in B(Y,X)$ such that $TT^-T = T$. Thus, by Lemma 2.8, $T^M = (I - \pi_{N(T)})T^-\pi_{R(T)}$ is a bounded homogeneous operator. Then, for any $x \in D(T)$, we have

$$\begin{aligned}\|T^M\delta T x\| &= \big\|(I - \pi_{N(T)})T^-\pi_{R(T)}\delta T x\big\| \\ &= \big\|(I - \pi_{N(T)})T^-\delta T x\big\| \\ &\leq \|T^-\|\|\delta T x\| \\ &\leq a\|T^-\|\|x\| + b\|T^-\|\|Tx\|.\end{aligned}$$

So, $T^M\delta T$ is a $T$-bounded linear operator.

(2) For any $y \in D(T^M)$,

$$\begin{aligned}\|\delta T T^M y\| &\leq a\|T^M y\| + b\|T T^M y\| \\ &\leq a\|T^M\|\|y\| + b\|\pi_{R(T)}\|\|y\| \\ &\leq \big(a\|T^M\| + 2b\big)\|y\|.\end{aligned}$$

This indicates that $\delta T T^M$ is a bounded homogeneous operator and that

$$\|\delta T T^M\| \leq \big(a\|T^M\| + 2b\big). \qquad \square$$

**Proposition 3.7.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T : D(\delta T) \to D(T^M)$ be a linear operator such that $D(T) \subset D(\delta T)$, $R(\delta T) \subseteq R(T)$, $\bar{T} = T + \delta T$. Assume that $\pi_{N(T)}$ is a linear metric projection. Then the following are equivalent:*

(1) *$I + \delta T T^M : D(T^M) \to D(T^M)$ is bijective,*
(2) *$T^M\bar{T}|_{R(T^M)} = (I + T^M\delta T)|_{R(T^M)} : R(T^M) \to R(T^M)$ is bijective,*
(3) *$D(T^M) = \bar{T}R(T^M) + N(T^M)$ and $N(\bar{T}) \cap R(T^M) = \{0\}$.*

*Proof.* Since $\pi_{N(T)}$ is a linear metric projection, we have that $T^M$ is quasiadditive on $R(T)$ by Lemma 2.10. Since $N(T^M) = F_Y^{-1}(R(T)^{\perp})$, we have $R(T) \cap N(T^M) = \{0\}$.

(1) $\Rightarrow$ (2). Assume that $W = I + \delta T T^M$ is bijective. For any $\xi \in R(T^M)$, there is a $z \in D(T^M)$ such that $\xi = T^M z$. Thus,

$$T^M\bar{T}\xi = (T^M T + T^M\delta T)T^M z = (I + T^M\delta T)T^M z = (I + T^M\delta T)\xi.$$

This shows that $T^M\bar{T}|_{R(T^M)} = (I + T^M\delta T)|_{R(T^M)}$.

Let $\xi \in R(T^M)$ and let $T^M\bar{T}\xi = 0$. Then $(I + T^M\delta T)\xi = 0$. Since $W$ is invertible, we have

$$(I - T^M W^{-1}\delta T)(I + T^M\delta T)\xi = 0,$$

(i.e., $\xi = 0$). So $T^M\bar{T}|_{R(T^M)}$ is injective.

For any $y \in R(T^M)$, there is a $z \in D(T^M)$ such that $y = T^M z$. Since $I + \delta T T^M$ is bijective, there is a $\xi \in D(T^M)$ such that $z = (I + \delta T T^M)\xi$. Thus,

$$y = T^M(I + \delta T T^M)\xi = T^M\bar{T}T^M\xi \in R(T^M\bar{T}|_{R(T^M)}).$$

This shows that $R(T^M) \subset R(T^M\bar{T}|_{R(T^M)})$. Hence, $R(T^M) = R(T^M\bar{T}|_{R(T^M)})$ and consequently $T^M\bar{T}|_{R(T^M)}$ is surjective.

(2) $\Rightarrow$ (3). For any $\xi \in D(T^M)$, there is an $\eta \in D(T^M)$ such that $T^M\xi = T^M\bar{T}T^M\eta$ since $T^M\bar{T}|_{R(T^M)}$ is surjective. Noting that $R(\delta T) \subseteq R(T)$ and $T^M$ is quasiadditive on $R(T)$, we have $T^M(\xi - \bar{T}T^M\eta) = 0$. Thus, $\zeta = \xi - \bar{T}T^M\eta \in N(T^M)$ and so $D(T^M) \subseteq \bar{T}R(T^M) + N(T^M) \subseteq D(T^M)$. Hence, $D(T^M) = \bar{T}R(T^M) + N(T^M)$.

For any $\xi \in N(\bar{T}) \cap R(T^M)$, there is $\eta \in D(T^M)$ such that $\xi = T^M\eta$ and $\bar{T}\xi = 0$. So $T^M\bar{T}T^M\eta = 0$. Since $T^M\bar{T}|_{R(T^M)}$ is a injective, we have $T^M\eta = 0$ (i.e., $\xi = 0$). This proves that $N(\bar{T}) \cap R(T^M) = \{0\}$.

(3) $\Rightarrow$ (1). For any $\xi \in D(T^M)$, there are $\xi_1 \in D(T^M)$, $\xi_2 \in N(T^M)$ such that $\xi = \bar{T}T^M\xi_1 + \xi_2$ since $D(T^M) = \bar{T}R(T^M) + N(T^M)$. Let $\eta = TT^M\xi_1 + \xi_2$. Then $(I + \delta T T^M)\eta = \xi$, that is, $I + \delta T T^M : D(T^M) \to D(T^M)$ is surjective.

To prove $I + \delta T T^M$ is injective, let $\zeta \in D(T^M)$ such that $(I + \delta T T^M)\zeta = 0$. Noting that $T^M$ is quasiadditive on $R(\delta T) \subset R(T)$, we have

$$0 = T^M(I + \delta T T^M)\zeta = T^M\bar{T}T^M\zeta$$

and consequently $\bar{T}T^M\zeta \in \bar{T}R(T^M) \cap N(T^M)$. Since $R(\delta T) \subset R(T)$, we have $\bar{T}R(T^M) \subset R(T)$. Noting that $R(T) \cap N(T^M) = \{0\}$, we have $\bar{T}T^M\zeta = 0$. Thus, $T^M\zeta \in N(\bar{T}) \cap R(T^M) = \{0\}$ and finally,

$$0 = (I + \delta T T^M)\zeta = \zeta. \qquad \square$$

**Corollary 3.8.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X, Y)$ with $R(T)$ closed, and let $\delta T : D(\delta T) \to D(T^M)$ be a linear operator such that $D(T) \subset D(\delta T)$, $R(\delta T) \subseteq R(T)$, $\bar{T} = T + \delta T$. Assume that $\pi_{N(T)}$ is a linear metric projection. If*

$$N(\bar{T}) \cap R(T^M) = \{0\}, \qquad D(T) = N(\bar{T}) + R(T^M) \quad \textit{and}$$
$$D(T^M) = R(\bar{T}) + N(T^M),$$

*then $I + \delta T T^M : D(T^M) \to D(T^M)$ is bijective.*

*Proof.* Since $D(T) = N(\bar{T}) + R(T^M)$, we have $R(\bar{T}) = \bar{T}R(T^M)$, and consequently $D(T^M) = \bar{T}R(^M) + N(T^M)$. Thus, $I + \delta T T^M : D(T^M) \to D(T^M)$ is bijective by Proposition 3.7. $\square$

**Proposition 3.9.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T$ be a $T$-bounded linear operator with $D(T) \subseteq D(\delta T)$. Assume that $R(T)$ is topological-complemented in $Y$ and that $T^M$ is quasi-additive on $R(\delta T)$. If $\|\delta T T^M\| < 1$, then $I + \delta T T^M$ is invertible, and*

$$\|(I + \delta T T^M)^{-1}\| \leq \frac{1}{1 - \|\delta T T^M\|}.$$

*Proof.* By Theorem 3.6, we have that $\delta T T^M$ is a bounded homogeneous operator. If $\|\delta T T^M\| < 1$, then the series

$$\sum_{n=0}^{\infty} \|(\delta T T^M)^n\| \leq \sum_{n=0}^{\infty} \|\delta T T^M\|^n = \left(1 - \|\delta T T^M\|\right)^{-1} < +\infty.$$

This shows the series $\sum_{n=0}^{\infty}(-1)^n(\delta T T^M)^n$ is convergent to $A$, say. Since $T^M$ is quasiadditive on $R(\delta T)$, we have that

$$(I + \delta T T^M)\sum_{i=0}^{n}(-1)^i(\delta T T^M)^i$$

converges to $(I + \delta T T^M)A = A(I + \delta T T^M)$ and to $I$ as $n \to +\infty$. That is $(I + \delta T T^M)^{-1} = A$ and consequently $\|(I + \delta T T^M)^{-1}\| \leq \frac{1}{1 - \|\delta T T^M\|}$. □

**Theorem 3.10.** *Let $X$, $Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T \in L(X,Y)$ be a $T$-bounded linear operator with $b < 1$ and $D(T) \subseteq D(\delta T)$, $R(\delta T) \subseteq R(T)$, $\bar{T} = T + \delta T$. Assume that $\pi_{N(T)}$ is a linear metric projection and that $R(T)$ is topological-complemented in $Y$. If $\|\delta T T^M\| < 1$, then*

(1) $\bar{T}^M = (I - \pi_{N(\bar{T})})T^M(I + \delta T T^M)^{-1}\pi_{R(T)}$,

(2) $\|\bar{T}^M\| \leq 2\frac{\|T^M\|}{1 - \|\delta T T^M\|}$,

(3) $\|\bar{T}^M - T^M\| \leq 4\{\frac{a\|T^M\|}{1-b} + \frac{\|\delta T T^M\|}{2} + \delta_1\}\frac{\|T^M\|}{1 - \|\delta T T^M\|}$.

*Here, $\delta_1 = \eta(F_X^{-1}(N(\bar{T})^{\perp}), F_X^{-1}(N(T)^{\perp}))$.*

*Proof.* Since $R(T)$ is topological-complemented in $Y$ and $R(\delta T) \subseteq R(T)$, then $T^M$ and $\bar{T}^M$ are bounded homogeneous operator by Lemma 2.8.

(1) By Theorem 3.6(2), we know that $\delta T T^M$ is a bounded homogenous operator. If $\|\delta T T^M\| < 1$, then $I + \delta T T^M$ is invertible by Proposition 3.9. Noting that $R(\delta T) \subseteq R(T)$, we have $T T^M \delta T = \delta T$. Thus, $\bar{T} T^M(I + \delta T T^M)^{-1}\bar{T} = \bar{T}$ by simple computation. By Lemma 2.8, we have $\bar{T}^M = (I - \pi_{N(\bar{T})})T^M(I + \delta T T^M)^{-1}\pi_{R(T)}$.

(2) It is clear that $\|\bar{T}^M\| \leq \frac{\|T^M\|}{1 - \|\delta T T^M\|}\|\pi_{R(T)}\|$ since $\|I - \pi_{N(\bar{T})}\| \leq 1$ and $\|(I + \delta T T^M)^{-1}\| \leq \frac{1}{1 - \|\delta T T^M\|}$.

(3) Since $T^M = (I - \pi_{N(T)})T^M\pi_{R(T)}$, we have

$$\begin{aligned}
&\bar{T}^M - T^M \\
&\quad = (I - \pi_{N(\bar{T})})T^M(I + \delta T T^M)^{-1}\pi_{R(T)} - (I - \pi_{N(T)})T^M\pi_{R(T)} \\
&\quad = \left\{(I - \pi_{N(\bar{T})})T^M - (I - \pi_{N(T)})T^M(I + \delta T T^M)\right\}(I + \delta T T^M)^{-1}\pi_{R(T)}
\end{aligned}$$

$$= \{(\pi_{N(T)} - \pi_{N(\bar{T})})T^M - T^M \delta T T^M\}(I + \delta T T^M)^{-1}\pi_{R(T)}$$
$$= (\pi_{N(T)} - \pi_{N(\bar{T})})T^M(I + \delta T T^M)^{-1}\pi_{R(T)} - T^M \delta T T^M (I + \delta T T^M)^{-1}\pi_{R(T)}.$$

For convenience, we set $\delta_1 = \eta(F_X^{-1}(N(\bar{T})^\perp), F_X^{-1}(N(T)^\perp))$ in the following. By Proposition 3.4(4) and Lemma 3.1, we have

$$\begin{aligned}
&\|(\pi_{N(\bar{T})} - \pi_{N(T)})T^M(I + \delta T T^M)^{-1}\pi_{R(T)}\| \\
&\le 2\Big\{\frac{a}{(1-b)\gamma(T)} + \delta_1\Big\}\|T^M(I + \delta T T^M)^{-1}\pi_{R(T)}\| \\
&\le 4\Big\{\frac{a}{(1-b)\gamma(T)} + \delta_1\Big\}\frac{\|T^M\|}{1 - \|\delta T T^M\|} \\
&\le 4\Big\{\frac{a\|T^M\|}{1-b} + \delta_1\Big\}\frac{\|T^M\|}{1 - \|\delta T T^M\|}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\bar{T}^M - T^M\| &\le \|(\pi_{N(\bar{T})} - \pi_{N(T)})T^M(I + \delta T T^M)^{-1}\pi_{R(T)}\| \\
&\quad + \|T^M \delta T T^M (I + \delta T T^M)^{-1}\pi_{R(T)}\| \\
&\le 4\Big\{\frac{a\|T^M\|}{1-b} + \delta_1\Big\}\frac{\|T^M\|}{1 - \|\delta T T^M\|} + \frac{2\|T^M\|\|\delta T T^M\|}{1 - \|\delta T T^M\|} \\
&= 4\Big\{\frac{a\|T^M\|}{1-b} + \frac{\|\delta T T^M\|}{2} + \delta_1\Big\}\frac{\|T^M\|}{1 - \|\delta T T^M\|}.
\end{aligned}$$

□

**Theorem 3.11.** *Let $X, Y$ be reflexive strictly convex Banach spaces, $T \in C(X,Y)$ with $R(T)$ closed, and let $\delta T \in L(X,Y)$ be a $T$-bounded linear operator with $D(T) \subseteq D(\delta T)$, $N(T) \subseteq N(\delta T)$, $\bar{T} = T + \delta T$. Assume that $\pi_{N(T)}$ is a linear metric projection and that $R(T)$ is topological-complemented in $Y$. If $T^M$ is quasiadditive on $R(\delta T)$ and $a\|T^M\| + 2b < 1$, then*

(1) $\bar{T}^M = (I - \pi_{N(T)})T^M(I + \delta T T^M)^{-1}\pi_{R(\bar{T})}$,
(2) $\|\bar{T}^M\| \le 2\frac{\|T^M\|}{1 - \|\delta T T^M\|}$,
(3) $\|\bar{T}^M - T^M\| \le \frac{\|T^M\|}{1 - \|\delta T T^M\|}(1 + a\|T^M\| + 2b)$.

*Proof.* By Theorem 3.6(2), we have that $\delta T T^M$ is a bounded homogenous operator and $\|\delta T T^M\| \le a\|T^M\| + 2b$.

(1) Since $a\|T^M\| + 2b < 1$, we have $b < \frac{1}{2}$. Thus, let $\bar{T}$ be a closed linear operator with $I + \delta T T^M$ invertible. Then $N(T) \subseteq N(\delta T)$ shows that $\delta T = \delta T T^M T$. So, it is easy to check that $\bar{T} T^M (I + \delta T T^M)^{-1}\bar{T} = \bar{T}$. By Lemma 2.8, we have

$$\bar{T}^M = (I - \pi_{N(T)})T^M(I + \delta T T^M)^{-1}\pi_{R(\bar{T})}.$$

(2) Obviously, $\bar{T}$ is a bounded homogenous operator and $\|\bar{T}^M\| \le 2\frac{\|T^M\|}{1 - \|\delta T T^M\|}$.

(3) For any $\xi \in D(T)$, $\pi_{R(\bar{T})}\xi \in R(\bar{T})$. Hence, there is an $x \in D(T)$ such that $\pi_{R(\bar{T})}\xi = \bar{T}x$. Thus,

$$\begin{aligned}
(I + \delta T T^M)^{-1}\pi_{R(\bar{T})}\xi &= (I + \delta T T^M)^{-1}\bar{T}x \\
&= (I + \delta T T^M)^{-1}(T + \delta T T^M T)x = Tx \in R(T).
\end{aligned}$$

Note that $T^M$ is quasiadditive on $R(T)$ and $R(\delta T)$. So is $I+\delta TT^M$. Hence, $(I+\delta TT^M)^{-1}$ is quasiadditive on $R(T)$ and $R(\delta T)$ since $I+\delta TT^M$ is invertible. Thus,

$$\begin{aligned}\bar{T}^M - T^M &= (I-\pi_{N(T)})T^M(I+\delta TT^M)^{-1}\pi_{R(\bar{T})} - (I-\pi_{N(T)})T^M \\ &= (I-\pi_{N(T)})T^M\big\{(I+\delta TT^M)^{-1}\pi_{R(\bar{T})} - I\big\} \\ &= (I-\pi_{N(T)})T^M(I+\delta TT^M)^{-1}\{\pi_{R(\bar{T})} - I - \delta TT^M\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\bar{T}^M - T^M\| &\le \big\|(I+\delta TT^M)^{-1}\big\|\|T^M\|\big(\|I-\pi_{R(\bar{T})}\| + \|\delta TT^M\|\big) \\ &\le \frac{\|T^M\|}{1-\|\delta TT^M\|}\big(1+\|\delta TT^M\|\big) \\ &\le \frac{\|T^M\|}{1-\|\delta TT^M\|}\big(1+a\|T^M\|+2b\big).\end{aligned}$$

□

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