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PERTURBATION THEORY FOR GAUGE-INVARIANT FIELDS

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PREFACE

In 1967 an excellent paper by V. N. Popov and L. D. Faddeev

appeared as a Kiev Report No. ITP 67-36. We regret that this

remarkable paper has not been available to a wider audience. We

have undertaken the task of making their work more accesible to the

English reading audience.

Due to recent developments in both dual resonance models and

the intense theoretical activity in constructing a renormalizable

theory of weak interactions we believe that this work is not only

"timely" but merits thoughtful consideration for those theorists

involved in constructing theories which possess gauge symmetries.

The present text is based on an informal translation of the original

Russian version that Professor M. T. Veltman arranged in Paris

in 1968. We have also consulted Dr. Yu. K. Pilipenko for clarification

of some passages. None of them are responsible for any errors.

Caveat emptor!

The reader is also referred to Faddeev's article "The Feynman

Integral for Singular Lagrangians" in Teoreticheskaya i Matematicheskaya

Fizika, Vol. 1, No. 1, p. 3 (1969) [English translation: Theoretical

and Mathematical Physics, Vol. 1, p. 1 (1970), Copyright by

Consultants Bureau], which was written after the present paper.

Finally we thank Professor L. D. Faddeev for permitting this

venture.

May 1972   David Gordon and Benjamin W. Lee

ABSTRACT

A method is developed for the manifestly convariant quantization
of gauge-invariant fields by means of a functional integration. It is
shown that for the fields with non-Abelian gauge groups (the Yang-
Mills and gravitational fields) fictitious particles appear naturally in
the diagram technique, which are not present in the initial Lagrangian.
An appearance of these particles restores the transversality of
scattering amplitudes and the unitarity of the S-matrix.

I. INTRODUCTION

It is well known that certain difficulties are involved in the quanti-

zation of gauge-invariant fields, and that they are due to the singular

nature of the Lagrangian concerned. There are many artificial tricks

invented in order to circumvent this difficulty in the case of quantum

electrodynamics; one of the best known is Fermi's method which

makes use of an indefinite metric.[1]

But this technique does not work in the case of a non-abelian

gauge group. It seems that Feynman was the first to observe this in

trying to develop a diagrammatic perturbation theory for the gravitational

and the Yang-Mills fields.[2] He noted, for instance, that diagrams

with closed loops depend non-trivially on the longitudinal parts of

Green's functions (in internal lines), and the scattering amplitudes

obtained were neither unitary nor transverse.

In order to get rid of these difficulties Feynman proposed some

modified rules of computing one-loop diagrams; similar and more

detailed descriptions of these Feynman rules were given by de Witt.[3]

As far as we are aware, however, these rules have not been generalized

for artibrary diagrams.

In this work we are going to analyze a quantization method which

permits the possibility of such a generalization (cf. de Witt, Ref. 3).

The method is based on a redefinition of the Feynman path integral.

When developed in perturbation series, this integral gives a relativistic invariant diagram technique, which in turn leads to scattering amplitudes fulfilling the unitarity condition. The striking feature of this technique is the presence of some new diagrams which restore unitarity and which have no analogue in the case where no gauge fields are involved.

In the case of Yang-Mills fields we shall also show the connection of our formalism with the Hamiltonian one; such a connection was also developed by Schwinger.[4] It is this possibility of passing to the Hamiltonian formalism which assures the unitarity of the S-matrix. At the end we will try to employ the formalism to the theory of gravitation.

Throughout this work we use $\delta_{\mu\nu}$ to denote the Minkowski tensor, in which the only non-vanishing elements are given by: $\delta_{00} = -\delta_{11} = -\delta_{22} = -\delta_{33} = 1$. Vector indices everywhere, with the exception of Section V, do not recognize any distinction between contravariant and covariant indices. Repeated Greek indices imply summation over the pseudo-Euclidean metric; repeated Latin indices imply summation over 1, 2, and 3. For example: $k^2 = k_\mu k_\mu = k_0^2 - k_1^2 - k_2^2 - k_3^2$; $\partial_\mu A_\mu = \partial_0 A_0 - \partial_1 A_1 - \partial_2 A_2 - \partial_3 A_3$; $k_i k_i = k_1^2 + k_2^2 + k_3^2$.

II. QUANTIZATION BY PATH INTEGRAL

Following Feynman, we write[5] an element of the S-matrix, up to

an infinite normalization factor, as a functional integral:

$$\langle in | out \rangle = \int exp\left\{ i S[B] \right\} \prod_x d B(x) ,$$ (1.1)

where $S[B] = \int \mathcal{L}(x) dx$ is the action functional, and the integration is

performed over all fields which converge to the in (out) states for

$x_0 \to \mp \infty$. Elements of the S-matrix can be expressed by Green's

functions which can be written also in the form of the functional

integrals over all the fields B(x) vanishing at infinity, the integrand

being a product $B(x_1) \ldots B(x_n)$ with a weighting factor exp $\{iS[B]\}$.

The Lagrangian as a rule is always the sum $\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1$, where

$\mathcal{L}_0$ is a quadratic form, $\mathcal{L}_1$ is a sum of higher powers of the field B,

and ϵ is some small parameter. After developing such an integral

in a series in powers of ϵ, we obtain the perturbation series, for which

there is a Feynman diagram corresponding to each term. The vertices

of a diagram are produced by perturbing term $\mathcal{L}_1$, whereas $\mathcal{L}_0$ gives

the form of the propagator corresponding to a line in a diagram. The

propagator is the inverse of the matrix of the quadratic form $\mathcal{L}_0$.

For gaugeless fields the form $\mathcal{L}_0$ is non-degenerate, and the propagator

is uniquely defined.

When there are gauge fields, the form $\mathcal{L}_0$ is degenerate and the

propagator is not uniquely defined. In quantum electrodynamics,
however, this ambiguity does not affect the physical results of the
theory. Feynman[2] was the first to remark that this is not true in the
case of non-abelian gauge fields (Yang-Mills fields, gravitation). The
trouble is that there are diagrams with closed loops, which do affect
unitarity, and which depend non-trivially on the choice of the propagator.

In what follows we proceed with some formal considerations which
give an explanation of these difficulties, and give a prescription for
circumventing them.

In gauge-invariant theories the action functional should not be
changed after replacing $B(x)$, by $B^{\Omega}(x)$, where $B^{\Omega}(x)$ is a result of applying
the element Ω of the gauge group to the field $B(x)$. In other words the
action is constant on the orbits of the gauge group, which are formed
by all $B^{\Omega}(x)$ for fixed B and Ω ranging over the group G. Hence the
integral is proportional to the "volume of an orbit" $\prod_x d\Omega(x)$
where the integration is performed over all the elements of the group G.
It seems natural to extract this infinite factor before proceeding to
develop a perturbation series.

There are different possibilities for giving a recipe for such an
extraction. The idea of one of them is to pass from integrating over
all possibile fields to the integral over a "hypersurface" in the manifold
of all fields, which intersect any orbit only once. This means that if
the equation of our hypersurface is $f[B] = 0$, then the equation

$f[B^\Omega] = 0$ has only one solution in $\Omega(x)$ for any $B(x)$.

We define a functional $\Delta_f[B]$ from the condition,

$$\Delta_f[B] \int \prod_x \delta(f[B^\Omega(x)]) \, d\Omega(x) = \text{const.}, \qquad (1.2)$$

by integrating the "infinite-dimensional" δ-function $\prod_x \delta(f[B^\Omega])$

over the group G. Below we shall give a few examples of computing

such an integral. Note that the functional $\Delta_f[B]$ is gauge-invariant,

i.e. $\Delta_f[B^\Omega] = \Delta_f[B]$.

Now put under the integral (1.1) the left-hand side of (1.2)

which does not depend on B. In the resulting integral over the variables

B and Ω perform the substitution $B^\Omega \to B$. The new expression obtained

for Eq. (1.1) is a product of the orbit volume $\prod_x d\Omega(x)$ and the integral

$$\int exp\{i\,S[B]\} \Delta_f(B) \prod_x \delta(f[B(x)]) \, dB(x), \qquad (1.3)$$

It is this redefined Feynman functional integral (1.3) that we propose to

utilize in developing a perturbation theory for gauge-invariant fields.

We define the generating functional of Green's functions:

$$Z_f[\eta] = \int exp\{i\,S[B] + i\int \eta(x)B(x)dx\} \Delta_f[B] \prod_x \delta(f[B]) \, dB \qquad (1.4)$$

The Green's functions themselves are logarithmic variational derivatives

of this functional:

$$\Gamma_f(x_1,\ldots,x_n) = \frac{\delta^n}{\delta\eta(x_1)\cdots\delta\eta(x_n)} \ln Z_f[\eta]\Big|_{\eta=0}.$$

The notations Z_f, Γ_f show that these expressions depend on the choice of a hypersurface $f[B] = 0$.

There is also another possibility of extracting the "orbit volume". Let $\Delta S[B]$ be some term noninvariant with respect to G, and such that the quadratic form $\mathscr{L}_0$ corresponding to the action $S + \Delta S$ shall be nondegenerate. Defining a gauge-invariant functional $\phi[B]$ by the definition

$$\phi[B] \int exp\left\{ i \Delta S[B^{\Omega}] \right\} \prod_x d\Omega(x) = \text{const.}, \quad (1.5)$$

then, in the same way as before, we shall express the integral (1.1) as a product of an orbit volume $\prod_x d\Omega(x)$ and the factor

$$\int exp\left\{ i(S+\Delta S) \right\} \phi[B] \prod_x dB(x). \quad (1.6)$$

Both prodecures described leave great freedom in choosing the hypersurface $f[B] = 0$ or an addition to the action, which amounts to the choice of a particular gauge.

In quantum electrodynamics one of the variants of the first procedure leads to the perturbation theory with purely transverse photon Green's functions (Landau gauge), whereas proceeding as in second case mentioned, we arrive at the perturbation theory with non-transverse Green's functions (in particular, the Feynman gauge), but in both cases the factors $\Delta_f[B]$ and $\phi[B]$ do not depend on $B(x)$.

On the contrary, in the case of a non-abelian gauge group (Yang-

Mills fields, gravitation) these factors depend non-trivially on B(x)

and give rise to some new diagrams for Green's functions. We shall

also remark that the computation of $\phi[B]$ in higher orders of perturbation

theory is much more difficult than that of $\Delta[B]$.

So our main idea is to extract from the integral (1.1) the factor

which is proportional to the orbit volume. This can be done in different

ways and leads to the integrals of the type (1.3) or (1.6) where the

factors $\Delta[B]$ or $\phi[B]$ appear, which depend on B in the case of a

non-abelian gauge fields. The following will illustrate this procedure

on concrete examples.

III. QUANTUM ELECTRODYNAMICS

This section has a methodological character. Here we show how all known results of quantum electrodynamics can be obtained by means of the general scheme of part I.

It is well known that the Lagrangian of electrodynamics:

$$\mathcal{L}(x) = \bar{\psi}(i\not{\partial} - m + e\not{A})\psi - \frac{1}{4}(\partial_\nu A_\mu - \partial_\mu A_\nu)^2 =$$
$$= \mathcal{L}_0 + \mathcal{L}_I \tag{2.1}$$

where $\mathcal{L}_I = \bar{\psi}\not{A}\psi$, is invariant with respect to the abelian gauge group:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu C(x) ; \quad \psi(x) \to e^{iec(x)}\psi(x) ; \quad \bar{\psi}(x) \to e^{-iec(x)}\bar{\psi}(x). \tag{2.2}$$

The first recipe of Sec. I is to choose a hypersurface f = 0. There are two well known special cases: the Lorentz gauge and the Coulomb (radiation) gauge:

$$f_L[A] \equiv \partial_\mu A_\mu = 0 \quad, \qquad f_R[A] \equiv \operatorname{div} \vec{A} = 0,$$

which gives us the equations of the "hyperplanes" in the manifold of all fields A(x). The corresponding integrals of the type (1.2)

$$\int \prod_x \delta(\partial_\mu A_\mu(x) - \Box C(x)) dc(x) \quad \text{for the Lorentz gauge,}$$

and

$$\int \prod_x \delta(\operatorname{div}\vec{A}(x) - \Delta C(x)) dc(x) \quad \text{for the Coulomb gauge,}$$

where Δ and $\Box$ mean the Laplace and d'Alembert operators respectively,

do not depend on $A(x)$, hence we can put without loss of generality

$\Delta_L = \Delta_R = 1$.

In order to proceed to relativistic perturbation theory it is useful

to find a generating functional of the form (1.4) for the non-perturbed

Green's functions:

$$Z^0[\eta] = \int \exp\left\{ i S + i \int (\bar{\eta}\psi + \bar{\psi}\eta + \eta_\mu A_\mu) dx \right\} \prod_x \delta(\partial_\mu A_\mu) dA_\mu d\psi_d$$

Here $\bar{\eta}$, η, η_μ are the sources of the fields ψ, $\bar{\psi}$, A_μ. This functional

can be computed by means of a translation

$$\psi \to \psi + \psi_0 \;,\; \bar{\psi} \to \bar{\psi} + \bar{\psi}_0 \;;\; A_\mu \to A_\mu + A_\mu^{(0)}$$

that does not affect $\delta(\partial_\mu A_\mu)$, i.e. $\partial_\mu A_\mu^{(0)} = 0$, and is equal to

$$\exp\left\{ -i \int \bar{\eta}(x) G(x-y)\eta(y) + \tfrac{1}{2}\eta_\mu(x) \mathcal{D}_{\mu\nu}^T(x-y) \eta_\nu(y) dx dy \right\} \quad (2.3)$$

where

$$G(x) = (2\pi)^{-4} \int \exp\{ ipx \} (\not{p} - m)^{-1} dp$$

and

$$\mathcal{D}_{\mu\nu}^T(x) = (2\pi)^{-4} \int \exp\{ i kx \} (k^2 + i\epsilon)^{-2} (-k^2 \delta_{\mu\nu} + k_\mu k_\nu) dk.$$

$$(2.4)$$

From (2.3) and (2.4) the Wick theorem follows, and hence, all usual

diagrammatic techniques with well known expressions for the electron

lines and vertices and purely transverse photon Green's functions

(Landau gauge).

In the second procedure we have to choose the additional term

ΔS. Choosing it to be

$$\Delta S = - \frac{1}{2\beta} \int [\partial_\mu A_\mu (x)]^2 dx ,$$

we see the corresponding factor $\phi [A]$ (cf. 1.5) does not depend on A

and therefore we can put $\phi = 1$. Hence we come once more to the

diagrammatic techniques with the transverse photon Green's functions

of the form

$$\left(k^2 + i\epsilon \right)^{-2} \left(- k^2 \delta_{\mu\nu} + (\beta-1) k_\mu k_\nu \right) \quad \text{(in k-space)},$$

which for the special case $\beta = 1$ is equal to the Feynman function:

$$\delta_{\mu\nu} \left(k^2 + i\epsilon \right)^{-1}$$

Both frameworks of perturbation theory obtained are well known

and lead to the same results for physical quantities.

Exact relations among Green's functions of electrodynamics can

be obtained by a change of variables in the functional integrals. As an

illustration of this we shall derive the Ward identity[6] and the relations

between Green's functions in different gauges:

$$G_L (x-y) = \langle \psi(x) \bar\psi (y) \rangle_L =$$

$$= N_L^{-1} \int \psi(x) \bar\psi(y) \exp \{ i S \} \prod_x \delta(\partial_\mu A_\mu) dA\, d\psi\, d\bar\psi ,$$

$$(2.5)$$

where

$$N_L = \int \exp \{ i S \} \prod_x \delta(\partial_\mu A_\mu) dA\, d\psi\, d\bar\psi .$$

A rotation of the spinor field into:

$$\psi(x) \rightarrow \exp\{ic(x)\}\psi(x) \quad ; \quad \bar{\psi}(x) \rightarrow \exp\{-ic(x)\}\bar{\psi}(x)$$

in the numerator of (2.5) gives rise to the factor:

$$\exp\left\{ie\left[c(x) - c(y) - \int c(z)\partial_\mu j_\mu \, dz\right]\right\},$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$. Differentiating with respect to c (z) and putting

afterwards c = 0 we obtain

$$G(x-y)\left[\delta(x-z) - \delta(y-z)\right] = \langle\psi(x)\bar{\psi}(y)\partial_\mu j_\mu(z)\rangle_L,$$

from which the Ward identity follows:

$$G^{-1}(p) - G^{-1}(q) = (p-q)_\mu \Gamma_\mu(p, q; p-q)$$

connecting the Green's function G(p) and the irreducible vertex part

$\Gamma_\mu(p, q; p-q)$. It is clear that this identity is valid in any

gauge of the photon Green's function, because in the derivation the

change of variables was made only on the spinor fields.

Let us look at the transition from the Coulomb gauge to the Lorentz

gauge for the electron Green's function:

$$G_R(x-y) = \langle\psi(x)\bar{\psi}(y)\rangle_R =$$

$$= N_R^{-1}\int\psi(x)\bar{\psi}(y)\exp\{iS\}\prod_x\delta(\text{div}\vec{A})\,dA\,d\psi\,d\bar{\psi}, \qquad (2.6)$$

where

$$N_R = \int\exp\{iS\}\prod_x\delta(\text{div}\vec{A})\,dA\,d\psi\,d\bar{\psi}.$$

In order to accomplish the transition we put under the integral signs

in the numerator and in the denominator of (2.6) the expression

$\int \prod_X \delta(\Box c - \partial_\mu A_\mu) dc(x)$, which does not depend on A. Then we perform

a transformation of the type (2.2) which does not affect the action S.

Under this transformation the arguments of the delta functions

change as $\delta(\Box c - \partial_\mu A_\mu) \rightarrow \delta(\partial_\mu A_\mu)$, $\delta(\text{div}\vec{A}) \rightarrow \delta(\text{div}\vec{A} + \Delta c)$.

In the factor $\exp\{ie[c(x) - c(y)]\}$ which emerges in the numerator we can

put for c(x) a solution of the equation $\Delta c(x) + \text{div}\vec{A} = 0$, i.e.,

$$c(x) = \frac{1}{4\pi} \int |\vec{x} - \vec{z}|^{-1} \text{div}\vec{A}(z) d^3z = \int \phi_i(x-z) A_i(z) dz,$$

where

$$\phi_i(x) = -\delta(x_0) \partial_i (4\pi|\vec{x} - \vec{z}|)^{-1}.$$

When this substitution is made the integrals over c(x) in the numerator

and denominator cancel each other. The formula that results is:

$$G_R(x-y) = \langle \psi(x)\bar{\psi}(y) \exp\{ie[\phi_i(x-z) - \phi_i(y-z)]A_i(z)dz\}\rangle$$

It expresses the Green's function in Coulomb gauge as a power series

of Green's functions $\langle \psi(x)\bar{\psi}(y) \prod_k A_k(z_k)\rangle_L$ in the Lorentz

gauge.[7]

IV. THE YANG-MILLS FIELDS

1. Yang-Mills field theories[8] are the simplest example of fields

associated with non-abelian gauge groups.   It is very convenient to

describe Yang-Mills vector fields related to any simple compact

Lie group, G,[9,10] in terms of matrices $B_\mu(x)$ which form a definite

representation of the Lie algebra characterizing the group.   It is clear

that such a representation may be defined by

$$\widetilde{B}_\mu(x) = \sum_{a=1}^{n} \tau_\mu^{a}(x) \, \tau_a \, ,$$

where τ_a are linearly independent matrices representing the Lie

algebra, are normalized according to the relation $\mathrm{Tr}\,(\tau_a \tau_b) = 2\,\delta_{ab}$

n is the number of group parameters, and $b_\mu^a(x)$ is a function with

vector index μ and isotopic index a.   As is known one may represent

the latter index to denote a matrix element of $B_\mu(x)$ by means of the

relation, $\quad (B_\mu)_{ab} = \sum_{c} (\tau_c)_{ab} \, \tau_\mu^{c} = \sum_{c} t_{abc} \, \tau_\mu^{c} \, ,$

where t_{abc} are the totally anti-symmetric structure constants of the

group G.

The gauge transformations are:

$$B_\mu \longrightarrow B_\mu^{\Omega} \equiv \Omega \, B_\mu \, \Omega^{-1} + \tfrac{1}{2} \, \partial_\mu \Omega \, \Omega^{-1}$$

where $\Omega(x)$ is an arbitrary matrix-function with the values in G.

The Lagrangian

$$\mathcal{L}(x) = \frac{1}{8} \, \mathrm{Tr} \, F_{\mu\nu} F_{\mu\nu} \, ,$$

where

$$F_{\mu\nu} \equiv \partial_\nu B_\mu - \partial_\mu B_\nu + \epsilon [B_\mu, B_\nu]$$

(3.1)

is invariant with respect to such transformations. It is also clear
that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ where $\mathcal{L}_0$ is a quadratic form and $\mathcal{L}_I$ is a
sum of trilinear and quartic forms in the fields B(x).

2. Now we shall proceed with the formalism developed in the
part 1, making use of the first recipe. As in electrodynamics the
most convenient gauges seem to be the Lorentz or Coulomb gauges:

$$f_L[B] \equiv \partial_\mu B_\mu = 0 \quad , \quad f_R[B] \equiv \text{div } \vec{B} = 0 .$$

Both equations are matrix equations and represent really n conditions
(n = dim G) instead of only one condition as in electrodynamics.

In order to compute the factor $\Delta_L[B]$ we shall remark that we
are interested only in its value for the transverse fields $\left(\partial_\mu B_\mu = 0\right)$.
For such fields the only solution of the equation $\partial_\mu B_\mu^\Omega = 0$ in Ω
will be the identity element of the group G, and so the
entire contribution to the integral (1.2) is given by the neighborhood
of the identity element. Therefore we can put

$$\Omega \approx 1 + u(x)$$

where u(x) is an element of the Lie algebra of the group, and retain in
$\partial_\mu B_\mu^\Omega$ only the terms linear in u(x):

$$\partial_\mu B_\mu^\Omega \approx \partial_\mu (B + \epsilon [u, B_\mu] + \partial_\mu u) = \Box u - \epsilon [B_\mu, \partial_\mu u],$$

where $\square$ is the d'Alembert operator.

The condition (1.2) reads as

$$\Delta_L [B] \int \prod_x \delta(Au) \, du = \text{const.},$$

where

$$Au = \square u - \epsilon [B_\mu, \partial_\mu u].$$

Formally $\Delta_L [B]$ is the determinant of the operator A. It is useful

to have also another realization of this operator. We introduce

instead of the matrices u(x) the column vector $u_a(x)$ in such a

way that $u = u_a \tau_a$, τ_a, being the matrices of the adjoined representation

of G. In this new representation the operator A is given by the matrix

$$(A)_{ab} = \square \delta_{ab} - \epsilon (B_\mu)_{ab} \partial_\mu$$

Extracting the trivial infinite factor $\det \square$ we are lead to a

following expression for $\ln \Delta_L [B]$:

$$\ln \Delta_L[B] = \ln \det (\square^{-1} A) = \text{tr} \ln (1 - \epsilon \, \square^{-1} B_\mu \partial_\mu) =$$

$$= \sum_{n=2}^{\infty} \frac{\epsilon^n}{n} \int dx_1 \cdots dx_n \, \text{tr} (B_{\mu_1}(x_1) \cdots B_{\mu_n}(x_n)) \cdot$$

$$\times \partial_{\mu_1} \mathcal{D}(x_1 - x_2) \cdots \partial_{\mu_n} \mathcal{D}(x_n - x_1).$$

(3.2)

where $\mathcal{D}(x)$ is the Feynman Green's function for the d'Alembert

operator

$$\mathcal{D}(x) = (2\pi)^{-4} \int exp\{ikx\} (k^2 + i\epsilon)^{-1} dk.$$

On the left-hand side of (3.2) the trace is understood in the operator

sense, whereas on the right-hand side the trace is taken only in the

matrix sense.

Analogous computations in the Coulomb gauge give the factor $\triangle_R [B]$. It is given by the expression,

$$\ln \triangle_R [B] = \text{tr} \ln (1 - \epsilon \triangle^{-1} B_i \partial_i) =$$

$$= \sum_{n=2}^{\infty} \frac{\epsilon^n}{n} \int d^3 x_1 \cdots d^3 x_n \ \text{tr} \ (B_{i_1} (x_1) \cdots B_{i_n} (x_n))$$

$$\times \ \partial_{i_1} \widetilde{\mathscr{D}} (\vec{x_1} - \vec{x_2}) \cdots \partial_{i_n} \widetilde{\mathscr{D}} (\vec{x_n} - \vec{x_1}), \qquad (3.3)$$

where $\widetilde{\mathscr{D}} (\vec{x})$ is a Green's function for the Laplace operator :

$$\widetilde{\mathscr{D}} (\vec{x}) = (2\pi)^{-3} \int \exp\{\vec{k} \cdot \vec{x}\} \ (k_i k_i)^{-1} d^3 k = -(4\pi |\vec{x}|)^{-1}$$

and the indices $i_1 \cdots i_n$ run over the values 1, 2, 3.

3. Relativistic invariant perturbation theory for computing the

Green's functions in the Lorentz gauge, viz,

$$\langle B_{\mu_1} (x_1) \cdots B_{\mu_n} (x_n) \rangle_L =$$

$$= N_L^{-1} \int B_{\mu_1} (x_1) \cdots B_{\mu_n} (x_n) \exp\{i S[B]\} \triangle_L [B] \prod_x \delta(\partial_\mu B_\mu) d$$

where

$$N_L = \int \exp\{i S[B]\} \triangle_L [B] \prod_x \delta(\partial_\mu B) dB,$$

arises naturally in the Lorentz gauge when developing the functional

$$\exp\{i S[B]\} \triangle_L [B] = \exp\{i S[B] + \ln \triangle_L [B]\}$$

in a power series in the parameter ϵ . The expression $\ln \triangle_L [B]$

can be interpreted as an addition to the action, but this addition does

not have the form $\int \mathcal{L}(x)dx$ associated with some Lagrangian. The

term of degree ϵ^n of this series in a striking way looks like a

contribution of a closed loop along which a scalar particle is propagating

and interacting with a vector field in a tri-linear way. This circumstance

permits a description of the perturbation theory in terms of additional

graphs containing the propagation of a fictitious scalar particle and a vertex

with one vector and two scalar ends. The elements of this diagrammatic

technique are shown in Fig. 1, the corresponding analytic expressions

are given by:

$$G^{ab}_{\mu\nu}(p) = \delta_{ab}(p^2+i0)^{-2}(-p^2\delta_{\mu\nu}+p_\mu p_\nu) \quad ; \quad G^{ab}(p) = -\delta_{ab}(p^2+i0)^{-1}$$

$$V^{abc}_{\mu,\nu\rho} = i\epsilon\, t_{abc}(p_{1\nu}\delta_{\mu\rho} - p_{1\rho}\delta_{\mu\nu}) \quad ; \quad V^{abc}_\mu = \frac{i\epsilon}{2} t_{abc}(p_3 - p_2)$$

$$V^{abcd}_{\mu\nu,\rho\sigma} = \epsilon^2 t_{abc}\, t_{cde}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})$$
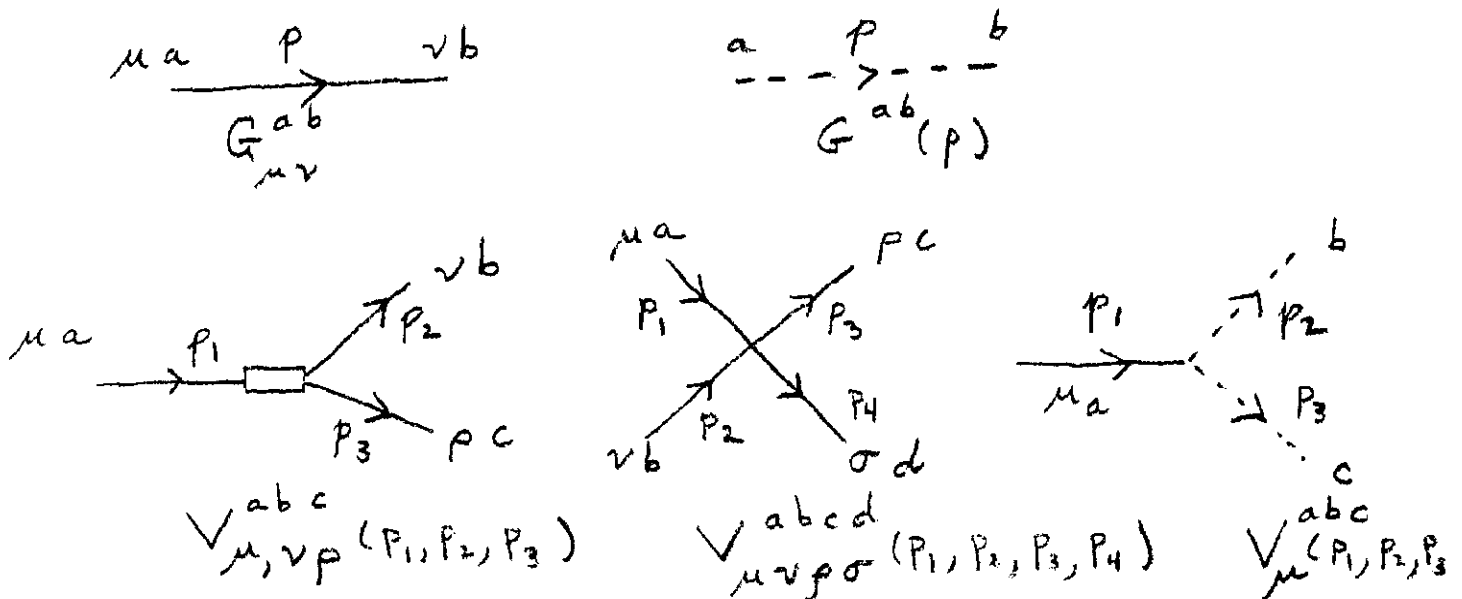
$$(3.4)$$

Figure 1.

In order to calculate the contribution of a given diagram we ought

to integrate the product of the expressions for its elements over the

independent momenta and sum over discrete indices, and then multiply

the result by $R^{-1}\left((2\pi)^{-4}i\right)^{\ell_i - \nu + 1}(-)^S$, where ν

is the number of vertices, ℓ_i the number of internal lines, S -

the number of closed scalar loops, and R is the rank of a symmetry

group of the diagram concerned.

In this way all the effects of the presence of a factor $\Delta_L[B]$

can be interpreted as the introduction of a fictitious scalar particle. The

propagation lines of this particle should be taken into consideration

only inside the diagrams so that they form closed loops. Moreover

these scalar particles behave in a way as fermions. The factor $(-1)^S$,

indeed, where S is the number of loops is characteristic of fermions.

The last fact becomes less queer if we remark that for $\Delta_L[B]$

one can rewrite the integral representation,

$$\Delta_L[B] = const. \int exp\left\{ i\, tr\, \bar{u}\left(\Box u - \epsilon[B_\mu, \partial_\mu u]\right)\right\} du\, d\bar{u}$$

in terms of the anticommuting fields u and ū. Putting this expression

in the integral (1.3) we are lead to the usual Feynman integral for a

two field system: The transverse vector field B_μ, $\partial_\mu B_\mu = 0$,

and scalar fermion field (ū, u).

4. In the second-type approach to Yang-Mills fields it is natrual

to choose an additional term to the action as in electrodynamics, i.e.,

$$\Delta S = \frac{1}{4\beta} \int tr \left(\partial_\mu B_\mu \right)^2 dx .$$

Then after expanding the functional $\phi[B] \, exp\{i(S+\Delta S\}$ in the

powers of ϵ we are lead to a non-transverse Green's function, in

particular, for $\beta = 1$, the Feynman function. Nevertheless in this

case it is no longer possible to account for all additional diagrams

generated by introducing into our diagrammatic technique just one

new line and one new vertex as was done in the previous case.

5. Sometimes the so-called first order formalism is more useful.

This formalism can be obtained if the Lagrangian (3.1) is re-expressed

as follows:

$$\mathcal{L}(x) = -\frac{1}{8} tr \, F_{\mu\nu} F_{\mu\nu} + \frac{1}{4} tr \left[\left(\partial_\nu B_\mu - \partial_\mu B_\nu \right) + \epsilon \left[B_\mu, B_\nu \right] \right] F_{\mu\nu}$$

$$(3.5)$$

and to integrate in the functional integral over the fields B_μ and $F_{\mu\nu}$

as if they were independent, extracting, as usual the "orbit volume".

Then for instance, for the Lorentz gauge we arrive at the integral,

$$\int exp\left\{ i \, S[B, F] \, \Delta_L[B] \prod_x \delta \left(\partial_\mu B_\mu \right) dB \, dF. \right. \qquad (3.6)$$

Here the integral over F can be computed exactly, thus leading us to the

formerly investigated formalism. However, if we do not do that, but

yet in the integral (3.6) expand the expression $exp\left\{ i \, S[B,F] \right\} \Delta_L[B]$

in powers of ϵ , we will obtain a new variant of the diagrammatic

perturbation theory with three lines, corresponding to functions $\langle B, B \rangle$,

$\langle B, F \rangle$, $\langle F, F \rangle$ and one vertex describing tri-linear interaction FBB.   The

elements of this diagrammatic technique are given in Fig. 2 and

formulae (3.7) ;   The field B is represented by a single line, whereas

F is represented by a double line:

$$G_{\mu\nu}^{ab}(p) = \delta_{ab} (p^2 + io)^{-2} (-\delta_{\mu\nu} p^2 + p_\mu p_\nu) ;$$

$$G_{\mu\nu,\rho}^{ab}(p) = i \delta_{ab} (p^2 + io)^{-1} (p_\nu \delta_{\mu\rho} - p_\mu \delta_{\nu\rho}) ;$$

$$G_{\mu\nu,\rho\sigma}^{ab}(p) = \delta_{ab} [\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} +$$

$$+ (p^2 + io)^{-1} (\delta_{\mu\sigma} p_\nu p_\rho + \delta_{\nu\rho} p_\mu p_\sigma - \delta_{\mu\rho} p_\nu p_\sigma - \delta_{\nu\sigma} p_\mu p_\rho)]$$

$$V_{\rho\sigma\mu\nu}^{abc} \equiv i \epsilon t_{abc} \delta_{\rho\mu} \delta_{\sigma\nu} \tag{3.7}$$
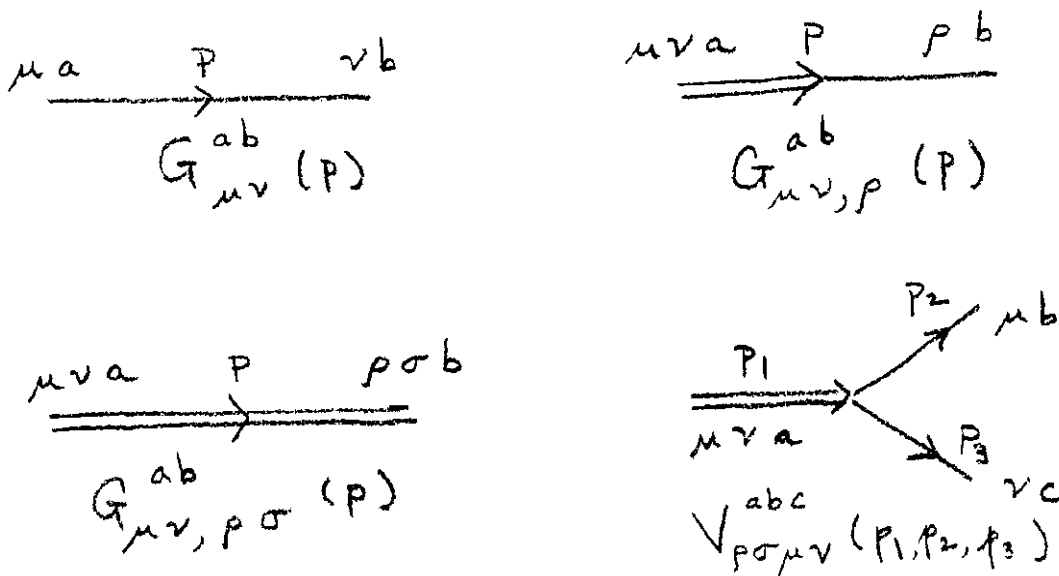

Figure 2.

The lines and vertices describing the propagation of the fictitious scalar

particles and their interaction with the vector ones are the same as

in the case of the second-order formalism, because the factor

depends only on B, but not on F.

6. In the first-order formalism discussed above we have three

full one-particle Green's functions which can be expressed in terms of

the corresponding non-perturbed functions (3.7) and three self-energy

parts, which in second order perturbation theory take the form:

$$\sum_{\mu\nu}^{ab}(k) = \frac{\delta_{ab}\,\epsilon^2}{12\pi^2}\left[(\delta_{\mu\nu}\,k^2 - k_\mu k_\nu)\ln\left(\frac{-k^2}{k_0^2}\right) - (ak^2+b)\delta_{\mu\nu} + c\,k_\mu k_\nu\right]$$

$$\sum_{\mu\nu,\sigma}^{ab}(k) = \frac{\delta_{ab}\,3i\epsilon^2}{16\pi^2}\left[k_\mu\delta_{\nu\sigma} - k_\nu\delta_{\mu\sigma}\right]\left(\ln\left(\frac{-k^2}{k_0^2}\right)+d\right)$$

$$\sum_{\mu\nu,\rho\sigma}^{ab}(k) = \frac{\delta_{ab}\,\epsilon^2}{16\pi^2}\left\{(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\left(\ln\left(\frac{k^2}{k_0^2}\right)+e\right)\right.$$

$$+ \frac{1}{2}(k^2+io)^{-1}\left(k_\mu k_\rho\delta_{\nu\sigma} + k_\nu k_\sigma\delta_{\mu\rho} - k_\mu k_\sigma\delta_{\nu\rho} - k_\nu k_\rho\delta_{\mu\sigma}\right)$$

Here k_o denotes some fixed 4-momentum, with $k_0^2 > 0$; and a, b, c, d, e

are some renormalization constants (indeed only first derivatives with

respect to momentum of $\sum_{\mu\nu,\rho\sigma}^{ab}(p)$, second derivatives of $\sum_{\mu\nu,\sigma}^{ab}(p)$

and third derivatives of $\sum_{\mu\nu}^{ab}(p)$ are uniquely defined).

In $\sum_{\mu\nu,\sigma}^{ab}$ there is also a contribution of an additional

diagram and only due to its appearance can we assume the function

$$\sum_{\mu \nu}^{a b}$$

to be purely transverse, i.e., proportional to

$k^2 \delta_{\mu\nu} - k_\mu k_\nu$, by choosing in some special way the constants

a, b, and c such that b = 0, and a = -c.

7. In order to establish a connection with the canonical quantization

method it is useful to start from an integral over B and F in Coulomb

gauge with sources:

$$Z_R[\eta] = \int exp \left\{ i S[B,F] + i \int tr \left(\eta_\mu B_\mu + \tfrac{1}{2} \eta_{\mu\nu} F_{\mu\nu} \right) dx \right\}$$
$$\times \Delta_R[B] \prod_x \delta \left(div \vec{B} \right) dB \, dF. \tag{3.8}$$

Following Schwinger[4] we shall choose for the dynamical variable the

transverse (in three-dimensional sense) components of the field B_i,

and F_{io}, i = 1, 2, 3. We also assume the sources present only

correspond to our dynamical variables; this means that

$$\eta_o = \eta_{ik} = \partial_i \eta_i = \partial_i \eta_{oi} = 0.$$

In three-dimensional notations our Lagrangian (3.5) takes the form

$$\mathcal{L}(x) = tr \left\{ -\tfrac{1}{2} F_{ik} F_{ik} + \tfrac{1}{4} F_{io} F_{io} + \right.$$
$$+ \tfrac{1}{4} F_{ik} \left(\partial_k B_i - \partial_i B_k + \epsilon [B_i, B_o] \right) -$$
$$\left. - \tfrac{1}{2} F_{io} \partial_o B_i - \tfrac{1}{2} B_o \left(\partial_i F_{io} + \epsilon [B_i, F_{io}] \right) \right\} .$$

The absence of the sources corresponding to F_{ik} and B_o permits the

integration in (3.8) over these fields, which is equivalent to changing

in the resulting integral over the remaining variables B_i, F_{io} the

matrix F_{ik} into

$$H_{ik} = \partial_k B_i - \partial_i B_k + \epsilon [B_i, B_k] ,$$

and also the δ -function into

$$\prod_X \delta \left(\partial_i F_{io} + \epsilon [B_i, F_{io}] \right).$$

Now, put into the integral (3.8) the factor $\int \prod_X \delta (\Delta c - \partial_i F_{io}) dc(x)$

that realy does not depend on F_{io} and perform a translation $F_{io} \rightarrow F_{io} +$

$+ \partial_i c$, thus inducing the transformations:

$$\prod_X \delta (\Delta c - \partial_i F_{io}) \longrightarrow \prod_X \delta (\partial_i F_{io})$$

$$\prod_X \delta (\partial_i F_{io} + \epsilon [B_i, F_{io}]) \rightarrow \prod_X \delta (\Delta c + \epsilon [B_i, \partial_i c] + \epsilon [B_i, F_{io}]).$$

Let c_o be a solution of the equation:

$$\Delta c + \epsilon [B_i, \partial_i c] = \epsilon [F_{io}, B_i],$$

which can be expressed by a Green's function depending on B (cf. Ref. 4)

$$c_o(x) = \epsilon \int \mathscr{D}(\vec{x}, \vec{y}; B) [F_{io}(y), B_i(y)] d^3 y;$$

$$\Delta \mathscr{D}(\vec{x}, \vec{y}; B) + \epsilon [B_i(x), \partial_i \mathscr{D}(\vec{x}, \vec{y}; B)] = \delta^3(\vec{x} - \vec{y})$$

After performing a translation $c \rightarrow c + c_o$ the δ -function

appears as, $\prod_X \delta (\Delta c + \epsilon [B_i, \partial_i c])$ and we can put c = 0 everywhere

except when equal to the argument of this function. The resulting integral

$$\int \prod_X \delta (\Delta c + \epsilon [B_i, \partial_i c]) dc(x)$$

cancels with $\Delta_R [B]$.

Hence the integral (3.8) can be transformed, up to an infinite factor,

to the form

$$Z_R[\eta] = \int exp \left\{ i S_R + i \int tr [\eta_i B_i + \eta_{io} F_{io}] dx \right.$$

$$\times \prod_X \delta (\partial_i B_i) \delta (\partial_i F_{io}) dB_i dF_{oi}, \qquad (3.9)$$

where

$$S_R = \int dx_0 \left\{ f_{io}^a \ \partial_0 \ f_i^a \ d^3x - H(f,h) \right\},$$

and

$$H = \frac{1}{2} \int d^3x \left[\frac{1}{2} h_{ik}^a h_{ik}^a + f_{io}^a f_{io}^a + \partial_i C_o^a \partial_i C_o^a \right].$$

The relations obtained look like the usual formulae of quantization

of a classical Hamiltonian system using Feynman method (cf. Ref. 11).

The transverse variables $f_{oi}^a(\vec{x})$ and $b_i^a(\vec{x})$ are playing the

role of canonically conjugate variables, whereas the functional $H[f,b]$

that of the Hamiltonian function. Following the general quantization scheme,

we ought to introduce the operators $\hat{b}_i^a(\vec{x}), \hat{f}_{oi}^a$ fulfilling the

transversality conditions:

$$\partial_i \hat{f}_{oi} = \partial_i \hat{b}_i = 0$$

as well as the commutation relations

$$\left[\hat{f}_{oi}^a(\vec{x}), \hat{b}_j^c(\vec{q}) \right] = i \, \delta_{ab} \, \delta_{ij}^T (\vec{x} - \vec{q})$$

where

$$\delta_{ij}^T(\vec{x}) = (2\pi)^{-3} \int exp\left\{ i\vec{k}\cdot\vec{x} \right\} \left(\delta_{ij} - \frac{k_i k_j}{k_s k_s} \right) d^3k$$

The energy operator $\hat{H}$ can be obtained by putting the operators $\hat{b}$

and $\hat{f}$ instead of the functions b and f into the functional H.

The canonical quantization of Yang-Mills fields given above is

equivalent to the one proposed by Schwinger in Ref. 4.

It is also known that the Feynman and the canonical quantization

proceedure are equivalent. In particular the functional integral (3.9)

is the generating functional for the Green-Schwinger function. Strictly speaking, the following relations occur:

$$\langle \Psi_0 | T(\hat{B}(x) \cdots \hat{F}(y) \cdots) | \Psi_0 \rangle = \langle B(x) \cdots F(y) \rangle_R \equiv$$

$$\equiv N_R^{-1} \int B(x) \cdots F(y) \cdots \exp\{iS_R\} \prod \delta(\partial_i F_{oi}) \delta(\partial_i B_i) dB \, dF,$$

where

$$N_R = \int \exp\{iS_R\} \prod \delta(\partial_i F_{oi}) \delta(\partial_i B_i) dB \, dF.$$

The left-hand side is an average over the physical vacuum of the chronological product of the Heisenberg operators $B(x). \ldots F(y). \ldots$

As a closing remark we indicate that the presence of the factor

$$\triangle_R[B]$$ in the primary integral (3.8) is necessary for casting it into canonical form.

8. After having established the equivalence of our quantization with more conventional one in terms of operators, we can define the unitary S-matrix using well known reduction formulae (cf. Ref. 12). For the sake of brevity we return to the second-order formalism, so that the Coulomb-gauge Green's functions are given by the following expressions:

$$\langle b_{i_1}^{a_1}(x_1) \cdots b_{i_m}^{a_m}(x_m) \rangle_R = N_R^{-1} \int b_{i_1}^{a_1}(x_1) \cdots b_{i_m}^{a_m}(x_m)$$

$$\times \exp\{i S[B]\} \triangle_R[B] \prod \delta(\text{div}\,\vec{B}) dB \qquad (3.10)$$

where

$$N_R = \int \exp\{i S[B]\} \triangle_R[B] \prod \delta(\text{div}\,\vec{B}) dB.$$

Take now their Fourier transforms;

$$G_R{}^{a_1 \cdots a_m}_{i_1 \cdots i_m}(P_1, \cdots, P_m) = - (2\pi)^4 \int \langle t_{i_1}^{a_1}(x_1) \cdots t_{i_n}^{a_n}(x_n) \rangle e^{i \sum k \cdot x} dx_1 \cdots dx$$

which are proportional to $\zeta(\Sigma P)$ due to translational invariance.

In order to obtain the transition amplitudes between an initial state

containing r particles and a final state containing s particles, r + s = n,

we have to go to the mass shell (i. e., to put $p^2 = 0$ for all the momenta p)

in the product of $G_R{}^{a_1 \cdots a_m}_{i_1 \cdots i_m}(P_1 \cdots P_m)$ and n factors

of the form $e_i(p) u(p) Z_R^{-1}$ for every particle. Here $e_i(p)$ is

a unit transverse polarization vector, $P_i e_i = 0$ and $u =$

$= (2\pi)^{-3/2} p^2 |2 P_0|^{-1/2} \theta(\pm P_0)$, where the last sign takes account of the

difference between the in-and outgoing particles; the renormalization

constant Z_R is defined by the residuum of the one-particle Green's

function $G_{ij}^{ab}(p)$ for $p^2 = 0$. More exactly, for $p^2 = 0$ this

function has the asymptotic form ;

$$G_{ij}^{ab}(p) \approx \frac{Z_R \delta_{ab}}{p^2 + i 0} \left(\delta_{ij} - \frac{P_i P_j}{P_s P_s} \right),$$

up to infrared singularities.

From the practical point of view it is useful to express the matrix

elements of the S-matrix in terms of the Green's functions in the

Lorentz gauge, for which we have developed above a covariant perturbation

theory. As a matter of fact this can be done and the corresponding

formulae have exactly the same form as the ones developed above in

the terms of Coulomb-gauge Green's functions. We need only change

the Coulomb gauge Green's functions to the Lorentz gauge Green's

functions, and put instead of the factor Z_R the residuum of $G^{ab}_{\chi\, L}(p^2)$

for $p^2 = 0$.

For a more detailed explanation we consider the connection between

these two types of Green's functions corresponding to different gauges.

We put into the demoninator and numerator of (3.10) the factor not

depending on B: $\quad \Delta_L[B]\int \prod_x \delta (\partial_\mu B_\mu^{\Omega^{-1}})\, d\Omega \quad$ and then

perform a translation $\quad B \longrightarrow B^\Omega$. In the numerator the following

integral over Ω appears:

$$\int B^\Omega_{i_1}(x_1)\cdots B^\Omega_{i_n}(x_n)\prod_x \delta\,(div\,\vec{B}^{\,\Omega})\,d\Omega,$$

from which can be extracted a factor $\quad B^{\Omega_o}_{i_1}\cdots B^{\Omega_o}_{i_n}\quad,$

where Ω_0 is a gauge transformation changing the field B satisfying

the Lorentz condition $\partial_\mu B_\mu = 0$ into a field transverse in the three-

dimensional sense, i.e., div $\vec{B} = 0$. This transformation depends on

B, so that B^{Ω_o} is a complicated functional of B. Here are the first

terms of this functional developed in powers of ϵ :

$$B^{\Omega_o} = B^T - \frac{\epsilon}{2}\left[\Delta^{-1}\,div\,\vec{B},\; B+B^T\right]^T + \cdots \quad (3.11)$$

Here T denotes the three-dimensional transverse part of the correspond-

ing vector. The rest of the integral cancels with $\Delta_R[B]$ and we are lead

to the expression of the form:

$$N_L^{-1}\int B^{\Omega_o}_{i_1}(x_1)\cdots B^{\Omega_o}_{i_n}(x_n)\,exp\{iS\}\Delta_L[B]\prod_x \delta(\partial_\mu B_\mu)dE$$

that can be used to calculated the Coulomb Green's functions in terms

of the Lorentz Green's functions. When expanding in terms of perturbation

series we can ascribe the extra vertices to the terms of (3.11), so

that any Coulomb Green's function is really an infinite series contain-

ing the integrals of Lorentz-gauge Green's functions with any number

of external lines.

In this way the connection between the Green's functions in different

gauges appears to be very complicated. Our scheme becomes much

simpler, however, if we go to the mass-shell: We ought to compare

only the terms having the asymptotic singular behaviour when $p^2 \to 0$.

It is clear that the pole for a given p will remain only if the whole effect

of extra vertices reduced to terms of the type of self energy insertions

into a corresponding external line. In the limit $p^2 \to 0$, the resulting

Lorentz and Coulomb Green's functions will differ by the factor σ^n

where σ is certain constant, and n denotes the number of external

lines. By comparing the one-particle functions we obtain $\sigma^2 = Z_R Z_L^{-1}$

so that $\sigma = \left(\dfrac{Z_R}{Z_L} \right)^{1/2}$. This ends our explainatory remarks concerning

the elements of the S-matrix.

All these considerations have an explicitly invariant character and

therefore apply to any Green's functions. In other words, the expressions

for the elements of S-matrix have the same form in any gauge. In

particular we can make use of the following formula for computing

the S-matrix of the Green's functions:

$$\langle \psi_{i_1}^{a_1}(x_1) \cdots \psi_{i_m}^{a_m}(x_m) \rangle_{\Delta S} =$$

$$= N_{\Delta S}^{-1} \int \psi_{i_1}^{a_1}(x_1) \cdots \psi_{i_m}^{a_m}(x_m) \, \exp \left\{ i(S + \Delta S) \right\} \varphi_{\Delta S}[B] \prod d\xi$$

where

$$N_{\Delta s} = \int exp\left\{i\left(S + \Delta s\right)\right\} \varphi_{\Delta s}[B]\, dB,$$

for which the relativistic invariant perturbation theory could be

developed as well. It can be shown that the recipe given by Feynman

in (2) for computing one-loop diagrams is equivalent to taking into

account some of the terms appearing in the expansion of the factor

$\varphi_{\Delta s}[B]$ in powers of ϵ . In the general case this expansion

gives rise to the infinite number of extra vertices, whereas in the

formalism of the first type it is sufficient to introduce only one extra

vertex.

As a closing remark: The computation of S-matrix elements in

perturbation theory without taking into account factors of the type

$\Delta_L[B]$ or $\varphi[B]$ would lead to the formally invariant amplitudes,

which would not, however, satisfy the unitarity condition. This fact

was indeed revealed by Feynman in Ref. 2.

V. THE THEORY OF GRAVITATION

Now we will examine one proposed scheme for quantizing the

theory of gravitation. Here we limit ourselves only to an outline of

the perturbation theory and compute the analogue of $\Delta_L [B]$.

The Lagrangian of the gravitation field,

$$\mathcal{L}(x) = (2 \varkappa c^2)^{-1} R(x) =$$
$$= (2 \varkappa c^2)^{-1} \sqrt{-g} \, g^{\mu\nu} \left(\partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\sigma\rho} \right)$$

is invariant with respect to general coordinate transformations which we

write here in an infinitesimal form:

$$\delta g^{\mu\nu} = - \delta x^\lambda \partial_\lambda g^{\mu\nu} + g^{\mu\nu} \partial_\lambda \delta x^\lambda + g^{\lambda\nu} \partial_\lambda \delta x^\mu ;$$
$$\delta \Gamma^\sigma_{\mu\nu} = - \delta x^\lambda \partial_\lambda \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\mu\lambda} \partial_\nu \delta x^\lambda - \Gamma^\sigma_{\nu\lambda} \partial_\mu \delta x^\lambda + \Gamma^\lambda_{\mu\nu} \partial_\lambda \delta x^\sigma -$$
$$- \partial_\mu \partial_\nu \delta x^\sigma .$$

Here δx^λ are arbitrary infintesimal functions.

The coordinate transformation depend on four arbitrary functions.

Therefore in our scheme a hypersurface in the manifold of all possible

fields should be given by four conditions, and we choose the harmonicity

conditions:[13]

$$\partial_\nu \left(\sqrt{-g} \, g^{\mu\nu} \right) = 0 .$$

If for the variables of functional integration we choose $\sqrt{-g} \, g^{\mu\nu}$ and

$\Gamma^\sigma_{\mu\nu}$ we will arrive at an integral of the form

$$\int exp \left\{ i S[g, \Gamma] \right\} \Delta_h [g] \prod \delta \left(\partial_\nu \sqrt{-g} \, g^{\mu\nu} \right) dg \, d\Gamma ,$$

(4.1)

where

$$dg \, d\Gamma = \prod_{\mu < \nu} d(\sqrt{-g} \, g^{\mu\nu}) \prod_{\sigma, \, \mu < \nu} d\,\Gamma_{\mu\nu}^{\sigma}.$$

The notation $\Delta_h(g)$ recalls the origin of this factor from the harmonicity conditions.

In spite of the fact that the integral over $\Gamma_{\mu\nu}^{\sigma}$ in (4.1) can be taken explicitly we will prefer not to do it and work in the first-order formalism.

The diagrammatic perturbation technique arises in a natural way if we put

$$\sqrt{-g} \, g^{\mu\nu} = \delta_{\mu\nu} + \varkappa \, h^{\mu\nu}, \qquad \Gamma_{\mu\nu}^{\sigma} = \varkappa \, \Pi_{\mu\nu}^{\sigma},$$

and then expand the integral (4.1) in the powers of $\varkappa$.

At that moment we get two kinds of vertices: One of them corresponding to interaction $\varkappa \, h^{\mu\nu} \left(\Pi_{\mu\sigma}^{\rho} \Pi_{\nu\rho}^{\sigma} - \Pi_{\mu\nu}^{\rho} \Pi_{\rho\sigma}^{\sigma} \right)$ and the other being generated by the factor $\Delta_h[g]$ which we shall now evaluate.

We must know $\Delta_h[g]$ only for the harmonic fields. For such fields

$$\partial_\nu \, \delta(\sqrt{-g} \, g^{\lambda\nu}) = \sqrt{-g} \, g^{\mu\nu} \partial_\mu \partial_\nu \, \delta x^\lambda = -\Box \, \delta x^\lambda + \varkappa \, h^{\mu\nu} \partial_\mu \partial_\nu \delta)$$

and the general formula (1.2) for the factor we are seeking takes the form:

$$\Delta_h[g] \int \prod_x \delta\left(\sqrt{-g} \, g^{\mu\nu} \partial_\mu \partial_\nu (\delta x^\lambda) \right) d(\delta x^\lambda) = \text{const.}$$

Proceeding thereafter as in the case for Yang-Mills fields we obtain

the result:

$$\ln \Delta_h [g] = 4 \operatorname{tr} \ln \left(1 - \Box^{-1} h_{\mu\nu} \partial_\mu \partial_\nu \right) =$$

$$= -4 \sum_{n=2}^{\infty} \frac{\varkappa^n}{n} \int dx_1 \cdots dx_n \; h^{\mu_1 \nu_1} \partial_{\mu_1} \partial_{\nu_1} \mathscr{L}(x_1 - x_2) \cdots$$

$$\times \; h^{\mu_n \nu_n} \partial_{\mu_n} \partial_{\nu_n} \mathscr{L}(x_n - x_1). \tag{4.2}$$

A general term in the expansion (4.2) describes a fictitious vector

particle propagating along a loop with n external lines and interacting

with the "harmonic" field $h^{\mu\nu}$ following the coupling rule $\varkappa \, h^{\mu\nu} \partial_\mu \chi^\rho \partial_\nu \chi^\rho$.

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