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Perturbation of Dirichlet forms and stability of fundamental solutions

Masaki Wada *

March 4, 2013

1 Notions and Notations

The jump type Markov process on $\mathbb{R}^d$ describes the random motion of particle in the state space $\mathbb{R}^d$. We can characterize this by means of Dirichlet form $\mathscr{E}$ as follows:

$$\mathscr{E}(u,v) = \iint_{\mathbb{R}^d\times\mathbb{R}^d}(u(y)-u(x))(v(y)-v(x))J(x,y)dxdy.$$

Here $J(x,y)$ is a symmetric function and describes the frequency of jump from x to y. In particular, if $J(x,y) \asymp 1/|x-y|^{d+\alpha}$ holds for some $0<\alpha<2$, we call the associated Markov process α-stable-like. If $J(x,y) \asymp \exp(-m|x-y|)/|x-y|^{d+\alpha}$ holds for some $0<\alpha<2$ and $m>0$, we call the associated Markov process relativistic α-stable-like. In the sequel, we deal with these two kinds of jump Markov process.

2 Preceding Results

The transition density function $p(t,x,y)$ is one of the important notions in order to analyze Markov processes. This is the probability with which the particle in x at time 0 exists in y at time t. Moreover it is known that $p(t,x,y)$ coincides with the fundamental solution of $\partial u/\partial t = \mathscr{L}u$, where $\mathscr{L}$ is a non-local operator satisfying

$$\mathscr{E}(u,v) = -\int_{\mathbb{R}^d}\mathscr{L}u(x)v(x)dx.$$

Z. Q. Chen, P. Kim and T. Kumagai showed that $p(t,x,y)$ admits the two-sided estimates as follows:

$$C_1\phi(C_2t, C_3|x-y|) \le p(t,x,y) \le C_4\phi(C_5t, C_6|x-y|),$$

where ϕ is an appropriate function and C_i's are positive constants [1, 2].

3 Problem in Consideration

In the sequel we assume that the Markov process is transient, namely it holds that

$$G(x,y) := \int_0^\infty p(t,x,y) < \infty.$$

First we define some classes of small measure μ.

Definition 1. *(i) A measure μ is said to be in Kato class if it holds that*

$$\lim_{a\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le a}G(x,y)\mu(dy) = 0 \tag{1}$$

(ii) A Kato class measure μ is said to be Green tight if it holds that

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^d}\int_{|y|>r}G(x,y)\mu(dy) = 0. \tag{2}$$

Here we consider the perturbation of Dirichlet form $\mathscr{E}$ by Green tight measure μ as follows:

$$\mathscr{E}^\mu(u,u) = \mathscr{E}(u,u) - \int_{\mathbb{R}^d}u^2d\mu. \tag{3}$$

It is known that $\mathscr{E}^\mu$ (or corresponding operator $\mathscr{L}^\mu := \mathscr{L}+\mu$) also admits the fundamental solution $p^\mu(t,x,y)$. We consider the condition on μ under which $p^\mu(t,x,y)$ admits the same two sided estimates as $p(t,x,y)$ up to the choice of positive constants. We call this phenomenon stability of fundamental solution.

4 Main Result

The main result is the necessary and sufficient condition on μ for the stability of fundamental solution. The precise statement is as follows:

Theorem 2. *(W. 2012)*
Assume that the Green tight measure μ satisfies

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x,y)\mu(dx)\mu(dy) < \infty.$$

Then the stability of fundamental solution holds if and only if

$$\inf\{\mathscr{E}(u,u) \mid u\in\mathscr{F}, \int_{\mathbb{R}^d}u^2d\mu = 1\} > 1, \tag{4}$$

where $\mathscr{F}$ is the domain of the Dirichlet form $\mathscr{E}$.

Note that the formula (4) describes the smallness of measure μ compared with the initial Dirichlet form $\mathscr{E}$. Furthermore, this result is the same as that in Takeda [3], which deals with the same problem in the framework of transient Brownian motion.

References

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*Mathematical Institute, Tohoku University. Mail: sb1d14@math.tohoku.ac.jp