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PERTURBATION ANALYSIS OF AN APPROXIMATION
TO THE HODGKIN-HUXLEY THEORY*

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1. Introduction. The purpose of this paper is to analyze equations which are approximations to a theory of nerve impulse conduction. The equations are a nonlinear parabolic system of third order, and the desired solutions are steady, propagating waves. The method of analysis is that of singular perturbations and relies on the fact that the third-order system contains a small parameter.

Although the approximate equations used here lose some of the physical meaning and mathematical richness of the more complete theory, they allow for a great deal more analytical manipulation and therefore more insight into the mathematical processes involved.

The Hodgkin–Huxley theory for the conduction of voltage pulses along the membranes of nerve cell axons [1] is a satisfactory, empirically supported theory. It represents the voltage changes that are measurable as the fixed-shape, constant-velocity propagating voltage pulse by a set of partial differential equations. One of these,

$$C \frac{\partial v}{\partial t} + I_m(v; m, h, n) = \frac{a}{2R_i} \frac{\partial^2 v}{\partial x^2} , \qquad (1.1)$$

describes the voltage difference, $v(x, t)$, across the axon membrane in terms of lumped electrical characteristics of the membrane: C, the cross-membrane capacitance; a, axon radius; R_i , specific resistivity of the fluid inside the membrane; and $I_m(v, m, n, h)$, the cross-membrane ionic current. x is distance along the membrane, and t is the time.

The ionic current, I_m , is represented as a sum of three currents, given in terms of products of conductivities and voltages:

$$I_m = g_{na}(v - \bar{v}_{na}) + g_K(v - \bar{v}_K) + \bar{g}_l(v - \bar{v}_l), \qquad (1.2)$$

with

$$g_{na} = m^3 h \bar{g}_{na} , \; g_K = n^4 \bar{g}_k . \qquad (1.3, 1.4)$$

The barred quantities are constants. The terms of the expression represent sodium, potassium, and leakage ion current densities. v here represents the voltage difference between the measurable voltage and the voltage of the rest state at which there is identically zero total ion current. $\bar{v}_{na}$, $\bar{v}_K$, and $\bar{v}_l$ are the equilibrium values of the voltage, determined under the condition that the membrane is permeable to only one of the ion species at a time. g_{na} and g_K are the nonlinear conductivities for the sodium and potassium current densities. They depend, as is shown, on the variables m, h, and n, which in turn satisfy the equations:

$$\partial m / \partial t = \frac{1}{\tau_m(v)} (m_\infty(v) - m), \qquad (1.5)$$

$$\partial h / \partial t = \frac{1}{\tau_h(v)} (h_\infty(v) - h), \qquad (1.6)$$

$$\partial n / \partial t = \frac{1}{\tau_n(v)} (n_\infty(v) - n). \qquad (1.7)$$

τ_m, τ_h, τ_n, m_∞, h_∞, and n_∞ are experimentally determined functions of the voltage, $v(x, t)$. $\bar{g}_l$ is the constant conductivity of the leakage ion current density, and $\bar{g}_{na}$ and $\bar{g}_K$ are the maximum conductivity values for g_{na} and g_K.

As will be mentioned shortly, this fifth-order partial differential equation system, Eqs. (1.1)–(1.7), when solved numerically, exhibits many of the experimentally measurable phenomena. However, it is a cumbersome set of equations to handle analytically: linearizations are not very useful and, although some qualitative treatment has begun [4], it would be helpful to have other approximations available.

One such approximation has been suggested by Fitz Hugh [5] and used by Nagumo et al. [6]. It replaces (1.1)–(1.7) by the equations

$$\partial v / \partial t + f(v) + z = \partial^2 v / \partial x^2, \qquad (1.8)$$

$$\partial z / \partial t = \epsilon v, \; f(v) = v(1 - v)(a - v). \qquad (1.9, 1.10)$$

The relationship between the systems (1.1)–(1.7) and (1.8)–(1.10) will be discussed. Before doing so, however, it is useful to describe the kind of solutions that have been obtained for the Hodgkin–Huxley equations.

When the six experimental functions are provided, the system (1.1)–(1.7) can be shown by numerical solution to exhibit a number of the measurable properties of the nerve impulse and its conduction along the axon:

1. For the spatially-independent (space-clamped) case, $\partial / \partial x = 0$, $v = 0$ when $t \to \pm \infty$, the time dependence of the pulse and its amplitude compare well with experiment [1].

2. For the steady, traveling pulse, $v(x, t) = v(x + \theta t)$, $v = 0$ when $x + \theta t \to \pm \infty$, the shape of the pulse and its velocity of propagation compare well with experiment [1]. The solution is found on the infinite line. It is not unique; there appear to be at least two traveling pulses of different velocity and amplitude [2].

3. For the initial-value problem, $v(x, 0) = 0$; $v(0, t) = v_0(t)$, $0 \le t \le t_0$; $v = 0$, $x \to \pm \infty$, a transient solution is shown to build up into a steady traveling pulse whose shape and velocity approach those found in item 2 [3].

4. A threshold can be found numerically such that the amplitude of the stimulus applied (e.g., $v_0(t)$ in item 3) and its duration of application determine whether or not a pulse will form and propagate [3].

5. The effects of changing the electrolytic environment and of alterations in axon geometry can be calculated [3, 7, 8].

6. The response of the system for small signal stimulus is calculable, and compares well with experiment [9, 10].

7. Similar equations for axons in which ion transport across the membrane only occurs at nodes along the axon (myelinated axons) yield numerical solutions that correspond well with experiment [11, 12].

8. Solutions with repeated pulses, finite trains of pulses and infinite trains of periodic pulses have been numerically determined [3].

From the experimental data and from the calculations just mentioned, some useful observations can be made of the Hodgkin–Huxley theory:

1. The sodium and potassium current densities can be correctly separated as in Eq. (1.2); each current has a linear dependence on voltage changes near its equilibrium voltage.

2. The variables m and h describe the turning on and the turning off, respectively, of the rate of inflow (into the membrane) of the sodium ions. m follows the voltage pulse as it rises and falls; h decreases as the voltage increases. The variable n describes the outflow of the potassium ions; n increases as the voltage increases (m, n, and h are normalized to be maximum at unity). Thus the time course of m is closely related to the rise of the impulse; n and h act to return the voltage toward the original state.

3. The responsiveness of m, h and n are measured by τ_m , τ_h , and τ_n . τ_m is much smaller in magnitude than either τ_h or τ_n .

4. The basic data used in determining I_m is called voltage clamp data. It is obtained by threading an electrode through a nerve axon and closing the circuit with an external voltage control circuit. This allows a step voltage to be applied and held constant, and also shorts out changes with respect to distance along the axon, $\partial v / \partial t = \partial v / \partial x = 0$.
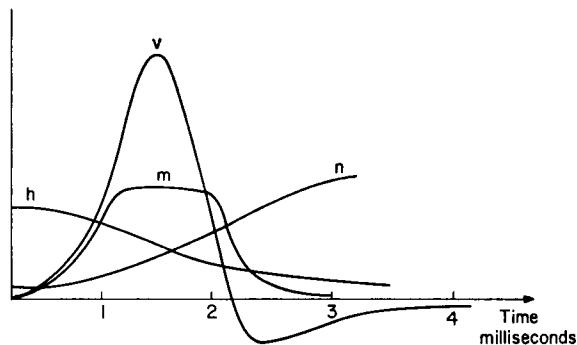

FIG. 1.1.   Time dependence of v and the ionic conductance variables m, n, and h, m, n, and h have maximum value unity. $v(t)$ is drawn on a different ordinate scale.
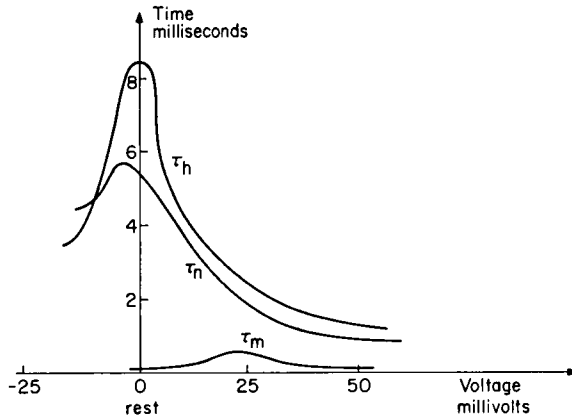
FIG. 1.2. Time dependence of the response times. τ_m, τ_n and τ_h given in milliseconds as a function o voltage.

The ionic current I_m is then measured as a function of time at each fixed voltage level. A typical set of voltage clamp data may look like Fig. 1.3.

It can be seen that there are two time scales involved in the pulse and reflected in the behavior of the ion current: a fast rise time that corresponds to turning on the sodium ion inflow (m) and the voltage rise (v), and a slower recovery that is related to turning off the sodium inflow (h) and the potassium outflow (n). If we make the assumption that these time scales are, in fact, very widely separated we might make the approximation in (1.5)–(1.7)

$$\tau_m = 0, \tau_h = \tau_n = \infty. \tag{1.11}$$
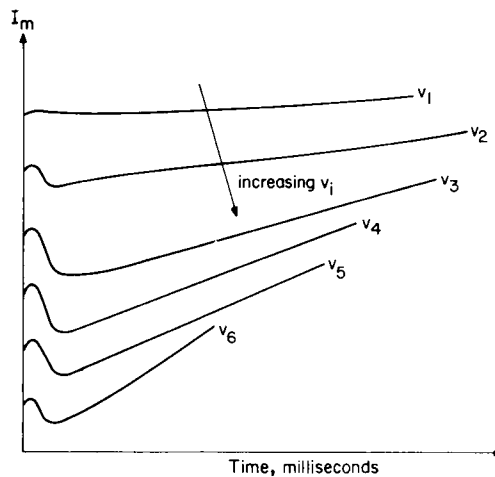
In this case

FIG. 1.3. Voltage clamp data: membrane current I_m as a function of time in milliseconds for various values of voltage v_i.

$$m = m_\infty(v); h = h_\infty(v = 0), n = n_\infty(v = 0), \tag{1.12}$$

and the resulting wave form for v and m is a step instead of a pulse, as shown in Fig. 1.4.

Setting $\tau_m = 0$ corresponds to translating the negative peaks of the voltage clamp curves (Fig. 1.3) to $t = 0$. An inspection of these peaks reveals that they behave very nearly like a cubic function of the voltage. The remainder of each voltage clamp curve has an increasing slope with increasing value of the voltage step. Putting together these notions, we may look for an approximation to I_m in the form:

$$I_m = C_1 g(v)(v - v_{na}) + C_2 z. \tag{1.13}$$

The function g will be quadratic in v and will represent the sodium ion turn-on behavior, and z will model the recovery process related to n and h. We will approximate the slow large-time behavior of the voltage clamp curves by

$$dz/dt = v. \tag{1.14}$$

For any fixed $v = v_0$ (if $z = 0$ at $t = 0$)

$$z = v_0 t, \tag{1.15}$$

$$I_m = C_1 g(v_0)(v_0 - v_{na}) + C_2 v_0 t. \tag{1.16}$$

Allowing $v = 0, I_m = 0$ to represent the rest state, we can account for the cubic character of the early sodium current density by taking

$$g(v) = (v/\bar{v}_{na})(a - (v/\bar{v}_{na})), \tag{1.17}$$

where a is a positive constant, chosen small enough so that the sodium current is approximated for $v > 0$.

$$I_m = C_1(v/\bar{v}_{na})(a - (v/\bar{v}_{na}))(v - \bar{v}_{na}) + C_2 z. \tag{1.18}$$

To determine the constants, one may try roughly matching this expression to the cubic behavior of the peak sodium inflow current density and to the increasing slope of the outflow current in the voltage clamp data. Suitable values are

$$C_1 = \frac{100 \text{ millimhos}}{\text{cm}^2}, \qquad C_2 = \frac{7 \text{ millimhos}}{\text{millisec cm}^2}.$$

We now have for the space-clamped (space-independent) case

$$C \frac{dv}{dt} + C_1(v/v_{na})(a - (v/\bar{v}_{na}))(v - \bar{v}_{na}) + C_2 z = 0, \tag{1.19}$$

$$dz/dt = v. \tag{1.20}$$

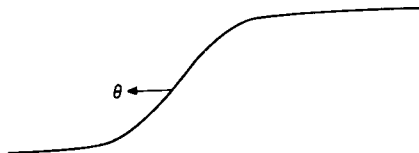C is the capacitance of the membrane, as in (1.1).

Fig. 1.4. Wave form for v and m when $\tau_m = 0, \tau_h = \tau_n = \infty$.

The two time scales we have mentioned earlier can now be associated with the fitting constants, C_1 and C_2. We can take $C/C_1 = T_1$ and $C_1/C_2 = T_2$. T_1 is related to the early sodium current as it charges the membrane capacitance; T_2 is the time scale of the outward potassium current and the sodium turn-off. We may now use T_1 to make the time scale non-dimensional and T_2 in non-dimensionalizing z:

$$t^* = t/T_1 , \ z^* = z/T_2\bar{v}_{na} , \ v^* = v/\bar{v}_{na} . \tag{1.21}$$

Then with

$$T_1/T_2 = \epsilon, \tag{1.22}$$

(1.19) and (1.20) become

$$(dv^*/dt^*) + v^*(a - v^*)(1 - v^*) + z^* = 0, \tag{1.23a}$$

$$dz^*/dt^* = \epsilon v^*. \tag{1.23b}$$

The ratio T_1/T_2 will be very small; for the given values of the constants, $\epsilon = .0007$. The perturbation analysis that we shall use depends on this small size of ϵ.

Dropping the stars and combining Eqs. (1.23a, b), we have

$$(d^2v/dt^2) + (df/dv)(dv/dt) + \epsilon v = 0, \tag{1.24}$$

where

$$f = v(a - v)(1 - v), \ df/dv = a - 2(1 + a)v + 3v^2. \tag{1.25a, b}$$

If we put $p = v - (1 + a)/3$, this equation becomes

$$\frac{d^2p}{dt^2} + \frac{dp}{dt}\left(p^2 - \frac{a^2 - a + 1}{9}\right) + \epsilon p = \epsilon\frac{1 + a}{3} , \tag{1.26}$$

which is a forced Van der Pol equation.

Fitz Hugh [5] observed that the nonlinearity of the Van der Pol equation provided a two-dimensional phase space analogue to the behavior of the reduced-dimensional Hodgkin–Huxley equations.

To complete the description of this approximate system we can add a longitudinal current term to describe propagation of the current along the axon:

$$\partial v/\partial t + v(a - v)(1 - v) + z = \partial^2 v/\partial x^2. \tag{1.27a}$$

$$\partial z/\partial t = \epsilon v. \tag{1.27b}$$

In taking the coefficient of v_{xx} to be unity, we are making the time scale of the longitudinal current density the same as the fast time of response in this approximate system.

For Eqs. (1.27) we may set a series of problems similar to those considered for the Hodgkin–Huxley equations:

(i) Space-clamped case: We assume $\partial/\partial x = 0$, so that v is a function of t only. Then

$$dv/dt + f(v) + z = 0, \ dz/dt = \epsilon v. \tag{1.28a, b}$$

As initial conditions one may take

$$v(0) = v_0 , \ z(0) = 0. \tag{1.28c, d}$$

(ii) Steady propagating waves: We put

$$\xi = x + \theta t, \tag{1.29}$$

where θ is a constant which without loss of generality we may assume to be positive. In the steady case v depends on ξ only and the equations are then

$$\frac{d^2v}{d\xi^2} - \theta \frac{dv}{d\xi} - f(v) - z = 0, \qquad \theta \frac{dz}{d\xi} = \epsilon v. \tag{1.30a, b}$$

We shall look for solutions which defined for all ξ and periodic in ξ. The single pulse (solitary wave) will be seen to be the limiting case of infinite period.

(iii) Mixed initial-value boundary-value problem: We prescribe the values at $x = 0$

$$v(0, t) = v_0(t), z(0, t) = 0 \tag{1.31a, b}$$

with suitable initial values, say

$$v = 0, z = 0 \text{ at } t = 0 \quad \text{for} \quad x > 0. \tag{1.31c, d}$$

Eqs. (1.27) were first studied by Nagumo [6]. For the space-clamped case and $\epsilon = 0$ (1.28) reduces to

$$dv/dt + v(1 - v)(a - v) + z = 0, dz/dt = 0, \tag{1.32, 1.33}$$

which has the solution

$$K \exp (-a(1 - a)t) = \frac{v^{(1-a)}(1 - v)^a}{(a - v)}, K = \text{constant.} \tag{1.34}$$

This solution has the property that for initial values above or below a, respectively, the solution will tend to one or to zero. Thus a has the character of a threshold. This will be seen to be true for Eqs. (1.27) as well. From the relationship to the peak inflow sodium current density, a will be seen to be positive. In an experimentally important case (squid axon) it has a value near 0.1.

Eqs. (1.27) are the approximation to the Hodgkin–Huxley equations that will be used in the remainder of this paper. We note that in the approximation used here the expression for the ion current density, I_m, has been greatly simplified: the parabola represented by Eq. (1.17), taken with $\tau_m = 0$, $\tau_h = \infty$ should be an approximation to
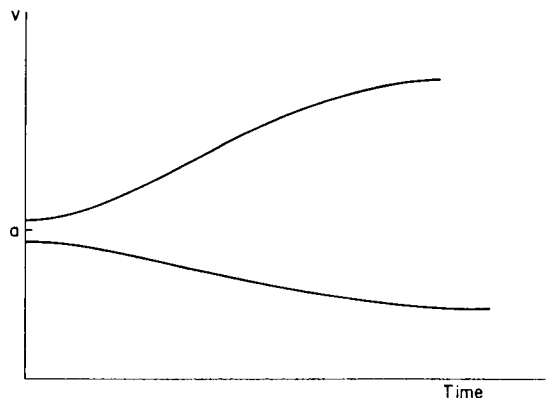

FIG. 1.5.   Behavior of $v(t)$ solutions of (1.32) for different initial conditions.

$m_\infty{}^3(v)$ $h_\infty(0)$ ($m_\infty(v)$ appears in (1.5)); in fact, the parabola fits only a portion of $m_\infty{}^3(v)$ $h_\infty(0)$. Similarly, the linear function taken for the recovery current term is only a crude fit to the current outflow portion of the voltage clamp curves. The functional dependence of m, n, and h on v is very much altered. We must expect, therefore, to find some of the behavior of nerve impulse conduction lacking.

In spite of these and other limitations, the equations do provide traveling pulses and periodic trains of traveling pulses. As we have mentioned earlier, the nature of the cubic nonlinearity relates this system to the Van der Pol equation. A singular perturbation analysis for relaxation oscillation solutions for the Van der Pol equation has been given by Carrier [13] and Dorodnitsyn [14] and is to be found in the book by Cole [15]. There are, however, important and interesting differences between the Van der Pol equation and the equations studied here.

Numerical solutions for problems (ii) and (iii) (p. 1.12 and 1.13) have been calculated first by Nagumo et al. [6] and also by James Cooley of IBM Research. For (ii) Cooley has found that well-formed propagating pulses are obtained for values of $0 < \epsilon$. For fixed ϵ, the solutions lie along the curves shown as dotted lines in Fig. 1.10. Solutions can be obtained for $1/2 > a \geq 0$ until ϵ reaches the value 0.015. Numerical solutions were difficult to obtain with a close to $1/2$. For ϵ extremely small, the pulses have a steep rise and a steep fall but have a flat, long top.

For (iii), with a small initial value given for v, Cooley has shown that after a transient phase, solutions reach well formed pulses and approach shapes and propagation velocities that correspond to those found for the same ϵ, and in the steady, propagating solutions.

Several qualitative studies have been made of problem (ii). Hastings [16] has shown that there are no bounded solutions for $a > 1/2$, and ϵ/θ^2 greater than a certain value. Greenberg [17] has studied the problem with an added term, linear in z, in the second equation. He shows that, with this term present, for $\epsilon > 0$ and certain other conditions satisfied there are propagating pulse-like solutions. Conley [18] has shown that pulse-like solutions exist for a set of values of ϵ and θ for $f(v)$ more general than the one considered here but with the condition that $\int_0^v f(v')\, dv'$ is negative for some values of v. This corresponds to a condition in the present analysis on the limiting solutions and the point of inflection of $f(v)$. McKean [19] has studied the phase plane behavior of (1.36)–(1.37), some of which is given again in our Appendix. He has also considered solutions for $\epsilon > 0$ for a piece-wise linear nonlinear current term which has a jump in place of the cubic function. Rinzel and Keller [20] studied this case in greater detail and have produced slow and fast traveling waves, both solitary and periodic. They have established stability for the fast waves and instability for the slow ones.

In subsequent chapters we shall use singular-perturbation methods to discuss steady progressing periodic waves. The calculations will be carried to order ϵ. Below we shall give a heuristic introduction to this problem and also give a survey of the principal results. We start with a discussion of the solitary wave for the case when θ is strictly of order unity. By this we mean that $\theta = \theta_0 + O(\epsilon)$ where $\theta_0 > 0$. As will be seen the method for the solitary wave generalizes easily to the general case of periodic waves. We shall also discuss slow waves for which $\theta = O(\epsilon)$.

Pulses for $\theta = O(1)$. If we put $\epsilon = 0$ in (1.30b) we obtain $dz/d\xi = 0$ since by assumption θ is bounded away from zero as ϵ tends to zero. For the pulse we assume that $z = 0$ at $\xi = -\infty$; hence z is everywhere zero. Solving (1.30a) with $z = 0$, we find a bounded solution which is zero at $\xi = -\infty$, namely

$$v = v_0(\xi) = 1/(1 + \exp(-\xi/\sqrt{2})). \tag{1.35}$$

As discussed in the Appendix, such a solution can exist only if

$$\theta_0 = \sqrt{2}\,(\tfrac{1}{2} - a). \tag{1.36}$$

This solution, however, does not represent a pulse but rather a front, as shown in Fig. 1.4. Near $\xi = 0$ it shows a sharp rise (which we shall call an upjump) from a value which is slightly greater than zero to a value which is almost unity. After the upjump it remains almost flat and rises slowly to the value one at $\xi = \infty$. Higher-order approximations will not improve the situation. The fact that the problem had two time scales of different orders of magnitude suggested a perturbation analysis of the problem. The same fact suggests that possibly different time scales should be used in different regions; in other words, that the problem should be treated as a singular perturbation problem. Indeed, we see that v_0 as given by (1.35) is not a uniformly valid approximation. The exact expression for z is $z(\xi) = \epsilon/\theta \int_{-\infty}^{\xi} v(\xi)d\xi$. At the upjump this quantity remains $O(\epsilon)$ but at a distance of $1/\epsilon$ from the upjump its value is of order unity and z can no longer be neglected in (1.30). The slow variation of v with ξ after the upjump suggests that to study large values of ξ we introduce a variable

$$\eta = \epsilon\xi. \tag{1.37}$$

(This amounts to using the long time scale T_2 instead of T_1.)

Putting $u(\eta, \epsilon) = v(\xi, \epsilon)$, $y(\eta, \epsilon) = z(\xi, \epsilon)$, we may rewrite (1.30) as

$$\epsilon^2 \frac{d^2u}{d\eta^2} - \epsilon\theta \frac{du}{d\eta} - f(u) - y = 0, \qquad \theta \frac{dy}{d\eta} = u. \tag{1.38a, b}$$

This system, with $\epsilon = 0$ and $\theta = \theta_0$, has solutions u_0 and y_0 given by (2.11a) and (2.9b) and illustrated by Fig. 2.1. The uppermost branch (Region I) of the multivalued solution should be used.

In the language of singular perturbation theory $u_0(\eta)$ and $y_0(\eta)$ represent outer solutions relative to the inner solutions $v_0(\xi)$ and $z_0(\xi)$ (where $z_0(\xi) \equiv 0$). The constants of integration of the outer solutions are determined by matching with the inner solutions. In our case the rules of matching yield the simple conditions

$$\lim_{\eta \downarrow 0} u_0(\eta) = v_0(\infty) = 1, \qquad y_0(0) = 0. \tag{1.39}$$

The function u_0 decreases with increasing η but is bounded from below by a positive value. Thus the combination of $v_0(\xi)$ and $u_0(\eta)$ still does not represent a uniformly valid solution for the pulse. Somehow, we must make the solution assume negative values. Now, corresponding to the upjump (1.35) there is also a solution of the same equation which represents a downjump from one to zero. Actually, however, we should use a slightly different equation. During the almost flat part, expressed by $u_0(\eta)$, the value of z has been growing. We may assume it to be constant during the downjump, which effectively takes place in a region of order ϵ on the η-scale. However, this constant, which we shall call $\tilde{K}_0$, is now greater than 0. Let the downjump take place near (that is, within $O(\epsilon)$), the value $\eta = \eta_d$. The ξ-variable has the correct scale for describing the downjump but it is convenient to shift the origin to near $\eta = \eta_d$ by introducing the variable ρ,

$$\epsilon(\rho - \rho_d) = \eta - \eta_d. \tag{1.40}$$

(The constant ρ_d will be determined by higher-order matching, as shown in Sec. 3). The first approximation to v near $\eta = \eta_d$ will be denoted by $\tilde{v}_0(\rho)$. Thus the equations are now

$$d\tilde{v}_0/d\rho = \tilde{w}_0 \tag{1.41a}$$

$$d\tilde{w}_0/d\rho = \theta_0 \tilde{w}_0 + f(\tilde{v}_0) + \tilde{K}_0 \tag{1.41b}$$

The essential fact is that the wave velocity θ_0 is now given; it must be the same as for the upjump. A downjump with given wave velocity is possible only for a specific value of $\tilde{K}_0$. Furthermore, the jump must occur from $u = \tilde{U}$ to $u = \tilde{V}$, where $\tilde{U}$ and $\tilde{V}$ are uniquely determined by $\tilde{K}_0$. The function u_0 is monotonely decreasing and the equation

$$u_0(\eta_d) = \tilde{U} \tag{1.42}$$

determines the location of the downjump uniquely. Corresponding to the upjump (1.35) the formula for the downjump is

$$v_0(\varphi) = \frac{\tilde{U} \exp(-\tilde{h}\rho) + V}{1 + \exp(-\tilde{h}\rho)} \ , \tag{1.43a}$$

where

$$\sqrt{2}\tilde{h} = \tilde{U} - \tilde{V}, \quad \tilde{U} = (2 + 2a)/3, \quad \tilde{V} = (2a - 1)/3. \tag{1.43b, c, d}$$

The constants $\tilde{U}$ and $\tilde{V}$ are the largest and smallest roots respectively of

$$f(u) + \tilde{K}_0 = 0, \tag{1.44a}$$

where

$$\tilde{K}_0 = -2f_{\inf} , \tag{1.44b}$$

$$f_{\inf} = \text{value of } f(u) \text{ at point of inflection} = \tfrac{1}{27}(1 + a)(2 - a)(2a - 1). \tag{1.44c}$$

Since θ_0 and u are positive it follows that $\tilde{K}_0 > 0$. The solution is possible only if we impose the restriction

$$f_{\inf} < 0 \tag{1.45a}$$

or, equivalently,

$$0 < a < \tfrac{1}{2}. \tag{1.45b}$$

To complete the solution we need a fourth piece, an outer solution which increases from $\tilde{V}$ at $\eta = \eta_d$ to zero at $\eta = \infty$. This solution is again given by (2.11a) although this time the branch where u is negative (cf. Fig. 2.1) should be chosen. It is convenient to shift the origin of the η-axis by defining a new outer variable κ which has the same relation to ρ as η has to ξ. Thus

$$\kappa = \epsilon\rho = \eta - \eta_d + \epsilon\rho_d . \tag{1.46}$$

The four pieces of the solution are indicated in Fig. 1.6. Thus we see how perturbation analysis can be used to obtain an approximate description of a pulse uniformly valid to order unity. We observe, however, that the crest of the pulse, described by $u_0(\eta)$, is very flat and long (in ξ), unlike the actual pulse shown in Fig. 1.6. However, as ϵ increases the pulse will be less flat and more like a proper nerve pulse.
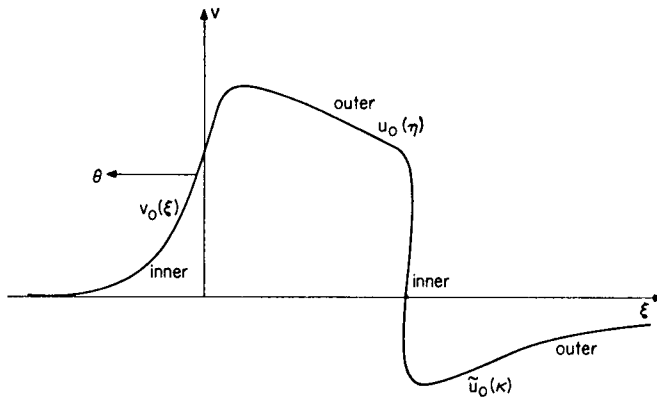
FIG. 1.6.   The four pieces of the perturbation solution.

Note that if one is interested only in determining θ_0 one need only consider the first of the four pieces of the solution, namely the upjump described by $v_0(\xi)$.

Phase-plane picture.   It is instructive to rephrase some of the above ideas in phase-plane language. Since we are dealing with a third-order autonomous equation, the solution may be described by a curve in a three-dimensional phase-space whose axes we may take to be v, $w = dv/d\xi$ and z. However, for upjumps and downjumps z changes little and phase plane analysis may be used. The phase plane is discussed in detail in the Appendix. The upjump looks as is shown in Fig. 1.7.

The singular points are (0.0), $(a, 0)$, $(1, 0)$. For the special value of θ_0 given by (1.36) the integral curve which leaves (0.0) in the first quadrant also passes through $(1, 0)$, as shown in Fig. 1.7. This corresponds to the function $v_0(\xi)$ given by (1.35). For most of this curve we find z to be practically zero. However, the curve arrives at $(1, 0)$ only when $\xi = \infty$. Thus the value of v would be near unity for a long period. This means that z gets appreciably greater than zero. The phase-plane solution no longer applies; instead the solution is described by $u_0(\eta)$ which corresponds to a rise in z and very slow change of v and $dv/d\xi$. When z comes near the value $\tilde{K}_0$ there is a downjump in v while z remains practically equal to $\tilde{K}_0$. This downjump is described by the lower branch of the curve passing from $(\tilde{U}, 0)$ to $(\tilde{V}, 0)$ in Fig. 1.8. Again, as v_0 approaches $\tilde{V}$, z starts changing appreciably, as described by $\tilde{u}_0(\kappa)$, and, instead of ending up at $v = \tilde{V}$, $w = 0$, $z = \tilde{K}_0$, the solution curve goes toward $v = 0$, $w = 0$, $z = 0$ as $\rho \to \infty$.
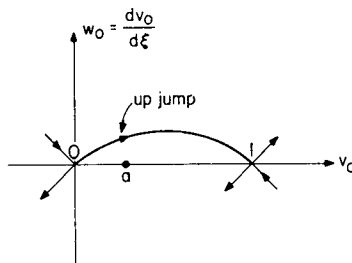

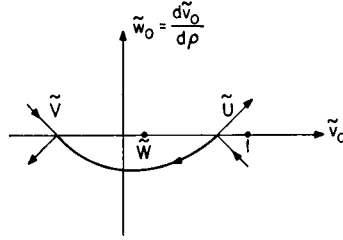FIG. 1.7.   Phase-plane picture of upjump $z = 0$.

FIG. 1.8.   Phase-plane picture of downjump, $z = \tilde{K}_0$.

Periodic solutions.   A slight generalization of the above leads to the construction of periodic solutions. It is necessary to retain the restrictions (1.45). We assume that rather than being zero during the upjump z has, to leading order, the value K_0 , where

$$0 < K_0 \leq - f_{\text{inf}} .\tag{1.47}$$

(As will be shown, solutions are not possible if $K_0 < 0$.) Then an upjump from V to U is possible, where V and U are the smallest and largest roots respectively of

$$f(v) + K_0 = 0,\tag{1.48}$$

and, generalizing (1.35), we find that the formula for the upjump is

$$v(\xi) = \frac{U \exp{(h\xi)} + V}{1 + \exp{(h\xi)}} = \frac{U + V \exp{(-h\xi)}}{1 + \exp{(-h\xi)}} ,\tag{1.49a}$$

where

$$\sqrt{2}h = (U - V).\tag{1.49b}$$

The wave velocity is now, to leading order,

$$\sqrt{2}\theta_0 = 3(U + V) - 2(1 + a).\tag{1.50}$$

After the upjump there is an upper branch $u_0(\eta)$ of the outer solution, at η_d a downjump $\tilde{v}_0(\rho)$ and then a negative branch $\tilde{u}_0(\kappa)$ of the outer solution. The constants $\tilde{U}$ and $\tilde{V}$ are still found from (1.44a) if we generalize (1.44b) to

$$\tilde{K}_0 = -K_0 - 2f_{\text{inf}} .\tag{1.51}$$

The essential difference from the case of the solitary wave is the following. By assumption V is now less than zero and the negative branch of the outer solution reaches this value for a finite value of κ,

$$\tilde{u}_0(\kappa_u) = V.\tag{1.52}$$

At this point there is another upjump and the process then repeats itself periodically. The period is, in the η-scale,

$$T_\eta = \eta_d + \kappa_u + o(1)\tag{1.53}$$

with the constants determined by (1.42) and (1.52).

Slow waves.   Eqs. (1.38) also admit solutions for which $\theta = O(\sqrt{\epsilon})$. These are discussed in Sec. 4. A pulse solution looks as is shown in Fig. 1.9. The inner solution
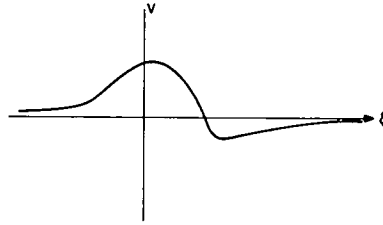
FIG. 1.9.   Pulse with velocity $= O(\sqrt{\epsilon})$.

uses ξ as a variable. For ξ large one needs an outer solution of amplitude $\sqrt{\epsilon}$ depending on the variable $\xi/\sqrt{\epsilon}$. The maximum of v is $O(1)$, the minimum is $O(\sqrt{\epsilon})$. However, the effective interval of integration for the positive part of v is $O(1)$, whereas that of the negative part is of order $\epsilon^{-1/2}$. In this way it is possible for $\int_{-\infty}^{\infty} v \, d\xi$ to be zero.

Thus for given values of a and ϵ two pulse solutions are possible, as shown in Fig. 1.10. It is believed that the larger value of θ corresponds to a stable solution and the smaller value to an unstable one.

2. Periodic solutions to lowest order. $\theta = O(1)$.

A. *Statement of mathematical problem.*   The equations to be solved are, expressed in the outer η-coordinate (cf. 1.38),

$$\epsilon^2 \frac{d^2 u}{d\eta^2} - \epsilon \theta \frac{du}{d\eta} - f(u) - y = 0, \qquad \theta \frac{dy}{d\eta} = u. \qquad (2.1\text{a, b})$$

Here

$$f(u) = u(u - a)(u - 1); \quad 0 < a < 1 \qquad (2.1\text{c})$$

As conditions we impose

$$u \text{ is defined and bounded for } -\infty < \eta < \infty, \qquad (2.2\text{a})$$

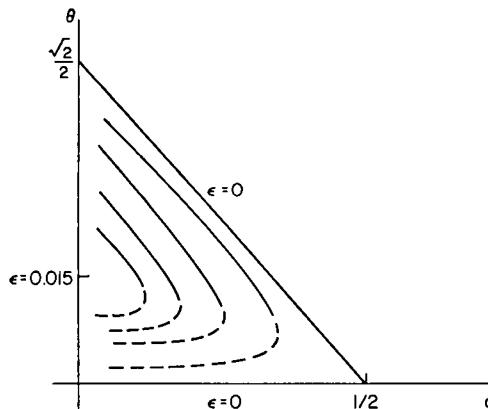$$u \text{ is periodic in } \eta. \qquad (2.2\text{b})$$

FIG. 1.10.   Wave velocity as a function of a parametrized by ϵ.

We include the solitary wave among the periodic solutions since it will turn out to be a limiting case as the period tends to infinity. We define U, V, W as the roots of

$$f(u) + K_0 = 0, \quad \text{with} \quad U < W < V. \tag{2.3}$$

The initial conditions at $\eta = 0$ are

$$u = (U + V)/2, \quad y = K = K_0 + \epsilon K_1, \quad K_1 = M + \sqrt{2} \log 2/\theta_0, \tag{2.4a, b, c}$$

$$du/d\eta > 0. \tag{2.4d}$$

Here K_0 and M are given constants and θ_0, the first approximation to θ, will be seen to be uniquely determined by K_0. The initial value of u may be considered as fixing the origin of the η-axis. The location of this origin is physically irrelevant since the equations are translation-invariant. The prescribed value of y is then the only true initial condition. This condition will uniquely determine the period and the wave velocity θ. Thus the problem cannot be solved for an arbitrary value of θ; in other words, θ should be regarded as an eigenvalue. The constant K_1 may of course be put equal to zero; however, it will be convenient for calculations to assume a general value. Condition (2.4d) implies that there is an upjump near $\eta = 0$.

As stated in Sec. 1, solutions will be possible only if K and a are in the range described by (2.28c) and (2.28d).

For the limiting case of the solitary wave we shall find

$$u(-\infty) = u(\infty) = 0. \tag{2.5}$$

We note that

$$u \text{ periodic} \Rightarrow y \text{ periodic.} \tag{2.6}$$

For the solitary wave we assume

$$y(-\infty) = 0, \tag{2.7a}$$

which implies

$$y(\infty) = \frac{1}{\theta} \int_{-\infty}^{\infty} u \, d\eta = 0. \tag{2.7b}$$

B. *Outer solutions to order unity.* The necessity for having inner and outer (expansions) was made plausible in Sec. 1. Actually it was shown that we needed two outer expansions, one for the upper branches of the solutions and one for the lower ones.

We assume that u and y have expansions of the form

$$u(\eta, \epsilon) = u_0(\eta) + \epsilon u_1(\eta) + \cdots, \tag{2.8a}$$

$$y(\eta, \epsilon) = y_0(\eta) + \epsilon y_1(\eta). \tag{2.8b}$$

Also, we put

$$\theta(\epsilon) = \theta_0 + \epsilon \theta_1 + \cdots. \tag{2.8c}$$

By assumption $\theta_0 > 0$. Actually, the expansions of u and y will be essentially different on the upper and lower branches. However, the equations which the terms of the expansions satisfy will be the same for both cases. To order unity we have

$$\theta_0(dy_0/d\eta) = u_0, \quad -f(u_0) = y_0 \tag{2.9a, b}$$

and hence

$$dy_0/d\eta = -f'(u_0)(du_0/d\eta) = u_0/\theta_0 . \qquad (2.9c)$$

To order ϵ we find

$$\theta_0(dy_1/d\eta) - u_1 = \theta_1(dy_0/d\eta), \qquad (2.10a)$$

$$f'(u_0)(du_1/dt) + y_1 = -\theta_0(du_0/d\eta). \qquad (2.10b)$$

The solution of (2.9c) is

$$\eta/\theta_0 = -\tfrac{3}{2}u_0^{\,2} + 2(1+a)u_0 - a\log|u_0| + C. \qquad (2.11a)$$

In addition, there is the solution

$$u_0 = 0. \qquad (2.11b)$$

Let $u_{\min}$ and $u_{\max}$ be the values of u for which $f(u)$ has a maximum and a minimum respectively. The slope $du_0/d\eta$ is infinite when u_0 assumes these values and u_0 becomes a multivalued function of η. Another source of multivaluedness if the occurrence of $\log|u_0|$ in (2.11a): a positive and a negative value of u_0 can give the same value of η. As a result there are four different branches of the solution depending on which region u_0 is in. The four regions are defined by

$$\text{Region I } : \quad u_0 \geq u_{\min} ,$$
$$\text{Region II} : \quad u_{\max} \leq u_0 \leq u_{\min} , \qquad (2.12)$$
$$\text{Region III} : \quad 0 \leq u_0 \leq u_{\max} ,$$
$$\text{Region IV} : \quad u_0 \leq 0.$$

The solution is plotted in Fig. 2.1. The curves shown may be translated an arbitrary amount along the η-axis.
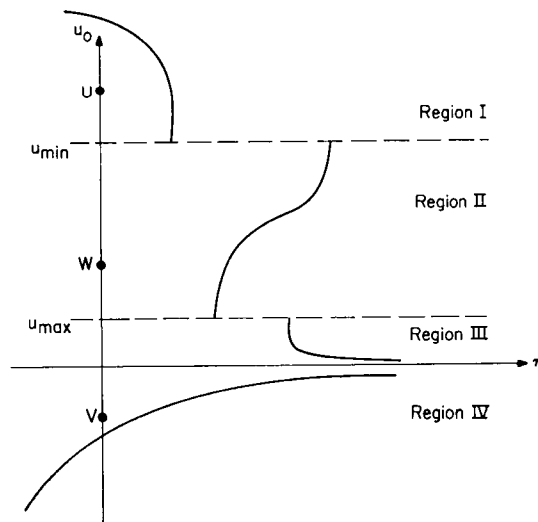

Fig. 2.1.   The four branches of u_0 .

The only branch which is defined for all values of η is in Region IV. However, on this branch u_0 tends to $-\infty$ as η tends to $-\infty$. Thus a solution u_0 which is bounded and which is defined for all η must jump discontinuously from one branch to another.

This confirms the assertion made in Sec. 1 that we are dealing with a singular perturbation problem. We regard u_0 as an outer solution and look for inner solutions which replace the discontinuities by rapid but smooth transitions. In the shock wave problem in fluid mechanics one can derive jump conditions from conservation laws, without studying the inner solution which corresponds to the discontinuity. In the present case, however, a detailed study of the inner solution is necessary for deriving jump conditions.

C. *Inner solution.* As will be studied in detail later, for $\theta_0 > 0$ there must be at least one upward jump followed by a downward jump (we determine its direction by considering increasing values of η). Without any loss of generality we may assume that there is an upward jump at $\eta = 0$. This is condition (2.4d). The form of the equations suggests that the correct inner variable is ξ (cf. (1.29) and (1.37))

$$\xi = \eta/\epsilon. \tag{2.13}$$

Using this coordinate one may write (2.1) as (1.30),

$$\frac{d^2v}{d\xi^2} - \theta \frac{dv}{d\xi} - f(v) - z = 0, \qquad \theta \frac{dz}{d\xi} = \epsilon v. \tag{2.14a, b}$$

Here $u(\eta; \epsilon) = v(\xi; \epsilon)$ and $y(\eta; \epsilon) = z(\xi; \epsilon)$. Assuming inner expansions near $\eta = 0$

$$v(\xi; \epsilon) = v_0(\xi) + \epsilon v_1(\xi) + \cdots, \tag{2.15a}$$

$$z(\xi; \epsilon) = z_0(\xi) + \epsilon z_1(\xi) + \cdots, \tag{2.15b}$$

we find

$$\frac{d^2v_0}{d\xi^2} - \theta_0 \frac{dv_0}{d\xi} - f(v_0) - K_0 = 0, \qquad \theta_0 \frac{dz_0}{d\xi} = 0. \tag{2.16a, b}$$

Since we assume $\theta_0 \neq 0$ we find that $z_0 = $ constant. The value of this constant is given by the initial condition (2.4b). Thus

$$z_0 = K_0 . \tag{2.17}$$

This equation has been used in writing down (2.16a).

To order ϵ we find

$$\frac{d^2v_1}{d\xi^2} - \theta_0 \frac{dv_1}{d\xi} - f'(v_0)v_1 = z_1 + \theta_1 \frac{dv_0}{d\xi}, \qquad \theta_0 \frac{dz_1}{d\xi} = v_0 . \tag{2.18a, b}$$

The second equation integrates to

$$z_1 = \frac{1}{\theta_0} \int_0^\xi v_0(s)\, ds + M + \frac{\sqrt{2}\, \ln 2}{\theta_0}. \tag{2.19}$$

According to standard theory of singular perturbations, the jump of an outer solution from $u_0 = A$ at $\eta = 0-$ to $u_0 = B$ at $\eta = 0+$ must be matched with a solution of (2.16) which tends to A as $\xi \to -\infty$ and to B as $\xi \to \infty$. This means that in the phase-plane of (2.16) the points where $dv_0/d\xi = 0$ and $v_0 = A$ and B respectively must be singular points.

Thus we must find solutions of (2.16) which originate at one singular point at $\xi = -\infty$ and reach another singular point at $\xi = \infty$. The singular points of (2.16) are obtained by solving the cubic equation

$$f(v_0) = -K_0 . \tag{2.20a}$$

We shall call the three roots U, V, W with the convention that

$$V < W < U. \tag{2.20b}$$

The value of $dv_0/d\xi$ at the singular points is zero. Obviously only real roots of (2.20a) are of interest. This places a first restriction on the range of K_0 . The position of the roots is illustrated in Fig. A.1 in the Appendix for the case $K_0 > 0$. If $K_0 \leq 0$, then W and U are still in Regions II and I respectively whereas V is in Region III or, for $K_0 = 0$, equal to zero.

As stated above, in addition to the upjump at $\eta = 0$ the outer solution must have at least one more discontinuity. For instance, assume that following the upjump at $\eta = 0$ there is a downjump at $\eta = \eta_d$. The inner variable ρ should now be defined by

$$\epsilon(\rho - \rho_d) = \eta - \eta_d . \tag{2.21}$$

(the choice of the constant ρ_d will be discussed later). Near $\eta = \eta_d$ the solution has the inner expansion

$$v \simeq \tilde{v}_0(\rho) + \epsilon \tilde{v}_1(\rho) + \cdots . \tag{2.22}$$

The value of z may have changed between $\eta = 0$ and $\eta = \eta_d$. Thus at η_d we have an expansion

$$z(\eta_d) = \tilde{K}_0 + \epsilon \tilde{K}_1 + O(\epsilon^2). \tag{2.23}$$

The values of $\tilde{K}_0$, $\tilde{K}_1$ may no longer be chosen arbitrary[1] but must be determined from the initial conditions at $\eta = 0$. It is obvious that $\tilde{v}_0$ will obey (2.16), with ξ replaced by ρ and K_0 by $\tilde{K}_0$. Thus the discussion below will apply to the upward jump at $\eta = 0$ as well as to any upward or downward jump at any other discontinuity in u_0 .

D. *Jump conditions.* The possible inner solutions are studied in the Appendix with the aid of standard phase-plane methods. It is found that for $\theta_0 > 0$ there are at most four possibilities of a solution going from one singular point to a different singular point as ξ increases, namely

$$V \rightarrow U, \ U \rightarrow V, \ W \rightarrow V, \ W \rightarrow U. \tag{2.24a, b, c, d}$$

The above restrictions were found from a study of the inner solution alone. If we now combine this with a study of the outer solution we can impose further restrictions. We shall prove:

The values of an outer solution cannot lie in Region II. (2.25a)

An easy corollary is

The jumps (2.24c, d) cannot occur. (2.25b)

---

[1] Thus, even if we put $M = 0$ in (2.4), the corresponding $\tilde{M}$ at $\eta = \eta_d$ may not be zero. In order to copy the calculations of the inner solution at $\eta = \eta_d$ from those at $\eta = 0$ it is convenient to have a general value of M at $\eta = 0$.

Proof. If $u_0(\eta_1)$ is in Region II for some η_1, then as η decreases it eventually has to jump out of Region II. This means that as η (and ξ) increases there is a jump *into* Region II. Since neither U nor V are in this region this is excluded according to (2.24). Furthermore a jump *from* W would mean that u_0 is in Region II just before the jump. Hence (2.24c) and (2.24d) are excluded.

From the Appendix we find that an *upward* jump from V to U fixes the value of θ_0 to be

$$\theta_0 = \sqrt{2}(U + V - 2W) \equiv G(U). \tag{2.26}$$

Thus, to lowest order, the wave velocity θ is determined from the given value K_0. Since the wave velocity must be the same for the entire η-range, any other jump must correspond to the same θ_0. In the Appendix it is proved that any other upward jump corresponding to θ_0 of (2.26) must have the same values of K_0, and hence of U and V, as before. After having reached U, u_0 is Region I and we know (cf. Fig. 2.1) that as η increases u_0 cannot reach V (which is in Region III or IV) continuously and also that a continuous one-valued solution exists only for a finite range of η. Since there is no upward jump possible from Region I there must be a downward jump. From the Appendix we know that there is exactly one downward jump $\tilde{U} \to \tilde{V}$ which gives the same θ_0 as (2.26). It corresponds to a value $\tilde{K}_0$ which has the property

$$\tilde{U} \leq U; \quad \tilde{V} \leq V; \quad \tilde{K}_0 \geq K_0, \tag{2.27a}$$

$$-K_0 - f_{\inf} = f_{\inf} + \tilde{K}_0, \quad \tilde{K}_0 \leq -f_{\inf}, \tag{2.27b}$$

where

$$f_{\inf} = \text{value of } f \text{ at its inflection point} = \tfrac{1}{27}(1 + a)(1 - 2a)(a - 2). \tag{2.27c}$$

E. *Further restrictions on parameters.* The study of the inner solution showed (see Appendix) that for an upjump to take place K_0 must be restricted to the range $f_{\inf} \leq -K_0 < f_{\max}$. By a study of the outer solution we shall now show

$$V \text{ and } \tilde{V} \text{ are in Region IV, i.e. } \tilde{V} < V \leq 0, \tag{2.28a}$$

$$u_0 \text{ takes values only in Regions I and IV}, \tag{2.28b}$$

$$K_0 \text{ is restricted to } 0 \leq K_0 \leq -f_{\inf}, \tag{2.28c}$$

$$0 < a < \tfrac{1}{2}. \tag{2.28d}$$

Proof. If V is not in Region IV then it must be inside Region III, in particular $V > 0$. Assume this, and follow the solution u_0 backwards, i.e. in the direction of decreasing η. From Fig. 2.1 we see that we can stay in Region III only for a limited range of η. To get out the solution has to jump. The only jump possible is from $\tilde{V}$ to $\tilde{U}$ (going backwards). However, as η decreases u_0 increases in Region III. On the other hand, $\tilde{V} < V$. This is a contradiction. Hence (2.28a) is proved.

If u_0 is in Region III it has to enter by a jump. Since neither U nor V are in Region III, this is impossible. We have already shown that u_0 cannot be in Region II. This proves (2.28b).

Finally, if $K_0 < 0$ then V is inside Region III which has been shown to be impossible. This proves (2.28c).

Originally (in (2.2c)) we restricted a to the region $0 < a < 1$. If $\tfrac{1}{2} < a < 1$ then

$f_{\text{inf}} > 0$, which is shown to be impossible by (2.28c). In the limiting case $a = \frac{1}{2}$ we have $K_0 = \tilde{K}_0 = 0$. As will be shown below, this is a degenerate case which does not correspond to a meaningful solution.

F. *Periodic solutions.* Using the results obtained above and in the Appendix, we may describe periodic solutions as follows. At $\eta = 0$, u_0 jumps from V to U, so that $u_0(0-) = V$, $u_0(0+) = U$. We may, for instance, pick a value of V in the allowable range. Then we determine K_0 by $K_0 = -f(V)$, and U and W are determined by (2.20). The wave velocity θ_0 is

$$\sqrt{2}\,\theta_0 = U + V - 2W = 3(v_{\text{inf}} - W). \tag{2.29a}$$

Here, v_{inf} is the value of v at which $f(v)$ has a point of inflection:

$$v_{\text{inf}} = (1 + a)/3. \tag{2.29b}$$

For K_0 approaching one of the end points in its allowed range we get the following limiting cases:

Limiting Case I:

$$V = v_{\text{inf}} - A^{1/2} < 0, \quad W = v_{\text{inf}}, \quad U = v_{\text{inf}} + A^{1/2} < 1, \tag{2.30a}$$

$$K_0 = -f_{\text{inf}}, \quad \theta_0 = 0, \tag{2.30b}$$

where

$$v_{\text{inf}} = (1 + a)/3, \quad 3A = 1 + a + a^2. \tag{2.30c}$$

Limiting Case II:

$$V = 0, \quad W = a < v_{\text{inf}}, \quad U = 1, \tag{2.31a}$$

$$\sqrt{2}\,\theta_0 = 1 - 2a, \quad K_0 = 0. \tag{2.31b}$$

As K_0 varies in the allowed range U, V, W, θ_0 vary monotonically between the extreme values given above.

After having attained the value U at $\eta = 0+$ the function u_0 is in Region I and decreases with increasing η according to the formula (cf. 2.11)

$$\eta/\theta_0 = (-3u_0{}^2/2) + 2(1 + a)u_0 - a \log u_0 + C, \tag{2.32a}$$

where

$$C = (3U^2/2) - 2(1 + a)U + a \log U. \tag{2.32b}$$

The solution of (2.32) which gives $u_0 > u_{\text{min}}$ (Region I) should be chosen. The value of y_0 increases according to

$$y_0 = K_0 + \frac{1}{\theta_0} \int_0^\eta u_0 \, d\eta \tag{2.33a}$$

or, more simply, if we treat y_0 as a function of u_0, (2.9b) gives

$$v_0 = -f(u_0) \tag{2.33b}$$

Let $\tilde{U}, \tilde{K}_0$, etc., be defined from U and K_0 as in the Appendix. Define η_d as the value of η at which u_0 has decreased to $\tilde{U}$:

$$u_0(\eta_d) = \tilde{U}. \tag{2.34}$$

From (2.33b) we see that

$$y_0(\eta_d) = -f(\tilde{U}) = \tilde{K}_0 . \tag{2.35}$$

The conditions for a downjump from $\tilde{U}$ to $\tilde{V}$ at $\eta = \eta_d$, consistent with keeping the same value of θ_0 , are thus fulfilled (cf. Eq. (A.24) in the Appendix). The values of $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ are found from U, V, W according to (A.23) and that of $\tilde{K}_0$ from (2.27b).

In the limiting cases we find

Limiting Case I:

$$\eta_d = 0, \quad U = \tilde{U}, \text{ etc.} \tag{2.36}$$

This is obviously a degenerate case.

Limiting Case II:

$$\tilde{K}_0 = -2f_{\text{inf}} , \tag{2.37a}$$

$$\tilde{V} = 2v_{\text{inf}} - 1 = \frac{2a - 1}{3} < 0, \quad \tilde{W} = 2v_{\text{inf}} - a = \frac{2 - a}{3} , \quad \tilde{U} = 2v_{\text{inf}} = \frac{2 + 2a}{3} < 1, \tag{2.37b}$$

$$C = -(\tfrac{1}{2} + 2a). \tag{2.37c}$$

Since $u_0(\eta_d+) = \tilde{V}$ it follows that u_0 is in Region IV after the downjump at η_d . The analytic formula is

$$\frac{\eta}{\theta_0} = (-3u_0{}^2/2) + 2(1 + a)u_0 - a \log |u_0| + \tilde{C}. \tag{2.38}$$

The value of $\tilde{C}$ is obtained by inserting the values η_d and $\tilde{V}$ for η and u_0 respectively. The solution of (2.38) which gives $u_0 < 0$ (Region IV) should be chosen. It is convenient to shift the origin of the outer variable by introducing κ as defined by (1.46) and to put $\tilde{u}_0(\kappa) = u_0(\eta)$.

As κ increases $\tilde{u}_0$ increases and finally reaches the value V at the point $\kappa = \kappa_u$:

$$\tilde{u}_0(\kappa_u) = V. \tag{2.39}$$

At κ_u the solution jumps to the value U and then continues periodically. Thus, to lowest order,

$$\text{Period} = \eta_d + \kappa_u . \tag{2.40}$$

As for the limiting cases, we see that Case I is meaningless. More generally, as K_0 increases to $-f_{\text{inf}}$ the period eventually becomes of order ϵ and the perturbation method does not make sense.

For the other extreme we find

$$\text{Limiting Case II is the solitary wave.} \tag{2.41}$$

As the parameter K_0 decreases towards zero the value of V increases towards zero. Then κ_u , and hence also the period, increase towards infinity. In this limiting case we have the solitary wave for which

$$u_0 = 0 = V \quad \text{for} \quad \eta < 0, \tag{2.42a}$$

$$u_0(0+) = U = 1, \tag{2.42b}$$

$$u_0 \text{ decreases from 1 to } \tilde{U} \text{ for } 0 < \eta < \eta_d , \tag{2.42c}$$

$$u_0(\eta_d+) = \tilde{V}, \tag{2.42d}$$

$$u_0 \text{ increases from } \tilde{V} \text{ to } 0, \text{ for } \eta_d < \eta < \infty . \tag{2.42e}$$

The values of $\tilde{U}$ and $\tilde{V}$ are given by (2.37).

The *inner solutions* corresponding to the outer periodic solutions discussed above are as follows. The inner solution for the upjump from V to U at $\eta = 0$ is given by (A.17) and the downjump from $\tilde{U}$ to $\tilde{V}$ at $\eta = \eta_d$ is given by (A.19) with U and V replaced by $\tilde{U}$ and $\tilde{V}$, respectively. At $\kappa = \kappa_u$ we have (A.17) again, etc., as demanded by periodicity.

Solution near $\eta = 0$ reconsidered. We assume that the outer solution jumps from V to U. For ξ large, according to (A.17),

$$v_0 = U + O(\exp{(-h\xi)}). \tag{2.43}$$

Matching requires that for η small the value of u_0 be $U + o(1)$. Thus if u_0 denotes the outer solution valid for $0 < \eta < \eta_d$ then u_0 is given by (2.11a) with the constant of integration C chosen so that

$$u_0 = U \quad \text{for} \quad \eta = 0. \tag{2.44}$$

From (2.9b) it follows that, at $\eta = 0$,

$$y_0 = -f(U) = K_0 . \tag{2.45}$$

Thus $v_0(\xi)$ and $z_0(\xi)$ satisfy the correct initial conditions (2.4) to order unity. The outer solution $u_0(\eta)$, being discontinuous at $\eta = 0$, does not satisfy the correct initial condition. On the other hand, (2.45) shows that $y_0(\eta)$, being essentially an integral of $u_0(\eta)$ and hence continuous at $\eta = 0$, does satisfy the correct condition.

3. Approximations valid to order ϵ

A. *Inner solution for upjump at $\eta = 0$. Determination of θ_1.* The initial conditions at $\xi = \eta = 0$ are given by (2.4). Assuming the inner expansions $v = v_0(\xi) + \epsilon v_1(\xi) + O(\epsilon^2)$ and $z = z_0(\xi) + \epsilon z_1(\xi) + O(\epsilon^2)$, we find that v_0 is given by (A.17), $z_0 \equiv K_0$, and z_1 is obtained from v_0 and (2.19). Summarizing and using the expression for v_0, we have

$$v_0 = \frac{U \exp{(h\xi)} + V}{1 + \exp{(h\xi)}} , \qquad \sqrt{2}\, h = U - V, \qquad z_0 = K_0 , \tag{3.1}$$

$$\begin{aligned} \theta_0 z_1 &= \sqrt{2} \ln{(1 + \exp{(h\xi)})} + V\xi + \theta_0 M \\ &= \sqrt{2} \ln{(1 + \exp{(-h\xi)})} + U\xi + \theta_0 M, \end{aligned} \tag{3.2}$$

$$\frac{d^2 v_1}{d\xi^2} - \theta_0 \frac{dv_1}{d\xi} - f'(v_0)v_1 = z_1 + \theta_1 \frac{dv_0}{d\xi}. \tag{3.3}$$

For (3.3) we have the initial condition

$$v_1(0) = 0. \tag{3.4}$$

The parameter θ_1 is an eigenvalue. Since $dv_0/d\xi$ is an eigensolution of the homogeneous equation corresponding to (3.3), the right-hand side of (3.3) is subject to an orthogonality condition. It is this condition which determines θ_1. By standard methods we find

$$\theta_1 \int_{-\infty}^{\infty} \exp\left(-\theta_0 s\right)\left(\frac{dv_0}{ds}\right)^2 ds + \int_{-\infty}^{\infty} \exp\left(-\theta_0 s\right)\frac{dv_0}{ds} z_1(s)\, ds = 0. \qquad (3.5)$$

In Sec. 2 we saw that if one is interested only in calculating θ_0 one need study only the upjump at $\xi = 0$. Similarly we see from (3.4) that θ_1 is obtained directly, without even solving for v_1. However, here we are also interested in the details of the solution and shall proceed to find v_1.

Explicit expression for v_1. In addition to the conditions (3.5) we shall impose conditions that $|v_1|$ can increase at most linearly at $\xi = \pm\infty$. This is necessary for matching with the outer solution. Thus we impose three conditions, at 0 and at $\pm\infty$, on the solution. It will, however, be seen that the results are self-consistent exactly if θ_1 is determined by (3.5). Thus we have an independent check on (3.5).

Using the fact that $dv_0/d\xi$ is an eigensolution, we put

$$v_1(\xi) = g(\xi)(dv_0/d\xi) \qquad (3.6)$$

and obtain, using "prime" to denote derivatives with respect to ξ,

$$v_0'g'' + (2v_0'' - \theta_0 v_0')g' = z_1 + \theta_1 v_0'. \qquad (3.7)$$

Integrating twice, using (3.4), we obtain

$$v_1(\xi) = v_0'(\xi)\int_0^{\xi} \frac{\exp\left(\theta_0 s\right)g(s)\, ds}{[v_0'(s)]^2} \qquad (3.8a)$$

with

$$g(s) = \int_0^s \exp\left(-\theta_0 r\right)v_0'(r)[z_1 + \theta_1 v_0'(r)]\, dr + C, \qquad (3.8b)$$

$C = $ constant of integration.

At $\pm\infty$ z_1 grows linearly. Obviously then $g(\infty)$ is finite. Since $\sqrt{2}(h - \theta_0) = \sqrt{2}(V - W) > 0$, we also find that $g(-\infty)$ is finite. Actually we see from (3.8) that to avoid exponential growth of v_1 at $\xi = \pm\infty$ we must have

$$g(-\infty) = 0, \quad g(\infty) = 0. \qquad (3.9a, b)$$

These equations represent two conditions on the constant of integration C. Actually, they are consistent, since if one condition is fulfilled the other one follows from (3.5) which simply states that $g(-\infty) = g(\infty)$. Thus we have verified (3.5) by a method independent of orthogonality arguments.

Behavior for $|\xi|$ large. We have now found an explicit expression for v_1. For matching we need to find asymptotic expressions of v_1 for $|\xi|$ large. These may be found from (3.8) but we may also proceed as follows. As ξ tends to infinity v_1 tends asymptotically to a function v_+ which obeys the equation

$$v_+'' - \theta_0 v_+' - f'(U)v_+ = (U/\theta_0)\xi + M. \qquad (3.10)$$

A particular solution is

$$v_+ = A_+\xi + B_+, \qquad (3.11a)$$

$$A_+ = -U/\theta_0 f'(U), \qquad (3.11b)$$

$$B_+ = U[f'(U)]^{-2} - M[f'(U)]^{-1}. \qquad (3.11c)$$

The general solution of the corresponding homogeneous equation is the sum of a term which increases exponentially as $\xi \to \infty$ and a term which decreases exponentially. (Note that $f'(U)$ is positive.) The coefficient of the first term must be zero since it could not be matched with an outer solution. The second term is irrelevant for matching. Hence (3.11) actually gives the asymptotic behavior of v_1 for ξ large, i.e.

$$v_1 = v_+ + \text{exponentially small terms.} \tag{3.12}$$

Similarly, we find as $\xi \to -\infty$

$$V_1 = v_- + \text{exponentially small terms,} \tag{3.13}$$

$$v_- = A_-\xi + B_- , \tag{3.14a}$$

$$A_- = V/\theta_0 f'(V), \tag{3.14b}$$

$$B_- = V[f'(V)]^{-2} - M[f'(V)]^{-1}. \tag{3.14c}$$

B. *Outer solution from $\eta = 0$ to $\eta = \eta_d$.* The outer equations to order ϵ were given by (2.10). From (2.9c) we find

$$d/du_0 = -(\theta_0 f'(u_0)/u_0)(d/d\eta). \tag{a}$$

Introducing u_0 as an independent variable in (2.10) gives

$$\frac{dy_1}{du_0} + \frac{\theta_1}{\theta_0}\frac{dy_0}{du_0} = -\frac{u_1 f'(u_0)}{u_0} = \frac{y_1}{u_0} - \frac{1}{f'(u_0)} \tag{b}$$

or

$$\frac{d}{du_0}\left(\frac{y_1}{u_0}\right) = -\frac{1}{u_0 f'(u_0)} + \frac{\theta_1}{\theta_0}\frac{f'(u_0)}{u_0}. \tag{3.15}$$

Anticipating the determination of the constant of integration, we find the solution to be

$$\frac{y_1}{u_0} = \frac{M}{U} + k(u_0) - k(U) \tag{3.16}$$

with

$$k(r) = -\frac{1}{a}\ln r - \frac{\ln (r - r_1)}{3r_1(r_1 - r_2)} + \frac{\ln (r - r_2)}{3r_2(r_1 - r_2)} + \frac{\theta_1}{\theta_0}\left[\frac{3}{2}r^2 - 2(1 + a)r + a \ln r\right].$$

Here r_1 and r_2 are the roots of $f'(r) = 0$, i.e.

$$r_{1,2} = \frac{1 + a \pm(1 + a^2 - a)^{1/2}}{3}.$$

From (2.10a) we find that the value of u_1 is

$$u_1 = \frac{u_0}{f'(u_0)}\left[\frac{1}{f'(u_0)} - \frac{y_1}{u_0}\right] = \frac{u_0}{f'(u_0)}\left[\frac{1}{f'(u_0)} - \frac{M}{U} - k(u_0) + k(U)\right]. \tag{3.17}$$

Matching. The choice of the constant of integration in (3.17) is determined by matching. For η small the outer expansion is, according to (2.9c) and (3.16),

$$y \simeq K_0 + (U\eta/\theta_0) + O(\eta^2) + \epsilon[M + O(\eta)] + O(\epsilon^2). \tag{3.18a}$$

For ξ large the inner expansion is, according to (3.1) and (3.2),

$$y \simeq K_0 + \epsilon[M + (U\xi/\theta_0) + O(\exp -h\xi)] + O(\epsilon^2). \tag{3.18b}$$

Thus the matching condition is fulfilled.

Note that v_1 did not play a role in the matching since it contributes only to the term of order ϵ^2. On the other hand, we see that the constant of integration in (3.16) occurs also in the expression for u_1 given by (3.17). If we match the inner and outer expansions of u rather than of y then the formula for v_1 must be used. The result is of course the same, as is easily checked.

C. *Inner solution for downjump at* $\eta = \eta_d$. *Determination of* $\tilde{v}_1$ *and* $\tilde{z}_1$. We assume that near $\eta = \eta_d$ the inner expansions are

$$v = \tilde{v}_0(\rho) + \epsilon\tilde{v}_1(\rho) + O(\epsilon^2), \tag{3.19a}$$

$$z = \tilde{z}_0(\rho) + \epsilon\tilde{z}_1(\rho) + O(\epsilon^2) \tag{3.19b}$$

with ρ determined by

$$\epsilon(\rho - \rho_d) = \eta - \eta_d. \tag{3.20}$$

The constant ρ_d will be determined by matching. According to (A.19),

$$\tilde{v}_0(\rho) = \frac{\tilde{U} \exp(-\tilde{h}\rho) + \tilde{V}}{1 + \exp(-\tilde{h}\rho)}. \tag{3.21a}$$

$$\sqrt{2}\,\tilde{h} = \tilde{U} - \tilde{V}. \tag{3.21b}$$

The function $\tilde{z}_0$ is constant $\tilde{z}_0 = \tilde{K}_0$.

Corresponding to (3.2) we find by integrating (2.18b)

$$\theta_0\tilde{z}_1 = -\sqrt{2} \ln(1 + \exp(\tilde{h}\rho)) + \tilde{U}\rho + \theta_0\tilde{M}. \tag{3.22}$$

The minus sign in the first term is due to the fact that we are dealing with a downjump. Note that now

$$\tilde{z}_1(0) = \tilde{M} - \sqrt{2} \ln 2/\theta_0. \tag{3.23}$$

At the upjump K_0 and M were given and used to determine θ_0 and θ_1. At the downjump these latter constants are now given and determine $\tilde{K}_0$ and $\tilde{M}$. Furthermore, at $\xi = 0$ we could arbitrarily specify the initial value. Now we have at $\rho = 0$

$$2\tilde{v}_0(0) = \tilde{U} + \tilde{V}. \tag{3.24}$$

However, we cannot prescribe a new initial value. Thus we have to adjust the value of the parameter ρ_d.

The function $\tilde{v}_1(\rho)$ is determined in the same way as $v_1(\xi)$. Corresponding to (3.8) we obtain

$$\tilde{v}_1(\rho) = \tilde{v}_0'(\rho) \int_0^\rho \frac{\exp(\theta_0 s)\tilde{g}(s)\,ds}{[\tilde{v}_0'(s)]^2}, \tag{3.25a}$$

$$\tilde{g}(s) = \int_0^s \exp(-\theta_0 r)\tilde{v}_0'(r)[\tilde{z}_1(r) + \theta_1\tilde{v}_0'(r)]\,dr + \tilde{C}. \tag{3.25b}$$

The constant of integration $\tilde{C}$ is determined by relations which correspond to (3.9),

$$\tilde{g}(-\infty) = 0, \quad \tilde{g}(\infty) = 0. \tag{3.26}$$

Determination of $\tilde{M}$. For the upjump M was given and θ_1 determined by (3.5). The corresponding relation now determines $\tilde{M}$ from θ_1 ,

$$\theta_0 \tilde{M} \int_{-\infty}^{\infty} \exp\,(-\theta_0 s)\tilde{v}_0{}'(s)\,ds$$

$$+ \int_{-\infty}^{\infty} \exp\,(-\theta_0 s)\tilde{v}_0'\,(s)[\theta_0 \theta_1 \tilde{v}_0'\,(s) + \tilde{U}s - \sqrt{2}\,\ln\,(1 + \exp\,(\breve{h}s))]\,ds = 0 \qquad (3.27)$$

Determination of ρ_d . By varying ρ_d we translate the inner solution at $\eta = \eta_d$ along the η-axis. Only in a special position will the inner solution match with the outer solution. Using formulas analogous to (3.18), we find

$$\rho_d = \theta_0\left[\frac{M}{U} - \frac{\tilde{M}}{\tilde{U}} - k(U) + k(\tilde{U})\right], \qquad (3.28)$$

where the function k is given by (3.16).

D. *Outer solution from downjump to upjump.* We now study the outer solution from the point where $u_0 = \tilde{V}$ to the point where u_0 has regained the value V and a new upjump to the value U takes place.

Any variable κ which differs from η by a constant may be used as an outer variable. It is convenient to let κ bear the same relation to ρ as η does to ξ. Thus we define

$$\kappa = \epsilon\rho \equiv \eta - \eta_d + \epsilon\rho_d \,. \qquad (3.29)$$

We assume outer expansions of the form

$$u \simeq \tilde{u}_0(\kappa) + \epsilon\tilde{u}_1(\kappa) + \cdots, y \simeq \tilde{y}_0(\kappa) + \epsilon\tilde{y}_1(\kappa). \qquad (3.30\text{a, b})$$

The computations proceed as in Sec. 3B and we find

$$-\frac{\kappa}{\theta_0} = a\,\ln\frac{\tilde{u}_0}{\tilde{V}} - 2(1 + a)(\tilde{u}_0 - \tilde{V}) + \tfrac{3}{2}(\tilde{u}_0{}^2 - \tilde{V}^2), \qquad (3.31\text{a})$$

$$\tilde{u}_1(\tilde{u}_0) = \frac{\tilde{u}_0}{f'(\tilde{u}_0)}\left[\frac{1}{f'(\tilde{u}_0)} - \frac{\tilde{M}}{\tilde{V}} - k(\tilde{u}_0) + k(\tilde{V})\right], \qquad (3.31\text{b})$$

$$\tilde{y}_0 = -f(\tilde{u}_0), \qquad (3.32\text{a})$$

$$\tilde{y}_1 = \tilde{u}_0[(\tilde{M}/\tilde{V}) + k(\tilde{u}_0) - k(\tilde{V})], \qquad (3.32\text{b})$$

In (3.31a) the branch of the solution for which $\tilde{u}_0$ is negative must be chosen.

E. *Inner solution for upjump between V and U.* Let κ_u be the value such that

$$\tilde{u}_0(\kappa_u) = V. \qquad (3.33)$$

At this point the outer solution has a discontinuous upjump to the value U. We define an inner variable σ by

$$\epsilon(\sigma - \sigma_u) = \kappa - \kappa_u \qquad (3.34)$$

To express periodicity in a convenient form we impose a special condition on σ_u . We require that, to first order, the inner solution around $\kappa = \kappa_u$ be

$$\text{Inner solution} = v_0(\sigma), \qquad (3.35)$$

where v_0 is exactly the same function which was used to express the inner solution $v_0(\xi)$ around $\eta = 0$.

Using the same methods of matching we find

$$\sigma_u = \theta_0 \left[-\frac{M}{V} + k(V) + \frac{\tilde{M}}{\tilde{V}} - k(\tilde{V}) \right]. \tag{3.36}$$

Period.   From the definitions of the variables we find

$$\epsilon\xi = \epsilon\sigma + T_0 + \epsilon T_1 \tag{3.37a}$$

where

$$T_0 = \eta_d + \kappa_u , \quad T_1 = -(\rho_d + \sigma_u). \tag{3.37b, c}$$

From the construction we conclude

The period of the motion, measured in the η-scale, is $T_0 + \epsilon T_1 + O(\epsilon^2)$ (3.38)

The constants in (3.38) may be evaluated from the following formulas: η_d is given by (2.34). The function u_0 is given by (2.11a) and lies in Region IV of Fig. 2.1. The constants $\tilde{K}$, $\tilde{U}$ and $\tilde{V}$ are given by (A.23). ρ_d and σ_u are determined from (3.28), (3.26), (3.16) and (3.27). κ_u is found from (2.38) by replacing u_0 by V and replacing η by $\kappa_u + \eta_d$.

4. Progressing waves with low velocity.   We shall study a pulse traveling with a velocity which is $o(1)$. By checking various possibilities one finds that only the following assumptions about the form of the expansions yield reasonable results.

Inner expansions.   We retain ξ as an inner variable and assume

$$z = \sqrt{\epsilon}\, z_1(\xi) + \cdots , \tag{4.1a}$$

$$v = v_0(\xi) + \sqrt{\epsilon}\, v_1(\xi) + \cdots , \tag{4.1b}$$

$$\theta = \sqrt{\epsilon}\, \theta_1 + \cdots . \tag{4.1c}$$

The equation for v_0 is thus

$$d^2 v_0/d\xi_0 - f(v_0) = 0. \tag{4.2}$$

To next order we have

$$\theta_1(dz_1/d\xi) = v_0 , \tag{4.3a}$$

$$\frac{d^2 v_1}{d\xi^2} - f'(v_0)v_1 = \theta_1 \frac{dv_0}{d\xi} + z_1 \tag{4.3b}$$

Outer expansions.   As outer variables we no longer use η, as in the case of $\theta = O(1)$, but ζ defined by

$$\zeta = \sqrt{\epsilon}\xi. \tag{4.4}$$

We assume the expansions

$$z = \sqrt{\epsilon}y_1(\zeta) + \cdots , \quad v = \sqrt{\epsilon}u_1(\zeta) + \cdots . \tag{4.5a, b}$$

The equations are then

$$au_1 + y_1 = 0, \quad \theta_1(dy_1/d\zeta) = u_1 \tag{4.6a, b}$$

Inner solutions. The solution for v_0 is given by (A.31). To find v_1 we proceed as in Sec. 3A. The homogeneous equation corresponding to (4.3b) has $dv_0/d\xi$ as a solution. This has two consequences. 1) We may use the method of variation of parameters. This, followed by use of an integrating factor, reduces the solution to a problem of quadrature. 2) An orthogonality condition must be imposed on the right-hand side of (4.3b). This, together with integration of (4.3a), gives a formula for θ_1 which also may be verified from matching conditions.

However, before carrying out this program we list some integrals which will be needed later.

Evaluation of Integrals. The following method, illustrated by the first example below, will be helpful in evaluating various integrals.

$$\sqrt{2} \int_{-\infty}^{\xi} (v_0')^2 \, d\xi = \int_0^{v_0} v_0 g(v_0) \, dv_0$$

$$= \frac{g^3(v_0)}{3} + \frac{b}{2} \left[(v_0 - b)g(v_0) - (b^2 - 2a) \ln |g(v_0) + v_0 - b| \right]$$

$$- \frac{(2a)^{3/2}}{3} - \frac{b}{2} \left[-b\sqrt{2a} - (b^2 - 2a) \ln (b - \sqrt{2a}) \right] \tag{4.7}$$

Here g is a polynomial and b is a constant, both defined by (A.28b). For $\xi \leq 0$ the positive square root should be used in evaluating g. We note that at $\xi = 0$, v_0 is v_m as given by (A.29) and $g(v_0) = 0$. Standard methods gives formulas for $\xi > 0$.

Similarly

$$\int_{-\infty}^{\xi} v_0^2 \, d\xi = \sqrt{2} \left[g(v_0) + b \ln |g(v_0) + v_0 - b| \right] - \sqrt{2} \left[\sqrt{2a} + b \ln (b - \sqrt{2a}) \right] \tag{4.8}$$

$$\theta_1 z_1 = \int_{-\infty}^{z} v_0 \, d\xi = \sqrt{2} \ln |g(v_0) + v_0 - b| - \sqrt{2} \ln (b - \sqrt{2a}) \tag{4.9}$$

Inner solutions (continued). The orthogonality condition discussed above gives us

$$\theta_1^2 = \frac{\displaystyle\int_{-\infty}^{\infty} v_0^2 \, d\xi}{\displaystyle\int_{-\infty}^{\infty} (v_0')^2 \, d\xi} = \frac{2\sqrt{2a} - 2b \ln \alpha}{\dfrac{(2a)^{3/2}}{3} - \dfrac{b^2}{2}\sqrt{2a} + \dfrac{b(b^2 - 2a)}{2} \ln \alpha}, \qquad \alpha = \left(\frac{b + \sqrt{2a}}{b - \sqrt{2a}} \right)^{1/2}.$$

$$\tag{4.10}$$

The positive value of the square root should be used for θ_1.

The formula for z_1 has already been given by (4.9). In this formula we have assumed $z_1(-\infty) = 0$. Since v_0 is positive, we must then have $z_1(\infty) > 0$. This will be discussed further in connection with matching.

The computation of v_1 proceeds as in Sec. 3.A. We obtain

$$v_1(\xi) = v_0'(\xi) \int_0^{\xi} \left[\frac{F(s)}{(v_0'(s))^2} - \frac{F(0)}{f^2(v_m)s^2} \right] ds - \frac{F(0)}{f^2(v_m)} \frac{v_0'(\xi)}{\xi} \tag{4.11}$$

where

$$F(s) = \theta_1 \int_{-\infty}^{s} [v_0'(r)]^2 \, dr - \frac{1}{\theta_1} \int_{-\infty}^{s} [v_0(r)]^2 \, dr + z_1(s)v_0(s)$$

and v_m is the maximal value of v_0 , as defined by (A.29). The constants of integration have been chosen so that matching is possible. In particular, we have $F(\infty) = 0$, which is equivalent to (4.10). One also finds

$$v_1(-\infty) = 0, \qquad v_1(+\infty) = -\frac{2\sqrt{2}}{a\theta_1} \ln \alpha, \qquad (4.12\text{a, b})$$

$$v_1(0) = -\frac{F(0)}{f(v_m)}, \qquad F(0) = z_1(0)v_m . \qquad (4.12\text{c})$$

To the expression for v_1 as given by (4.11) we may add a $c_2 v_0'(\xi)$, $c_2 =$ arbitrary constant. This, however, is obtained from the solution for v_0 if we replace ξ by $\xi + \epsilon^{1/2}c_2$. Actually, there is a class of solutions for $v(\xi)$, all obtainable from any one of them by a translation of the ξ-axis. For convenience of calculation we have chosen the origin of the ξ-axis such that $v_0(0) = v_m$ and $v_1(0)$ has the value given by (4.12c).

Outer solutions. The function v_0 is everywhere positive. Its integral from $\xi = -\infty$ to $\xi = \infty$ is of order unity. Thus, if $z(-\infty) = 0$, then, according to (2.14b), $z(\infty) = O(\sqrt{\epsilon})$ if we base our computations on v_0 . Since, for the pulse we must have $z(\infty) = 0$ we need additional terms to correct the error. Adding $\sqrt{\epsilon}v_1$ is not sufficient to solve the problem since $v_1(\infty) \neq 0$. We therefore introduce an outer solution for ξ large which matches with the inner solution for moderate values of ξ and which behaves correctly at $\xi = +\infty$. The equations are (4.6) which have the solutions

$$y_1 = -\beta a \exp(-\zeta/a\theta_1), \qquad (4.13\text{a})$$

$$u_1 = \beta \exp(-\zeta/a\theta_1). \qquad (4.13\text{b})$$

We shall show that matching requires the constant of integration β to be

$$\beta = -(2\sqrt{2}/a\theta_1) \ln \alpha. \qquad (4.14)$$

As ξ tend to infinity $\sqrt{\epsilon}z_1$ tends to $\sqrt{\epsilon}/\theta_1 \int_{-\infty}^{\infty} v_0 \, d\xi = (\sqrt{\epsilon} \, 2\sqrt{2}/\theta_1) \ln \alpha$. As ζ tends to zero $\sqrt{\epsilon}y_1$ tends to the same value if (4.14) is assumed. Thus the choice of β is correct.

Appendix. *Phase-plane analysis of the inner equation.*[2] The equation for the leading term of an inner solution was given in Sec. 2 to be (2.16a). Omitting the subscript zero we write this equation as

$$v'' - \theta v' - f(v) - K = 0 \qquad (\text{A.1a})$$

where

$$f(v) = v(v - a)(v - 1). \qquad (\text{A.1b})$$

Here "prime" denotes differentiation with respect to ξ. The analysis here is of course valid if ξ is replaced by any of the other inner variables used in Sec. 3.

Assume that at some value of η the outer solution changes discontinuously from v_- to v_+ . Matching then requires that the corresponding inner solution approach these values at $-\infty$ and $+\infty$ respectively. This means that in the phase-plane of (A.1) the

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[2] A phase-plane analysis of (A.1) was given in [19]. The present analysis is, however, more complete. It is also written from the point of view of perturbation analysis. Thus, for every solution of (A.1) we consider whether it can be matched with a corresponding outer approximation.
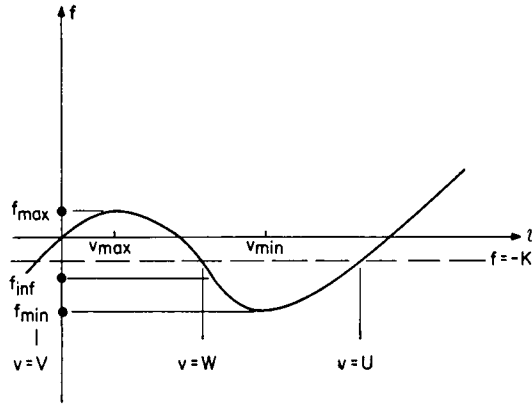
FIG. A1.   The cubic function $f(v)$ with critical points at V, W, and U.

solution must go from one singular point to another. The importance of studying (A.1) in the phase-plane is thus clear.

Being a cubic, the function $f(v)$ is symmetric about its reflection point $(v_{\text{inf}}, f_{\text{inf}})$, where

$$v_{\text{inf}} = (1 + a)/3, \; f_{\text{inf}} = \tfrac{1}{27}(1 + a)(1 - 2a)(a - 2). \qquad \text{(A.2a, b)}$$

To see this we transfer the origin to this point by the translation

$$\varphi = f - f_{\text{inf}} \; ; \; \omega = v - v_{\text{inf}} \qquad \text{(A.3a)}$$

and verify that

$$\varphi(-\omega) = -\varphi(\omega). \qquad \text{(A.3b)}$$

To find the singular points of (A.1) we need to know the solutions of the algebraic equation, V, W, U:

$$f(v) + K = 0, \quad V < W < U. \qquad \text{(A.4)}$$

For K positive this corresponds to shifting the v-axis downward to the broken line in Figure A1. The intersection of the curve of $f(v)$ with the new v-axis then gives three roots.

The matching condition requires that (A.4) have two, and hence three real roots. This requires K to be restricted to the range

$$f_{\text{min}} < -K < f_{\text{max}}. \qquad \text{(A.5)}$$

Actually, as shown by (2.28), the conditions of matching with the outer solution restrict the useful range still further. Explicit formulas for V and W in terms of U are

$$V = \tfrac{1}{2}(1 + a - U - R), \, W = \tfrac{1}{2}(1 + a - U + R), \qquad \text{(A.6a, b)}$$

with

$$R^2 = (1 - a)^2 + 2(1 + a)U - 3U^2, R > 0. \qquad \text{(A.6c)}$$

There is obviously a one-to-one correspondence between the values of the largest root

U in the range for which R is real and $\neq 0$ and K in the range (A.5). In this range any one of the quantities K, U, V, W determines the others uniquely.

Local solutions near singular points. To analyze (A.1) we will consider the equivalent system of first-order equations

$$dv/d\xi = w, \quad dw/d\xi = \theta w + (v - V)(v - W)(v - U) \qquad \text{(A.7a, b)}$$

where, for the time being, θ is just some arbitrary nonnegative constant, and $f(V) = f(W) = f(U) = -K$, $V < W < U$. The singular points in the (v, w) plane are $(V, 0)$, $(W, 0)$, and $(U, 0)$.

If we linearize (A.1) at $(V, 0)$ we obtain, with $v^* = v - V$,

$$dv^*/d\xi = w, \quad dw/d\xi = \theta w + (V - W)(V - U)v^* \qquad \text{(A.8a, b)}$$

Solutions will be linear combinations of $\exp(\alpha_i \xi)$, where

$$\alpha_i = \frac{\theta \pm [\theta^2 + 4(W - V)(U - V)]^{1/2}}{2}, \quad j = 1, 2. \qquad \text{(A9)}$$

Since $(W - V)(U - V) > 0$ the roots α_i are real and distinct: $\alpha_1 < 0 < \alpha_2$. Hence $(V, 0)$ is a saddle point with the two asymptotes having slopes α_1 and α_2. We note that $d\alpha_1/d\theta > 0$, $d\alpha_2/d\theta > 0$, i.e. α_1 and α_2 are increasing functions of θ.

Similarly for the point $(W, 0)$, the characteristic roots β are

$$\beta_i = \frac{\theta \pm [\theta^2 - 4(W - V)(U - W)]^{1/2}}{2}. \qquad \text{(A.10)}$$

For $\theta^2 < 4(W - V)(U - W)$ we have a spiral point except for $\theta = 0$ when it is center. For $\theta^2 = 4(W - V)(U - W)$ we have an improper node with all trajectories entering the singular point at the same slope, $((W - V)(U - V))^{1/2}$. For $\theta^2 > 4(W - V)(U - W)$ we have a node with two characteristic directions having slopes $\beta_2 > \beta_1 > 0$.

At $(U, 0)$ the characteristic roots γ are

$$\gamma_i = \frac{\theta \pm [\theta^2 + 4(U - V)(U - W)]^{1/2}}{2}. \qquad \text{(A.11)}$$

The roots γ_i are real and $\gamma_1 < 0 < \gamma_2$. Hence $(U, 0)$ is also a saddle point with the two asymptotes having slopes γ_1 and γ_2. The singularities at $(V, 0)$ and at $(U, 0)$ are thus of the same type (Fig. A.2). Note that $d\gamma_1/d\theta > 0$, $d\gamma_2/d\theta > 0$.
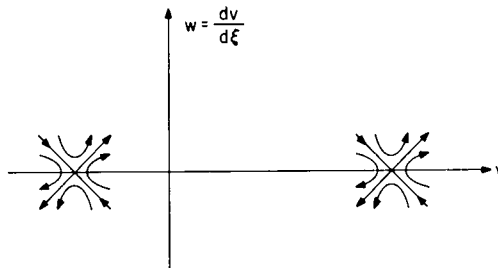

FIG. A2.   Local solutions at $(V, 0)$ and $(U, 0)$.

Global solutions through $(V, 0)$ *and* $(U, 0)$. We look for solutions which pass through both $(V, 0)$ and $(U, 0)$. We write (A.1) as

$$w(dw/dv) \; = \; \theta w \; + \; (v - V)(v - W)(v - U) \tag{A.12}$$

and try to find polynomial solutions $w(v)$. Such a solution must be of the form $w = \lambda(v - V)(U - v)$, where

$$\lambda = \pm 1/\sqrt{2}, \; \theta = \lambda G(U), \; G(U) = U + V - 2W. \tag{A.13a, b}$$

As pointed out in Sec. 1, we may assume without loss of generality that $\theta \geq 0$.

From the symmetry of the curve $f(v)$ about its inflection point it follows $G(U)$ is zero when $-K$ has the value f_{inf}. If we shift the line $f = -K$ upward in Fig. A1 the roots U and W come farther apart, the roots V and W come closer and the value of U increases. Thus

$$G(U) \text{ increases monotonically with } -K \text{ and with } U. \tag{A.14}$$

We distinguish three cases:

$$\text{Case I} \; : \; -K > f_{\text{inf}}, \, G(U) > 0, \tag{A.15a}$$

$$\text{Case II} : \; -K < f_{\text{inf}}, \, G(U) < 0, \tag{A.15b}$$

$$\text{Case III}: \; -K = f_{\text{inf}}, \, G(U) = 0. \tag{A.15c}$$

These cases will be discussed in detail below.

Case I. Since by assumption $\theta \geq 0$, λ must be given the positive values in (A.13a). Hence

$$\sqrt{2}w = (v - V)(U - v), \; \sqrt{2}\theta = G(U). \tag{A.16a, b}$$

Integration gives

$$v(\xi) \; = \; \frac{U \exp(h\xi) + V}{1 + \exp(h\xi)}, \qquad \sqrt{2}h = (U - V). \tag{A.17}$$

Strictly speaking, in (A.17) ξ should be replaced by $\xi + C$, $C = $ constant of integration. However, the constant of integration corresponds to a shift of the origin of the inner variable; this shift is discussed in Sec. 3. This solution can be matched with an outer solution which has a discontinuous increase from V to U, at $\eta = \eta_0$. We call the solution (A.17) an *upjump* at η_0.

Case II. Here $\theta \geq 0$ requires that λ be given the negative value in (A.13a). Similar to Case I we find

$$\sqrt{2} \, w = -(v - V)(U - v), \qquad \sqrt{2} \, \theta = -G(U), \tag{A.18a, b}$$

$$v(\xi) \; = \; \frac{U \exp(-h\xi) + V}{1 + \exp(-h\xi)}. \tag{A.19}$$

This solution represents a *downjump*. It is the inner solution corresponding to a discontinuous decrease from U to V in the outer solution.

Before discussing Case III we shall investigate the uniqueness of the solutions obtained and also how several upjumps and downjumps can be fitted into the same outer solution.

Uniqueness. We have seen that if w is a polynomial in v then there is one and only one solution going from V to U (Case I) or from U to V (Case II). If we remove the restriction that w be a polynomial, we may ask whether the value of θ is still uniquely determined by U and whether there are several solutions for the same θ. The answer is given by

$$\text{No solutions exist for } \theta \neq \theta_c = \lambda G(U). \tag{A.20}$$

$$\text{For } \theta = \theta_c \text{ one and only one solution exists.} \tag{A.21}$$

Proof. It is obviously sufficient to discuss Case I only. By "solution" we then mean a solution of (A.12) which goes from V at $\xi = -\infty$ to U at $\xi = +\infty$. The value of U must be chosen such that $G(U) > 0$. Now let $\theta > \theta_c$. We note that at any point (v, w), $w \neq 0$, dw/dv is an increasing function of θ. Since the slopes at $(V, 0)$ and $(U, 0)$ for trajectories entering or leaving those points are increasing functions of θ, we have a phase plane diagram that looks like Fig. A3. In order for the trajectory for $\theta > \theta_c$ to pass through $(V, 0)$ and $(U, 0)$ it would have to cross the trajectory for $\theta = \theta_c$ and, at the point of crossing, the slope of the trajectory for $\theta > \theta_c$ would be less than the slope of the trajectory for $\theta = \theta_c$, which is a contradiction. Similarly, for $\theta < \theta_c$ we would get a contradiction if we assumed a trajectory passing through $(V, 0)$ and $(U, 0)$. This proves (A.20).

Assume now $\theta = \theta_c$. From Fig. A2 we see that there are only two solutions leaving $(V, 0)$. These are determined by the two asymptotes shown with arrows in the outgoing direction. The asymptote with $w \geq 0$, $dw/dv > 0$ corresponds to the polynomial solution. The other asymptote has $w \leq 0$, $dw/dv > 0$. As seen from (A.7), these conditions must then hold for the entire range of the corresponding solution which hence never can reach $(U, 0)$. This proves (A.21).

Relations between jumps in the same solution. Assume that an outer solution has a discontinuity at $\eta = \eta_1$ with limiting values V and U and an other discontinuity at $\eta = \eta_2$ with limiting values $\tilde{V}$ and $\tilde{U}$. Each discontinuity corresponds to either an upjump or a downjump as described by Case I or Case II above. We know that U determines V and that $\tilde{U}$ determines $\tilde{V}$. Is there any relation between U and $\tilde{U}$? Such a relation is imposed by the obvious consistency requirement that both jumps correspond to the same wave velocity θ. This gives immediately

If two upjumps (downjumps) occur in the same solution the limiting
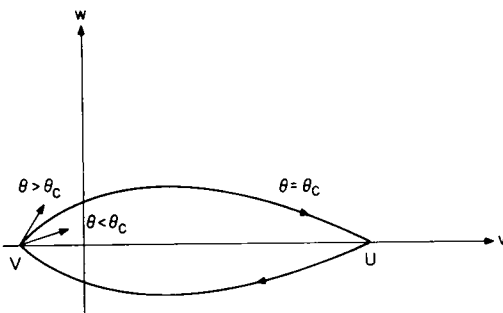
$$\text{values are the same.} \tag{A.22}$$

FIG. A3. Polynomial solution for $\theta = \theta_c$.

Proof. Since both jumps occur in the same direction, equality of wave velocity demands that $G(U) = G(\tilde{U})$. Since G is a monotonely increasing function of U the only solution of this equation is $U = \tilde{U}$ which proves the theorem.

Assume now that the two jumps have different directions. The solution of this case depends on the following algebraic lemma.

LEMMA: Let K be in the range $f_{\inf} \leq -K < f_{\max}$ and let the roots (A.23) of $f(v) + K = 0$ be $V < W < U$. Then there exists a unique $\check{K}$ in the range $f_{\min} < -\check{K} \leq f_{\inf}$ such that, if $\tilde{V}, \tilde{W}, \tilde{U}$ are the roots of $f(v) + \check{K} = 0$ then

$$G(U) = -G(\tilde{U}), \tag{a}$$

$\check{K}$ is determined by

$$\check{K} + f_{\inf} = -K - f_{\inf}, \tag{b}$$

and the following relations hold:

$$v_{\inf} - \tilde{V} = U - v_{\inf}, \tag{c}$$

$$v_{\inf} - V = \tilde{U} - v_{\inf}, \tag{d}$$

$$v_{\inf} - W = \tilde{W} - v_{\inf}. \tag{e}$$

Proof. The definition of $\check{K}$ by (b) states that the lines $f = -K$ and $f = -\check{K}$ are located symmetrically above and below the line $f = f_{\inf}$ (see Fig. A4). Relations (c), (d) and (e) are then immediate consequences of the symmetry of the cubic (see A.3). These relations and the definition of $G(U)$, (A.16), then prove (a). The $\check{K}$ is unique. If $\check{\tilde{K}}$ has the same property as $\check{K}$ then $G(\tilde{U}) = G(\check{\tilde{U}})$, hence $\tilde{U} = \check{\tilde{U}}$ and $\check{K} = \check{\tilde{K}}$.

From the algebraic Lemma we prove

If an upjump from V to U corresponds to a wave velocity θ there is a   (A.24) unique downjump from $\tilde{U}$ to $\tilde{V}$ which gives the same wave velocity.

Relations (A.23) hold between the parameters of the two jumps.

Proof. We determine $\tilde{U}$ from U as in (A.23). Relation (A.23a) then states that the wave velocities are the same. Uniqueness follows from monotonicity of $G(U)$.

Global solutions from $(V, 0)$ to $(V, 0)$. As K tends to $-f_{\inf}$ the value of θ, as computed above, tends to zero. However, as discussed in Sec. 2, the limiting solution has no meaning. Suppose now instead we put θ equal to zero in (A.1a) and look for global solutions different from those studied above. The equation (A.1) is now
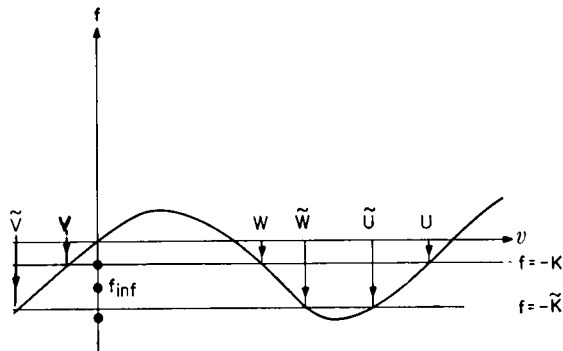

FIG. A4. Values of K giving same wave velocity.

$$w(dw/dv) = f(v) + K. \tag{A.25}$$

Hence

$$w^2 = \tfrac{1}{2}v^4 - \frac{2(1 + a)}{3}v^3 + av^2 + 2Kv + C. \tag{A.26}$$

The constant of integration is determined from

$$w(V) = 0. \tag{A.27}$$

The singular points in the phase-plane are as described earlier. Since $\theta = 0$, W is a center. We consider the solution passing through the point $(V, 0)$ and corresponding to the asymptote with $dw/dv > 0$, $w \geq 0$. The solution curve must lie below the curve going from $(V, 0)$ to $(U, 0)$, $w \geq 0$, obtained when $\theta = \theta_c$ (cf. proof of (A.21)). It thus has to cross the v-axis for $v \leq U$. However, for $V \leq v < W$, $w = 0$, solutions cross the v-axis in the upward direction. Furthermore, the solutions do not go to $(W, 0)$, since this is a center, and not to $(U, 0)$ because of the uniqueness of θ_c (we assume $-K \neq f_{\inf}$, so that $\theta_c > 0$). Thus the solution crosses the v-axis for $W < v < U$. The slope at the crossing is infinite and the solution is symmetrical about the v-axis. Thus, qualitatively, the solution is the closed curve in Figure A5.

We shall only study the case $K = 0$ and hence $V = 0$, $U = 1$. The general case may be studied in a very similar way by introducing $v^* = v - V$. Then $f(v) + K$, has a zero at $v^* = 0$. For $K = 0$ (A.26) reduces to

$$w^2 = \tfrac{1}{2}v^2 g^2(v) \tag{A.28a}$$

$$g^2(v) = v^2 - 2bv + 2a, \quad b = \tfrac{2}{3}(1 + a). \tag{A.28b}$$

The maximum value of v occurs at the smallest root of $g(v) = 0$ and is

$$v_m = b - (b^2 - 2a)^{1/2}. \tag{A.29}$$

The symmetry of the solution is best exhibited if we require

$$v = v_m \quad \text{at} \quad \xi = 0. \tag{A.30}$$

From (A.28) we find ξ as a function of v by integrating

$$\xi = -\frac{1}{\sqrt{a}} \ln \left| \left(\frac{g(v) + \sqrt{2a}}{v} - \frac{b}{\sqrt{2a}} \right) \frac{\sqrt{2a}}{(b^2 - 2a)^{1/2}} \right| \tag{A.31}$$
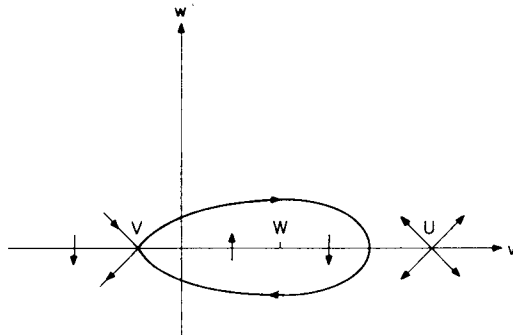

FIG. A5.   Qualitative picture of solutions for $\theta = 0$.

for $\xi \leq 0$. Inverting we obtain

$$v = \frac{4a(b^2 - 2a)^{1/2} \exp(\sqrt{a}\xi)}{((b^2 - 2a)^{1/2} \exp(\sqrt{a}\xi) + b)^2 - 2a}.$$ (A.32)

This formula is valid for all values of ξ and has the symmetry

$$v(\xi) = v(-\xi).$$ (A.33)

Completion of phase-plane analysis. We shall now complete the phase-plane analysis of (A.1), and in particular find more global solutions going from one singular point to another. However, none of these can be matched with an outer solution. This follows from the discussion of the outer solution in Sec. 2. Thus it is shown that, except for the global solutions discussed earlier in this Appendix, there are no other solutions of (A.1) which can be used as inner solutions in an asymptotic analysis of the original problem (2.1). Though the results are negative, it is necessary to carry out the analysis in order to show that we have found all solutions of (A.1) which are useful for our purpose. However, since the formulas derived below will not be used we shall omit many details in the proof.

Completion of classification of bounded solutions. We shall prove the following statements about solutions to (A1) for $G(U) > 0$, i.e. for $(U + V)/2 > W$:

(a) For $\theta = 0$ there are no solutions going from one singular point to another other than that which goes from $(V, 0)$ to $(V, 0)$.

(b) For $\theta > 0$ the only bounded solutions go from one singular point to another.

(c) For $\theta > 0$ there is a solution going from $(W, 0)$ to $(V, 0)$.

(d) For $\theta > \theta_c$ there is a solution going from $(W, 0)$ to $(V, 0)$; for $0 \leq \theta \leq \theta_c$ there is no solution going from $(W, 0)$ to $(U, 0)$.

(e) For $\theta > 0$ there is no solution going from a singular point back to itself.

Proof: Since we will only be interested in bounded solutions of (A.1), we need not consider those solutions which enter or leave $(V, 0)$ for $v \leq V$ or those which enter or leave $(U, 0)$ for $v \geq U$.

(a): We have seen previously that for $\theta = 0$ there is a solution going from $(V, 0)$ to $(V, 0)$. Since $(W, 0)$ is a center, the only other possibility for a solution to go from one singular point to another is for a solution leaving $(U, 0)$ for $v \leq U$ to come back to $(U, 0)$ for $v \leq U$ (cf. Fig. A5). However, this solution must cross the v-axis in the upward direction. While the solution is beneath the v-axis, it moves to the left, and the only possible place for an upcrossing is between $(V, 0)$ and $(W, 0)$. Since this solution cannot cross the solution going from $(V, 0)$ to $(V, 0)$, it cannot get to the v-axis between $(V, 0)$ and $(W, 0)$. Thus this solution is confined to be below the v-axis. Statement (a) is thus proven.

(b): We shall first show that for $\theta \neq 0$, there are no periodic solutions. We write (A.7) as the vector equation

$$\begin{pmatrix} dv/d\xi \\ dw/d\xi \end{pmatrix} = \mathbf{F}(v, w)$$

where $\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ with $F_1 = w$ and $F_2 = \theta w + (v - V)(v - W)(v - U)$. Since div $\mathbf{F} \equiv \partial F_1/\partial v + \partial F_2/\partial w = \theta \neq 0$, then by a theorem of Bendixson, (A.7) cannot have any periodic solution for $\theta \neq 0$. (The fact that div $\mathbf{F} \neq 0$ for any v and w implies that there
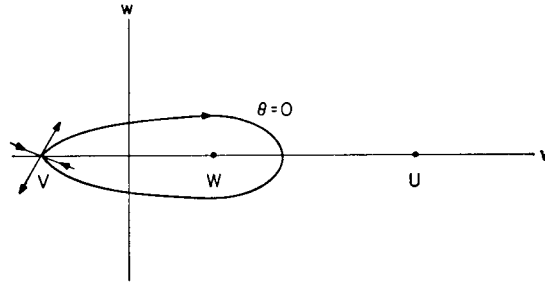
FIG. A6.

is no closed curve having the property that the vector **F** is tangent to the curve at every point on the curve.)

Since there are no periodic solutions, there can be no limit cycles, and thus every solution confined to a bounded region for $t > t_0$ $(t < t_0)$, for some t_0, must approach a singular point as $t \to \infty$ $(t \to -\infty)$. (The only other possibility is that the solution approach a solution going between singular points, which, by inspection of the direction field of (A.7), cannot happen.)

(c): From Fig. A6 we see that the solution entering $(V, 0)$ from $w \leq 0$ is, for large ξ, inside the solution going from $(V, 0)$ to $(V, 0)$ for $\theta = 0$. As we follow the solution back for decreasing ξ, it is seen that this solution must remain inside. Thus the solution must have come from $(W, 0)$.

(d): For $\theta > \theta_c$, we see from Fig. A7 that a solution entering $(U, 0)$ from $v \leq U$ has to be confined below the solution going from $(V, 0)$ to $(U, 0)$ for $\theta = \theta_c$, above the v-axis between $(W, 0)$ and $(U, 0)$, and above the solution going from $(W, 0)$ to $(V, 0)$. Thus the solution must have come from $(W, 0)$.

For $\theta \leq \theta_c$, we see from Fig. A8 that a solution could not go from $(W, 0)$ to $(U, 0)$ as it would have to intersect the solution from $(V, 0)$ to $(U, 0)$ for $\theta = \theta_c$, which it cannot do.

(e): Since solutions always leave $(W, 0)$, there are no solutions from $(W, 0)$ back to itself.

Since no solution for $\theta > 0$ which is outside the solution from $(V, 0)$ to $(V, 0)$ for $\theta = 0$ can get inside (cf. Fig. A7), there can be no solution from $(V, 0)$ back to itself.

Since there is no way for the solution leaving $(U, 0)$ for $v \leq U$ to cross the v-axis in the upward direction, there can be no solution from $(U, 0)$ back to itself. (The solution from $(U, 0)$ cannot get inside the solution from $(V, 0)$ to $(V, 0)$ for $\theta = 0$.)
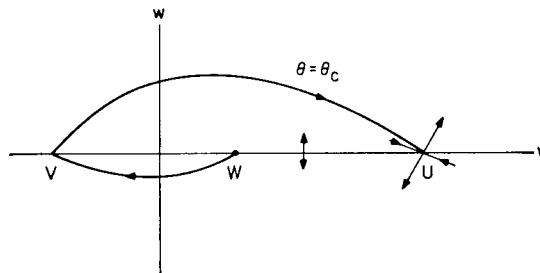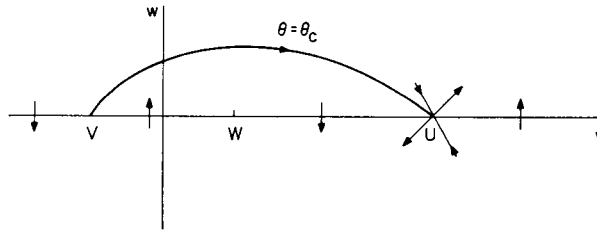

FIG. A7.

FIG. A8.

For $G(U) = 0$ all but statement (a) are true. In this case statement (a) should be replaced by: for $\theta = 0$ there are no solutions going from one singular point to another other than a solution which goes from $(V, 0)$ to $(U, 0)$ and a solution which goes from $(U, 0)$ to $(V, 0)$. (This is just limiting case III discussed earlier: $K = -f_{\text{inf}}$, $\theta = 0$.)

To obtain solutions of (A.1) for $G(U) < 0$, i.e. for $(U + V)/2 > W$, we note the following. Eqs. (A.10) are invariant under the transformation $w \to -w, v \to U + V - v$, $W \to U + V - W$. Thus we can obtain all solutions of (A.1) for $G(U) < 0$ by taking all solutions for $G(U) > 0$ and reflecting them about both the v-axis and the line $v = (U + V)/2$, i.e. the line through the midpoint of the interval $[V, U]$.

We can summarize the above by stating that the only solutions of (A.1) that go from one singular point to another are those which go from:

(i) $V \to V$ for $\theta = 0$, $G(U) > 0$,

(ii) $U \to U$ for $\theta = 0$, $G(U) < 0$,

(iii) $W \to V$ for $\theta > 0$, $G(U) \geq 0$,

            for $\theta > -\dfrac{1}{\sqrt{2}} G(U)$, $G(U) \leq 0$,

(iv) $W \to U$ for $\theta > \dfrac{1}{\sqrt{2}} G(U)$, $G(U) \geq 0$,

            for $\theta > 0$, $G(U) \leq 0$,

(v) $V \to U$ for $\theta = \dfrac{1}{\sqrt{2}} G(U)$, $G(U) \geq 0$,

(vi) $U \to V$ for $\theta = -\dfrac{1}{\sqrt{2}} G(U)$, $G(U) \leq 0$.

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