Progress of Theoretical Physics, Vol. 50, No. 6, December 1973

Perturbation Calculation of Gravitational Potentials

Hiroshi OKAMURA, Tadayuki OHTA,*
Toshiei KIMURA** and Kichiro HIIDA***

Department of Physics, University of Tokyo, Tokyo
and
Department of Physics, Faculty of General Education
Kogakuin University, Hachioji, Tokyo
**Department of Mathematical Physics, College of Technology*
Seikei University, Musashino, Tokyo
***Research Institute for Theoretical Physics, Hiroshima University*
Takehara, Hiroshima-ken
****Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo*

(Received April 23, 1973)

The static gravitational potential for four-body system in the order of G^3 is calculated in the quantized theory. It is derived from the S-matrix elements for scattering of four spinless particles, among which gravitons are exchanged. Feynman's and de Donder's gauges are used for graviton propagator. The potential has contributions from the transverse-traceless part of the graviton field. The potential is different from that obtained in classical theory from the physically acceptable metric tensor derived in our previous paper under new coordinate conditions. Since all other known coordinate conditions such as de Donder's ones lead to divergent metric tensor at spatial infinity in the post-post-Newtonian approximation, it is not unreasonable that the two potentials become different first in this order of approximation.

§ 1. Introduction

There are two methods for obtaining gravitational potentials for many-body system. One is to solve Einstein's equation in general theory of relativity by expanding the metric tensor $g_{\alpha\beta}$*) in the inverse powers of the velocity of light,**) and then to get the potentials from the metric tensor. The other method, which is familiar to particle physicists, is to calculate S-matrix elements for particle scattering in quantum theory by using propagators, and to obtain the potentials in the limit $\hbar\rightarrow 0$**) from the S-matrix elements. Up to the post-Newtonian approximation the two methods lead to the same results.[1),2)]

Since Einstein's theory is invariant under general coordinate transformation, there is a freedom of choosing coordinate conditions.***) An example is de

*) Greek indices run from 1 to 4. Latin indices run from 1 to 3.

**) Hereafter the unit $\hbar=c=1$ is used.

***) Throughout this paper the terminologies "coordinate conditions" and "gauge" are used in classical and quantized theories, respectively.

Donder's coordinate conditions:[3)]

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta})=0\,, \tag{1·1}$$

where $g=\det(g_{\alpha\beta})$. In the post-post-Newtonian and higher orders of the approximation, the conditions (1·1) and other known conditions give the metric tensor which diverges at infinitely remote place from the region where the matter exists.[4),5)] In a previous paper[4)] we found the metric tensor which is Minkowskian at spatial infinity up to the post-post-Newtonian order. We call it the physically acceptable one. There is a class of coordinate conditions which give the acceptable metric tensor. Then a natural question arises: Whether or not coordinate conditions such as de Donder's ones lead to right physical results in the post-post-Newtonian and higher orders of approximation.

In order to answer this question, we would like to calculate in this paper the gravitational potentials in the post-post-Newtonian approximation by using the graviton propagators in Feynman's and de Donder's gauges. The potentials consist of three parts; the potentials in the order of G^3, G^2p^2 and Gp^4, G and p being Newton's gravitational constant and the momentum of a massive particle, respectively. These potentials are to be compared with those obtained from the physically acceptable metric tensor. The detailed calculations for the latter will be published in a subsequent paper.[6)]

In our perturbation calculations we first put

$$g_{\alpha\beta}=\delta_{\alpha\beta}+\kappa h_{\alpha\beta}\,, \tag{1·2}$$

where $h_{\alpha\beta}$ is considered as the graviton field and $\kappa^2=32\pi G$. We use the propagator

$$\langle 0|T(h_{\alpha\beta}(x), h_{\gamma\delta}(y))|0\rangle\,, \tag{1·3}$$

where T and $|0\rangle$ denote the time-ordering operator and the vacuum state, respectively. It is usually assumed that physical observables are independent of the definition of the graviton field. For example we may define the graviton field $\beta^{\alpha\beta}$ by

$$\sqrt{-g}\,g^{\alpha\beta}=\delta^{\alpha\beta}-\kappa(\beta^{\alpha\beta}-\tfrac{1}{2}\delta^{\alpha\beta}\beta_\gamma{}^\gamma)\,. \tag{1·4}$$

In this case we use the propagator

$$\langle 0|T(\beta^{\alpha\beta}(x), \beta^{\gamma\delta}(y))|0\rangle\,. \tag{1·5}$$

The field $h_{\alpha\beta}$ can be expressed in terms of $\beta_{\alpha\beta}$ as

$$h_{\alpha\beta}=\beta_{\alpha\beta}+\kappa[\tfrac{1}{8}\{\beta_{\gamma\gamma}\beta_{\delta\delta}-2\beta_{\gamma\delta}\beta_{\gamma\delta}\}\delta_{\alpha\beta}-\tfrac{1}{2}\beta_{\gamma\gamma}\beta_{\alpha\beta}+\beta_{\alpha\gamma}\beta_{\beta\gamma}]+O(\kappa^2)\,. \tag{1·6}$$

It would be worth while to note that both the propagators (1·3) and (1·5) are considered to be independent of κ in perturbation calculations. This does not contradict with (1·6), but only means that definitions of the graviton field are different in (1·2) and (1·4).

In the next section we first calculate the S-matrix elements for the scattering of four spinless particles using the progagator (1·3) in the Feynman's and de Donder's gauges, and then derive the static four-body gravitational potentials in the order of G^3 from the S-matrix elements. Since the field $h_{\alpha\beta}$ couples with the conserved energy-momentum tensor, it is guaranteed that the use of the graviton propagators in these two gauges leads to the same results. It is shown that the transverse-traceless part of the graviton field contributes to the static potential.

As was shown in the previous papers,[2),4)] one-graviton-exchange potential in Feynman's and de Donder's gauges coincides up to the order of Gp^4 with the corresponding part of the potentials obtained from the physically acceptable metric tensor in the classical theory. The static potential in the order of G^3 obtained in § 2, however, does not coincide with that obtained in the classical theory.[6)] In order to confirm that the calculations in § 2 is right, we calculate again in § 3 the static potential by using the propagator (1·5) for the field $\beta^{\alpha\beta}$. It is shown that the potential coincides with that obtained in § 2.

In the last section we discuss the reason why the static potential in the order of G^3 obtained in §§ 2 and 3 does not coincide with that obtained by the classical method. It is shown that the potential in the order of G^2p^2 cannot be determined uniquely as far as Feynman's and de Donder's gauges are used. In Appendices 1 and 2, the Lagrangian density is expressed in terms of the graviton fields $h_{\alpha\beta}$ and $\beta_{\alpha\beta}$, respectively.

§ 2. The static potential in the order of G^3

In this section we calculate the static part of four-body gravitational potential in the order of G^3, $V_4(G^3)$. We start with the Lagrangian density

$$L=\frac{1}{16\pi G}\sqrt{-g}R-\frac{1}{2}\sqrt{-g}\sum_i\left\{g^{\alpha\beta}\frac{\partial\phi_i}{\partial x^\alpha}\frac{\partial\phi_i}{\partial x^\beta}+m_i^2\phi_i^2\right\}, \tag{2·1}$$

where R is the scalar curvature and ϕ_i denotes a scalar field with the mass m_i. The explicit form of the Lagrangian density expressed in terms of the graviton field $h_{\alpha\beta}$ is presented in Appendix 1. With this Lagrangian density, we calculate the S-matrix elements for collisions of four spinless particles by graviton exchange in relativistic perturbation theory.

The propagator (1·3) can be expressed as

$$-\frac{i}{2(2\pi)^4}\int d^4k e^{ikx}\frac{X_{\alpha\beta,\gamma\delta}(k)}{k^2-i\varepsilon}, \tag{2·2}$$

where the numerator $X_{\alpha\beta,\gamma\delta}(k)$ depends on the gauge. It is given by

$$X^F_{\alpha\beta,\gamma\delta}(k)=\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{2·3}$$

in Feynman's gauge, and by

$$X^D_{\alpha\beta,\gamma\delta}(k)=X^F_{\alpha\beta,\gamma\delta}(k)-\frac{1}{\boldsymbol{k}^2}(k_\alpha k_\gamma\delta_{\beta\delta}+k_\alpha k_\delta\delta_{\beta\gamma}+k_\beta k_\gamma\delta_{\alpha\delta}+k_\beta k_\delta\delta_{\alpha\gamma}) \qquad (2\cdot 4)$$

in de Donder's gauge. Since the field $h_{\alpha\beta}$ couples with conserved energy-momentum tensor, it is evident that the difference between these two numerators has no contribution to the S-matrix elements.

When the propagator (2·2) with (2·3) is used to calculate the S-matrix elements, they depend on the energy transfer k_0. The potentials obtained from these matrix elements become ambiguous. To overcome this difficulty we may use the freedom of gauge transformation in gravitational theory and can show that one-graviton-exchange potential is determined up to the order of Gp^4. Then the factor $X_{\alpha\beta,\gamma\delta}(k)$ is given by[2)]

$$\begin{aligned} X_{\alpha\beta,\gamma\delta}(k)=\frac{k^2}{\boldsymbol{k}^2}\Big[&\delta_{\alpha4}\delta_{\beta4}\delta_{\gamma4}\delta_{\delta4}-\delta_{\alpha4}\delta_{\beta4}\bar{\delta}_{\gamma\delta}-\delta_{\gamma4}\delta_{\delta4}\bar{\delta}_{\alpha\beta}+\delta_{\alpha4}\delta_{\gamma4}\bar{\delta}_{\beta\delta}+\delta_{\beta4}\delta_{\delta4}\bar{\delta}_{\alpha\gamma}\\ &+\delta_{\alpha4}\delta_{\delta4}\bar{\delta}_{\beta\gamma}+\delta_{\beta4}\delta_{\gamma4}\bar{\delta}_{\alpha\delta}-\frac{k_ik_j}{\boldsymbol{k}^2}\Big\{\frac{x}{4}(\delta_{\alpha i}\delta_{\gamma j}\delta_{\beta4}\delta_{\delta4}+\delta_{\beta i}\delta_{\delta j}\delta_{\alpha4}\delta_{\gamma4}\\ &+\delta_{\alpha i}\delta_{\delta j}\delta_{\beta4}\delta_{\gamma4}+\delta_{\beta i}\delta_{\gamma j}\delta_{\alpha4}\delta_{\delta4})+\frac{(1-x)}{2}(\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma4}\delta_{\delta4}+\delta_{\gamma i}\delta_{\delta j}\delta_{\alpha4}\delta_{\beta4})\Big\}\Big]\\ &+\Big\{\Big(\bar{\delta}_{\alpha\gamma}-\delta_{\alpha i}\delta_{\gamma j}\frac{k_ik_j}{\boldsymbol{k}^2}\Big)\Big(\bar{\delta}_{\beta\delta}-\delta_{\beta k}\delta_{\delta l}\frac{k_kk_l}{\boldsymbol{k}^2}\Big)+\Big(\bar{\delta}_{\alpha\delta}-\delta_{\alpha i}\delta_{\delta j}\frac{k_ik_j}{\boldsymbol{k}^2}\Big)\\ &\times\Big(\bar{\delta}_{\beta\gamma}-\delta_{\beta k}\delta_{\gamma l}\frac{k_kk_l}{\boldsymbol{k}^2}\Big)-\Big(\bar{\delta}_{\alpha\beta}-\delta_{\alpha i}\delta_{\beta j}\frac{k_ik_j}{\boldsymbol{k}^2}\Big)\Big(\bar{\delta}_{\gamma\delta}-\delta_{\gamma k}\delta_{\delta l}\frac{k_kk_l}{\boldsymbol{k}^2}\Big)\Big\}, \end{aligned} \qquad (2\cdot 5)$$

where x is a real parameter, suffices i, j, k and l take the values 1, 2 and 3, and $\bar{\delta}_{\alpha\beta}=\delta_{\alpha\beta}-\delta_{\alpha4}\delta_{\beta4}$. Using only the nature that the field $h_{\alpha\beta}$ couples with a conserved quantity, we can rewrite (2·3) and (2·4) as (2·5).[2)] For this reason we call hereafter (2·2) with (2·5) the propagator in Feynman's and de Donder's gauges.

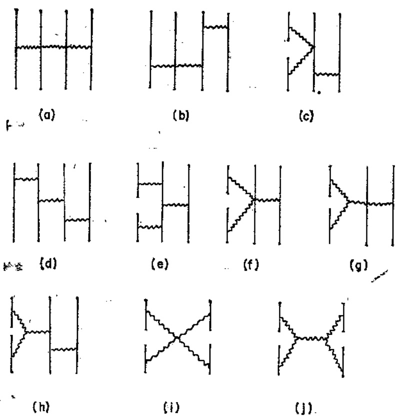

Fig. 1. The diagrams which contribute to $V_4(G^3)$. Straight lines represent the spinless particles and the wavy lines represent the exchanged gravitons.

The diagrams which contribute to the four-body potential in the order of G^3 are shown in Fig. 1. It should be noted that these diagrams represent not only the diagrams themselves but also the same types of diagrams. For example, the diagram (b) represents also the diagram shown in Fig. 2. The same is true for the diagrams (c), (d) and (h).

We consider first the contribution of the diagram (f) to the S-matrix elements. Let p_i and q_i be the initial and the final mo-

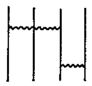

Fig. 2. The other diagram represented by Fig. 1(b).

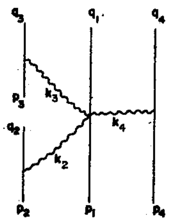

Fig. 3. An example of assigning the momenta to the diagram 1(f).

menta of the particle with the mass m_i $(i=1,2,3,4)$. There are four different ways in assigning the mass m_i and the momenta p_i and q_i to the four straight lines. An example is shown in Fig. 3. The S-matrix element for Fig. 3 is given by

$$\frac{iG^3}{16\pi^5}\frac{m_1m_2m_3m_4}{\boldsymbol{k}_2^2\boldsymbol{k}_3^2\boldsymbol{k}_4^2}\delta^{(4)}\Big(\sum_{i=1}^{4}p_i-\sum_{i=1}^{4}q_i\Big)$$
$$\times\left\{(171-30x+3x^2)-2(1-x)^2\left\{\frac{(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)^2}{\boldsymbol{k}_2^2\boldsymbol{k}_3^2}+\frac{(\boldsymbol{k}_3\cdot\boldsymbol{k}_4)^2}{\boldsymbol{k}_3^2\boldsymbol{k}_4^2}+\frac{(\boldsymbol{k}_4\cdot\boldsymbol{k}_2)^2}{\boldsymbol{k}_4^2\boldsymbol{k}_2^2}\right\}\right].\tag{2·6}$$

From this S-matrix element, the four-body potential is calculated to be

$$-\frac{G^3}{2}\frac{m_1m_2m_3m_4}{r_{12}r_{13}r_{14}}\Big[(87-18x+3x^2)-\frac{(1-x)^2}{2}$$
$$\times\left\{\frac{(\boldsymbol{r}_{12}\cdot\boldsymbol{r}_{13})^2}{r_{12}^2r_{13}^2}+\frac{(\boldsymbol{r}_{13}\cdot\boldsymbol{r}_{14})^2}{r_{13}^2r_{14}^2}+\frac{(\boldsymbol{r}_{14}\cdot\boldsymbol{r}_{12})^2}{r_{14}^2r_{12}^2}\right\}\Big],\tag{2·7}$$

where $\boldsymbol{r}_{ij}=\boldsymbol{r}_i-\boldsymbol{r}_j$ and $r_{ij}=|\boldsymbol{r}_{ij}|$, $\boldsymbol{r}_i$ being the position vector of i-th particle. When $x\neq1$, this potential depends on the angle between $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$.

Here we discuss the nature of the angle-dependent potentials obtained from the diagrams in Fig. 1. The diagrams (a), (b), (d) and (e) do not contribute to the angle-dependent potential. The contributions from the diagrams (c), (f), (g), (h) and (i) contain angle-dependent potentials, which vanish individually only when $x=1$, as was illustrated in (2·7). The contribution from the remaining diagram (j) also contains angle-dependent potentials, which are separated into two parts: One comes from the transverse-traceless part of the graviton field $h_{\alpha\beta}$, and is independent of x. The other is free from the transverse-traceless part, and vanishes when $x=1$.

Hereafter we take the value $x=1$. Then the four-body potential obtained from all the diagrams involved in Fig. 1(f) is given by

$$V_{4f}(G^3)=-36V(\curlywedge),\tag{2·8}$$

where

$$V(\text{⅄}) = G^3 m_1 m_2 m_3 m_4 \left(\frac{1}{r_{12} r_{13} r_{14}} + \frac{1}{r_{21} r_{23} r_{24}} + \frac{1}{r_{31} r_{32} r_{34}} + \frac{1}{r_{41} r_{42} r_{43}} \right). \qquad (2 \cdot 9)$$

The symbol (⅄) represents the configuration of the four related particles, as is shown by the right-hand side of (2·9).

Next we consider the contribution from the diagram (e) in Fig. 1 to the S-matrix. There are twenty-four combinations in assigning the mass m_i and the momenta p_i and q_i to four straight lines. An example is shown in Fig. 4. The S-matrix element for Fig. 4 is given by

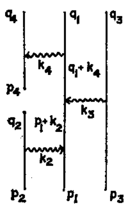

Fig. 4. An example of assigning the momenta to the diagram 1(e).

$$S_e = \frac{iG^3}{\pi^5} Y(k_{20}, k_{40}) \frac{1}{\{(p_1+k_2)^2 + m_1^2\}\{(q_1+k_4)^2 + m_1^2\}}, \qquad (2 \cdot 10)$$

where

$$Y(k_{20}, k_{40}) = \delta^{(4)}\left(\sum_{i=1}^{4} p_i - \sum_{i=1}^{4} q_i\right) \frac{m_1 m_2 m_3 m_4}{\boldsymbol{k}_2^2 \boldsymbol{k}_3^2 \boldsymbol{k}_4^2} \{m_1^2 + 4m_1(k_{20} + k_{40}) + 14 k_{20} k_{40}\}. \qquad (2 \cdot 11)$$

In order to derive the four-body potential from the S-matrix element (2·10) with (2·11), we recall the relation between an S-matrix element and the corresponding potential V in old-fashioned perturbation theory. It is given by

$$\langle f|(S-1)|i\rangle = -2\pi i \delta(E_f - E_i)$$
$$\times \left\{ \langle f|V|i\rangle + \sum_n \frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_i - E_n} + \sum_n \sum_m \frac{\langle f|V|n\rangle\langle n|V|m\rangle\langle m|V|i\rangle}{(E_i - E_m)(E_i - E_n)} + \cdots \right\}, \qquad (2 \cdot 12)$$

where the letters i and f denote the initial and the final state, respectively. The S-matrix element (2·10) contains the terms corresponding to the second and the third terms on the right-hand side of (2·12), though the matrix element (2·6) from the diagram (f) in Fig. 1 contains only the first term. Now we rewrite (2·10) as

$$S_e = (S_e - S_e^{B1} - S_e^{B2} + S_e^{BB}) + \{(S_e^{B1} - S_e^{BB}) + (S_e^{B2} - S_e^{BB})\} + S_e^{BB}, \quad (2 \cdot 13)$$

in which

$$S_e^{B1} = \frac{iG^3}{\pi^5} Y(k_{20}, k'_{40}) \frac{1}{4m_1^2} \left\{ \frac{1}{2m_1} + \frac{1}{k'_{20} - k_{20}} \right\} \frac{1}{k'_{40} - k_{40}},$$

$$S_e^{B2} = \frac{iG^3}{\pi^5} Y(k'_{20}, k_{40}) \frac{1}{4m_1^2 (k'_{20} - k_{20})} \left\{ \frac{1}{2m_1} + \frac{1}{k'_{40} - k_{40}} \right\} \tag{2·14}$$

and

$$S_e^{BB} = \frac{iG^3}{\pi^5} Y(k'_{20}, k'_{40}) \frac{1}{4m_1^2 (k'_{20} - k_{20})(k'_{40} - k_{40})},$$

where

$$k'_{20} = \sqrt{(\boldsymbol{p}_1 + \boldsymbol{k}_2)^2 + m_1^2} - p_{10}$$

and

$$k'_{40} = \sqrt{(\boldsymbol{q}_1 + \boldsymbol{k}_4)^2 + m_1^2} - q_{10}.$$

Then the first, the second and the third terms on the right-hand side of (2·13) correspond to those of (2·12), respectively.

The potential for many-body system is included in the first terms in (2·12) and (2·13). From (2·10), (2·11) and (2·14) we get

$$S_e - S_e^{B1} - S_e^{B2} + S_e^{BB} = \frac{41iG^3}{16\pi^5} \delta^{(4)} \left(\sum_{i=1}^{4} p_i - \sum_{i=1}^{4} q_i \right) \frac{m_1 m_2 m_3 m_4}{\boldsymbol{k}_2^2 \boldsymbol{k}_3^2 \boldsymbol{k}_4^2}, \tag{2·15}$$

which leads to the four-body static potential

$$-\frac{41G^3}{4} \frac{m_1 m_2 m_3 m_4}{r_{12} r_{13} r_{14}}. \tag{2·16}$$

As was mentioned above there are twenty-four diagrams represented by Fig. 1(e). Thus the contribution from Fig. 1(e) to $V_4(G^3)$ is

$$V_{4e}(G^3) = -\frac{123}{2} V(\ ). \tag{2·17}$$

There are three diagrams represented by Fig. 1(j). One of them is shown in Fig. 5. The S-matrix element for this diagram is

$$S_j = -\frac{iG^3}{16\pi^5} \delta^{(4)} \left(\sum_{i=1}^{4} p_i - \sum_{i=1}^{4} q_i \right) \frac{m_1 m_2 m_3 m_4}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2 \boldsymbol{k}_3^2 \boldsymbol{k}_4^2}$$

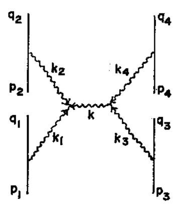

Fig. 5. One of the diagram represented by Fig. 1(j).

$$\times\left\{33\boldsymbol{k}^2-9(\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+\boldsymbol{k}_3^2+\boldsymbol{k}_4^2)-\frac{15}{\boldsymbol{k}^2}(\boldsymbol{k}_1^2+\boldsymbol{k}_2^2)(\boldsymbol{k}_3^2+\boldsymbol{k}_4^2)\right\}$$

$$+\frac{iG^3}{\pi^5}\delta^{(4)}\left(\sum_{i=1}^4 p_i-\sum_{i=1}^4 q_i\right)\frac{m_1m_2m_3m_4}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2\boldsymbol{k}_3^2\boldsymbol{k}_4^2\boldsymbol{k}^2}$$

$$\times\{K(k_1,k_3)K(k_2,k_4)+K(k_1,k_4)K(k_2,k_3)-K(k_1,k_2)K(k_3,k_4)\}, \tag{2·18}$$

where

$$K(k_i,k_j)=(\boldsymbol{k}_i\cdot\boldsymbol{k}_j)-\frac{(\boldsymbol{k}_i\cdot\boldsymbol{k})(\boldsymbol{k}_j\cdot\boldsymbol{k})}{\boldsymbol{k}^2}. \tag{2·19}$$

The second term on the right-hand side of (2·18) is the contribution from the part where the internal graviton with the momentum $\boldsymbol{k}$ is in the transverse-traceless state. Recalling that Fig. 1(j) represents three diagrams, we get the following contribution from Fig. 1(j) to $V_4(G^3)$:

$$V_{4j}(G^3)=\frac{3}{2}V(⅄)-\frac{15}{4}V(\circ\!-\!\circ\!-\!\circ\!-\!\circ)+V_4^{TT}, \tag{2·20}$$

where

$$V(\circ\!-\!\circ\!-\!\circ\!-\!\circ)=G^3m_1m_2m_3m_4\Big(\frac{1}{r_{12}r_{23}r_{34}}+\frac{1}{r_{12}r_{24}r_{43}}+\frac{1}{r_{13}r_{32}r_{24}}+\frac{1}{r_{13}r_{34}r_{42}}$$
$$+\frac{1}{r_{14}r_{42}r_{23}}+\frac{1}{r_{14}r_{43}r_{32}}+\frac{1}{r_{21}r_{13}r_{34}}+\frac{1}{r_{21}r_{14}r_{43}}+\frac{1}{r_{23}r_{31}r_{14}}$$
$$+\frac{1}{r_{24}r_{41}r_{13}}+\frac{1}{r_{31}r_{12}r_{24}}+\frac{1}{r_{32}r_{21}r_{14}}\Big) \tag{2·21}$$

and V_4^{TT} denotes the potential which comes from the second term on the right-hand side of (2·18) and similar two terms. The presence of V_4^{TT} part in the static potential $V_{4j}(G^3)$ means that the contribution from the transverse-traceless part of the graviton field appears first in the post-post-Newtonian approximation. The V_4^{TT} is the angle-dependent potential mentioned before, and is independent of the gauge parameter x. It cannot be expressed by a linear combination of the potentials $V(⅄)$ and $V(\circ\!-\!\circ\!-\!\circ\!-\!\circ)$. The same potential V_4^{TT} is obtained[6] from the physically acceptable metric tensor in the classical theory.

For the remaining seven diagrams in Fig. 1, the S-matrix elements can be calculated similarly. In order to obtain the gravitational potential from the S-matrix element for the diagram (d), we must subtract, from the S-matrix element, the terms corresponding to the second and the third terms on the right-hand side of (2·12), as we have done in the case of the diagram (e). The S-matrix elements for the diagrams (b), (c) and (h) contain the terms corresponding to the second term, but do not contain those corresponding to the third term

Table I. The contribution to $V_4(G^3)$ from each diagram in Fig. 1, when the graviton field is defined by $h_{\alpha\beta}$. The coefficients of the potentials $V($⅄$)$ and $V($ o-o-o-o $)$ are written in the table. The definition of these potentials is given by (2·9) and (2·21). The potential $V_4{}^{TT}$ is obtained from the diagram (j), though it is not included in the table.

Diagram	a	b	c	d	e	f	g	h	i	j	Total
$V($⅄$)$	0	0	108	0	$-\frac{123}{2}$	-36	$\frac{33}{2}$	$-\frac{21}{2}$	-6	$\frac{3}{2}$	12
$V($ o-o-o-o $)$	-55	94	0	-40	0	0	29	-25	0	$-\frac{15}{4}$	$-\frac{3}{4}$

on the right-hand side of (2·12). In this case it is easier to obtain gravitational potentials than in the former case. For the other diagrams (a), (g) and (i), the potentials can be calculated without subtraction. We summarize the result in Table I, where contributions to $V_4(G^3)$ from each diagram in Fig. 1 are shown. From Table I we have the static potential in the order of G^3,

$$V_4(G^3) = 12V(\text{⅄}) - \tfrac{3}{4}V(\text{o-o-o-o}) + V_4^{TT}. \tag{2·22}$$

§ 3. Recalculation of the static potential in the order of G^3

The static potential $V_4(G^3)$ was calculated in the previous section. There the graviton field was defined by $h_{\alpha\beta}$ in (1·2), and the propagator (2·2) with (2·5) was used. In a separate paper[6] the gravitational potentials in the post-post-Newtonian approximation will be calculated by the classical method from the physically acceptable metric tensor $g_{\alpha\beta}$. Then it will be found that the potential $V_4(G^3)$ obtained by the classical method does not coincide with that given by (2·22), though in the order of Gp^4 the same potential is obtained by the two methods. We think that this difference is very important. In order to confirm that the potential (2·22) is correct as far as Feynman's and de Donder's gauges are used in quantized theory, we calculate in this section the potential $V_4(G^3)$ by defining the graviton field by $\beta^{\alpha\beta}$ in (1·4).

The Lagrangian density (2·1) is expressed in terms of the field $\beta_{\alpha\beta}$ in Appendix 2. As is shown by (A2·2) and by (A1·5) with (A1·6), the free Lagrangian density for the field $\beta_{\alpha\beta}$ has exactly the same form as that for the field $h_{\alpha\beta}$. Feynman's and de Donder's gauges are used again in this section. Then the propagator for the field $\beta_{\alpha\beta}$ is given again by (2·2) with (2·5). All the diagrams in Fig. 1 are necessary to calculate $V_4(G^3)$. Since the calculation is similar to that in the previous section, we confine ourselves to summarize the result in Table II. As in the previous section, $V_{4j}(G^3)$ also contains $V_4{}^{TT}$. We find from Table II that the static part of the four-body potential in the order of G^3 remains unchanged and is again given by (2·22).

Table II. The contribution to $V_4(G^3)$ from each diagram in Fig. 1, when the graviton field is defined by $\beta_{\alpha\beta}$. The coefficients of the potentials $V(\curlywedge)$ and $V($o-o-o-o$)$ are written in the table.

Diagram	a	b	c	d	e	f	g	h	i	j	Total
$V(\curlywedge)$	0	0	18	0	$-\frac{123}{2}$	12	-24	$\frac{159}{2}$	48	-60	12
$V($o-o-o-o$)$	-4	28	0	-40	0	0	-16	41	0	$-\frac{39}{4}$	$-\frac{3}{4}$

§ 4. Discussion

The gravitational potentials in the post-post-Newtonian approximation consist of three parts: The potentials in the order of Gp^4, G^2p^2 and G^3. It has been shown[2),4)] that the potential in the order of Gp^4 obtained in quantized theory in Feynman's and de Donder's gauges coincides with that obtained in the classical theory from the physically acceptable metric tensor. In §§ 2 and 3 we have calculated the gravitational potential $V_4(G^3)$ for four-body system in the order of G^3 in Feynman's and de Donder's gauges. The potential in this order, except for V_4^{TT}, is angle-independent only when $x=1$. Then the potential $V_4(G^3)$ is given by (2·22). It is interesting to compare (2·22) with the corresponding potential obtained from the physically acceptable metric tensor in the classical theory. In general the latter potential depends on several parameters, but there is a unique choice of parameters, which makes the latter potential angle independent except V_4^{TT}. It is given by

$$V_4(G^3) = -\tfrac{3}{2}V(\curlywedge) - \tfrac{3}{4}V(\text{o-o-o-o}) + V_4^{TT}. \qquad (4\cdot1)$$

A discrepancy is found in the coefficients of the potential $V(\curlywedge)$ in (2·22) and (4·1), though the remaining parts coincide.

Recall that the potential (2·22) is obtained in Feynman's and de Donder's gauges, but the potential (4·1) is obtained under the coordinate conditions used in the previous paper.[4)] There we put the metric tensor $g_{\alpha\beta}$ in the form

$$g_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta}^{(2)} + h_{\alpha\beta}^{(3)} + \cdots, \qquad (4\cdot2)$$

where the numbers in superscripts with parentheses denote the order of c^{-1}. The simplest form of the coordinate conditions which lead to the physically acceptable metric tensor is given in the lowest order of c^{-1} by

$$\partial_i h_{4i}^{(3)} + \tfrac{1}{2}\partial_4 h_{ii}^{(2)} = 0$$

and

$$\partial_i h_{44}^{(2)} + \partial_j h_{ij}^{(2)} = 0. \qquad (4\cdot3)$$

Up to the post-Newtonian approximation the gravitational potentials obtained in both classical and quantized theories are identical, when parameters such as x are properly chosen. In that order of approximation the metric tensor becomes Minkowskian under any known coordinate conditions at a place infinitely remote from the region where the matter exists. In the post-post-Newtonian approximation, however, the situation is quite different. At spatial infinity the metric tensor diverges under de Donder's coordinate conditions, while the physically acceptable metric tensor becomes Minkowskian. It would be natural that the discrepancy between (2·22) and (4·1) first appears in the post-post-Newtonian approximation. Thus there is a possibility that the potential (4·1) is obtained in quantum theory by using the propagator (1·3) in which the field $h_{\alpha\beta}$ satisfies the conditions corresponding to (4·3). This possibility will be investigated on another occasion.

There remains to estimate the potential in the order of G^2p^2. We calculate the S-matrix elements for the scattering of three particles shown by the diagrams in Fig. 6. Using the propagator (2·2) with (2·5), we can get the potentials

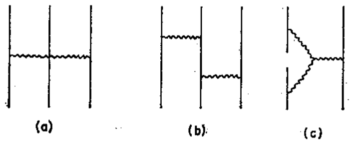

Fig. 6. The diagrams for the potential in the order of G^2p^2.

uniquely from the S-matrix elements for the diagrams (a) and (b). The situation is different for the diagram (c). In this diagram there exists a three-graviton vertex. This vertex shows that the graviton couples with the energy-momentum tensor of the graviton itself. Thus the S-matrix element for the diagram (c) contains, in general, the terms proportional to the energies transferred by the gravitons, and the terms proportional to the square of the energies. We have checked that there remain terms proportional to the energy transfers in the S-matrix element, though the terms proportional to the square of the transferred energies happen to cancel out. Consequently it is impossible to determine the potential uniquely in the order of G^2p^2 from the S-matrix element for the diagram (c). It is an interesting problem to study whether or not we can eliminate this dependence on energy transfers by using the propagator (1·3) derived under the conditions (4·3).

Appendix 1

The expression of the Lagrangian density (2·1) in terms of $h_{\alpha\beta}$

From the definitions (1·4) and $g_{\alpha\beta}g^{\beta\gamma}=\delta_\alpha{}^\gamma$, we have

$$g^{\alpha\beta}=\delta^{\alpha\beta}-\kappa h^{\alpha\beta}-\kappa^2 h^{\alpha\gamma}h_\gamma{}^\beta-\kappa^3 h^{\alpha\gamma}h^{\beta\delta}h_{\gamma\delta}+\kappa^4 h^{\alpha\gamma}h^{\beta\delta}h_{\gamma\varepsilon}h_\delta{}^\varepsilon+O(\kappa^5) \qquad (\text{A1}\cdot 1)$$

and

$$\sqrt{-g}=1+\frac{\kappa}{2}h_\alpha{}^\alpha+\frac{\kappa^2}{8}\{(h_{\alpha\alpha})^2-2h_{\alpha\beta}h_{\alpha\beta}\}$$
$$+\kappa^3\left\{\frac{1}{6}h_{\alpha\beta}h_{\alpha\gamma}h_{\beta\gamma}-\frac{1}{8}h_{\alpha\beta}h_{\alpha\beta}h_{\gamma\gamma}+\frac{1}{48}(h_{\alpha\alpha})^3\right\}+O(\kappa^4). \qquad (\text{A1}\cdot 2)$$

The Lagrangian density containing only the graviton field is given by

$$L_g=\frac{2}{\kappa^2}\sqrt{-g}g^{\alpha\beta}(\Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma}-\Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\gamma\delta}), \qquad (\text{A1}\cdot 3)$$

where

$$\Gamma^\gamma_{\alpha\delta}=\tfrac{1}{2}g^{\gamma\mu}(\partial_\alpha g_{\mu\delta}+\partial_\delta g_{\alpha\mu}-\partial_\mu g_{\alpha\delta}). \qquad (\text{A1}\cdot 4)$$

It is expanded in terms of $h_{\alpha\beta}$ as

$$L_g=I\left\{1+\frac{\kappa}{2}h_{\alpha\alpha}+\frac{\kappa^2}{8}(h_{\alpha\alpha}h_{\beta\beta}-2h_{\alpha\beta}h_{\alpha\beta})\right\}$$
$$+\kappa I_{\alpha\beta}\left(h_{\alpha\beta}-\kappa h_{\alpha\gamma}h_{\beta\gamma}+\frac{\kappa}{2}h_{\alpha\beta}h_{\gamma\gamma}\right)+\kappa^2 h_{\alpha\beta}h_{\gamma\delta}\Big(-h_{\gamma\rho,\alpha}h_{\delta\rho,\beta}$$
$$+h_{\gamma\delta,\alpha}h_{\rho\rho,\beta}+h_{\delta\rho,\alpha}h_{\beta\rho,\gamma}-h_{\beta\delta,\alpha}h_{\rho\rho,\gamma}-h_{\beta\rho,\alpha}h_{\gamma\delta,\rho}-h_{\alpha\beta,\gamma}h_{\delta\rho,\rho}$$
$$+2h_{\alpha\rho,\gamma}h_{\beta\delta,\rho}-\frac{1}{2}h_{\alpha\gamma,\rho}h_{\beta\delta,\rho}+\frac{1}{2}h_{\alpha\beta,\rho}h_{\gamma\delta,\rho}\Big)+O(\kappa^3), \qquad (\text{A1}\cdot 5)$$

where

$$I=-\tfrac{1}{2}h_{\beta\gamma,\alpha}h_{\beta\gamma,\alpha}+\tfrac{1}{2}h_{\beta\beta,\alpha}h_{\gamma\gamma,\alpha}+h_{\beta\gamma,\alpha}h_{\alpha\gamma,\beta}-h_{\alpha\beta,\alpha}h_{\gamma\gamma,\beta},$$
$$I_{\alpha\beta}=\tfrac{1}{2}h_{\gamma\delta,\alpha}h_{\gamma\delta,\beta}-\tfrac{1}{2}h_{\gamma\gamma,\alpha}h_{\delta\delta,\beta}-h_{\alpha\delta,\gamma}h_{\beta\gamma,\delta}+h_{\gamma\delta,\gamma}h_{\alpha\beta,\delta}$$
$$+h_{\alpha\delta,\gamma}h_{\beta\delta,\gamma}-h_{\alpha\beta,\gamma}h_{\delta\delta,\gamma}+h_{\beta\gamma,\alpha}h_{\delta\delta,\gamma}+h_{\delta\delta,\alpha}h_{\beta\gamma,\gamma}-2h_{\gamma\delta,\alpha}h_{\beta\delta,\gamma} \qquad (\text{A1}\cdot 6)$$

and

$$h_{\alpha\beta,\gamma}=\frac{\partial}{\partial x^\gamma}h_{\alpha\beta}.$$

The Lagrangian density for the scalar field with mass m interacting with the graviton field is given by

$$L_m=-\frac{1}{2}\sqrt{-g}\left(g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}+m^2\phi^2\right)$$
$$=-\frac{1}{2}\left(\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\alpha}+m^2\phi^2\right)+\frac{\kappa}{2}\left\{\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}h_{\alpha\beta}-\frac{1}{2}\left(\frac{\partial\phi}{\phi x^\alpha}\frac{\partial\phi}{\partial x^\alpha}+m^2\phi^2\right)h_{\beta\beta}\right\}$$
$$+\frac{\kappa^2}{4}\left[\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}(h_{\alpha\beta}h_{\gamma\gamma}-2h_{\alpha\gamma}h_{\beta\gamma})-\frac{1}{4}\left(\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\alpha}+m^2\phi^2\right)\right.$$
$$\left.\times\{(h_{\beta\beta})^2-2h_{\beta\gamma}h_{\beta\gamma}\}\right]$$

$$+\frac{\kappa^3}{16}\Big[\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}\{h_{\alpha\beta}(h_{\gamma\gamma}h_{\delta\delta}-2h_{\gamma\delta}h_{\gamma\delta})-4h_{\alpha\gamma}h_{\beta\gamma}h_{\delta\delta}+8h_{\alpha\gamma}h_{\beta\delta}h_{\gamma\delta}\}$$

$$-\frac{1}{6}\Big(\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\alpha}+m^2\phi^2\Big)(8h_{\alpha\gamma}h_{\alpha\delta}h_{\gamma\delta}-6h_{\beta\gamma}h_{\beta\gamma}h_{\delta\delta}+h_{\beta\beta}h_{\gamma\gamma}h_{\delta\delta})\Big]+O(\kappa^4). \qquad \text{(A1·7)}$$

Appendix 2

The expression of the Lagrangian density (2·1) in terms of $\beta_{\alpha\beta}$

Using the definitions (1·4) and (1·6) for the fields $h_{\alpha\beta}$ and $\beta^{\alpha\beta}$, we can express $h_{\alpha\beta}$ in terms of $\beta_{\alpha\beta}$ as

$$h_{\alpha\beta}=\beta_{\alpha\beta}+\kappa\Big\{\frac{1}{8}(\beta_{\gamma\gamma}\beta_{\delta\delta}-2\beta_{\gamma\delta}\beta_{\gamma\delta})\delta_{\alpha\beta}-\frac{1}{2}\beta_{\gamma\gamma}\beta_{\alpha\beta}+\beta_{\alpha\gamma}\beta_{\beta\gamma}\Big\}$$

$$+\kappa^2\Big\{\frac{1}{12}(-\beta_{\gamma\gamma}\beta_{\delta\delta}\beta_{\rho\rho}+3\beta_{\gamma\delta}\beta_{\gamma\delta}\beta_{\rho\rho}-2\beta_{\gamma\delta}\beta_{\gamma\rho}\beta_{\delta\rho})\delta_{\alpha\beta}$$

$$+\frac{1}{8}(3\beta_{\gamma\gamma}\beta_{\delta\delta}-2\beta_{\gamma\delta}\beta_{\gamma\delta})\beta_{\alpha\beta}-\beta_{\delta\delta}\beta_{\alpha\gamma}\beta_{\beta\gamma}+\beta_{\alpha\gamma}\beta_{\beta\delta}\beta_{\gamma\delta}\Big\}+O(\kappa^3). \qquad \text{(A2·1)}$$

The substitution of this relation into (A1·5) with (A1·6) and into (A1·7) leads to

$$L_g=-\frac{1}{2}\beta_{\beta\gamma,\alpha}\beta_{\beta\gamma,\alpha}+\frac{1}{2}\beta_{\beta\beta,\alpha}\beta_{\gamma\gamma,\alpha}+\beta_{\beta\gamma,\alpha}\beta_{\alpha\gamma,\beta}-\beta_{\alpha\beta,\alpha}\beta_{\gamma\gamma,\beta}$$

$$+\kappa\Big\{\Big(\frac{1}{2}\beta_{\gamma\delta,\alpha}\beta_{\gamma\delta,\beta}+\beta_{\alpha\delta,\gamma}\beta_{\beta\gamma,\delta}-\beta_{\alpha\delta,\gamma}\beta_{\beta\delta,\gamma}+\frac{1}{2}\beta_{\alpha\beta,\gamma}\beta_{\delta\delta,\gamma}-\beta_{\beta\gamma,\alpha}\beta_{\delta\delta,\gamma}\Big)\beta_{\alpha\beta}$$

$$+\Big(\frac{1}{4}\beta_{\beta\gamma,\alpha}\beta_{\beta\gamma,\alpha}-\frac{1}{4}\beta_{\beta\beta,\alpha}\beta_{\gamma\gamma,\alpha}-\frac{1}{2}\beta_{\beta\gamma,\alpha}\beta_{\alpha\gamma,\beta}+\frac{1}{2}\beta_{\alpha\beta,\alpha}\beta_{\gamma\gamma,\beta}\Big)\beta_{\delta\delta}\Big\}$$

$$+\kappa^2\Big\{\Big(\beta_{\gamma\rho,\alpha}\beta_{\delta\rho,\beta}-\frac{1}{2}\beta_{\gamma\delta,\alpha}\beta_{\rho\rho,\beta}-\frac{1}{2}\beta_{\alpha\gamma,\rho}\beta_{\beta\delta,\rho}+\frac{1}{4}\beta_{\alpha\beta,\rho}\beta_{\gamma\delta,\rho}\Big)\beta_{\alpha\beta}\beta_{\gamma\delta}$$

$$+\Big(\frac{1}{4}\beta_{\gamma\gamma,\alpha}\beta_{\delta\delta,\beta}-\beta_{\beta\gamma,\alpha}\beta_{\delta\delta,\gamma}+\beta_{\alpha\delta,\gamma}\beta_{\beta\gamma,\delta}-\beta_{\alpha\delta,\gamma}\beta_{\beta\delta,\gamma}+\beta_{\alpha\beta,\gamma}\beta_{\delta\delta,\gamma}\Big)$$

$$\times\beta_{\alpha\rho}\beta_{\beta\rho}+\Big(-\frac{1}{2}\beta_{\gamma\delta,\alpha}\beta_{\gamma\delta,\beta}+\beta_{\beta\gamma,\alpha}\beta_{\delta\delta,\gamma}-\beta_{\alpha\delta,\gamma}\beta_{\beta\gamma,\delta}+\beta_{\alpha\delta,\gamma}\beta_{\beta\delta,\gamma}$$

$$-\frac{3}{4}\beta_{\alpha\beta,\gamma}\beta_{\delta\delta,\gamma}\Big)\beta_{\alpha\beta}\beta_{\rho\rho}+\Big(\frac{1}{4}\beta_{\beta\gamma,\alpha}\beta_{\alpha\gamma,\beta}-\frac{1}{4}\beta_{\alpha\beta,\alpha}\beta_{\gamma\gamma,\beta}-\frac{1}{8}\beta_{\beta\gamma,\alpha}\beta_{\beta\gamma,\alpha}$$

$$+\frac{3}{16}\beta_{\beta\beta,\alpha}\beta_{\gamma\gamma,\alpha}\Big)\beta_{\rho\rho}\beta_{\sigma\sigma}-\frac{1}{8}\beta_{\beta\beta,\alpha}\beta_{\gamma\gamma,\alpha}\beta_{\rho\sigma}\beta_{\rho\sigma}\Big\}+O(\kappa^3). \qquad \text{(A2·2)}$$

The Lagrangian density for the scalar field is given by

$$L_m = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\alpha}+m^2\phi^2\right)+\kappa\left\{\frac{1}{2}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}\beta_{\alpha\beta}-\frac{1}{4}\left(\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\alpha}+m^2\phi^2\right)\beta_{\beta\beta}\right\}$$
$$+\kappa^2 m^2\phi^2\left\{-\frac{1}{16}(\beta_{\alpha\alpha})^2+\frac{1}{8}\beta_{\alpha\beta}\beta_{\alpha\beta}\right\}+\kappa^3 m^2\phi^2\left\{\frac{1}{96}(\beta_{\alpha\alpha})^3-\frac{1}{16}\beta_{\alpha\alpha}\beta_{\beta\tau}\beta_{\beta\tau}\right.$$
$$\left.+\frac{1}{12}\beta_{\alpha\beta}\beta_{\beta\tau}\beta_{\tau\alpha}\right\}+O(\kappa^4). \tag{A2·3}$$

References

1) Y. Iwasaki, Lett. Nuovo Cim. **1** (1971), 783; Prog. Theor. Phys. **46** (1971), 1587.
   K. Hiida and M. Kikugawa, Prog. Theor. Phys. **46** (1971), 1610.
2) K. Hiida and H. Okamura, Prog. Theor. Phys. **47** (1972), 1743.
3) T. de Donder, *La gravifique einsteinienne* (Paris, 1921).
4) T. Ohta, H. Okamura, T. Kimura and K. Hiida, Prog. Theor. Phys. **50** (1973), 492.
5) V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon Press, 1964).
   A. Papapetrou, Proc. Phys. Soc. **A64** (1951), 57, 302.
   M. Carmeli, Nuovo Cim. **37** (1965), 842.
6) T. Ohta, H. Okamura, T. Kimura and K. Hiida, to be submitted.