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Perturbation solution to unsteady flow in a porous channel with expanding or contracting walls in the presence of a transverse magnetic field *

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Abstract An incompressible flow in a porous channel with expanding or contacting walls in the presence of a transverse magnetic field is considered. Using similarity transformations, the governing equations are reduced to the nonlinear ordinary differential equations. The exact similar solutions for the different cases of the expansion ratio and the Hartmann number are obtained with a singular perturbation method, and the associated behavior is discussed in detail.

Key words porous channel, transverse magnetic field, expanding or contracting walls

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1 Introduction

The flow through channels and tubes with porous walls is of great importance in both technological and biophysical flows such as soil mechanics, transpiration cooling, food preservation, cosmetic industry, blood flow, and artificial dialysis. A large number of theoretical investigations dealing with a steady incompressible laminar flow with either injection or suction at the boundary layer have been performed during the last few decades. Especially, Suryaprakashrao, Terrill, and Shrestha[1-4] investigated the problems of the steady laminar flow of electrically conducting viscous fluid through porous walls of a channel with an applied transverse magnetic field. They got the valid solutions for the different cases of the Hartmann number *Ha* and the Reynolds number *Re*.

However, the above models of channel flows do not take account of wall motion. Because of the applications in the modeling of pulsating diaphragms, sweat cooling or heating, filtration, and grain regression during the solid-propellant combustion, the flows in a porous channel with deformable walls also gained much attention. Uchida and Aoki[5] first examined the viscous

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flow inside an impermeable tube with a contracting cross-section. Ohki[6] investigated an unsteady flow in a semi-infinite tube with a porous and elastic wall whose length varies with time but the cross-section does not vary. To simulate the laminar flow field in the cylindrical solid rocket motors, Goto and Uchida[7] analyzed the laminar incompressible flow in a semi-infinite porous pipe whose radius varies with time. Bujurke et al.[8] obtained a series solution for an unsteady flow in a contracting and expanding pipe. Majdalani et al.[9] obtained an exact similar solution for the viscous flow with slowly contracting or expanding walls and weak permeability. Dauenhauer and Majdalani[10] obtained numerical solutions and Majdalani and Zhou[11] got numerical and asymptotical solutions for moderate-to-large Reynolds numbers.

In this paper, the solution for the large injection Reynolds number *Re* and arbitrary Hartmann number *Ha* in a porous channel with expanding or contracting walls is presented. In order to reduce the Navier-Stokes equations, we use the similarity transformation in the space and time that Uchida and Aoki[5] ever used to reduce the Navier-Stokes equations into a single nonlinear equation. By making the walls motionless, our solutions embrace the previous formulations shown in [4]. The effects of the expansion ratio α and the Hartmann number *Ha* on the skin friction are studied and shown graphically.

2 The model of the problem

In our study, we consider a channel with a rectangular cross-section. The distance $2b(t)$ between the porous walls is much smaller than the lengths of the other two sides. This enables us to treat the problem as a two-dimensional flow case. Both the walls have the same permeability υ_{w} and expand or contract uniformly at a same time-dependent rate $\dot{b}(t)$. Because the body length L is unrestricted, the channel is assumed to be of semi-infinite length. One end is closed by a compliant membrane.

We adopt a constant magnetic field of the strength H_0, which is perpendicular to the walls and fixed relative to them. The following assumptions are made in this analysis:

(a) The induced magnetic and electric fields produced by the motion of the electrically conducting fluid are negligible.

(b) No external electric field is applied.

As shown in Fig. 1, a coordinate system may be chosen with the origin at the center of the channel. The axial and normal velocity components are defined as $\bar{u}$ and $\bar{\upsilon}$, which are parallel to the $\bar{x}$- and $\bar{y}$-axes, respectively.

Under these assumptions, the continuity and momentum equations are

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{\upsilon}}{\partial \bar{y}} = 0, \tag{1}$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\upsilon}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \upsilon\Big(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\Big) - \frac{\sigma B_0^2}{\rho}\bar{u}, \tag{2}$$

$$\frac{\partial \bar{\upsilon}}{\partial t} + \bar{u}\frac{\partial \bar{\upsilon}}{\partial \bar{x}} + \bar{\upsilon}\frac{\partial \bar{\upsilon}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} + \upsilon\Big(\frac{\partial^2 \bar{\upsilon}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\upsilon}}{\partial \bar{y}^2}\Big), \tag{3}$$

where ρ, υ, and $\bar{p}$ are the dimensional density, the kinematic viscosity, and the pressure.

The boundary conditions are

$$\bar{u}(\bar{x}, b) = 0, \quad \bar{\upsilon}(b) = -\upsilon_{\mathrm{w}} = -A\dot{b}, \tag{4}$$

$$\frac{\partial \bar{u}}{\partial \bar{y}}(\bar{x}, 0) = 0, \quad \bar{\upsilon}(0) = 0, \quad \bar{u}(0, \bar{y}) = 0, \tag{5}$$

where $A = \dfrac{\upsilon_{\mathrm{w}}}{\dot{b}}$ is the injection coefficient, which is the measure of the wall permeability.

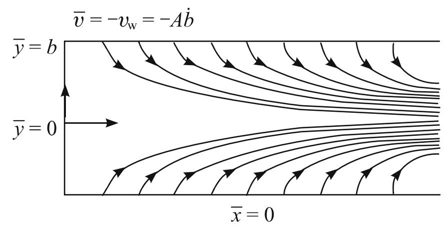

Fig. 1 The two-dimensional channel with expanding or contracting porous walls

3 Reduction of the flow equation

We introduce the Stokes stream function

$$\bar{\psi} = \upsilon b^{-1}\bar{x}\bar{F}(y,t), \tag{6}$$

where $y = \dfrac{\bar{y}}{b}$. Hence,

$$\bar{u} = \frac{\upsilon\bar{x}}{b^2}\bar{F}_y(y,t), \qquad \bar{v} = -\frac{\upsilon}{b}\bar{F}(y,t). \tag{7}$$

Eliminating the pressure terms in Eqs. (2) and (3), we can obtain the vorticity transport equation

$$\frac{\partial\bar{\xi}}{\partial t} + \bar{u}\frac{\partial\bar{\xi}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{\xi}}{\partial\bar{y}} = \upsilon\Big(\frac{\partial^2\bar{\xi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\xi}}{\partial\bar{y}^2}\Big) + \frac{\sigma B_0^2}{\rho}\frac{\partial\bar{u}}{\partial\bar{y}}, \tag{8}$$

where

$$\bar{\xi} = \frac{\partial\bar{v}}{\partial\bar{x}} - \frac{\partial\bar{u}}{\partial\bar{y}}. \tag{9}$$

Substituting Eqs. (7) and (9) into Eq. (8), we obtain a differential equation for $\bar{F}$,

$$\bar{F}_{yyyy} + \alpha(y\bar{F}_{yyy} + 3\bar{F}_{yy}) + \bar{F}\bar{F}_{yyy} - \bar{F}_y\bar{F}_{yy} - b^2\upsilon^{-1}\bar{F}_{yyt} - Ha^2\bar{F}_{yy} = 0, \tag{10}$$

where α is the wall expansion ratio defined by

$$\alpha = \frac{b\dot{b}}{\upsilon}, \tag{11}$$

and Ha is the Hartmann number defined by $B_0 b(\dfrac{\sigma}{\rho\upsilon})^{\frac{1}{2}}$. Note that the expansion ratio will be positive for expansion and negative for contraction. An integration of Eq. (10) produces

$$\bar{F}_{yyy} + \alpha(y\bar{F}_{yy} + 2\bar{F}_y) + \bar{F}\bar{F}_{yy} - \bar{F}_y^2 - b^2\upsilon^{-1}\bar{F}_{yt} - Ha^2\bar{F}_y = k_0, \tag{12}$$

where k_0 is a space-invariant parameter. The boundary conditions given by Eqs. (4) and (5) can be translated into

$$\bar{F}_{yy}(0) = 0, \quad \bar{F}(0) = 0, \quad \bar{F}_y(1) = 0, \quad \bar{F}(1) = Re, \tag{13}$$

where Re is the Reynolds number defined by $Re = \frac{bv_{\mathrm{w}}}{\upsilon}$. Note that Re is positive for injection and negative for suction. Equations (12) and (13) can be normalized via

$$u = \frac{\bar{u}}{v_{\mathrm{w}}}, \quad v = \frac{\bar{v}}{v_{\mathrm{w}}}, \quad x = \frac{\bar{x}}{b}, \quad F = \frac{\bar{F}}{Re}, \tag{14}$$

where F is the characteristic mean flow function. Then, the normalized equations become

$$F_{yyy} + \alpha(yF_{yy} + 2F_y) + Re(FF_{yy} - F_y^2) - b^2\upsilon^{-1}F_{yt} - Ha^2F_y = k, \tag{15}$$

$$F_{yy}(0) = 0, \quad F(0) = 0, \quad F_y(1) = 0, \quad F(1) = 1, \tag{16}$$

where $k = \frac{k_0}{Re}$.

A similar solution with respect to space and time can be developed by the transformation that Uchida and Aoki[5] ever described: α is constant and $F = F(y)$. It leads to $F_{yt} = 0$. Under these assumptions, Eq. (15) becomes

$$F''' + \alpha(yF'' + 2F') + Re(FF'' - {F'}^2) - Ha^2F' = k, \tag{17}$$

$$F''(0) = 0, \quad F(0) = 0, \quad F'(1) = 0, \quad F(1) = 1. \tag{18}$$

Here, when $\alpha = 0$, it is the case that [4] has obtained.

4 Solution for large injection Reynolds number *Re* and arbitrary *Ha*

To obtain the outer and inner expansions of Eq. (17) subject to the conditions (18), for the case of large $\frac{Ha^2}{Re}$ ($Ha \gg 1$) corresponding to the large injection at the walls, (17) may be written as

$$-\varepsilon F''' - \varepsilon\alpha(yF'' + 2F') + {F'}^2 - FF'' + \varepsilon^{-\frac{1}{2}}\gamma F' = \beta^2 + \beta\gamma\varepsilon^{-\frac{1}{2}} - 2\alpha\beta\varepsilon, \tag{19}$$

where

$$\varepsilon = \frac{1}{Re}, \quad \gamma = Ha^2\varepsilon^{\frac{3}{2}}, \quad -\varepsilon k = \beta^2 + \beta\gamma\varepsilon^{-\frac{1}{2}} - 2\alpha\beta\varepsilon. \tag{20}$$

In Eq. (19), the parameter $\gamma\varepsilon^{-\frac{1}{2}} \gg 1$. Hence, the terms containing a multiple of $\gamma\varepsilon^{-\frac{1}{2}}$ will not disappear in the limiting case of $\varepsilon \to 0$.

4.1 The outer solution

The outer solution satisfying the boundary conditions

$$F(0) = 0, \quad F''(0) = 0 \tag{21}$$

is suggested to be

$$F = \beta y, \tag{22}$$

where

$$\beta = \beta_0 + \beta_1\varepsilon^{\frac{1}{2}} + \beta_2\varepsilon + \cdots = \sum_{i=0}^{\infty}\beta_i\varepsilon^{\frac{i}{2}}. \tag{23}$$

This is the outer solution that is valid in the region between the edge of the boundary layer and the center of the channel. In this solution, β_i ($i = 0, 1, 2, \cdots$) can be determined by the obtained inner solution.

4.2 The inner solution

Subject to $F(1) = 1$, an inner solution is sought and has the form

$$F = 1 + \varepsilon^{\frac{1}{2}} w(t, \varepsilon), \tag{24}$$

where $t = \varepsilon^{-\frac{1}{2}}(1 - y)$ is a stretching variable. Substituting (24) into (19) yields

$$-\gamma w' + \varepsilon^{\frac{1}{2}}(w''' + w'^2 - ww'') - w''(1 + \varepsilon\alpha) + \varepsilon^{\frac{3}{2}}\alpha(2w' + tw'') = \beta^2\varepsilon^{\frac{1}{2}} + \beta\gamma - 2\alpha\beta\varepsilon^{\frac{3}{2}}. \tag{25}$$

From (18), the boundary conditions to be satisfied by $w(t, \varepsilon)$ are

$$w(0) = 0, \quad w'(0) = 0. \tag{26}$$

A further substitution of

$$w = \sum_{n=0}^{\infty} w_n(\eta)\varepsilon^{\frac{n}{2}} \tag{27}$$

into (25) gives rise to the general term

$$w_n'' + w_n' = \gamma w_{n-1}''' - \frac{\beta_n}{\gamma} - \alpha w_{n-2}'' + \frac{\alpha}{\gamma}(2w_{n-3}' + w_{n-3}''\eta + \frac{2\beta_{n-3}}{\gamma}) + \sum_{r=s=0}^{r+s=n-1}(w_r' w_s' - w_r w_s'' - \frac{\beta_r\beta_s}{\gamma^2}), \tag{28}$$

where the prime ($'$) denotes the differentiation with respect to the variable

$$\eta = \gamma t = \gamma\varepsilon^{-\frac{1}{2}}(1 - y). \tag{29}$$

The boundary conditions (26) become

$$w_n(0) = 0, \quad w_n'(0) = 0. \tag{30}$$

The equation for w_0 is

$$w_0'' + w_0' = -\frac{\beta_0}{\gamma}. \tag{31}$$

Subject to the boundary conditions (30), the solution of (31) is

$$w_0 = \frac{\beta_0}{\gamma}(1 - \eta - \mathrm{e}^{-\eta}). \tag{32}$$

The equation for w_1 is

$$w_1'' + w_1' = (\beta_0 - \frac{\beta_0^2}{\gamma^2})\mathrm{e}^{-\eta} - \frac{\beta_0^2}{\gamma^2}\eta\mathrm{e}^{-\eta} - \frac{\beta_1}{\gamma}. \tag{33}$$

The solution for w_1 satisfying the boundary conditions (30) is

$$w_1 = \frac{\beta_1}{\gamma}(1 - \eta) + \beta_0 - \frac{2\beta_0^2}{\gamma^2} + [-\frac{\beta_1}{\gamma} - \beta_0(1 + \eta) + \frac{\beta_0^2}{2\gamma^2}(\eta + 2)^2]\mathrm{e}^{-\eta}. \tag{34}$$

The function w_2 satisfies the equation

$$w_2'' + w_2' = [\beta_1 + \beta_0\gamma(\eta - 2) + \frac{\beta_0^2}{2\gamma}(-3\eta^2 + 2\eta + 2) - \frac{\beta_0\beta_1}{\gamma^2}(2\eta + 2) + \frac{\beta_0^3}{2\gamma^3}(\eta^3 + \eta^2 + 2\eta - 2)]\mathrm{e}^{-\eta} + \frac{\beta_0^3}{\gamma^3}\mathrm{e}^{-2\eta} - \frac{\beta_2}{\gamma} + \frac{\alpha\beta_0}{\gamma}\mathrm{e}^{-\eta}. \tag{35}$$

Subject to the boundary conditions (30), the solution of (35) is

$$\begin{aligned} w_2 = {}& \frac{\beta_2}{\gamma}(1-\eta) + \frac{9}{2}\frac{\beta_0^3}{\gamma^3} + \beta_1 - \beta_0\gamma - \frac{\beta_0^2}{\gamma} - 4\frac{\beta_0\beta_1}{\gamma^2} + \frac{\alpha\beta_0}{\gamma} \\ & - \frac{\alpha\beta_0}{\gamma}\mathrm{e}^{-\eta}(1+\eta) - [\frac{\beta_2}{\gamma} + \beta_1(\eta+1) + \frac{1}{2}\beta_0\gamma(\eta^2 - 2\eta - 2) \\ & - \frac{\beta_0^2}{2\gamma}(\eta^3 + 2\eta^2 + 2\eta + 2) - \frac{\beta_0\beta_1}{\gamma^2}(\eta^2 + 4\eta + 4) \\ & + \frac{1}{24}(\frac{\beta_0}{\gamma})^3(3\eta^4 + 16\eta^3 + 60\eta^2 + 96\eta + 120)]\mathrm{e}^{-\eta} \\ & + \frac{1}{2}(\frac{\beta_0}{\gamma})^3\mathrm{e}^{-2\eta}. \end{aligned} \tag{36}$$

The constants β_n are determined by matching the outer solution and inner solutions, and they are given by

$$\beta_0 = 1, \quad \beta_1 = \frac{1}{\gamma}, \quad \beta_2 = 1 - \frac{1}{\gamma^2}. \tag{37}$$

Hence, the second-order inner solution satisfing the boundary conditions is

$$\begin{aligned} F(\eta) = {}& 1 + \frac{\varepsilon^{\frac{1}{2}}}{\gamma}(1 - \eta - \mathrm{e}^{-\eta}) + \varepsilon[\frac{1}{\gamma^2}(1-\eta) + 1 - \frac{2}{\gamma^2}] + \varepsilon[-\frac{1}{\gamma^2} \\ & - (1+\eta) + \frac{1}{2\gamma^2}(\eta+2)^2]\mathrm{e}^{-\eta} + \varepsilon^{\frac{3}{2}}[\frac{1}{\gamma}(1 - \frac{1}{\gamma^2})(1-\eta) \\ & + \frac{1}{2\gamma^3} - \gamma + \frac{\alpha}{\gamma} - \frac{\alpha}{\gamma}\mathrm{e}^{-\eta}(1+\eta)] - \varepsilon^{\frac{3}{2}}[\frac{1}{\gamma}(1 - \frac{1}{\gamma^2}) \\ & + \frac{1}{\gamma}(\eta+1) + \frac{1}{2}\gamma(\eta^2 - 2\eta - 2) - \frac{1}{2\gamma}(\eta^3 + 2\eta^2 + 2\eta + 2)]\mathrm{e}^{-\eta} \\ & - \varepsilon^{\frac{3}{2}}[-\frac{1}{\gamma^3}(\eta^2 + 4\eta + 4) + \frac{1}{24\gamma^3}(3\eta^4 + 16\eta^3 + 60\eta^2 \\ & + 96\eta + 120)]\mathrm{e}^{-\eta} + \frac{\varepsilon^{\frac{3}{2}}}{2\gamma^3}\mathrm{e}^{-2\eta}, \end{aligned} \tag{38}$$

and its derivatives are

$$\begin{aligned} F'(\eta) = {}& \frac{\varepsilon^{\frac{1}{2}}}{\gamma}(-1 + \mathrm{e}^{-\eta}) - \frac{\varepsilon}{\gamma^2} + \varepsilon[-1 + \frac{1}{\gamma^2}(\eta+3) + (1+\eta) \\ & - \frac{1}{2\gamma^2}(\eta+2)^2]\mathrm{e}^{-\eta} + \varepsilon^{\frac{3}{2}}[-\frac{1}{\gamma}(1 - \frac{1}{\gamma^2}) + \frac{\alpha}{\gamma}\mathrm{e}^{-\eta}\eta] \\ & - \varepsilon^{\frac{3}{2}}[-\frac{1}{\gamma}(1 - \frac{1}{\gamma^2}) - \frac{1}{2}\gamma(\eta^2 - 4\eta) + \frac{1}{2\gamma}(\eta^3 - \eta^2 \\ & - 4\eta)]\mathrm{e}^{-\eta} - \varepsilon^{\frac{3}{2}}[\frac{1}{\gamma^3}(\eta^2 + 2\eta) - \frac{1}{24\gamma^3}(3\eta^4 + 4\eta^3 \\ & + 12\eta^2 - 24\eta + 24)]\mathrm{e}^{-\eta} - \frac{\varepsilon^{\frac{3}{2}}}{\gamma^3}\mathrm{e}^{-2\eta}, \end{aligned} \tag{39}$$

$$F''(\eta) = -\frac{\varepsilon^{\frac{1}{2}}}{\gamma}\mathrm{e}^{-\eta} + \varepsilon[2 - \frac{1}{\gamma^2}(2\eta+4) - (1+\eta) + \frac{1}{2\gamma^2}(\eta+2)^2]\mathrm{e}^{-\eta}$$
$$+ \frac{\varepsilon^{\frac{3}{2}}\alpha}{\gamma}\mathrm{e}^{-\eta}(1-\eta) - \varepsilon^{\frac{3}{2}}[\frac{1}{2\gamma}(-\eta^3+4\eta^2+2\eta-2) - \frac{1}{\gamma^3}$$
$$+ \frac{1}{2}\gamma(\eta^2-6\eta+4)]\mathrm{e}^{-\eta} + \frac{2\varepsilon^{\frac{3}{2}}}{\gamma^3}\mathrm{e}^{-2\eta} + \varepsilon^{\frac{3}{2}}[\frac{1}{\gamma^3}(\eta^2-2)$$
$$- \frac{1}{24\gamma^3}(3\eta^4-8\eta^3-48\eta+48)]\mathrm{e}^{-\eta}, \tag{40}$$

$$F''(y) = \gamma^2\varepsilon^{-1}F''(\eta). \tag{41}$$

In fact, the inner solution (38) is valid not only inside but also outside the boundary layer. However, in the process of matching, the exponential terms are negligible. It is observed that when $\alpha = 0$, this is the second-order solution given by Shrestha[4]. The second derivative $F''(y)$ evaluated at $y = 1$ is

$$F''(1) = -\varepsilon^{-\frac{1}{2}}\gamma + \gamma^2 - 2 + \frac{\varepsilon^{\frac{1}{2}}}{\gamma}(-2\gamma^4 + \gamma^2 - 1 + \gamma^2\alpha). \tag{42}$$

Figures 2 and 3 show the effects of the expansion ratio, the Hartmann number, and the Reynolds number on the skin friction. It can be observed in Fig. 2 that whether the walls of the channel are expanding or contracting or not, there is only a little difference of $F''(1)$ as $\frac{Ha^2}{Re}$ is small. However, when $\frac{Ha^2}{Re}$ increases, the difference between them also increases. When $\alpha < 0$, $-F''(1)$ is bigger than that when $\alpha = 0$, and when $\alpha > 0$, $-F''(1)$ is smaller than that when $\alpha = 0$. That is to say, when the walls are expanding, $-F''(1)$ becomes bigger, and when the walls are contracting, $-F''(1)$ becomes smaller. It is also observed that in Fig. 2, when Ha increases, $-F''(1)$ also increases.

Figure 3 illustrates that when Re is large enough, the distances between different curves for different α become small. In this problem, the sufficiently large Reynolds number can dominate over wall expansion or contraction. When Re is small enough, the Hartmann number has much more influence on the flow. $-F''(1)$ also decreases as Re increases.

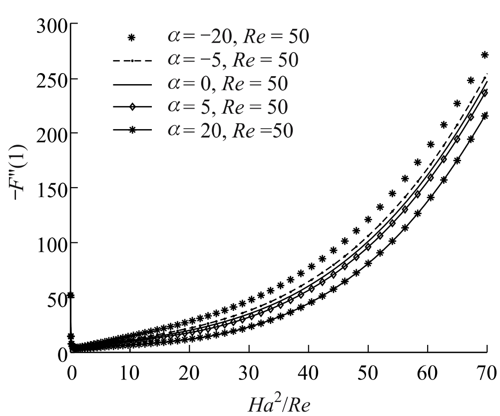

Fig. 2 The skin friction for $Re = 50$ and different α as $\frac{Ha^2}{Re}$ increases

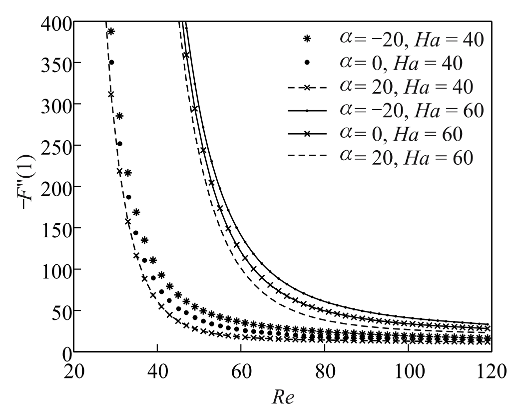

Fig. 3 The skin friction for different Ha and α as Re increases

5 Conclusions

In this paper, an exact similar solution for an unsteady flow in a semi-infinite channel with porous, expanding or contracting walls in the presence of a transverse magnetic field is obtained.

It is easily verified that the equation is the Berman problem with an applied magnetic field when the channel is stationary. It can be observed that the expansion ratio has important influence on the skin friction. The skin friction increases when the Reynolds number decreases or the Hartmann number increases. However, when the Reynolds number is large enough, the effect of the expansion ratio can be negligible.

References

[1] Suryaprakasarao, U. Laminar flow in channels with porous walls in the presence of a transverse magnetic field. *Applied Science Rresearch B* **9**(1), 374–382 (1962)

[2] Terrill, R. M. and Sherstha, G. M. Laminar flow in a uniformly porous channel with an applied transverse magnetic field. *Applied Science Research B* **12**(1), 203–211 (1964)

[3] Terrill, R. M. and Sherstha, G. M. Laminar flow through channels with porous walls and with an applied transverse magnetic field. *Applied Science Research B* **11**(1), 134–144 (1964)

[4] Sherstha, G. M. Singular perturbation problems of laminar flow through channels in a uniformly porous channel in the presence of a transverse magnetic field. *The Quarterly Journal of Mechanics and Applied Mathematics* **20**(2), 233–246 (1967)

[5] Uchida, S. and Aoki, H. Unsteady flows in a semi-infinite contracting or expanding pipe. *Journal of Fluid Mechanics* **82**(2), 371–387 (1977)

[6] Ohki, Morimatsu. Unsteady flows in a porous, elastic, circular tube—part 1: the wall contracting or expanding in an axial direction. *Bulletin of the JSME* **23**(179), 679–686 (1980)

[7] Goto, M. and Uchida, S. Unsteady flow in a semi-infinite expanding pipe with injection through wall. *Journal of the Japan Society for Aeronautical and Space Science* **38** (434), 131–138 (1980)

[8] Bujurke, N. M., Pai, N. P., and Jayaraman, G. Computer extended series solution for unsteady flow in a contracting or expanding pipe. *IMA Journal of Applied Mathematics* **60**(2), 151–165 (1998)

[9] Majdalani, J., Zhou, C., and Dawson, C. D. Two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability. *Journal of Biomechanics* **35**(10), 1399–1403 (2002)

[10] Dauenhauer, C. E. and Majdalani, J. Exact self-similarity solution of the Navier-Stokes equations for a porous channel with orthogonally moving walls. *Physics of Fluids* **15**(6), 1485–1495 (2003)

[11] Majdalani, J. and Zhou, C. Moderate-to-large injection and suction driven channel flows with expanding or contracting walls. *ZAMM: Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* **83**(3), 181–196 (2003)