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Perturbation of positive semigroups

By

Wolfgang Arendt and Abdelaziz Rhandi

Introduction. The purpose of this note is to study perturbations of generators of positive semigroups by positive operators.

Let E be a complex Banach lattice and A be a linear operator on E with domain $D(A)$. We say that A is resolvent positive if there exists $w \in \mathbb{R}$ such that $(\lambda - A): D(A) \longrightarrow E$ is bijective and $(\lambda - A)^{-1}$ is a positive operator on E for all $\lambda > w$. Note that the generator of a positive semigroup is resolvent positive.

Assume that A generates a positive semigroup (by which we always mean a C_0-semigroup) and $B: D(\mathrm{A}) \longrightarrow E$ is linear and positive such that $A + B$ (with domain $D(A + B) = D(A)$) is resolvent positive.

Then it was shown by Desch [8] that $A + B$ generates a positive semigroup whenever E is a space L^1. A simple proof is given by Voigt [20].

If E is an L^p-space, $1 < p < \infty$, then the assertion is false, in general (see [4]). However, we show in Section 1 that in the case where the semigroup generated by A is holomorphic, also $A + B$ generates a holomorphic semigroup without any restriction on the space.

Furthermore, we prove in Section 2 that $A + B$ generates a semigroup whenever B is a positive rank-one perturbation of A. This is remarkable in view of a recent result of Desch-Schappacher [9]. If the semigroup generated by A is not holomorphic, there always exists a (necessarily non positive) rank-one perturbation B such that $A + B$ is not a generator.

In Section 3 we give a criterion for perturbation by multiplication operators which, in view of the Sobolev embedding theorems, is particularly useful for elliptic operators. As an illustrating example we consider Schrödinger operators.

In Section 4 the results are applied to systems of evolution equations, which obtained special attention recently (see [14]).

Concerning terminology and basic results we follow [17] and [13].

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1. Perturbation of holomorphic semigroups. Let E be a complex Banach lattice (see [17]) and let A be the generator of a positive semigroup $T = (T(t))_{t \geqq 0}$ on E. We consider a positive linear operator $B: D(A) \longrightarrow E$ (i.e. $Bf \geqq 0$ for all $f \in D(A)_+ := D(A) \cap E_+$).

In this section we prove the following perturbation result.

Theorem 1.1. *Assume that the semigroup generated by A is holomorphic.*
If $A + B$ is resolvent positive, then $A + C$ generates a holomorphic semigroup whenever $C: D(A) \longrightarrow E$ is a linear mapping satisfying

$$|Cu| \leqq Bu \quad (u \in D(A)_+). \tag{1.1}$$

Remark. In particular, $A + B$ generates a positive holomorphic semigroup.
Using the classical perturbation result one would obtain this under the hypothesis that $\lim_{\lambda \to \infty} \|BR(\lambda, A)\| = 0$, whereas the assumption that $A + B$ be resolvent positive can be rephrased by saying $\lim_{\lambda \to \infty} r(BR(\lambda, A)) < 1$ (where $r(S)$ denotes the spectral radius of a bounded operator), see [20, Theorem 1.1].

Remark. We emphasize that it does not suffice to assume the existence of the resolvent of $Z + Y$ on a half-plane (without any norm or order condition). In order to see this it suffices to take the generator Z of a holomorphic semigroup on a Banach space G with empty spectrum and $Y = -2Z$. Then $Y: (D(Z), \|\cdot\|_Z) \longrightarrow G$ is continuous, $Z + Y$ has empty spectrum, but $Z + Y$ does not generate a semigroup. As a concrete example one may take the generator of the Riemann-Liouville semigroup on $L^p(0, 1)\,(1 \leqq p < \infty)$, see [10, Sec 23.16].

In the situation of Theorem 1.1 the semigroup generated by $A + C$ is dominated by the one generated by $A + B$. More generally, the following holds.

Theorem 1.2. *Assume that $A + B$ generates a positive semigroup $(U(t))_{t \geqq 0}$. If $C: D(A) \longrightarrow E$ is linear and satisfies* (1.1), *then $A + C$ generates a semigroup $(V(t))_{t \geqq 0}$ satisfying*

$$|V(t) f| \leqq U(t)|f| \quad (f \in E) \quad \textit{for all } t \geqq 0. \tag{1.2}$$

We will use the following notation. For an operator Z we denote by

$$s(Z) = \sup \{\mathrm{Re}(\lambda): \lambda \in \sigma(Z)\}$$

the *spectral bound* of Z. If Z is resolvent positive, then

$$0 \leqq R(\mu, Z) \leqq R(\lambda, Z) \quad \text{for } s(Z) < \lambda \leqq \mu \tag{1.3}$$

(see for example [13, B-II-Lemma 1.9]).
If Z generates a semigroup $(S(t))_{t \geqq 0}$ we denote by $w(Z)$ the *growth bound* (or *type*) of S; i.e.

$$\begin{aligned} w(Z) &= \inf \left\{ w \in \mathbb{R}: \sup_{t \geqq 0} \|\exp(-wt) S(t)\| < \infty \right\} \\ &= \inf \left\{ w > s(Z): \sup_{\lambda > w, n \in \mathbb{N}} \|(\lambda - w)^n R(\lambda, Z)^n\| < \infty \right\}. \end{aligned}$$

Now we establish some auxiliary results.

Lemma 1.3. *If $Q, R: E \longrightarrow E$ are linear such that $|Qf| \leqq Rf$ for all $f \in E_+$, then*

$$|Qf| \leqq R|f| \quad \text{for all } f \in E.$$

For a proof we refer to [17, p. 234].

Lemma 1.4. *Assume that $A + B$ is resolvent positive and let $C: D(A) \longrightarrow E$ be linear and satisfy* (1.1). *Let $\lambda \in \mathbb{C}$ such that $\lambda_0 := \operatorname{Re}(\lambda) > \max\{w(A), s(A+B)\}$. Then*

$$|[CR(\lambda, A)]^n u| \leqq [BR(\lambda_0, A)]^n |u| \quad (u \in E), \quad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

Moreover, $r(CR(\lambda, A)) < 1$, $\lambda \in \varrho(A + C)$ and

$$R(\lambda, A + C) = R(\lambda, A) \sum_{n=0}^{\infty} [CR(\lambda, A)]^n. \tag{1.5}$$

Proof. By a result of Voigt [20] one has $r(BR(\lambda_0, A)) < 1$ and

$$R(\lambda_0, A + B) = R(\lambda_0, A) \sum_{n=0}^{\infty} [BR(\lambda_0, A)]^n. \tag{1.6}$$

Let $\mu > s(A)$. Then by (1.1) $CR(\mu, A)$ and $BR(\mu, A): E \longrightarrow E$ are linear and satisfy

$$|CR(\mu, A)u| \leqq BR(\mu, A)u \quad \text{for } u \in E_+.$$

So by Lemma 1.3,

$$|CR(\mu, A)u| \leqq BR(\mu, A)|u| \quad (u \in E). \tag{1.7}$$

Since $R(\lambda, A) = \int_0^\infty \exp(-\lambda t)\, T(t)\, dt$, one has $|R(\lambda, A)u| \leqq R(\lambda_0, A)|u|$ for all $u \in E$.
Hence

$$\begin{aligned} |CR(\mu, A) R(\lambda, A)u| &\leqq BR(\mu, A)\, |R(\lambda, A)u| \\ &\leqq BR(\mu, A) R(\lambda_0, A)\, |u| \quad (u \in E). \end{aligned} \tag{1.8}$$

We consider $D(A)$ with the graph norm $\|u\|_A = \|u\| + \|Au\|$.

Let $\mu > s(A)$. Then $BR(\mu, A): E \longrightarrow E$ is continuous as a positive linear mapping (see [17, 5.3 p. 84]). It follows from (1.7) that $CR(\mu, A): E \longrightarrow E$ is continuous. Since $\mu - A$ is an isomorphism from $(D(A), \|.\|_A)$ onto E, it follow that $C: (D(A), \|.\|_A) \longrightarrow E$ is continuous as well.

For $f \in D(A)$ one has $\lim_{\mu \to \infty} \|\mu R(\mu, A) f - f\|_A = 0$.

So we conclude from (1.8)

$$\begin{aligned} |CR(\lambda, A)u| &= \lim_{\mu \to \infty} |\mu CR(\mu, A) R(\lambda, A)u| \\ &\leqq \lim_{\mu \to \infty} \mu BR(\mu, A) R(\lambda_0, A)|u| \\ &= BR(\lambda_0, A)\,|u| \quad (u \in E). \end{aligned}$$

This is (1.4) for $n = 1$. For $n \in \mathbb{N}$ the inequality follows by iteration.

As a consequence of (1.4) one has

$$r(CR(\lambda, A)) \leqq r(BR(\lambda_0, A)) < 1$$

and so

$$(I - CR(\lambda, A))^{-1} = \sum_{n=0}^{\infty} [CR(\lambda, A)]^n \quad \text{exists}.$$

Consequently, $(\lambda - (A + C)) = (I - CR(\lambda, A))(\lambda - A)$ is invertible and (1.5) holds. □

For the proof of Theorem 1.1 we recall that a densely defined operator Z generates a holomorphic semigroup if and only if there exist $M \geqq 0$, $w > w(Z)$ such that

$$\|\lambda R(\lambda, Z)\| \leqq M \quad (\mathrm{Re}(\lambda) \geqq w). \tag{1.9}$$

(This follows for example from [13, A-II-Theorem 1.14].)

Proof of Theorem 1.1. There exist $M \geqq 0$, $w > \max\{w(A), s(A + B)\}$ such that $\|\lambda R(\lambda, A)\| \leqq M\,(\mathrm{Re}(\lambda) \geqq w)$.

It follows from Lemma 1.4 that

$$\begin{aligned}\left|\sum_{n=0}^{\infty} [CR(\lambda, A)]^n u\right| &\leqq \sum_{n=0}^{\infty} [BR(\mathrm{Re}(\lambda), A)]^n |u| \\ &\leqq \sum_{n=0}^{\infty} [BR(w, A)]^n |u|. \quad (u \in E, \mathrm{Re}(\lambda) \geqq w)\end{aligned}$$

(where (1.3) was used for the last inequality). Hence

$$C = \sup_{\mathrm{Re}(\lambda) \geqq w} \left\|\sum_{n=0}^{\infty} [CR(\lambda, A)]^n\right\| < \infty.$$

Using this it follows that

$$\|\lambda R(\lambda, A + C)\| = \left\|\lambda R(\lambda, A) \sum_{n=0}^{\infty} [CR(\lambda, A)]^n\right\| \leqq Mc \quad (\mathrm{Re}(\lambda) \geqq w).$$

So $A + C$ generates a holomorphic semigroup. □

Proof of Theorem 1.2. Let $w > \max\{w(A), w(A + B)\}$.

It follows from Lemma 1.5 that

$$\begin{aligned}|R(\lambda, A + C)u| &= \left|R(\lambda, A) \sum_{n=0}^{\infty} [CR(\lambda, A)]^n u\right| \\ &\leqq R(\lambda, A) \sum_{n=0}^{\infty} [BR(\lambda, A)]^n |u| \\ &= R(\lambda, A + B)|u| \quad (u \in E) \quad \text{for all } \lambda > w.\end{aligned}$$

Iterating this one obtains

$$|R(\lambda, A+C)^n u| \leqq R(\lambda, A+B)^n |u| \quad (u \in E) \tag{1.10}$$

for all $\lambda > w$. Since $w > w(A+B)$ one has

$$\sup_{\substack{\lambda > w \\ n \in \mathbb{N}}} \|[(\lambda - w) R(\lambda, A+B)]^n\| < \infty .$$

It follows from (1.10) that

$$\sup_{\substack{\lambda > w \\ n \in \mathbb{N}}} \|[(\lambda - w) R(\lambda, A+C)]^n\| < \infty .$$

So by the Hille-Yosida theorem $A + C$ generates a semigroup $(V(t))_{t \geqq 0}$. Letting $\lambda = 1/t$ in (1.10) one obtains

$$\begin{aligned} |V(t)u| &= \lim_{n \to \infty} |(I - (t/n)(A+C))^{-n} u| \\ &\leqq \lim_{n \to \infty} (I - (t/n)(A+B))^{-n} |u| = U(t)|u| \end{aligned}$$

$(u \in E,\ t \geqq 0)$. □

Remark. If in Theorem 1.2 the operator C is positive, then by (1.5) $R(\lambda, A+C) \geqq 0$ for large λ and so $V(t) \geqq 0$ for $t \geqq 0$. We would like to mention the following more general result of Bidard-Zerner [6]. Assume that Z_1, Z_2, Z_3 are (unbounded) operators on E such that $D(Z_1) = D(Z_2) = D(Z_3)$ and $Z_1 f \leqq Z_2 f \leqq Z_3 f$ for all $f \in D(Z_1)_+$.

Assume that $\lambda \in \varrho(Z_1) \cap \varrho(Z_3) \cap \mathbb{R}$ such that $R(\lambda, Z_1) \geqq 0$, $R(\lambda, Z_3) \geqq 0$. Then $\lambda \in \varrho(Z_2)$ and $R(\lambda, Z_2) \geqq 0$.

2. Perturbation on *AL*-spaces and perturbation by finite rank operators. Let E be a (real or complex) Banach lattice, A the generator of a positive semigroup $(T(t))_{t \geqq 0}$ on E and $B: D(A) \longrightarrow E$ a positive linear mapping. In this section we allow $(T(t))_{t \geqq 0}$ to be arbitrary but assume restrictive conditions on E or the perturbation B.

Recall that E is an AL-space of $\|u + v\| = \|u\| + \|v\|$ whenever $u, v \in E_+$ (see [17]). Any space $L^1(\mu)$ is an AL-space.

The following result is due to Desch [8].

Theorem 2.1. *Assume that E is an AL-space.*
If $A + B$ is resolvent positive, then $A + B$ generates a positive semigroup.

This result is no longer true on $L^p (1 < p < \infty)$ or $C_0(\Omega)$ (Ω locally-compact but not compact); see [4] and Section 4. However, we obtain perturbation results valid in any space if we consider perturbations of finite rank.

By $D(A)'_+$ we denote the cone of all positive linear forms on $D(A)$ (i.e. linear mappings $\varphi: D(A) \longrightarrow \mathbb{R}$ satisfying $\varphi(u) \geqq 0$ whenever $u \in D(A)_+$).

Theorem 2.2. *Suppose that there exist $\varphi \in D(A)'_+$, $g \in E_+$ such that*

$$Bf = \varphi(f)g \quad (f \in D(A)).$$

Then $A + B$ generates a positive semigroup on E.

Example 2.3. Let $E = L^p(0, 1)$, $1 \leqq p < \infty$ and let A be defined by $Af = -f'$, $D(A) = \{f \in L^p(0, 1): \exists f' \in L^p(0, 1)$ such that $f(x) = \int_0^x f'(y)\,dy\ (x \in (0, 1))\}$.

Then A generates a positive semigroup. Let μ be a bounded positive measure on $[0, 1]$, $g \in E_+$ and define $B: D(A) \longrightarrow E$ by $Bf = \int_0^1 f(x)\,d\mu(x)g$. Then $A + B$ generates a positive semigroup.

Theorem 2.2 can be extended to perturbations of the following type. The mapping $B: D(A) \longrightarrow E$ is called a *regular finite rank perturbation of A* if there exist $\varphi_i \in \operatorname{span} D(A)'_+$, $g_i \in E\ (i = 1 \ldots n)$ such that

$$Bf = \sum_{i=1}^{n} \varphi_i(f)g_i \quad (f \in D(A)).$$

Corollary 2.4. *If B is a regular finite rank perturbation of A, then $A + B$ generates a semigroup.*

In view of Theorem 1.2 this is an immediate consequence of Theorem 2.2.

Theorem 2.2 and its corollary are remarkable in the context of results by Desch and Schappacher [9].

Let Z be the generator of a semigroup on a Banach space G. Consider $D(Z)$ with the graph norm and denote by $D(Z)'$ its dual space. An operator $C: D(Z) \longrightarrow G$ is called a rank 1 perturbation of Z if $D(C) = D(Z)$ and there exist $\varphi \in D(Z)'$ and $g \in G$ such that $Cf = \varphi(f)g$.

Then the following is proved in [9].

1. If Z generates an analytic semigroup, then so does $Z + C$ for any rank-one perturbation C.
2. If $Z + C$ generates a semigroup for all rank-one perturbations C of Z, then the semigroup generated by Z is analytic.

So Corollary 2.4 shows in particular, that $D(A)'_+ - D(A)'_+ \neq D(A)'$ if the semigroup generated by A is not analytic (in other words, the cone $D(A)_+$ is not normal in the ordered Banach space $D(A)$). On the other hand, if is easy to see that $D(A)_+$ is normal if A generates a multiplication semigroup (in the sense of [13, C-II-Sec 5]).

Proof of Theorem 2.2. There exist $\varphi \in D(A)'_+$, $g \in E_+$ such that $Bf = \varphi(f)g$ for all $f \in D(A)$.

a) We first show that $A + B$ is resolvent positive. Since $\lim_{\mu \to \infty} \|R(\mu, A)g)\|_A = 0$, it follows that there exists $w \in \mathbb{R}$ such that $\varphi(R(\mu, A)g) < 1$ for all $\mu \geqq w$. Define $R(\mu) \in L(E)$ by $R(\mu)f = R(\mu, A)f + [\varphi(R(\mu, A)f)/(1 - \varphi(R(\mu, A)g))]R(\mu, A)g\ (f \in E)$ for all $\mu \geqq w$.

Then $R(\mu)$ is a positive operator, and it is easy to see that $R(\mu) = (\mu - A - B)^{-1}$.

b) Now Voigt's proof of Theorem 2.1 can be adapted to the situation considered here. We merely indicate the necessary alterations in [20, Section 2].

At first, one assumes that there exists $\lambda > s(A)$ such that $\| BR(\lambda, A)\| < (1/2)$. Let $\alpha > 0$. Then for $f \in D(A)_+$ one has

$$\begin{aligned}\int_0^\alpha \| B\exp(-\lambda t)\, T(t) f\| \, dt &= \int_0^\alpha \exp(-\lambda t)\, \varphi(T(t) f)\, dt \, \|g\| \\ &\leqq \varphi(R(\lambda, A) f)\, \|g\| \\ &= \| BR(\lambda, A) f\| \\ &\leqq \| BR(\lambda, A)\| \, \|f\|.\end{aligned}$$

Arguing as in the proof of [20, Lemma 2.1] one concludes

$$\int_0^\alpha \| B\exp(-\lambda t)\, T(t) f\| \, dt \leqq \gamma \, \|f\| \quad (f \in D(A)) \quad \text{where} \quad \gamma := 2\, \| BR(\lambda, A)\| < 1.$$

So $A + B$ generates a semigroup by [21, Theorem 1]. The general case follows by replacing B by $(1/n)B$ and A by $A + (j/n)B$ successively ($j = 0, \ldots, n-1$) where $n \in \mathbb{N}$ such that $\| BR(\lambda, A + B)\| < (n/2)$ for a fixed $\lambda > s(A + B)$. □

Remark (Ordered Banach spaces). The lattice property is not essential in the results (but convenient, in particular, for domination properties). One may consider more generally a positive semigroup on an ordered Banach space with generating and normal cone (see [5]). Then Theorem 2.2 remains valid.

Also Desch's theorem (Theorem 2.1) holds, if we suppose that the norm is additive on the positive cone (generalizing AL-spaces). Our proof (resp. Voigt's proof for Desch's theorem) go through without alterations.

Theorem 1.1 remains valid for $B = C$; however, some modifications (using [5], Corollary 1.7.5 for example) are necessary.

3. Perturbation by multiplication operators. Let (Ω, μ) be a measure space, $1 \leqq p < \infty$ and let $T = (T(t))_{t \geqq 0}$ be a positive semigroup on $L^p(\Omega, \mu)$ with generator A. We assume that

(3.1) $\qquad D(A) \subset L^q(\Omega), \quad \text{where } p < q \leqq \infty.$

Let $(1/r) + (1/q) = (1/p)$ (so that $L^r L^q \subset L^p$).

Theorem 3.1. *Let $V \in L^r(\Omega, \mu)$ and $B: D(A) \longrightarrow E$ be given by $Bf = Vf$.*

a) *If T is holomorphic, then $A + V$ generates a holomorphic semigroup on $L^p(\Omega, \mu)$.*
b) *If $p = 1$, then $A + V$ generates a semigroup on $L^1(\Omega, \mu)$.*

We will show that B is relatively bounded with respect to A with relative bound 0; i.e.

(3.2) $$\lim_{\lambda \to \infty} \| VR(\lambda, A) \| = 0 .$$

So a) actually follows by the classical perturbation result and b) from Desch's theorem.

Proof. Let $F = \{V \in L^r(\Omega, \mu): \lim_{\lambda \to \infty} \| VR(\lambda, A) \| = 0\}$.

Since $L^\infty \cap L^r \subset F$, it suffices to show that F is closed. Let $V_n \in F$, $V \in L^r$ such that $\| V_n - V \|_{L^r} \longrightarrow 0$ $(n \longrightarrow \infty)$. By the closed graph theorem the embedding (3.1) is continuous if $D(A)$ carries the graph norm. Moreover,

$$\begin{aligned} \| VR(\lambda, A) \| &\leqq \| (V - V_n) R(\lambda, A) \| + \| V_n R(\lambda, A) \| \\ &\leqq \| V - V_n \|_{L^r} \| R(\lambda, A) \|_{L(L^p, L^q)} + \| V_n R(\lambda, A) \| \\ &\leqq \text{const} \| V - V_n \|_{L^r} \| R(\lambda, A) \|_{L(L^p, D(A))} + \| V_n R(\lambda, A) \| \\ &\leqq \text{const} \| V - V_n \|_{L^r} + \| V_n R(\lambda, A) \| \end{aligned}$$

for all $\lambda > s(A)$, $n \in \mathbb{N}$.

Hence

$$\lim_{\lambda \to \infty} \| VR(\lambda, A) \| = 0$$

(and $(\lambda - A - B)^{-1} = (\lambda - A)^{-1} \sum_{n=0}^{\infty} [VR(\lambda, A)]^n$ exists and is positive for λ sufficiently large). □

Remark. One cannot omit condition (3.1). To see this it suffices to take $0 \leqq m \in L^p \backslash L^\infty$, $Af = -mf$ with $D(A) = \{f \in L^p : mf \in L^p\}$ and $V = 2m$.

We consider concrete examples.

1. Schrödinger semigroups. a) Let $1 < p < \infty$ and define A on $L^p(\mathbb{R}^N)$ by $D(A) = W^{2,p}(\mathbb{R}^N)$, $Af = \Delta f$. Then A generates the Gaussian semigroup which is holomorphic and positive. We conclude from Theorem 3.1 that $A + V$ generates a holomorphic semigroup on $L^p(\mathbb{R}^N)$ whenever $0 \leqq V \in L^r(\mathbb{R}^N)$ for some r satisfying $r \geqq \max\{p, (N/2)\}$ if $p \neq (N/2)$ and $r > (N/2)$ if $p = (N/2)$. In fact, by the Sobolev imbedding theorem one has

$$W^{2,p} \subset \begin{cases} L^\infty & \text{if } (N/2) < p \\ \bigcup\limits_{p \leqq q < \infty} L^q & \text{if } (N/2) = p \\ L^q & \text{for } (1/q) = ((1/p) - (2/N)) \quad \text{if } (N/2) > p . \end{cases}$$

So the claim follows from Theorem 3.1.

b) Kato [11] defines the operator $\Delta + V$ on $L^1(\mathbb{R}^N)$ considering a well-known class of potentials K_N, where

$$K_N = \left\{V \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : VD(A_1) \subset L^1 \quad \text{and} \quad \lim_{\lambda \to \infty} \|VR(\lambda, A_1)\| = 0\right\}$$

and A_1 is the Laplacian on $L^1(\mathbb{R}^N)$ (i.e. $D(A_1) = \{f \in L^1 : \Delta f \in L^1\}$, $A_1 f = \Delta f$).

So by Theorem 1.1 $A_1 + V$ generates a positive holomorphic semigroup on L^1 whenever $0 \leqq V \in K_N$.

It is shown by Kato [11] that this semigroup interpolates on $L^p(\mathbb{R}^N)$ $(1 \leqq p < \infty)$. This gives another proof of the result of a) in the case $p > (N/2)$ since then $L^p(\mathbb{R}^N) \subset K_N$ (see [2, Proposition 4.3]).

Remark. A perturbation theory for a larger class $\hat{K}_N \supset K_N$ has been developped by Voigt [19].

2. Bounded domain. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary of class C^∞.

Then the operator A defined by $D(A) = W_0^{1,p} \cap W^{2,p}$, $Af = \Delta f$, generates a positive holomorphic semigroup on $L^p(\Omega)$ $(1 < p < \infty)$ (see [1]).

By the same argument as in 1.a) one sees that $A + V$ generates a positive holomorphic semigroup on $L^p(\Omega)$ whenever $0 \leqq V \in L^r(\Omega)$ where $r \geq \max\{p, (N/2)\}$ if $p \neq (N/2)$ and $r > (N/2)$ if $p = (N/2)$.

3. Elliptic operators. In Example 1 and 2 one can replace the Laplacian by any strictly elliptic real differential operator of second order with (sufficiently) regular coefficients. Indeed, those operators, with domain $W^{2,p}(\mathbb{R}^N)$ (resp. $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$), generate a positive holomorphic semigroup on $L^p(\mathbb{R}^N)$ (resp. $L^p(\Omega)$), $1 < p < \infty$.

4. Systems of evolution equations. Let A and D be generators of positive semigroups on a Banach lattice E (resp. F), and let $B: D(D) \longrightarrow E$ and $C: D(A) \longrightarrow F$ be linear and positive. We consider the operator

$$\mathscr{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

on $E \times F$ with domain $D(\mathscr{A}) = D(A) \times D(D)$.

Proposition 4.1. *The operator $\mathscr{A}$ is resolvent positive if and only if there exists $\lambda > \max\{s(A), s(D)\}$ such that $r(BR(\lambda, D)CR(\lambda, A)) < 1$.*

Remark. One has $r(BR(\lambda, D)CR(\lambda, A)) = r(CR(\lambda, A)BR(\lambda, D))$.

Proof. a) Assume that $\mathscr{A}$ is resolvent positive. Let $\lambda > s(\mathscr{A})$. We write $\mathscr{A} = \mathscr{A}_1 + \mathscr{B}$ where

$$\mathscr{A}_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad \mathscr{B} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},$$

both with domain $D(A) \times D(D)$. It follows from [20, Theorem 1.1] that

$$\mathscr{B} R(\lambda, \mathscr{A}_1) = \begin{bmatrix} 0 & BR(\lambda, D) \\ CR(\lambda, A) & 0 \end{bmatrix} \text{ has spectral radius } < 1 .$$

In particular,

$$Q(\lambda) := \begin{bmatrix} I & -BR(\lambda, D) \\ -CR(\lambda, A) & I \end{bmatrix}$$

is invertible and has positive inverse. It follows from [15, Lemma 2.1 and (2.1)] that $\Delta(\lambda) := (I - BR(\lambda, D)\, CR(\lambda, A))$ is invertible and $\Delta(\lambda)^{-1} \geqq 0$. Since $BR(\lambda, D)\, CR(\lambda, A)$ is a positive operator one concludes $r(BR(\lambda, D)\, CR(\lambda, A)) < 1$ (by [16, App. 2.3]).

b) Assume that $\lambda_0 > \max\{s(A), s(D)\}$ such that $r(BR(\lambda_0, D)\, CR(\lambda_0, A)) < 1$. Since $R(\lambda, D)$ and $R(\lambda, A)$ are decreasing in λ (by (1.3)), it follows that $r(BR(\lambda, D)\, CR(\lambda, A)) < 1$ for all $\lambda \geqq \lambda_0$. Consequently, by [15, Lemma 2.1] the operator $Q(\lambda)$ is invertible for $\lambda \geqq \lambda_0$ and $Q(\lambda)^{-1} \geqq 0$ (this can be seen from [15, (2.1)]). Thus

$$(\lambda - \mathscr{A}) = Q(\lambda) \begin{bmatrix} \lambda - A & 0 \\ 0 & \lambda - D \end{bmatrix}$$

has positive inverse for $\lambda \geqq \lambda_0$. □

Corollary 4.2. *$\mathscr{A}$ is resolvent positive whenever B or C is bounded.*

Using the results of Section 1 and 2 one obtains the following conclusion.

Theorem 4.3. *Assume that $r(BR(\lambda, D)\, CR(\lambda, A)) < 1$ for some $\lambda > \max\{s(A), s(D)\}$.*
a) *If the semigroups generated by A and D are holomorphic, then $\mathscr{A}$ generates a holomorphic positive semigroup.*
b) *If E and F are AL-spaces, then $\mathscr{A}$ generates a positive semigroup.*

Remark. A systematic investigation of matrices of unbounded operators is given by Nagel [14], [15]. The problem under which condition a matrix of unbounded operators generates a positive semigroup is treated by [7].

We conclude by an example where $\mathscr{A} + \mathscr{B}$ does not generate a semigroup.
Let $F = C_0(0, 1] = \{f \in C[0, 1] : f(0) = 0\}$, $Af = -f'$,

$$D(A) = \{f \in C^1[0, 1] : f'(0) = f(0) = 0\}.$$

Then A generates a positive contraction semigroup on F.
Let $B : D(A) \longrightarrow F$ be given by

$$Bf(x) = (1/x) f(x) \quad \text{if } x \neq 0 \quad \text{and } Bf(0) = 0 .$$

Then $\mathscr{A} + \mathscr{B} = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$ on $E = F \times F$ is resolvent positive but does not generate a semigroup $\left(\text{where } \mathscr{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \text{ and } \mathscr{B} = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right)$.

Proof. We have

$$(\lambda - \mathscr{A} - \mathscr{B})^{-1} = \begin{bmatrix} R(\lambda, A) & 0 \\ R(\lambda, A)\, B R(\lambda, A) & R(\lambda, A) \end{bmatrix}$$

for $\lambda > s(A) = -\infty$, where $(R(\lambda, A) f)(x) = \exp(-\lambda x) \int_0^x \exp(\lambda y) f(y)\, dy$.

For $f \in F$ we compute

$$\begin{aligned}(R(\lambda, A) B R(\lambda, A) f)(x) &= \exp(-\lambda x) \int_0^x \exp(\lambda y)(B R(\lambda, A) f)(y)\, dy \\ &= \exp(-\lambda x) \int_0^x \exp(\lambda y)(1/y) \exp(-\lambda y) \int_0^y \exp(\lambda z) f(z)\, dz\, dy \\ &= \exp(-\lambda x) \int_0^x (1/y) \int_0^y \exp(\lambda z) f(z)\, dz\, dy \\ &= \exp(-\lambda x) \int_0^x \exp(\lambda z) f(z) \int_z^x (1/y)\, dy\, dz \\ &= \int_0^x \exp(-\lambda(x - z)) f(z) \log(x/z)\, dz \\ &= \int_0^x \exp(-\lambda t) \log(x/(x - t)) f(x - t)\, dt \\ &= \int_0^x \exp(-\lambda t)(W(t) f)(x)\, dt\end{aligned}$$

where

(4.1) $$(W(t) f)(x) = \log(x/(x - t)) f(x - t) \quad \text{if } x > t \quad \text{and}$$
$$(W(t) f)(x) = 0 \quad \text{if } x \leqq t.$$

If $\mathscr{A} + \mathscr{B}$ were the generator of a semigroup U on $E = F \times F$, then by the uniqueness of Laplace transforms it would be of the form

$$U(t)(f, g) = \begin{bmatrix} T(t) f & 0 \\ W(t) f & T(t) g \end{bmatrix}.$$

But (4.1) does not define a bounded operator on $C_0(0, 1]$. □

The example demonstrates the sharpness of the Hille-Yosida theorem. In fact, let $\mathscr{R}(\lambda) = (\lambda - \mathscr{A} - \mathscr{B})^{-1}$ and $R(\lambda) = R(\lambda, A)$.

Then

(4.2) $$\sup_{\lambda > 0} \|(\lambda \mathscr{R}(\lambda))^n\| \leqq \text{const} \cdot n$$

for all $n \in \mathbb{N}$.

However, $\mathscr{A} + \mathscr{B}$ does not generate a semigroup, i.e.

$$\sup_{\substack{\lambda > w \\ n \in \mathbb{N}}} \| [(\lambda - w) \mathscr{R}(\lambda)]^n \| = \infty \quad \text{for all } w \geqq 0.$$

Proof of (4.2). It is easy to show by induction that

$$\mathscr{R}(\lambda)^n = \begin{bmatrix} R(\lambda)^n & 0 \\ \sum_{k=1}^{n} R(\lambda)^{n+1-k} B R(\lambda)^k & R(\lambda)^n \end{bmatrix}.$$

But $R(\lambda)^{n+1-k} B R(\lambda)^k \leqq R(\lambda)^{n+1-k} B R(0) R(\lambda)^{k-1}$ for all $\lambda \geqq 0$.

Hence

$$\lambda^n \| R(\lambda)^{n+1-k} B R(\lambda)^k \| \leqq \lambda^n \| R(\lambda)^{n+1-k} \| \, \| B R(0) \| \, \| R(\lambda)^{k-1} \| \leqq \| B R(0) \|$$

since $\| \lambda R(\lambda) \| \leqq 1$. Thus

$$\| \lambda^n \mathscr{R}(\lambda)^n \| \leqq \text{const} \cdot n \quad \text{for all } \lambda > 0. \quad \square$$

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Anschrift der Autoren:

Wolfgang Arendt
Abdelaziz Rhandi
Laboratoire de Mathématiques
Université de Franche-Compté
25030 Besançon-Cedex/France

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