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Perturbation theory for infinite-dimensional integrable systems on the line. A case study.

by

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Contents

1. Introduction

In this paper we consider perturbations

$$\begin{gathered} iq_t+q_{xx}-2|q|^2q-\varepsilon|q|^lq=0, \\ q(x,t=0)=q_0(x)\to 0 \quad \text{as } |x|\to\infty \end{gathered} \tag{1.1}$$

of the defocusing nonlinear Schrödinger (NLS) equation

$$\begin{gathered} iq_t+q_{xx}-2|q|^2q=0, \\ q(x,t=0)=q_0(x)\to 0 \quad \text{as } |x|\to\infty. \end{gathered} \tag{1.2}$$

Here $\varepsilon>0$ and $l>2$. The particular form of the perturbation $\varepsilon|q|^lq$ in (1.1) is not special, and it will be clear to the reader that the analysis goes through for any perturbation of the form $\varepsilon\Lambda'(|q|^2)q$, as long as $\Lambda:\mathbf{R}_+\to\mathbf{R}_+$ is sufficiently smooth, $\Lambda'(s)\geqslant 0$ and $\Lambda(s)$

A more detailed, extended version of this paper is posted on `http://www.ml.kva.se/publications/acta/webarticles/deift`. Throughout this paper we refer to the web version as [DZW].

vanishes sufficiently fast as $s \downarrow 0$. (For further discussion, see §2 and Remark 3.29 below. See also [DZW, §3].)

As is well known, the NLS equation is completely integrable, and we view the problem at hand as an example of the perturbation theory of infinite-dimensional integrable systems on the line. For systems of type (1.1), (1.2) in the spatially periodic case, resonances, or equivalently, small divisors, play a decisive role. Using KAM-type methods, various authors (see, in particular, [CrW], [Ku1], [Ku2], and also [Cr]) have shown that, under perturbation, the behavior of the unperturbed system persists on certain invariant tori which have a Cantor-like structure: on the remainder of the phase space, the KAM methods give no information. For systems such as (1.1), (1.2) on the line, however, the situation is very different. As time goes on, solutions of these systems disperse in space and the effect of resonances/small divisors is strongly muted, and indeed, one of the main results of our analysis is that, under perturbation, the behavior of the NLS equation (1.2) persists on open sets in phase space (see Theorems 1.29, 1.30, 1.32, 1.34, and the corollary to Theorem 1.29, below): no excisions on the complement of a Cantor-like set are necessary.

In order to understand the long-time behavior of solutions to (1.1) or (1.2), it is useful to consider the scattering theory of solutions of the equation

$$\begin{aligned} &iq_t+q_{xx}-2\varepsilon|q|^l q=0, \quad \varepsilon>0,\ l>2, \\ &q(x,t=0)=q_0(x)\to 0 \quad \text{as } |x|\to\infty \end{aligned} \tag{1.3}$$

with respect to the free Schrödinger equation

$$\begin{aligned} &iq_t+q_{xx}=0, \\ &q(x,t=0)=q_0(x)\to 0 \quad \text{as } |x|\to\infty. \end{aligned} \tag{1.4}$$

Many people have worked on the scattering theory of such equations, beginning with the seminal papers of Ginibre and Velo [GV1], [GV2] and Strauss [St] (see [O] for a (relatively) recent survey). Suppose that in a region $|x/t|\leqslant M$, a solution $q(x,t)$ of (1.3) behaves as $t\to\infty$ like a solution of the free equation. Then

$$q(x,t)\sim t^{-1/2}\beta(x/t)e^{ix^2/4t}, \quad |x/t|\leqslant M, \tag{1.5}$$

for some function $\beta(\cdot)$. In particular, $|q(x,t)|^l\sim 1/t^{l/2}$ and substituting this relation into (1.3) we obtain an equation of the form $iq_t+q_{xx}-(\text{const}/t^{l/2})q\approx 0$. If $l>2$, then the interaction is short range, the assumption (1.5) is consistent, and solutions of (1.3) indeed look asymptotically like solutions of the free equation (1.4). More precisely, in [MKS], the definitive paper of the genre, the authors have proved the following result. Let $U_t^{(l)}(q_0)$

and $U_t^F(q_0)$ denote the solutions of (1.3) and (1.4) with initial data q_0 respectively, and let l be any fixed number greater than 2. Then for all initial data in the unit ball of a weighted Sobolev space, and for $0<\varepsilon\leqslant\varepsilon(l)$ sufficiently small, the wave operator

$$W_l^+(q_0)=\lim_{t\to\infty} U_{-t}^F\circ U_t^{(l)}(q_0) \tag{1.6}$$

exists and is one-to-one onto an open ball. Furthermore W_l^+ conjugates the flows,

$$U_t^F\circ W_l^+ = W_l^+\circ U_t^{(l)}. \tag{1.7}$$

The case $l=2$, corresponding to the NLS equation (set $\varepsilon=1$ by scaling), is, however, critical. The potential term $|q|^2\sim 1/t$ is now long range, leading to a $\log t$ phase shift in the asymptotic form of the solution (1.5). And indeed one can show (see [ZaM], [DIZ], [DZ2]) that solutions of the NLS equation, with initial data that decay sufficiently rapidly and are sufficiently smooth, have asymptotics as $t\to\infty$ of the form

$$q(x,t)=t^{-1/2}\alpha(x/2t)\,e^{ix^2/4t-i\nu(x/2t)\log 2t}+O\left(\frac{\log t}{t}\right), \tag{1.8}$$

where the functions α and ν can be computed explicitly in terms of the initial data q_0 (see (1.26) et seq. below). In particular, the wave operator $W_{l=2}^+$ cannot exist. The above asymptotic form for NLS was first obtained in [ZaM], but without the error estimate.

In the language of field theory, the phase shift $\nu(x/2t)\log 2t$ in (1.8) plays the role of a counterterm needed to renormalize solutions of the NLS equation to solutions of the free equation (1.4). A precise and explicit form of renormalization theory for solutions of the NLS equation can be obtained by using the familiar scattering theory/inverse scattering theory for the ZS–AKNS system [ZaS], [AKNS] associated to NLS,

$$\partial_x\psi=U(x,z)\psi=\left(iz\sigma+\begin{pmatrix}0 & q\\ \bar q & 0\end{pmatrix}\right)\psi,\quad \sigma=\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}. \tag{1.9}$$

As is well known, the NLS equation is equivalent to an isospectral deformation of the operator

$$\partial_x-\left(iz\sigma+\begin{pmatrix}0 & q\\ \bar q & 0\end{pmatrix}\right).$$

As described in §2 below, for each $z\in\mathbf{C}\setminus\mathbf{R}$, one constructs solutions $\psi(x,z)$ of (1.9) of the type considered in [BC] with the properties: $m(x,z)\equiv\psi(x,z)e^{-ixz\sigma}$ is bounded in x and tends to I, the identity matrix, as $x\to-\infty$. For each fixed x, the (2×2)-matrix function $m(x,z)$ solves the following Riemann–Hilbert problem (RHP) in z:

(1.10) $m(x,z)$ is analytic in $\mathbf{C}\backslash\mathbf{R}$, and $m_+(x,z)=m_-(x,z)v_x(z)$, $z\in\mathbf{R}$, where $m_\pm(x,z)=\lim_{\varepsilon\downarrow 0} m(x,z\pm i\varepsilon)$ and

$$v_x(z)=\begin{pmatrix} 1-|r(z)|^2 & r(z)e^{izx} \\ -\overline{r(z)}e^{-izx} & 1 \end{pmatrix}$$

for some function $r=r(z)$ called the reflection coefficient of q, and $\lim_{z\to\infty} m(x,z)=I$.

The sense in which the limits in RHP's of type (1.10) are achieved will be made precise in §2. The reflection coefficient satisfies the important a priori bound $\|r\|_{L^\infty(dz)}<1$. If we expand out the limit for $m(x,z)$ as $z\to\infty$,

$$m(x,z)=I+\frac{m_1(x)}{z}+O\left(\frac{1}{z^2}\right), \tag{1.11}$$

then we obtain an expression for q,

$$q(x)=-i(m_1(x))_{12}. \tag{1.12}$$

The direct scattering map $\mathcal{R}$ is obtained by mapping $q\mapsto r$ as follows: $q\mapsto m(x,z)=m(x,z;q)\mapsto v_x(z)\mapsto r=\mathcal{R}(q)$. Given r, the inverse scattering map $\mathcal{R}^{-1}$ is obtained by solving the RHP (1.10) and mapping to q via (1.12) as follows: $r\mapsto\text{RHP}\mapsto m(x,z)=m(x,z;r)\mapsto m_1(x)\mapsto q=\mathcal{R}^{-1}(r)$. As discussed in §2, the basic fact is that the scattering map $q\mapsto r=\mathcal{R}(q)$ is bijective for q and r in suitable spaces. Also, and this is the truly remarkable discovery in the subject [ZaS], the map $\mathcal{R}$ linearizes the NLS equation. More precisely, if $q(t)$ solves the NLS equation (1.2), then $r(\,\cdot\,;q(t))=\mathcal{R}(q(t))$ evolves according to a simple multiplier,

$$r(z;q(t))=e^{-iz^2t}r(z,t=0). \tag{1.13}$$

Alternatively, if we take the inverse Fourier transform $\check{r}(t)=\left(1/\sqrt{2\pi}\right)\int_{\mathbf{R}} e^{ixz}r(z;q(t))\,dz$, then $\check{r}(t)$ solves the free Schrödinger equation

$$i\check{r}_t+\check{r}_{xx}=0. \tag{1.14}$$

Said differently, the map

$$q\mapsto T(q)=\mathcal{F}^{-1}\circ\mathcal{R}(q) \tag{1.15}$$

renormalizes solutions of NLS to solutions of the free Schrödinger equation. Furthermore, we clearly have the intertwining relation

$$T\circ U_t^{\text{NLS}}=U_t^F\circ T, \tag{1.16}$$

where $U_t^{\mathrm{NLS}}(q_0)$ denotes the solution of NLS and $U_t^F(q_0)$ denotes the solution of the free equation as before. Thus we see that also in the case $l=2$, it is possible to conjugate solutions of the nonlinear equation (1.3) to solutions of the free Schrödinger equation, but now the conjugating map is not given by an (unmodified) wave operator W_l^+ as in the case $l>2$. In the language of field theory, the map T renormalizes solutions of NLS to solutions of the free Schrödinger equation.

In a similar way, we do not expect that solutions of the perturbed NLS equation (1.1) should behave asymptotically like solutions of the free equation. Rather we expect that (1.1) is a "short-range" perturbation of (1.2) and that solutions of (1.1) should behave as $t\to\infty$ like solutions of the NLS equation, or more precisely, we expect that the wave operator

$$W^+(q)\equiv\lim_{t\to\infty}U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q) \tag{1.17}$$

exists, where $U_t^\varepsilon(q)$ denotes the solution of (1.1) with initial data q. Then W^+ intertwines U_t^ε and U_t^{NLS}, and $T_\varepsilon=T\circ W^+$ renormalizes solutions of (1.1) to solutions of the free equation

$$T_\varepsilon\circ U_t^\varepsilon=U_t^F\circ T_\varepsilon. \tag{1.18}$$

The key idea in this paper, motivated by (1.13) and by the expectation that (1.1) is a short-range perturbation of (1.2), is to use the map $q\mapsto r=\mathcal{R}(q)$ as a change of variables for (1.1). Suppose $q(t)$, $t\geqslant0$, solves (1.1) with $q(t=0)=q_0$. Then as we show in §2, under the change of variables $q(t)\mapsto r(z;q(t))=\mathcal{R}(q(t))(z)$, equation (1.1) takes the form

$$\partial_t r=-iz^2r+\varepsilon\int_{-\infty}^{\infty}e^{-iyz}(m_-^{-1}Gm_-)_{12}\,dy,\quad r|_{t=0}=\mathcal{R}(q_0), \tag{1.19}$$

where

$$G=G(q)=-i|q|^l\begin{pmatrix}0&q\\-\bar q&0\end{pmatrix}, \tag{1.20}$$

and $m_-e^{ixz\sigma}$ corresponds to the boundary value of the Beals–Coifman-type solution ψ defined above. Emphasizing the dependence on x, z and $q(t)$, the equation becomes

$$\partial_t r(z;q(t))=-iz^2r(z;q(t))+\varepsilon\int_{-\infty}^{\infty}e^{-iyz}(m_-^{-1}(y,z;q(t))\,G(q(y,t))\,m_-(y,z;q(t)))_{12}\,dy, \tag{1.21}$$

where $r(\,\cdot\,;q(t))|_{t=0}=\mathcal{R}(q_0)$. This equation was first obtained, essentially in the same form, by Kaup and Newell [K1], [KN]. Observe that for $\varepsilon=0$, (1.21) reduces, after integration, to (1.13), as it should. In the perturbative situation, $\varepsilon>0$, the really critical aspect of (1.21) is that the nonlinear part of the equation scales like $|q|^{l+1}$ as $|q|\to0$. This means that the inverse Fourier transform $\check r(x,t)=\left(1/\sqrt{2\pi}\,\right)\int_{-\infty}^{\infty}e^{ixz}r(z;q(t))\,dz$ solves an equation of the form

$$i\check r_t+\check r_{xx}-\varepsilon\mathcal{H}(\check r)=0, \tag{1.22}$$

where the (nonlocal) perturbation $\varepsilon\mathcal{H}(\check{r})$ scales like $|\check{r}|^{l+1}$ as $|\check{r}|\to 0$. In other words, slow decaying terms like $|\check{r}|^2\check{r}$ are removed under the map $q\mapsto r\to\check{r}$, and as in (1.3), we may expect that solutions of (1.22) will converge to solutions of the free equation $i\check{r}_t+\check{r}_{xx}=0$ as $t\to\infty$. In other words for solutions $r(z;q(t))$ of (1.21), we expect that as $t\to\infty$

$$r(z;q(t))\sim e^{-iz^2t}r_\infty(z) \tag{1.23}$$

for some function $r_\infty(z)$. But then

$$\mathcal{R}(U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q))=e^{-iz^2(-t)}\mathcal{R}(U_t^\varepsilon(q))=e^{iz^2t}r(z;q(t))\to r_\infty \quad \text{as } t\to\infty,$$

i.e., $W^+(q)=\lim_{t\to\infty}U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q)$ exists (and equals $\mathcal{R}^{-1}(r_\infty)$).

The body of this paper is concerned with analyzing (1.21) and ensuring that the above program indeed goes through. Although the natural condition for the theory is $l>2$ as in [MKS], for technical reasons we will need $l>\frac{7}{2}$. From the preceding calculations it is clear that we should remove the oscillation from $r(z;q(t))$ and consider

$$r(t)=r(z,t)\equiv e^{iz^2t}r(z;q(t)) \tag{1.24}$$

directly instead of $r(z;q(t))$. At the technical level (see in particular Theorem 4.16 and the discussion in §4 leading up to this result) this reduces to controlling the solutions m of RHP's of type (1.10) with jump matrices of the form

$$v_{x,t}(z)=\begin{pmatrix} 1-|r(z)|^2 & r(z)e^{i(xz-tz^2)} \\ -\overline{r(z)}e^{-i(xz-tz^2)} & 1 \end{pmatrix}, \tag{1.25}$$

uniformly as $|x|,t\to\infty$. Such oscillatory RHP's can be analyzed by the nonlinear steepest descent method introduced by the authors in [DZ1] (see also [DIZ] and [DZ2]). This method has now been extended by many authors to a wide variety of problems in mathematics and mathematical physics (see, for example, the recent summary in [DKMVZ]). In [DZ1] (and also [DIZ], [DZ2]), the potential q, and hence the reflection coefficient r, lies in Schwartz space. A considerable complication in the present paper comes from the fact that now we can only assume that r has a finite amount of smoothness and decay. Also, as is well known, the theory for the RHP (1.10) is simplest in L^2. However, it is clear from (1.21), that if we want to consider solutions $r(z;q(t))$ in an $L^2(dz)$-space, we need to control $m_-(y,z;q(t))$, or more precisely $m_-(y,z;q(t))-I$, in an $L^4(dz)$-space. In [DZ5], and also in [DZW, §4], we develop the L^p-theory of the RHP (1.10), and a summary of the results relevant to this paper is given in §2(c) below. These L^p-results are of independent interest and require the introduction of several new techniques in Riemann–Hilbert theory.

Consider the weighted Sobolev space $H^{k,j}=\{f:f,\partial_x^k f, x^j f\in L^2(\mathbf{R})\}$, $k,j\geqslant 0$, with norm $\|f\|_{H^{k,j}}=(\|f\|_{L^2}^2+\|\partial_x^k f\|_{L^2}^2+\|x^j f\|_{L^2}^2)^{1/2}$. Let $H_1^{k,j}=H^{k,j}\cap\{\|f\|_{L^\infty}<1\}$, $k\geqslant 1$, $j\geqslant 0$. As noted in §2, a basic result of Zhou [Z1] is that $\mathcal{R}$ is bi-Lipschitz from $H^{k,j}$ onto $H_1^{j,k}$ for $k\geqslant 0$, $j\geqslant 1$. This result illustrates, in particular, the well-known Fourier-like character of the scattering map in a precise sense. We will consider solutions of (1.1), (1.2) only in $H^{1,1}$, but it will be clear to the reader that our method goes through in $H^{k,k}$, for any $k\geqslant 1$ commensurate with the smoothness of the perturbation $\varepsilon|q|^l q$ in (1.1). Throughout the paper we assume that $\varepsilon>0$ to ensure that (1.1) has global solutions for all initial data (see Theorem 2.31). For definiteness, we note by the above that $\mathcal{R}$ maps $H^{1,1}$ onto $H_1^{1,1}$.

The asymptotic form of solutions $q(x,t)$ of NLS given in (1.8) above remains true in $H^{1,1}$, but with a weaker error estimate. More precisely (see [DZ4] or [DZW, Appendix III]), suppose that $q(x,t)$ solves (1.2) with initial data $q(x,t=0)=q_0(x)$ in $H^{1,1}$, then $r=\mathcal{R}(q_0)\in H_1^{1,1}$, and for some $0<\varkappa<\frac{1}{4}$, as $t\to\infty$,

$$q(x,t)=q_{\mathrm{as}}(x,t)+O(t^{-(1/2+\varkappa)}), \tag{1.26}$$

where

$$\begin{aligned} q_{\mathrm{as}}(x,t)&=t^{-1/2}\alpha(z_0)e^{ix^2/4t-i\nu(z_0)\log 2t},\\ \nu(z_0)&=-\frac{1}{2\pi}\log(1-|r(z_0)|^2),\\ |\alpha(z_0)|^2&=\tfrac{1}{2}\nu(z_0),\\ \arg\alpha(z_0)&=\frac{1}{\pi}\int_{-\infty}^{z_0}\log(z_0-z)\,d(\log(1-|r(z)|^2))+\tfrac{1}{4}\pi+\arg\Gamma(i\nu(z_0))+\arg r(z_0). \end{aligned}$$

Here Γ is the gamma-function, $z_0=x/2t$ is the stationary phase point, $\partial_z|_{z_0}(xz-tz^2)=0$, and the error term $O(t^{-(1/2+\varkappa)})$ is uniform for all $x\in\mathbf{R}$. The proof of (1.26) in $H^{1,1}$ requires finer control of oscillatory factors than is needed in [DIZ], [DZ2], where the data has higher orders of smoothness and decay.

Our results are the following. Set

$$\mathcal{B}_\varepsilon^+\equiv\{q\in H^{1,1}:W^+(q)=\lim_{t\to\infty}U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q)\text{ exists in }H^{1,1}\}. \tag{1.27}$$

Observe that if $q\in\mathcal{B}_\varepsilon^+$, then $U_t^\varepsilon(q)\in\mathcal{B}_\varepsilon^+$ and $W^+\circ U_t^\varepsilon(q)=\lim_{s\to\infty}U_t^{\mathrm{NLS}}\circ U_{-(t+s)}^{\mathrm{NLS}}\circ U_{t+s}^\varepsilon(q)=U_t^{\mathrm{NLS}}\circ W^+(q)$, i.e., $W^+\circ U_t^\varepsilon=U_t^{\mathrm{NLS}}\circ W^+$.

For any $\eta>0$, $0<\varrho<1$ set

$$\mathcal{B}_{\eta,\varrho}=\mathcal{R}^{-1}\{r\in H_1^{1,1}:\|r\|_{H^{1,1}}<\eta,\ \|r\|_{L^\infty}<\varrho\}. \tag{1.28}$$

Fix $l>\frac{7}{2}$.

THEOREM 1.29. (i) *For each* $\varepsilon>0$, $\mathcal{B}_\varepsilon^+$ *is a nonempty, open, connected set in* $H^{1,1}$ *and* W^+ *is Lipschitz from* $\mathcal{B}_\varepsilon^+\to H^{1,1}$. *Moreover* $U_t^\varepsilon(\mathcal{B}_\varepsilon^+)\subset\mathcal{B}_\varepsilon^+$ *for all* $t\in\mathbf{R}$.

(ii) *Given any* $\eta>0$, $0<\varrho<1$, *there exists* $\varepsilon_0=\varepsilon_0(\eta,\varrho)$ *such that* $\mathcal{B}_{\eta,\varrho}\subset\mathcal{B}_\varepsilon^+$ *for all* $0\leqslant\varepsilon<\varepsilon_0$. *In particular,* $\bigcup_{\varepsilon>0}\mathcal{B}_\varepsilon^+=H^{1,1}$.

(iii) *For* $q\in\mathcal{B}_\varepsilon^+$, *for some* $\varkappa>0$, *as* $t\to\infty$,

$$\|U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty(dx)}\sim\frac{1}{t^{1/2}},\quad \|U_t^\varepsilon(q)-U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty(dx)}=O\biggl(\frac{1}{t^{1/2+\varkappa}}\biggr).$$

The following result (cf. (1.8) above) is an immediate consequence of (1.26) and Theorem 1.29 (iii).

COROLLARY (to Theorem 1.29). *For* $q\in\mathcal{B}_\varepsilon^+$, *as* $t\to\infty$,

$$U_t^\varepsilon(q)=q_{\mathrm{as}}(x,t)+O(t^{-1/2-\varkappa})\quad \textit{for some } \varkappa>0,$$

where

$$\begin{aligned}
q_{\mathrm{as}}(x,t)&=t^{-1/2}\alpha(z_0)\,e^{ix^2/4t-i\nu(z_0)\log 2t},\\
\nu(z_0)&=-\frac{1}{2\pi}\log(1-|r_+(z_0)|^2),\\
|\alpha(z_0)|^2&=\tfrac{1}{2}\nu(z_0),\\
\arg\alpha(z_0)&=\frac{1}{\pi}\int_{-\infty}^{z_0}\log(z_0-z)\,d(\log(1-|r_+(z)|^2))+\tfrac{1}{4}\pi+\arg\Gamma(i\nu(z_0))+\arg r_+(z_0).
\end{aligned}$$

Here Γ *is the gamma-function,* $z_0=x/2t$, $r_+=\mathcal{R}(W^+(q))$, *and the error term is uniform for all* $x\in\mathbf{R}$.

The above corollary shows that the long-time behavior of solutions of the NLS equation $iq_t+q_{xx}-2|q|^2q-\varepsilon\Lambda'(|q|^2)q=0$ is *universal* for a very general class of perturbation $\varepsilon\Lambda'(|q|^2)q$.

Remarks. In a very interesting recent paper [HN], Hayashi and Naumkin have proved a version of (iii) above using powerful, new PDE/Fourier techniques. In [HN] the initial data q is required to have small norm. Of course, for systems of type (1.1) where the nonlinear terms have different orders of homogeneity, the problem with finite norm $\|q\|_{H^{1,1}}$ and ε small, which we treat in this paper, cannot be reduced in general to a problem with small norm and $\varepsilon=1$.

The proof in §3 below that $W^+(q)$ exists requires that the size of the perturbation in (1.1) (as measured in $H^{1,1}$-norm) has to be small relative to the initial data. This can be achieved either by making the initial data itself small, or for large data, making $\varepsilon>0$ small. This means that for a given $\varepsilon>0$, $\mathcal{B}_\varepsilon^+$ contains a (small) $H^{1,1}$-ball, and this is the main content of Theorem 1.29 (i). On the other hand, for any given initial data $q=q_0\in H^{1,1}$, $q\in\mathcal{B}_\varepsilon^+$ for some sufficiently small $\varepsilon>0$, and this is the main content of Theorem 1.29 (ii).

THEOREM 1.30. *For any $\varepsilon>0$, there exists a Lipschitz map $\widehat{W}^+ : H^{1,1}\to\mathcal{B}_\varepsilon^+$ such that:*

(i) $W^+\circ\widehat{W}^+=1$, $\widehat{W}^+\circ W^+=1_{\mathcal{B}_\varepsilon^+}$;

(ii) (*Conjugation of the flow.*) *For $q\in\mathcal{B}_\varepsilon^+$ and for all $t\in\mathbf{R}$,*

$$U_t^\varepsilon(q)=\widehat{W}^+\circ U_t^{\mathrm{NLS}}\circ W^+(q)=T_\varepsilon^{-1}\circ U_t^F\circ T_\varepsilon(q), \tag{1.31}$$

where $T_\varepsilon=\mathcal{F}^{-1}\circ\mathcal{R}\circ W^+$ as in (1.18) *above.*

Set $\mathcal{B}_\varepsilon^-=\{q\in H^{1,1} : W^-(q)=\lim_{t\to-\infty}U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q)$ exists in $H^{1,1}\}$. Clearly $\mathcal{B}_\varepsilon^-$ has similar properties to $\mathcal{B}_\varepsilon^+$. The following result shows how to relate the asymptotic behavior of solutions $U_t^\varepsilon(q)$ of (1.1) in the distant past to the asymptotic behavior of the solution in the distant future, in certain cases.

THEOREM 1.32. *Suppose $q\in\mathcal{B}_\varepsilon^+\cap\mathcal{B}_\varepsilon^-$ and set $q^\pm=W^\pm(q)$. Define the scattering operator*

$$S(q^-)\equiv W^+\circ\widehat{W}^-(q^-)=q^+.$$

Then as $t\to\pm\infty$, $\|U_t^\varepsilon(q)-U_t^{\mathrm{NLS}}(q^\pm)\|_{L^\infty(dx)}=O(1/|t|^{1/2+\varkappa})$ for some $\varkappa>0$.

Using the fact that if $q(x,t)$ is a solution of (1.1), then $\overline{q(x,-t)}$ is also a solution, the asymptotic behavior of $U_t^\varepsilon(q)$ in the above theorem can be made explicit as $t\to-\infty$, as in the case $t\to+\infty$ in the corollary to Theorem 1.29.

Our final result, which is perhaps unexpected, shows that (1.1) is completely integrable on the nonempty, open, connected, invariant set $\mathcal{B}_\varepsilon^+$. As noted in §2, in addition to the reflection coefficient $r(z)=r(z;q)$, scattering theory for the ZS–AKNS operator

$$\partial_x-\left(iz\sigma+\begin{pmatrix}0 & q\\ \bar{q} & 0\end{pmatrix}\right)$$

also involves a transmission coefficient $t(z)=t(z;q)$. In ZS–AKNS scattering theory, $r(z)$ and $t(z)$ are given in terms of natural parameters $a(z)$ and $b(z)$ where $t(z)=1/a(z)$ and $r(z)=-\overline{b(z)/a(z)}$ (see §2). As is well known (see §3), equations (1.1) and (1.2) are Hamiltonian with respect to the following (nondegenerate) Poisson structure on suitably smooth functions $H,K,\ldots$:

$$\{H,K\}(q)=\int_{\mathbf{R}}\left(\frac{\delta H}{\delta\alpha}\frac{\delta K}{\delta\beta}-\frac{\delta H}{\delta\beta}\frac{\delta K}{\delta\alpha}\right)dx, \tag{1.33}$$

where $q=\alpha+i\beta=\operatorname{Re}q+i\operatorname{Im}q$.

THEOREM 1.34. *Fix* $\varepsilon>0$. *Then on* $\mathcal{B}_\varepsilon^+$ *the functions* $-(1/2\pi)\log|a(z;W^+(q))|$, $z\in\mathbf{R}$, *provide a complete set of commuting integrals for the perturbed* NLS *equation* (1.1). *Together with the function* $\arg b(z';W^+(q))$, $z'\in\mathbf{R}$, *these integrals constitute action–angle variables for the flow*: *for* $z,z'\in\mathbf{R}$,

$$\begin{gathered}\left\{-\frac{1}{2\pi}\log|a(z;W^+(q)|,\arg b(z';W^+(q)\right\}=\delta(z-z'),\\ \left\{-\frac{1}{2\pi}\log|a(z;W^+(q)|,-\frac{1}{2\pi}\log|a(z';W^+(q)|\right\}=0,\\ \{\arg b(z;W^+(q),\arg b(z';W^+(q))\}=0.\end{gathered}\tag{1.35}$$

Remark. The theorem of course remains true if we replace $\mathcal{B}_\varepsilon^+$ with $\mathcal{B}_\varepsilon^-$. Also note that when $\varepsilon=0$, the above result reduces to the standard action–angle theory for NLS (see, for example, [FaT]).

As shown in §3, the proof of Theorem 1.34 follows directly from the fact that W^+ is symplectic. The fact that any Hamiltonian system whose solutions are asymptotically "free" (or "integrable"), is itself automatically completely integrable, was first pointed out, many years ago, to one of the authors by Jürgen Moser. For example, if $(x(t),y(t))$ solves a Hamiltonian system

$$\dot{x}=H_y,\quad \dot{y}=-H_x,\quad (x(t=0),y(t=0))=(x_0,y_0)\tag{1.36}$$

in $\mathbf{R}^{2n}$, and if for suitable constants $(x_\infty,y_\infty)\in\mathbf{R}^{2n}$

$$x(t)=y_\infty t+x_\infty+o(1),\quad y(t)=y_\infty+o(1)\tag{1.37}$$

as $t\to\infty$, then the wave operator $W^+(x_0,y_0)=\lim_{t\to\infty}U_{-t}^0\circ U_t(x_0,y_0)=(x_\infty,y_\infty)$ exists, where $U_t(x_0,y_0)$ denotes the solution of (1.36) and $U_t^0(x_0',y_0')$ denotes the solution of the free particle motion

$$\dot{x}=H_x^0,\quad \dot{y}=-H_x^0,\quad (x(t=0),y(t=0))=(x_0',y_0'),\tag{1.38}$$

where $H^0(x,y)=\frac{1}{2}\|y\|^2$. Necessarily W^+, as a limit of a composition of symplectic maps, is also symplectic. Clearly the momenta y provide n commuting integrals for the free flow, and so, using the intertwining property for W^+, $U_t^0\circ W^+=W^+\circ U_t$, we see that y_∞, the asymptotic momenta for solutions of (1.36), provide a complete set of commuting integrals for the system. We note in passing that, because of the above comments, it follows from the results of McKean and Shatah [MKS] that equation (1.3) is also completely integrable on an open (invariant) set in phase space.

It is an instructive exercise to apply these ideas to the Toda lattice, which is generated by the Hamiltonian $H_T=\frac{1}{2}\sum_{k=1}^n y_k^2+\sum_{k=1}^{n-1}e^{x_k-x_{k+1}}$, on $\mathbf{R}^{2n}$. Solutions of this system are free in the sense of (1.37), and it is easy to relate the asymptotic momenta y_∞ to the well-known integrals for the Toda lattice given by the eigenvalues of the associated Lax operator. We refer the reader to [Mo1] for details.

The outline of the paper is as follows. In §2(a) we give some basic information on RHP's and introduce, in particular, the rigorous definition of the RHP (1.10). In §2(b) we discuss the solution of (1.1) in $H^{1,1}$ and show how to derive equation (1.21). Finally, in §2(c), we present uniform L^p-bounds, $p\geqslant 2$, for solutions of RHP's of type (1.10) with jump matrices $v_{x,t}$ of form (1.25). In §3 we prove the main Theorems 1.29, 1.30, 1.32 and 1.34 using fairly standard methods together with estimates from the key Lemma 6.4 of §6. In §4 we prove various smoothing estimates for solutions of the NLS equation and also for the solutions of the associated RHP's. The main results of the section are presented in Theorem 4.16. The time decay in (4.17)–(4.21) is obtained by using and extending steepest descent ideas from [DZ1], [DIZ] and [DZ2]. We note that related, but weaker, smoothing estimates for NLS were obtained in [Z2]. Also, certain smoothing estimates for KdV were obtained by Kappeler [Ka], using the Gelfand–Levitan–Marchenko equation. In §5 we supplement the estimates in §4 and place them in a form directly applicable to the analysis of the evolution equation (1.21). The principal technical tool in this section is a Sobolev-type theory using the modified derivative operator $L=\partial_z-i(x-2zt)\,\mathrm{ad}\,\sigma$ in place of the bare derivative ∂_z. The operator L is closely related (see e.g. Lemma 5.14) to the operator $\tilde{L}=ix\,\mathrm{ad}\,\sigma-2t\partial_x$, which is very close in turn to the operator $L_{\mathrm{MSh}}=x-2it\partial_x$ considered by McKean and Shatah in [MKS]. Finally, in §6 we use results from the previous sections to prove basic a priori estimates for solutions of (1.21) (or more precisely, for solutions of the equivalent equation (6.3)). The main results of the section are given in Lemma 6.4.

The theory of perturbations of integrable systems has generated a vast literature, and we conclude with a brief survey of results which are closest to ours and which have not yet been mentioned in the text. We will focus, in particular, on problems in 1+1 dimensions.

Equations of the form (1.21) for a variety of systems of type

$$q_t+K_0(q,q_x,q_{xx},\dots)=\varepsilon K_1(q,q_x,\dots), \tag{1.39}$$

where $q_t+K_0(q,q_x,q_{xx},\dots)=0$ is integrable, were first derived in [K1], [KN] and [KM]. In these papers the authors used equations of form (1.21), expanded formally in powers of ε, to obtain information on solutions (in particular, soliton-type solutions) of (1.39) for times of order $\varepsilon^{-\alpha}$ for some $\alpha>0$. Recently, Kivshar et al. [KGSV], and also Kaup [K2], have extended the method in [K1], [KN], [KM] to obtain information for times of order $\varepsilon^{-\alpha}$ for large values of α.

Results similar to [K1], [KN] and [KM] have been obtained by many authors, dating back to [A], [MLS], [W], using the multi-scale/averaging method directly on the perturbed equation (1.39) (for further information see [AS]). We also refer the reader to the interesting paper [Br] in which the author obtains similar results to those of Kivshar et al., using standard perturbation methods.

The result (1.7) of McKean and Shatah provides a very interesting infinite-dimensional example illustrating the case when a given nonlinear equation $\dot{x}=f(x)$, with equilibrium point $x=0$, say, can be conjugated to its linearization $\dot{y}=f'(0)y$ at the point. Equation (1.31) above now provides another such example. The subject of conjugation has a large literature; see, for example, [P], [Si], [H], [N], among many others. The literature is devoted almost exclusively to the finite-dimensional case.

As we have noted above, the map $q\mapsto\mathcal{R}(q)$ can also be viewed as a renormalization transformation taking solutions of (1.2) to solutions of the normal form equation (1.4). Kodama was the first to apply normal form ideas to nearly integrable (1+1)-dimensional systems, and in [Ko], in the case of KdV, he obtained a normal form transformation up to order ε^2. Kodama's transformation has been generalized recently by Fokas and Liu [FL].

In a different direction, Ozawa [O] considered solutions of generalized NLS equations

$$iq_t+q_{xx}-\lambda|q|^2q-\mu|q|^{p-1}q=0, \quad -\infty<x<\infty, \tag{1.40}$$

where $\lambda\in\mathbf{R}\setminus\{0\}$, $\mu\in\mathbf{R}$ and $p>3$. Under certain additional technical restrictions (e.g. $\mu\geqslant 0$ if $p\geqslant 5$), Ozawa used PDE methods to prove that modified Dollard-type wave operators $\widehat{W}^{\pm}$ (see, for example, [RS]) for (1.40) exist on a dense subset of a neighborhood of zero in $L^2(\mathbf{R})$ or $H^{1,0}(\mathbf{R})$. This means that solutions of (1.40) with initial data in $\operatorname{Ran}\widehat{W}^{\pm}$ behave, as $t\to\pm\infty$ respectively, like solutions of the NLS equation

$$iq_t+q_{xx}-\lambda|q|^2q=0. \tag{1.41}$$

In the case $\lambda>0$, these results are clearly related to Theorem 1.30 above.

Finally we mention the fundamental work of Zakharov on normal form theory for nonlinear wave systems [Za]. A particularly illuminating exposition of the consequences of Zakharov's theory in the context of a class of (1+1)-dimensional dispersive wave equations can be found in the recent paper of Majda et al. [MMT].

Some of the results of this paper were announced in [DZ3]. In future publications we plan to extend the methods of this paper to analyze perturbations of a variety of integrable systems, including systems with soliton solutions. Of course, when solitons are present, smoothing estimates of the form (4.20) can no longer be valid. However,

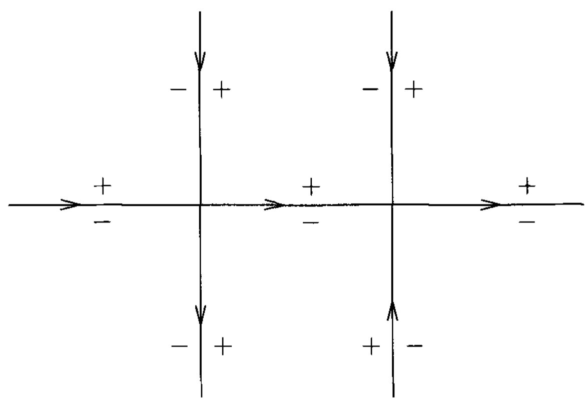

Fig. 2.1

after subtracting out the contribution of the solitons, we still expect an estimate of the form (4.20) to be true, but perhaps with a smaller power of t-decay.

Notational remarks. Throughout the text constants $c>0$ are used generically. Statements such as $\|f\|\leqslant 2c(1+e^c)\leqslant c$, for example, should not cause any confusion. Throughout the text, c always denotes a constant independent of x, t, η and ϱ.

Throughout the paper we use $\Diamond$ to denote a dummy variable. For example, $e^{\Diamond}g$ denotes the function f defined as $f(z)=e^z g(z)$.

2. Preliminaries

This section is in three parts:

Part (a). Give some basic information on RHP's, with particular reference to special features of RHP's occurring in this paper. A general reference text for RHP's is, for example, [CG].

Part (b). Discuss the solution of (1.1) in $H^{1,1}$ and show how to derive the basic dynamical equation (1.21).

Part (c). Present L^p-bounds, $p\geqslant 2$, for solutions of RHP's of type (1.10) with jump matrices $v_{x,t}$ of form (1.25). The key property of these bounds is that they are uniform in x and t.

Considerably more detail on parts (a), (b), (c) can be found in [DZW, §§2, 3, 4].

Part (a). Consider an oriented contour $\Sigma\subset\overline{\mathbf{C}}$. By convention we assume that as we traverse an arc of the contour in the direction of the orientation, the (+)-side (resp. (−)-side) lies to the left (resp. right), as indicated in Figure 2.1. Let $v:\Sigma\to GL(k,\mathbf{C})$ be a $k\times k$ jump matrix on Σ: as a standing assumption throughout the text, we always assume that $v, v^{-1} \in L^\infty(\Sigma \to GL(k, \mathbf{C}))$. For $1<p<\infty$, let

$$Ch(z) = C_\Sigma h(z) = \int_\Sigma \frac{h(s)}{s-z} \frac{ds}{2\pi i}, \quad h \in L^p(\Sigma),$$

define the Cauchy operator C on Σ. We say that a pair of $L^p(\Sigma)$-functions $f_\pm \in \partial \operatorname{Ran} C$ if there exists a (unique) function $h \in L^p(\Sigma)$ such that $f_\pm = C^\pm h$, the nontangential boundary values of Ch from the $(\pm)$-sides of Σ. In turn we call $f(z) = Ch(z)$, $z \in \bar{\mathbf{C}} \setminus \Sigma$, the *extension* of $f_\pm = C^\pm h \in \partial \operatorname{Ran} C$ off Σ. We refer the reader to [DZW, Appendix I] for the relevant analytic properties of the operators C and $C^\pm$. Note in particular that $C^\pm$ are bounded from $L^p(\Sigma) \to L^p(\Sigma)$ for $1<p<\infty$. A standard text on the subject is, for example, [Du].

Formally, a $(k \times k)$-matrix-valued function analytic in $\bar{\mathbf{C}} \setminus \Sigma$ solves the (normalized) RHP (Σ, v) if $m_+(z) = m_-(z)v(z)$ for $z \in \Sigma$, where $m_\pm$ denote the nontangential boundary values of m from the $(\pm)$-side, and $m(z) \to I$ in some sense, as $z \to \infty$. More precisely, we make the following definition.

Definition 2.2. Fix $1<p<\infty$. We say that $m_\pm$ solves the (normalized) RHP $(\Sigma, v)_p$ if $m_\pm - I \in L^p(\Sigma) \cap \partial \operatorname{Ran} C$ and $m_+(z) = m_-(z)v(z)$, a.e. $z \in \Sigma$.

In the above definition, we also say that the extension m of $m_\pm$ off Σ solves the RHP. Clearly m solves the RHP in the above formal sense with $m_\pm - I \in L^2$.

Mostly, we are interested in $p=2$, in which case we will drop the subscript and simply write (Σ, v). Let $v = (v^-)^{-1} v^+ = (I - w^-)^{-1}(I + w^+)$ be a factorization of v with $v^\pm, (v^\pm)^{-1} \in L^\infty$, and let C_w, $w = (w^-, w^+)$, denote the associated singular integral operator

$$C_w h = C^+(hw^-) + C^-(hw^+) \tag{2.3}$$

acting on L^p-matrix-valued functions h. As $w^\pm \in L^\infty$, C_w is clearly bounded from $L^p \to L^p$ for all $1<p<\infty$. The operator C_w plays a basic role in the solution of the RHP $(\Sigma, v)_p$. Indeed, suppose that in addition $w^\pm = \pm(v^\pm - I) \in L^p$, and let $\mu \in I + L^p(\Sigma)$ solve the equation

$$(1 - C_w)\mu = I, \tag{2.4}$$

or more precisely, suppose that $h = \mu - I$ solves the equation

$$(1 - C_w)h = C_w I = C^+ w^- + C^- w^+$$

in L^p. Then a simple calculation shows that

$$m_\pm = I + C^\pm(\mu(w^+ + w^-)) = \mu v^\pm \tag{2.5}$$

and hence solves the RHP $(\Sigma, v)_p$ for any factorization $v = (I - w^-)^{-1}(I + w^+)$, as long as $w^\pm = \pm(v^\pm - I) \in L^p \cap L^\infty$. Such factorizations of v play the role of parametrices for the RHP in the sense of the theory of pseudo-differential operators, and different factorizations are used freely throughout the text in order to achieve various analytical goals. Moreover, using simple identities, it is easy to see ([DZW, §2]) that bounds obtained using one factorization $v=(v^-)^{-1}v^+$ imply similar bounds for any other factorization $v=(\tilde{v}^-)^{-1}\tilde{v}^+$.

If $k=2$, $p=2$ and $\det v(z)=1$ a.e. on Σ, then the solution m (or equivalently $m_\pm$) of the normalized RHP $(\Sigma, v)=(\Sigma, v)_2$ is unique. Also $\det m(z)\equiv 1$ (see [DZW, §2]). These results apply in particular to the RHP (1.10), and will be used without further comment throughout the paper.

The principal objects of study for the RHP (1.10) are eigensolutions $\psi=\psi(x,z)$ of the ZS–AKNS operator $\partial_x - U(x,z)$ (see (1.9)),

$$(\partial_x - U(x,z))\psi = \left(\partial_x - \left(iz\sigma + \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix}\right)\right)\psi = 0. \tag{2.6}$$

Setting

$$m = \psi e^{-ixz\sigma}, \tag{2.7}$$

equation (2.6) takes the form

$$\partial_x m = iz \operatorname{ad}\sigma(m) + Qm, \quad Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{2.8}$$

where $\operatorname{ad} A(B)=[A,B]=AB-BA$. Under exponentiation we have

$$e^{\operatorname{ad} A}B = \sum_{n=0}^{\infty} \frac{(\operatorname{ad} A)^n(B)}{n!} = e^A B e^{-A}.$$

The theory of ZS–AKNS ([ZS], [AKNS]) is based on the following two Volterra integral equations for real z,

$$m^{(\pm)}(x,z) = I + \int_{\pm\infty}^{x} e^{i(x-y)z \operatorname{ad}\sigma} Q(y) m^{(\pm)}(y,z)\, dy \equiv I + K_{q,z,\pm} m^{(\pm)}. \tag{2.9}$$

By iteration, one sees that these equations have bounded solutions continuous for both x and real z when $q\in L^1(\mathbf{R})$. The matrices $m^{(\pm)}(x,z)$ are the unique solutions of (2.8) normalized to the identity as $x\to\pm\infty$. The following are some relevant results of ZS–AKNS theory:

(2.10a) There is a continuous matrix function $A(z)$ for real z, $\det A(z)=1$, defined by $\psi^{(+)}=\psi^{(-)}A(z)$, where $\psi^{(\pm)}=m^{(\pm)}e^{ixz\sigma}$ and A has the form

$$A(z) = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}.$$

(2.10b) a is the boundary value of an analytic function, also denoted by a, in the upper half-plane $\mathbf{C}_+$: a is continuous and nonvanishing in $\overline{\mathbf{C}_+}$, and $\lim_{z\to\infty} a(z)=1$.

(2.10c)

$$a(z)=\det(m_1^{(+)}, m_2^{(-)})=1-\int_{\mathbf{R}} q(y)m_{21}^{(+)}(y,z)\,dy=1+\int_{\mathbf{R}} \overline{q(y)}\,m_{12}^{(-)}(y,z)\,dy,$$

$$b(z)=e^{ixz}\det(m_1^{(-)}, m_1^{(+)})=-\int_{\mathbf{R}} \overline{q(y)}\,e^{iyz}m_{11}^{(+)}(y,z)=-\int_{\mathbf{R}} \overline{q(y)}\,e^{iyz}m_{11}^{(-)}(y,z)\,dy,$$

where

$$m^{(\pm)}=(m_1^{(\pm)}, m_2^{(\pm)})=\begin{pmatrix} m_{11}^{(\pm)} & m_{12}^{(\pm)} \\ m_{21}^{(\pm)} & m_{22}^{(\pm)} \end{pmatrix}.$$

(2.10d) The *reflection coefficient* r is defined by $-\bar{b}/\bar{a}$. As $\det A=1$, $|a|^2-|b|^2=1$, so that $|a|\geqslant 1$ and $|r|^2=1-|a|^{-2}<1$. Together with (2.10b), this implies that $\|r\|_{L^\infty(\mathbf{R})}<1$. The *transmission coefficient* $t(z)$ is defined by $1/a(z)$. Thus

$$\|t\|_{L^\infty(\mathbf{R})}\leqslant 1 \quad \text{and} \quad |r(z)|^2+|t(z)|^2=1.$$

The basic scattering/inverse scattering result of ZS–AKNS is that the reflection map $\mathcal{R}: q\mapsto\mathcal{R}(q)\equiv r$ is one-to-one and onto for $q(x)$ and $r(z)$ in suitable spaces. In this paper we will study the map $\mathcal{R}$ by means of the associated RHP for the first-order system (2.6) introduced by Beals and Coifman. In [BC] the authors consider solutions $m=m(x,z)$ of (2.8) for $z\in\mathbf{C}\setminus\mathbf{R}$ with the following properties:

$$m(x,z)\to I \quad \text{as } x\to-\infty, \tag{2.11}$$

$$m(x,z) \text{ is bounded} \quad \text{as } x\to+\infty. \tag{2.12}$$

Such solutions exist and are unique, and for fixed x, they are analytic for z in $\mathbf{C}\setminus\mathbf{R}$ with boundary values $m_\pm(x,z)=\lim_{\varepsilon\downarrow 0} m(x,z\pm i\varepsilon)$ on the real axis. Moreover, $m_\pm$ are related to the ZS–AKNS solution $m^{(-)}$ through

$$m_\pm(x,z)=m^{(-)}(x,z)e^{ixz\,\mathrm{ad}\,\sigma}v^\pm(z), \quad z\in\mathbf{R}, \tag{2.13}$$

where

$$v^+=\begin{pmatrix} 1 & 0 \\ -\bar{r} & 1 \end{pmatrix}, \quad v^-=\begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \tag{2.14}$$

(see e.g. [Z1]). Note that the asymptotic relation (2.11) fails for $m_\pm(x,z)$, $z\in\mathbf{R}$, but (2.12) remains true both as $x\to+\infty$, $x\to-\infty$.

Using the notation $B_x \equiv e^{ixz\,\mathrm{ad}\,\sigma} B$, we have from (2.13), (2.14) the jump relation

$$m_+(x,z) = m_-(x,z)v_x(z), \quad z \in \mathbf{R}, \tag{2.15}$$

where

$$v = v(z) = (v^-(z))^{-1}v^+(z) = \begin{pmatrix} 1-|r(z)|^2 & r(z) \\ -\overline{r(z)} & 1 \end{pmatrix}. \tag{2.16}$$

Thus

$$v_x = (v_x^-)^{-1}v_x^+, \tag{2.17}$$

where we always take $v_x^\pm \equiv (v^\pm)_x$. In addition, if q has sufficient decay, for example if q lies in the space $\{q : \int_{\mathbf{R}}(1+x^2)|q(x)|^2\,dx<\infty\} \subset L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, then for each $x\in\mathbf{R}$, $m_\pm = m_\pm(x,\cdot) \in I + L^2(\mathbf{R})$ solves the normalized RHP $(\Sigma, v_x) = (\Sigma, v_x)_2$ in the precise sense of Definition 2.2,

$$\begin{aligned} m_+(x,z) &= m_-(x,z)v_x(z), \quad z\in\mathbf{R}, \\ m_\pm(x,\cdot) - I &\in \partial \operatorname{Ran} C, \end{aligned} \tag{2.18}$$

with contour $\Sigma = \mathbf{R}$ oriented from left to right (see e.g. [Z1]; see also [DZW, Appendix II]). This is the Beals–Coifman RHP associated with (2.6) and the NLS equation.

Define $w_x = (w_x^-, w_x^+)$ through $v_x^\pm = I \pm w_x^\pm$, and let C_{w_x} be the associated singular integral operator as in (2.3). Then (see §2(c) below), $1 - C_{w_x}$ is invertible in $L^2(\mathbf{R})$. Let μ be the (unique) solution of $(1 - C_{w_x})\mu = I$, $\mu \in I + L^2(\mathbf{R})$. Then as noted above, the boundary values $m_\pm$ of

$$m(z) = m(x,z) \equiv I + \int_{\mathbf{R}} \frac{\mu(x,s)(w_x^+(s) + w_x^-(s))}{s-z}\frac{ds}{2\pi i}, \quad z \in \mathbf{C}\setminus\mathbf{R}, \tag{2.19}$$

lie in $I + L^2(\mathbf{R})$ and satisfy the RHP (2.18). Set

$$\mathbf{Q} = \mathbf{Q}(x) = \int_{\mathbf{R}} \mu(x,s)(w_x^+(s) + w_x^-(s))\,ds. \tag{2.20}$$

Then a simple computation shows that

$$Q = \frac{1}{2\pi}\,\mathrm{ad}\,\sigma(\mathbf{Q}) \tag{2.21}$$

is the potential

$$\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}$$

in the ZS–AKNS equation (2.8).

As indicated above, the map $\mathcal{R}$ is a bijection for q and $\mathbf{r}$ in suitable spaces. In particular, the methods in [ZS], [AKNS] and [BC] imply that $\mathcal{R}: \mathcal{S}(\mathbf{R}) \to \mathcal{S}_1(\mathbf{R}) =$

$\mathcal{S}(\mathbf{R})\cap\{r:\|r\|_{L^\infty(\mathbf{R})}<1\}$, taking $q\mapsto r=\mathcal{R}(q)$, is a smooth bijection with a smooth inverse $\mathcal{R}^{-1}$. However, for perturbation theory, it is important to consider $\mathcal{R}$ as a mapping between Banach spaces. The following result plays a central role.

First we need some definitions. Throughout this paper we denote by $|A|$ the Hilbert–Schmidt norm of a matrix $A=(A_{ij})$, $|A|=(\sum_{i,j}|A_{ij}|^2)^{1/2}$. A simple computation shows that $|\cdot|$ is a Banach norm, $|AB|\leqslant|A|\,|B|$. For a matrix-valued function $f(x)$ on $\mathbf{R}$, define the weighted Sobolev space

$$H^{k,j}=\{f:f,\partial_x^k f,x^j f\in L^2(\mathbf{R})\},\quad k,j\geqslant 0, \tag{2.22}$$

with norm

$$\|f\|_{H^{k,j}}=(\|f\|_{L^2}^2+\|\partial_x^k f\|_{L^2}^2+\|x^j f\|_{L^2}^2)^{1/2}, \tag{2.23}$$

where the L^2-norm of a matrix function f is defined as the L^2-norm of $|f|$. Also define

$$H_1^{k,j}=\{f\in H^{k,j}:\|f\|_{L^\infty}<1\},\quad k\geqslant 1,\ j\geqslant 0. \tag{2.24}$$

Observe that by standard computations, if $f\in H^{k,j}$, then $x^i\partial_x^{l-i}f\in L^2(\mathbf{R})$ and

$$\|x^i\partial_x^{l-i}f\|_{L^2}\leqslant c\|f\|_{H^{l,l}} \tag{2.25}$$

for $0\leqslant i\leqslant l\equiv\min(k,j)$.

Recall that a map F from a subset D of a Banach space $\mathcal{B}$ into $\mathcal{B}$ is (locally) Lipschitz if D is covered by a collection of (relatively) open sets $\{N\}$ with the following property: for each N there exists a positive number $L(N)$ such that

$$\|F(q_1)-F(q_2)\|_{\mathcal{B}}\leqslant L(N)\|q_1-q_2\| \tag{2.26}$$

for all $q_1,q_2\in N\subset D$.

PROPOSITION 2.27 [Z1]. *The map $\mathcal{R}$ is bi-Lipschitz from $H^{k,j}$ onto $H_1^{j,k}$ for $k\geqslant 0$, $j\geqslant 1$.*

A proof of this proposition in the case $k=j=1$, which is of central interest in this paper, is given in [DZW, Appendix II]).

Part (b). The spaces $H^{k,j}$ are particularly well-suited to the NLS equation. Indeed, as is well known (see below), if $q(t)$, $t\geqslant 0$, solves the NLS equation with initial data $q(0)$, then $\mathcal{R}(q(t))$ evolves in the simple fashion

$$\mathcal{R}(q(t))=\mathcal{R}(q(0))e^{-itz^2}=re^{-itz^2}. \tag{2.28}$$

On the other hand, a straightforward computation shows that for $k \geqslant j$, multiplication by e^{-itz^2} is a bijection from $H_1^{j,k}$ onto itself (the key fact is that if $f \in H^{j,k}$ for $j \leqslant k$, then $z^l \partial_z^{j-l} f \in L^2$ for $0 \leqslant l \leqslant j$, by (2.25)). Hence the NLS equation is soluble in $H^{k,j}$ for $1 \leqslant j \leqslant k$,

$$q(t=0) \stackrel{\mathcal{R}}{\longmapsto} r \longmapsto re^{-itz^2} \stackrel{\mathcal{R}^{-1}}{\longmapsto} q(t)$$

Moreover, as $\mathcal{R}^{-1}$ is Lipschitz, it is easy to verify that the map $t \mapsto q(t) = \mathcal{R}^{-1}(re^{-it\lozenge^2})$ is continuous from $\mathbf{R}$ to $H^{k,j}$.

The perturbed equation (1.1) is a particular example of the general form

$$\begin{aligned} iq_t + q_{xx} &= V'(|q|^2)q, \\ q(t=0) &= q_0, \end{aligned} \tag{2.29}$$

where V is a smooth function from $\mathbf{R}_+$ to $\mathbf{R}_+$ with $V(0) = V'(0) = 0$. We say that $q(t)$, $t \geqslant 0$, is a (weak, global) solution of (2.29) if $q \in C([0, \infty), H^{k,k})$ for some $k \geqslant 1$ and

$$q(t) = e^{-iH_0 t} q_0 - i \int_0^t e^{-iH_0(t-s)} V'(|q(s)|^2) q(s)\, ds, \tag{2.30}$$

where $H_0 = -\partial_x^2$ is negative Laplacian regarded as a self-adjoint operator on $L^2(\mathbf{R})$. Observe that if $k \geqslant 2$, then q solves (2.29) in the L^2-sense, and if $k \geqslant 3$, then $q = q(x,t)$ is a classical solution.

The following result, which is far from optimal, is sufficient for our purposes. For more information on solutions of (2.29) see, for example, [O] and the references therein.

THEOREM 2.31. *Let $\varepsilon > 0$ and suppose that Λ is a C^2-map from $\mathbf{R}_+$ to $\mathbf{R}_+$. Suppose in addition that $\Lambda(0) = \Lambda'(0) = 0$ and $\Lambda(s), \Lambda'(s) \geqslant 0$ for $s \in \mathbf{R}_+$. Then for $V(s) = s^2 + \varepsilon\Lambda(s)$, equation (2.30) has a unique (weak, global) solution q in $H^{1,1}$. Moreover, for each $t > 0$, the map $q_0 \mapsto q_0(t) = q(t; q_0)$ is a bi-Lipschitz map from $H^{1,1}$ to $H^{1,1}$.*

The proof of Theorem 2.31 is standard and uses the conserved quantities

$$\int |q(x,t)|^2\, dx = \int |q_0(x)|^2\, dx, \tag{2.32}$$

$$\int |\partial_x q(x,t)|^2\, dx + \int V(|q(x,t)|^2)\, dx = \int |\partial_x q_0(x)|^2\, dx + \int V(|q_0(x)|^2)\, dx \tag{2.33}$$

to obtain a global solution in the familiar way. If V has additional smoothness, then (2.30) has a solution in $H^{k,k}$ for values of $k > 1$. Equation (1.1) corresponds to the choice

$$V(s) = s^2 + \frac{2\varepsilon}{l+2} s^{(l+2)/2},$$

and throughout this paper by the solution of (1.1) we mean the unique (weak, global) solution of (2.30) in $H^{1,1}$.

As $\mathcal{R}$ is a bijection from $H^{1,1}$ onto $H_1^{1,1}$, solutions $q(t)$ of (1.1) induce a flow on reflection coefficients $r=\mathcal{R}(q(0))$ in $H_1^{1,1}$ via $t\mapsto r(\cdot\,;q(t))=\mathcal{R}(q(t))$, $t\geqslant 0$. The equation for this flow can be derived as follows (cf. [KN]). Simple algebraic manipulations show that (2.29) with $V(s)=s^2+\varepsilon\Lambda(s)$ is equivalent to the commutator relation

$$[\partial_x-U,\partial_t-W]=\varepsilon G(q), \tag{2.34}$$

where

$$\begin{aligned} U&=iz\sigma+\begin{pmatrix}0&q\\ \bar q&0\end{pmatrix},\\ W&=-iz^2\sigma-z\begin{pmatrix}0&q\\ \bar q&0\end{pmatrix}+\begin{pmatrix}-i|q|^2&i\partial_x q\\ -i\partial_x\bar q&i|q|^2\end{pmatrix},\\ G&=-i\Lambda'(|q|^2)\begin{pmatrix}0&q\\ -\bar q&0\end{pmatrix}. \end{aligned} \tag{2.35}$$

Applying (2.34) to $\psi^{(-)}=m^{(-)}e^{ixz\sigma}$, using variation of parameters, and evaluating the constant of integration at $x=-\infty$, one obtains the equation

$$(\partial_t-W)\psi^{(-)}=iz^2\psi^{(-)}\sigma+\psi^{(-)}\int_{-\infty}^{x}(\psi^{(-)})^{-1}G\psi^{(-)}\,dy. \tag{2.36}$$

Using the relation $\psi^{(+)}=\psi^{(-)}A$ to substitute for $\psi^{(-)}$ in terms of $\psi^{(+)}$, and letting $x\to+\infty$, one obtains an equation for A^{-1}, and hence for a and b:

$$\begin{aligned} \partial_t\bar a&=\varepsilon\bar a\left(\int_{-\infty}^{\infty}(\psi^{(-)})^{-1}G\psi^{(-)}\,dy\right)_{11}-\varepsilon\bar b\left(\int_{-\infty}^{\infty}(\psi^{(-)})^{-1}G\psi^{(-)}\,dy\right)_{21},\\ -\partial_t\bar b&=iz^2\bar b+\bar a\varepsilon\left(\int_{-\infty}^{\infty}(\psi^{(-)})^{-1}G\psi^{(-)}\,dy\right)_{12}-\bar b\varepsilon\left(\int_{-\infty}^{\infty}(\psi^{(-)})^{-1}G\psi^{(-)}\,dy\right)_{22}. \end{aligned} \tag{2.37}$$

Substituting m^- for $m^{(-)}$ via relation (2.13), we obtain finally an equation for $r=-\bar b/\bar a$:

$$\partial_t r=-iz^2r+\varepsilon\int_{-\infty}^{\infty}e^{-iyz}(m_-^{-1}Gm_-)_{12}\,dy, \tag{2.38}$$

or emphasizing the dependence on x, z and $q(t)$,

$$\partial_t r(z;q(t))=-iz^2r(z;q(t))+\varepsilon\int_{-\infty}^{\infty}e^{-iyz}(m_-^{-1}(y,z;q(t))G(q(y,t))m_-(y,z;q(t)))_{12}\,dy, \tag{2.39}$$

where $r(z;q(t))=(\mathcal{R}(q(t)))(z)$. Defining $r(t)$ via

$$r(t)(z)=e^{itz^2}r(z;q(t)) \tag{2.40}$$

as in (1.24), and integrating, we obtain

$$r(t)(z)=r_0(z)+\varepsilon\int_0^t ds\, e^{iz^2s}\int_{-\infty}^{\infty} dy\, e^{-iyz}(m_-^{-1}(y,z;q(s))G(q(y,s))m_-(y,z;q(s)))_{12}, \tag{2.41}$$

where

$$r_0(z)=\mathcal{R}(q_0)(z), \quad q_0=q(t=0). \tag{2.42}$$

Note that if $q(t)=U_t^{\mathrm{NLS}}(q_0)$ solves NLS (i.e. the case $\varepsilon=0$), then

$$r(t)(z)=e^{iz^2t}r(z;U_t^{\mathrm{NLS}}(q_0))=r(z;q_0)$$

and the $H^{1,0}(dz)$-norm of $r(t)$ is constant and hence bounded in t. As we will see, for the general evolution $q(t)=U_t^{\varepsilon}(q_0)$, at least when $\varepsilon>0$ is small, the heart of the analysis lies in the fact that the $H^{1,0}(dz)$-norm of $r(t)(z)=e^{itz^2}r(z;q(t))$ also remains bounded as $t\to\infty$.

Note that if $\varepsilon=0$, so that $G=0$, then $r(z;q(t))=e^{-iz^2t}\mathcal{R}(q_0)$, which is the well-known evolution for the reflection coefficient under the NLS flow as described above. Replacing $r=\mathcal{R}(q_0)$ with $e^{-iz^2t}r$ in (2.16) we obtain the RHP

$$\begin{aligned} m_+(z)&=m_-(z)v_\theta(z), \quad z\in\mathbf{R},\\ m_\pm-I&\in\partial\operatorname{Ran}C, \end{aligned} \tag{2.43}$$

for the solution $q(x,t)=(\mathcal{R}^{-1}(e^{-i\diamond^2t}r))(x)$ of NLS, $q(0)=q_0$, where

$$\theta=xz-tz^2 \quad\text{and}\quad v_\theta=e^{i\theta\,\mathrm{ad}\,\sigma}v=\begin{pmatrix}1-|r|^2 & e^{i\theta}r\\ -e^{-i\theta}\bar r & 1\end{pmatrix}. \tag{2.44}$$

Observe that x and t play the role of external parameters for the RHP. For future reference, we note from (2.20), (2.21) that if $\mu=\mu(x,t,z)$ solves $(1-C_{w_\theta})\mu=I$, then

$$Q=\begin{pmatrix}0 & q(x,t)\\ \bar q(x,t) & 0\end{pmatrix}=\frac{\mathrm{ad}\,\sigma}{2\pi}\left(\int_{\mathbf{R}}\mu(x,t,z)(w_\theta^+ + w_\theta^-)\,dz\right). \tag{2.45}$$

In order to write (2.41) as an equation for $r(t)=e^{iz^2t}\mathcal{R}(q(t))$ we must substitute $\mathcal{R}^{-1}(e^{-iz^2t}r)$ for q on the right-hand side of the equations. Recall that $m_-(y,z;q)$, which is the boundary value from $\mathbf{C}_-$ of the Beals–Coifman eigensolution for q, can also be viewed as the boundary value from $\mathbf{C}_-$ of the solution of the RHP (2.18) with $r=\mathcal{R}(q)$.

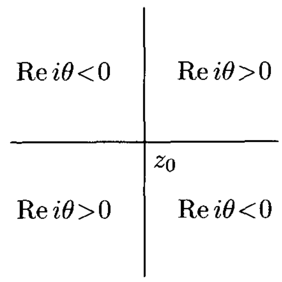

Fig. 2.51. Signature table for Re $i\theta$.

With this understanding it is natural to write $m_-(y,z;q)=m_-(y,z;r)$ where $r=\mathcal{R}(q)$. With this notation (2.41) becomes

$$r(t)(z)=r_0(z)+\varepsilon\int_0^t ds\, e^{iz^2 s}\int_{-\infty}^{\infty} dy\, e^{-iyz}(m_-^{-1}(y,z;e^{-i\Diamond^2 s}r(s)) \\ \times G(\mathcal{R}^{-1}(e^{-i\Diamond^2 s}r(s))(y))m_-(y,z;e^{-i\Diamond^2 s}r(s)))_{12}, \tag{2.46}$$

where $r(t{=}0)=r_0$ and $r\in C([0,\infty),H_1^{1,1})$. Uniqueness for solutions of (2.46) follows from the Lipschitz estimate (6.6) in §6. The basic result on (1.1), (2.41), (2.46) is the following.

PROPOSITION 2.47. *If $q\in C([0,\infty),H^{1,1})$ solves (1.1) with $\varepsilon>0$, $l>2$, then $r(t)(z)=e^{itz^2}\mathcal{R}(q(t))(z)$ solves (2.41). Conversely, suppose that $r\in C([0,\infty),H_1^{1,1})$ solves (2.46). Then $q(t)\equiv\mathcal{R}^{-1}(e^{-i\Diamond^2 t}r(t))$ solves (1.1) in $C([0,\infty),H^{1,1})$.*

For later reference we note the following useful property of functions in $H^{1,1}$.

LEMMA 2.48. *If $r\in H^{1,1}$ then $zr^2\in L^1\cap L^\infty$ and $\|\Diamond r^2(\Diamond)\|_{L^p}\leqslant c_p\|r\|_{H^{1,1}}^2$, $1\leqslant p\leqslant\infty$.*

Part (c). The operator $1-C_{w_\theta}$ associated with the RHP (2.43) with factorization

$$v_\theta=\begin{pmatrix}1 & -re^{i\theta}\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ -\bar{r}e^{-i\theta} & 1\end{pmatrix}$$

is invertible in L^2 with a bound independent of x and t (see [DZW, (4.2)]),

$$\|(1-C_{w_\theta})^{-1}\|_{L^2(\mathbf{R})\to L^2(\mathbf{R})}\leqslant c(1-\varrho)^{-1} \tag{2.49}$$

for some absolute constant c and for all $r\in L^\infty$ satisfying $\|re^{i\theta}\|_{L^\infty}=\|r\|_{L^\infty}\leqslant\varrho<1$. Furthermore if $m_\pm$ solves the RHP (2.43), then (see [DZW, (4.7)])

$$\|m_\pm-I\|_{L^2(\mathbf{R})}\leqslant c(1-\varrho)^{-1}\|r\|_{L^2(\mathbf{R})} \tag{2.50}$$

for all $x,t\in\mathbf{R}$. In the analysis of (2.46), we will need similar uniform bounds in L^p for $p>2$. Standard RHP arguments (see e.g. [CG]) imply that $1-C_{w_\theta}$ is invertible for all $1<p<\infty$, but a priori the bounds on $\|(1-C_{w_\theta})^{-1}\|_{L^p(\mathbf{R})\to L^p(\mathbf{R})}$ and $\|m_\pm-I\|_{L^p(\mathbf{R})}$ may grow as $x,t\to\infty$. It is one of the basic technical results of [DZ5] (see also [DZW]) that for $p>2$, there exists bounds, uniform in $x,t\in\mathbf{R}$, on these two norms.

Following the steepest descent method introduced in [DZ1], and applied to the NLS equation in [DIZ], [DZ2], we expect the RHP to "localize" near the stationary phase point $z_0=x/2t$ for $\theta=xz-tz^2$, $\theta'(z_0)=0$. Furthermore, the signature table of $\operatorname{Re} i\theta$ should play a crucial role. The basic idea of the method is to deform the contour $\Gamma=\mathbf{R}$ so that the exponential factors $e^{i\theta}$ and $e^{-i\theta}$ are exponentially decreasing, as dictated by Figure 2.51. In order to make these deformations we must separate the factors $e^{i\theta}$ and $e^{-i\theta}$ algebraically, and this is done using the upper/lower and lower/upper factorizations of v,

$$\begin{aligned} v &= \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\bar{r}/(1-|r|^2) & 1 \end{pmatrix} \begin{pmatrix} 1-|r|^2 & 0 \\ 0 & 1/(1-|r|^2) \end{pmatrix} \begin{pmatrix} 1 & r/(1-|r|^2) \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.52}$$

The upper/lower factorization is appropriate for $z>z_0$, and the lower/upper factorization is appropriate for $z<z_0$. The diagonal terms in the lower/upper factorization can be removed by conjugating v,

$$\breve{v}=\delta_-^{\sigma_3} v \delta_+^{-\sigma_3}, \quad \sigma_3 = \text{Pauli matrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\sigma, \tag{2.53}$$

by the solution $\delta_\pm$ of the scalar, normalized RHP $(\mathbf{R}_- + z_0, 1-|r|^2)$,

$$\begin{aligned} \delta_+ &= \delta_-(1-|r|^2), \quad z\in\mathbf{R}_- + z_0, \\ \delta_\pm - 1 &\in \partial \operatorname{Ran} C, \end{aligned} \tag{2.54}$$

where the contour $\mathbf{R}_- + z_0$ is oriented from $-\infty$ to z_0. The properties of δ can be read off from the following elementary proposition, which will be used repeatedly throughout the text that follows, and whose proof is left to the reader.

PROPOSITION 2.55. *Suppose $r\in L^\infty(\mathbf{R})\cap L^2(\mathbf{R})$ and $\|r\|_{L^\infty}\leqslant\varrho<1$. Then the solution $\delta_\pm$ of the scalar, normalized RHP* (2.54) *exists, is unique and is given by the formula*

$$\delta_\pm(z) = e^{C^\pm_{\mathbf{R}_- + z_0}\log(1-|r|^2)} = e^{\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\log(1-|r(s)|^2)}{s-z_\pm}\,ds}, \quad z\in\mathbf{R}. \tag{2.56}$$

The extension δ of $\delta_\pm$ off $\mathbf{R}_- + z_0$ is given by

$$\delta(z) = e^{C_{\mathbf{R}_- + z_0}\log(1-|r|^2)} = e^{\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\log(1-|r(s)|^2)}{s-z}\,ds}, \quad z\in\mathbf{C}\setminus(\mathbf{R}_- + z_0), \tag{2.57}$$

and satisfies for $z\in\mathbf{C}\setminus(\mathbf{R}_- + z_0)$,

$$\begin{aligned} &\delta(z)\overline{\delta(\bar{z})} = 1, \\ &(1-\varrho)^{1/2} \leqslant (1-\varrho^2)^{1/2} \leqslant |\delta^{\pm 1}(z)| \leqslant (1-\varrho^2)^{-1/2} \leqslant (1-\varrho)^{-1/2}, \\ &|\delta^{\pm 1}(z)| \leqslant 1 \quad \textit{for } \pm\operatorname{Im} z > 0. \end{aligned} \tag{2.58}$$

For real z,

$$|\delta_+(z)\delta_-(z)|=1 \tag{2.59}$$

and, in particular, $|\delta(z)|=1$ *for* $z>z_0$, $|\delta_+(z)|=|\delta_-^{-1}(z)|=(1-|r(z)|^2)^{1/2}$, $z<z_0$, *and*

$$\begin{gathered}\Delta\equiv\delta_+\delta_-=e^{\frac{1}{i\pi}\,\mathrm{P.V.}\int_{-\infty}^{z_0}\frac{\log(1-|r(s)|^2)}{s-z}\,ds},\\ |\Delta|=|\delta_+\delta_-|=1,\quad \|\delta_\pm-1\|_{L^2(dz)}\leqslant\frac{c\|r\|_{L^2}}{1-\varrho}.\end{gathered}\tag{2.60}$$

We obtain the following factorizations for $\breve{v}$:

$$\breve{v}=\breve{v}_-^{-1}\breve{v}_+=\begin{pmatrix}1 & r\delta^2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\bar{r}\delta^{-2} & 1\end{pmatrix},\quad z>z_0, \tag{2.61}$$

$$\breve{v}=\breve{v}_-^{-1}\breve{v}_+=\begin{pmatrix}1 & 0\\ -\bar{r}\delta_-^{-2}/(1-|r|^2) & 1\end{pmatrix}\begin{pmatrix}1 & r\delta_+^2/(1-|r|^2)\\ 0 & 1\end{pmatrix},\quad z<z_0, \tag{2.62}$$

which imply in turn the factorizations for $\breve{v}_\theta=e^{i\theta\,\mathrm{ad}\,\sigma}\breve{v}$:

$$\breve{v}_\theta=\breve{v}_{\theta-}^{-1}\breve{v}_{\theta+}=\begin{pmatrix}1 & re^{i\theta}\delta^2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\bar{r}e^{-i\theta}\delta^{-2} & 1\end{pmatrix},\quad z>z_0, \tag{2.63}$$

$$\breve{v}_\theta=\breve{v}_{\theta-}^{-1}\breve{v}_{\theta+}=\begin{pmatrix}1 & 0\\ -\bar{r}e^{-i\theta}\delta_-^{-2}/(1-|r|^2) & 1\end{pmatrix}\begin{pmatrix}1 & re^{i\theta}\delta_+^2/(1-|r|^2)\\ 0 & 1\end{pmatrix},\quad z<z_0. \tag{2.64}$$

Using Figure 2.51 we observe the crucial fact that the analytic continuations to $\mathbf{C}_+$ of the exponentials in the factors on the right in (2.63) and (2.64) are exponentially decreasing, whereas the same is true for the exponentials on the left, when continued to $\mathbf{C}_-$.

For later reference, observe that (2.62) and (2.64) can also be written in the form

$$\breve{v}=\begin{pmatrix}1 & 0\\ -\bar{r}\delta_+^{-1}\delta_-^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & r\delta_+\delta_-\\ 0 & 1\end{pmatrix},\quad z<z_0, \tag{2.65}$$

and

$$\breve{v}_\theta=\begin{pmatrix}1 & 0\\ -\bar{r}e^{-i\theta}\delta_+^{-1}\delta_-^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & re^{i\theta}\delta_+\delta_-\\ 0 & 1\end{pmatrix},\quad z<z_0, \tag{2.66}$$

respectively.

The basic result is the following. For any jump matrix v let C_v denote the associated operator C_w with the trivial factorization $v=I^{-1}v$, i.e. $v^+=v$, $v^-=I$. As noted earlier, L^p-bounds for $(1-C_{w_\theta})^{-1}$ imply similar L^p-bounds for any other factorization $v_\theta=(v_\theta^-)^{-1}v_\theta^+$. Hence by (2.49),

$$\|(1-C_{v_\theta})^{-1}\|_{L^2\to L^2}\leqslant c_2(1-\varrho)^{-1}\equiv K_2$$

for the trivial factorization $v_\theta=I^{-1}v_\theta$ above.

PROPOSITION 2.67. *Suppose* $r \in H_1^{1,0}$, $\|r\|_{H^{1,0}} \leqslant \lambda$, $\|r\|_{L^\infty} \leqslant \varrho < 1$. *Then for any* $x, t \in \mathbf{R}$, *and for any* $2 < p < \infty$, $(1-C_{v_\theta})^{-1}$ *and* $(1-C_{\check{v}_\theta})^{-1}$ *exist as bounded operators in* $L^p(\mathbf{R})$ *and satisfy the bounds*

$$\|(1-C_{v_\theta})^{-1}\|_{L^p \to L^p},\ \|(1-C_{\check{v}_\theta})^{-1}\|_{L^p \to L^p} \leqslant K_p, \tag{2.68}$$

where $K_p = c_p(1+\lambda)^8(1-\varrho)^{-37}$. *The constants* c_p *may be chosen so that* K_p *is increasing with* p *and* $K_p \geqslant K_2$.

As above, the bounds in (2.68) imply similar L^p-bounds for $(1-C_{w_\theta})^{-1}$ for any other factorization $v_\theta = (v_\theta^-)^{-1} v_\theta^+$. We will use this fact throughout the paper without further comment. Bounds on $\|m_\pm - I\|_{L^p(\mathbf{R})}$ of type (2.50) for $p>2$ are immediate consequences of (2.68).

The proof of Proposition 2.67 is given in [DZ5] and also in [DZW, §4].

3. Proofs of the main theorems

Notation. We refer the reader to (4.1) below for the definition of the symbol

$$\begin{bmatrix} k & l \\ i & j \end{bmatrix}$$

and to the beginning of §5 for the definition of Δ.

In this section we use the estimates for F and ΔF in Lemma 6.4 in §6 below to prove Theorems 1.29, 1.30, 1.32 and 1.34 in the Introduction.

Suppose $l > \frac{7}{2}$ and choose n sufficiently large and p sufficiently close to 2, $2 < p \leqslant 4$, so that

$$\frac{l}{2} - \frac{1}{2n} - \frac{3}{4} > \frac{l}{2} + \frac{1}{2p} - \frac{1}{2n} - 1 > 1. \tag{3.1}$$

Let $\eta > 0$ and $0 < \varrho < 1$. Then it follows from (6.5) and (6.6) that for $t \geqslant 0$ and $r, r_1, r_2 \in \{f : \|f\|_{H^{1,1}} \leqslant \eta,\ \|f\|_{L^\infty} \leqslant \varrho\}$,

$$\|F(t,r)\|_{H^{1,1}} \leqslant c \frac{\eta^{d_1}(1+\eta)^{d_2}}{(1+t)^{1+d_3}(1-\varrho)^{d_4}}, \tag{3.2}$$

$$\|F(t,r_2) - F(t,r_1)\|_{H^{1,1}} \leqslant c \frac{\eta^{e_1}(1+\eta)^{e_2}}{(1+t)^{1+e_3}(1-\varrho)^{e_4}} \|r_2 - r_1\|_{H^{1,1}}, \tag{3.3}$$

where c is a positive constant and

$$d_1 = l+1, \quad d_2 = l+29, \quad d_3 = \frac{l}{2} - \frac{1}{2n} - \frac{7}{4} > 0, \quad d_4 = 5l+111, \tag{3.4}$$

$$e_1 = l, \quad e_2 = l+38, \quad e_3 = \frac{l}{2} + \frac{1}{2p} - \frac{1}{2n} - 2 > 0, \quad e_4 = 5l + \frac{452}{3}. \tag{3.5}$$

If $l\geqslant 4$, these constants can be reduced considerably. Indeed for $l\geqslant 4$, we may take

$$d_1=l+1, \quad d_2=l+17+\varepsilon, \quad d_3=\frac{l}{2}-\frac{23}{12}>0, \qquad d_4=5l+57+\varepsilon, \tag{3.6}$$

$$e_1=l, \qquad e_2=l+22+\varepsilon, \quad e_3=\frac{l}{2}+\frac{1}{2p}-\frac{13}{6}>0, \quad e_4=5l+\frac{236}{3}+\varepsilon, \tag{3.7}$$

for $p>2$, p sufficiently close to 2, and for any $\varepsilon>0$.

Remark. These large constants should perhaps be compared with the large constants that appeared in the early papers in KAM theory (see, for example, [Mo2]). Just as the sizes of the KAM constants have been reduced by various researchers over the years, we anticipate that the constants in (3.4), (3.5), (3.6) and (3.7) will also be reduced when finer estimates on the inverse spectral map, $r\mapsto\mathcal{R}^{-1}(r)$, become available.

Observe from (6.3) that the basic dynamical equation (2.46) takes the form

$$r(t)=r_0+\varepsilon\int_0^t F(s,r(s))\,ds, \tag{3.8}$$

where F is given by (6.1), (6.2). The proofs of Theorems 1.29, 1.30 and 1.32 follow by applying (6.5) and (6.6) to (3.8) in the standard way.

We begin with the proof of Theorem 1.29. Fix $l>\frac{7}{2}$. Suppose that $\eta>0$ and $0<\varrho<1$ are given, and suppose that $\|r_0\|_{H^{1,1}}<\eta$ and $\|r_0\|_{L^\infty}<\varrho$. By the results of §2, equation (3.8) has a (unique) global solution $r\in C([0,\infty),H^{1,1})$, $r(t{=}0)=r_0$. Let

$$T=\sup\{\tau:\|r(t)\|_{H^{1,1}}\leqslant 2\eta,\ \|r(t)\|_{L^\infty}\leqslant\tfrac{1}{2}(1+\varrho)\text{ for all }0\leqslant t\leqslant\tau\}. \tag{3.9}$$

Clearly $T>0$; suppose $T<\infty$. Then by (3.2), for all $t\leqslant T$,

$$\begin{aligned}\|r(t)\|_{H^{1,1}}&\leqslant\|r_0\|_{H^{1,1}}+\varepsilon c\int_0^t\frac{(2\eta)^{d_1}(1+2\eta)^{d_2}2^{d_4}}{(1+s)^{1+d_3}(1-\varrho)^{d_4}}\,ds\\&\leqslant\eta+\frac{\varepsilon c(2\eta)^{d_1}(1+2\eta)^{d_2}2^{d_4}}{d_3(1-\varrho)^{d_4}}\leqslant\frac{3}{2}\eta\end{aligned} \tag{3.10}$$

provided that $\varepsilon\leqslant\varepsilon_1(\eta,\varrho)=d_3(1-\varrho)^{d_4}/c\eta^{d_1-1}(1+2\eta)^{d_2}2^{d_4+d_1+1}$. Similarly, using the fact that $\|r\|_{L^\infty}\leqslant\|r\|_{H^{1,1}}$,

$$\|r(t)\|_{L^\infty}\leqslant\tfrac{1}{3}(1+2\varrho)<\tfrac{1}{2}(1+\varrho) \tag{3.11}$$

provided that $\varepsilon\leqslant\varepsilon_2(\eta,\varrho)=d_3(1-\varrho)^{d_4+1}/3c\eta^{d_1}(1+2\eta)^{d_2}2^{d_1+d_4}$. But then by continuity $\|r(t)\|_{H^{1,1}}\leqslant 2\eta$, $\|r(t)\|_{L^\infty}\leqslant\frac{1}{2}(1+\varrho)$ for all $0\leqslant t\leqslant\tau$ for some $\tau>T$, which is a contradiction.

Hence $T=\infty$ and $\|r(t)\|_{H^{1,1}}\leqslant 2\eta$, $\|r(t)\|_{L^\infty}\leqslant\frac{1}{2}(1+\varrho)$ for all $t\geqslant 0$. It follows then that for $t_2>t_1>0$

$$\|r(t_2)-r(t_1)\|_{H^{1,1}}\leqslant\frac{\varepsilon c(2\eta)^{d_1}(1+2\eta)^{d_2}2^{d_4}}{(1-\varrho)^{d_4}d_3}\left(\frac{1}{(1+t_1)^{d_3}}-\frac{1}{(1+t_2)^{d_3}}\right),$$

and so $\{r(t)\}$ is Cauchy and $\Omega^+(r_0)=\lim_{t\to\infty}r(t)$ exists in $H_1^{1,1}$. But then as $\mathcal{R}$ is bi-Lipschitz and $\mathcal{R}(U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q_0))=r(t)$, $r(t=0)=r_0=\mathcal{R}(q_0)$, the wave operator

$$W^+(q_0)=\lim_{t\to\infty}U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q_0)=\mathcal{R}^{-1}\circ\Omega^+\circ\mathcal{R}(q_0)\tag{3.12}$$

exists in $H^{1,1}$ provided that $\varepsilon\leqslant\varepsilon_0(\eta,\varrho)=\min(\varepsilon_1(\eta,\varrho),\varepsilon_2(\eta,\varrho))$. Thus in the notation of the Introduction, $\mathcal{B}_{\eta,\varrho}\subset\mathcal{B}_\varepsilon^+$ for $\varepsilon\leqslant\varepsilon_0(\eta,\varrho)$. This proves (ii) in Theorem 1.29.

As noted in the Introduction, from the relation $U_{-s}^{\mathrm{NLS}}\circ U_s^\varepsilon\circ U_t^\varepsilon=U_t^{\mathrm{NLS}}\circ U_{-(t+s)}^{\mathrm{NLS}}\circ U_{t+s}^\varepsilon$, $s,t\in\mathbf{R}$, it follows that $U_t^\varepsilon\mathcal{B}_\varepsilon^+\subset\mathcal{B}_\varepsilon^+$ and the intertwining relation $W^+\circ U_t^\varepsilon=U_t^{\mathrm{NLS}}\circ W^+$ is satisfied on $\mathcal{B}_\varepsilon^+$.

As $U_t^\varepsilon(q_0)=U_t^{\mathrm{NLS}}(q_0)=0$ for $q_0=0$, it follows that $\mathcal{B}_\varepsilon^+\neq\varnothing$ for all $\varepsilon>0$. We now show that $\mathcal{B}_\varepsilon^+$ is open. Suppose that $r_\infty\equiv\Omega^+(r_0)$ exists for some r_0. Set $\eta=\|r_\infty\|_{H^{1,1}}$, $\varrho=\|r_\infty\|_{L^\infty}$. Then if $r(t)$ is the solution of (3.8) with $r(0)=r_0$, there exists $T>0$ such that $\|r(t)\|_{H^{1,1}}<\frac{3}{2}\eta$, $\|r(t)\|_{L^\infty}<\frac{1}{3}(1+2\varrho)$ for $t\geqslant T$. We can assume in addition that T is sufficiently large so that

$$\frac{\varepsilon c2^{d_1+d_4}\eta^{d_1-1}(1+2\eta)^{d_2}}{(1-\varrho)^{d_4}d_3(1+T)^{d_3}}<\frac{1}{2}\quad\text{and}\quad\frac{\varepsilon c2^{d_1+d_4}\eta^{d_1}(1+2\eta)^{d_2}}{(1-\varrho)^{d_4}d_3(1+T)^{d_3}}<\frac{1-\varrho}{6}.$$

Now choose $\gamma>0$ sufficiently small so that if $\|\tilde r_0-r_0\|_{H^{1,1}}<\gamma$, and $\tilde r(t)$ is the solution of (3.8) with $\tilde r(t=0)=\tilde r_0$, then $\|\tilde r(T)\|_{H^{1,1}}<\frac{3}{2}\eta$, $\|\tilde r(T)\|_{L^\infty}<\frac{1}{3}(1+2\varrho)$. Such a $\gamma>0$ clearly exists as $q_0\mapsto U_T^\varepsilon(q_0)$ is continuous in $H^{1,1}$. Arguing as in (3.9) above, we conclude that $\|\tilde r(t)\|_{H^{1,1}}<2\eta$, $\|\tilde r(t)\|_{L^\infty}<\frac{1}{2}(1+\varrho)$ for all $t\geqslant T$, and hence, as before, $\{\tilde r(t)\}$ is Cauchy. Thus $\Omega^+(\tilde r_0)$ exists in $H^{1,1}$ for all $\|\tilde r_0-r_0\|_{H^{1,1}}<\gamma$. This proves that $\mathcal{B}_\varepsilon^+$ is open.

Now suppose that $r_0,\tilde r_0\in\mathcal{R}(\mathcal{B}_\varepsilon^+)$ so that $\Omega^+(r_0)=\lim_{t\to\infty}r(t)$, $\Omega^+(\tilde r_0)=\lim_{t\to\infty}\tilde r(t)$ exist. Let

$$\eta=\max\bigl(\sup_{t\geqslant 0}\|r(t)\|_{H^{1,1}},\sup_{t\geqslant 0}\|\tilde r(t)\|_{H^{1,1}}\bigr),\quad\varrho=\max\bigl(\sup_{t\geqslant 0}\|r(t)\|_{L^\infty},\sup_{t\geqslant 0}\|\tilde r(t)\|_{L^\infty}\bigr).$$

Clearly $\eta<\infty$ and $\varrho<1$, and it follows from (3.3) that for all $t\geqslant 0$,

$$\|\tilde r(t)-r(t)\|_{H^{1,1}}\leqslant\|\tilde r_0-r_0\|_{H^{1,1}}+\frac{\varepsilon c\eta^{e_1}(1+\eta)^{e_2}}{(1-\varrho)^{e_4}}\int_0^t\frac{\|\tilde r(s)-r(s)\|_{H^{1,1}}}{(1+s)^{1+e_3}}\,ds$$

and hence by integration $\|\tilde r(t)-r(t)\|_{H^{1,1}}\leqslant(1+\varkappa e^\varkappa)\|\tilde r_0-r_0\|_{H^{1,1}}$, where

$$\varkappa=\frac{\varepsilon c\eta^{e_1}(1+\eta)^{e_2}}{e_3(1-\varrho)^{e_4}}.$$

Thus

$$\|\Omega^+(\tilde r_0)-\Omega^+(r_0)\|_{H^{1,1}} \leqslant (1+\varkappa e^{\varkappa})\|\tilde r_0 - r_0\|_{H^{1,1}}. \tag{3.13}$$

Now it is easy to see from the previous calculations that the map

$$r_0 \mapsto \big(\sup_{t\geqslant 0}\|r(t;r_0)\|_{H^{1,1}}, \sup_{t\geqslant 0}\|r(t;r_0)\|_{L^\infty}\big)$$

is a continuous map from $\mathcal{R}(\mathcal{B}_\varepsilon^+)$ to $\mathbf{R}^2$, and hence for each $\eta>0$, $0<\varrho<1$, the set $N_{\eta,\varrho}=\{q\in\mathcal{B}_\varepsilon^+ : \sup_{t\geqslant 0}\|r(t;\mathcal{R}(q_0))\|_{H^{1,1}}<\eta, \sup_{t\geqslant 0}\|r(t;\mathcal{R}(q_0))\|_{L^\infty}<\varrho\}$ is open. Clearly

$$\mathcal{B}_\varepsilon^+ = \bigcup_{\substack{\eta>0\\ 0<\varrho<1}} N_{\eta,\varrho}.$$

We conclude from (3.13) that W^+ is (locally) Lipschitz. More precisely (see (2.26)), if $q_0, \tilde q_0 \in N_{\eta,\varrho}$ for some $\eta>0$, $0<\varrho<1$, then $\|W^+(\tilde q_0)-W^+(q_0)\|_{H^{1,1}} \leqslant L(N_{\eta,\varrho})\|\tilde q_0 - q_0\|_{H^{1,1}}$ for some constant $L(N_{\eta,\varrho})$.

This completes the proof of (i) in Theorem 1.29, apart from the fact that $\mathcal{B}_\varepsilon^+$ is connected, which we will prove a little further on.

The fact that $\|U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty} \sim t^{-1/2}$ as $t\to\infty$ follows directly from (1.26). Alternatively, by (4.20) for $q\in\mathcal{B}_\varepsilon^+$ we have $\|U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty}=O((1+t)^{-1/2})$ as $t\to\infty$. Then an argument using the conservation of the L^2-norm of $q(t)$ (see [DZW, §8]) shows that in fact $\|U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty}\sim t^{-1/2}$ as $t\to\infty$.

Finally suppose that $q\in\mathcal{B}_\varepsilon^+$ and let $r(t)$ solve (3.8) with $r_0=\mathcal{R}(q)$. Set $r_\infty=\Omega^+(r_0)=\lim_{t\to\infty} r(t)$. Then by (4.21), for any $p>2$, as $t\to\infty$,

$$\|\widetilde{\mathbf{Q}}(e^{-i\lozenge^2 t}r(t))-\widetilde{\mathbf{Q}}(e^{-i\lozenge^2 t}r_\infty)\|_{L^\infty(dx)} \leqslant \frac{c}{(1+t)^{1/2p+1/4}}\|r(t)-r_\infty\|_{H^{1,1}}$$

for some constant c. Unravelling the definitions, this implies that

$$\|U_t^\varepsilon(q)-U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty(dx)} \leqslant \frac{c}{(1+t)^{1/2p+1/4}}\|r(t)-r_\infty\|_{H^{1,1}}.$$

But inserting (3.2) into (3.8), we easily see that as $t\to\infty$, $\|r(t)-r_\infty\|_{H^{1,1}}=O(1/t^{d_3})$. Choosing $p>2$ appropriately, it follows that $\|U_t^\varepsilon(q)-U_t^{\mathrm{NLS}}(W^+(q))\|_{L^\infty(dx)}=O(t^{-1/2-\varkappa})$ for some $\varkappa>0$. (Clearly we choose p so that $\varkappa$ is arbitrarily close to d_3.) This completes the proof of the second part of (iii) in Theorem 1.29.

We now consider Theorem 1.30. Fix $l>\frac{7}{2}$ and $\varepsilon>0$. Let $\eta>0$ and $0<\varrho<1$ be given and suppose that $r_\infty\in H_1^{1,1}$ with $\|r_\infty\|_{H^{1,1}}<\eta$, $\|r_\infty\|_{L^\infty}<\varrho$. It then follows from (3.2) and (3.3) that for $T>0$ sufficiently large the map $Z(r)(t)=r_\infty-\varepsilon\int_t^\infty F(s,r(s))\,ds$,

$t \geqslant T$, is a strict contraction, $\|Z(\tilde{r})-Z(r)\|_X \leqslant L\|\tilde{r}-r\|_X$, $L<1$, on the Banach space $X=C([T,\infty), H^{1,1}) \cap \{\sup_{t\geqslant T} \|r(t)\|_{H^{1,1}} \leqslant 2\eta, \sup_{t\geqslant T} \|r(t)\|_{L^\infty} \leqslant \frac{1}{2}(1+\varrho)\}$. Hence Z has a (unique) fixed point $r\in X$,

$$r(t) = Z(r)(t) = r_\infty - \varepsilon \int_t^\infty F(s, r(s))\, ds. \tag{3.14}$$

It follows directly from (3.2) and (3.14) that $\lim_{t\to\infty} r(t) = r_\infty$ exists in $H_1^{1,1}$. Set $q(T) = \mathcal{R}^{-1}(r(T))$ and let $q(t)$, $t\leqslant T$, be the (unique) solution of (1.1) in $H^{1,1}$ with $q(t=T)=q(T)$. Such a solution exists for all t by the methods of §2, which also imply that $\tilde{r}(t) \equiv \mathcal{R}(q(t))$ solves (3.8),

$$\tilde{r}(t) = \tilde{r}_0 + \varepsilon \int_0^t F(s, \tilde{r}(s))\, ds \tag{3.15}$$

for all $t\geqslant 0$, where $\tilde{r}_0 = \mathcal{R}(q(t=0))$. In particular for $t \geqslant T$, as $\tilde{r}(T) = \mathcal{R}(q(T)) = r(T)$, we have $\tilde{r}(t) = r(T) + \varepsilon \int_T^t F(s, \tilde{r}(s))\, ds$. But from (3.14), also for $t\geqslant T$, $r(t) = r(T) + \varepsilon \int_T^t F(s, r(s))\, ds$, and hence by uniqueness (again use (3.3)) we must have

$$r(t) = \tilde{r}(t), \quad t \geqslant T. \tag{3.16}$$

Set $\widehat{\Omega}^+(r_\infty) \equiv \tilde{r}_0$, which is clearly well defined (independently of T). As before, it is easy to check that $\widehat{\Omega}^+$ is Lipschitz on $H_1^{1,1}$. Now $\mathcal{R}(\tilde{r}_0) \in \mathcal{B}_\varepsilon^+$. Indeed if $\tilde{r}(t)$ solves (3.15), then $\tilde{r}(t) = r(t)$ for $t \geqslant T$ by (3.16), and so $\lim_{t\to\infty} \tilde{r}(t)$ exists in $H_1^{1,1}$. Moreover $\lim_{t\to\infty} \tilde{r}(t) = r_\infty$, and so $\Omega^+(\tilde{r}_0) = r_\infty$. It follows that if we set $\widehat{W}^+ = \mathcal{R}^{-1} \circ \widehat{\Omega}^+ \circ \mathcal{R}$, then $\widehat{W}^+$ maps $H^{1,1}$ into $\mathcal{B}_\varepsilon^+$ and

$$W^+ \circ \widehat{W}^+ = 1. \tag{3.17}$$

Conversely if $q_0 \in \mathcal{B}_\varepsilon^+$, and $r(t)$ solves (3.8) with $r_0 = \mathcal{R}(q_0)$, then it follows that $r(t) = r_\infty - \varepsilon \int_t^\infty F(s, r(s))\, ds$, $t\geqslant 0$, where $r_\infty = \Omega^+(r_0)$. But then the preceding arguments show that $\widehat{\Omega}^+(r_\infty) = r_0$. Thus

$$\widehat{W}^+ \circ W^+ = 1_{\mathcal{B}_\varepsilon^+}. \tag{3.18}$$

This proves (i) and (ii) of Theorem 1.30. The proof of (iii), conjugation of the flows, is immediate from the above intertwining relation and (3.17). This completes the proof of Theorem 1.30.

From (3.18) we see that $\mathcal{B}_\varepsilon^+ = \widehat{W}^+(H^{1,1})$, and as $H^{1,1}$ is connected, it follows, in particular, that $\mathcal{B}_\varepsilon^+$ is connected. This completes the proof of Theorem 1.29.

The set $\mathcal{B}_\varepsilon^- = \{q \in H^{1,1} : W^-(q) = \lim_{t\to-\infty} U_{-t}^{\mathrm{NLS}} \circ U_t^\varepsilon(q) \text{ exists in } H^{1,1}\}$ clearly has similar properties to $\mathcal{B}_\varepsilon^+$. The proof of Theorem 1.32 follows immediately by unravelling the definitions and using the proof of (iii) in Theorem 1.29.

Finally we consider Theorem 1.34. As is well known (see, for example, [FaT]), equation (1.1) and the NLS equation are Hamiltonian with respect to the symplectic structure on suitably smooth functions $H, K, \ldots$,

$$\{H, K\}(q) = \int_{\mathbf{R}} \left(\frac{\delta H}{\delta \alpha} \frac{\delta K}{\delta \beta} - \frac{\delta H}{\delta \beta} \frac{\delta K}{\delta \alpha} \right) dx, \tag{3.19}$$

where $q = \alpha + i\beta = \operatorname{Re} q + i \operatorname{Im} q$. Indeed

$$K^{\varepsilon}(q) = \frac{1}{2} \int_{\mathbf{R}} \left(|\partial_x q|^2 + |q|^4 + \frac{2\varepsilon}{l+2} |q|^{l+2} \right) dx$$

generates (1.1), $\partial q / \partial t = \{q, K^{\varepsilon}\} = i(q_{xx} - 2|q|^2 q - \varepsilon |q|^l q)$, and

$$K^{\mathrm{NLS}}(q) = \frac{1}{2} \int_{\mathbf{R}} (|\partial_x q|^2 + |q|^4)\, dx$$

generates NLS, $\partial q / \partial t = \{q, K^{\mathrm{NLS}}\} = i(q_{xx} - 2|q|^2 q)$.

The action–angle variables for NLS are given in terms of the matrix

$$A = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$$

of §2 (see [FaT]). One has

$$\begin{gathered} \left\{ -\frac{1}{2\pi} \log |a(z)|, \arg b(z') \right\} = \delta(z - z'), \\ \left\{ -\frac{1}{2\pi} \log |a(z)|, -\frac{1}{2\pi} \log |a(z')| \right\} = 0, \\ \{\arg b(z), \arg b(z')\} = 0. \end{gathered} \tag{3.20}$$

Using the relations $|a|^2 - |b|^2 = 1$, $r = -\bar{b}/\bar{a}$, and the identity

$$K^{\mathrm{NLS}}(q) = -\frac{1}{4\pi} \int z^2 \log(1 - |r(z)|^2)\, dz$$

(cf. the proof of Lemma 5.24), we compute for solutions $q(t)$ of NLS,

$$\begin{aligned} \frac{d}{dt} \left(-\frac{1}{2\pi} \log |a(z'; q(t))| \right) &= \left\{ -\frac{1}{2\pi} \log |a(z')|, K^{\mathrm{NLS}}(q(t)) \right\} \\ &= \left\{ -\frac{1}{2\pi} \log |a(z')|, \frac{1}{2\pi} \int z^2 \log |a(z)|\, dz \right\} \\ &= -\frac{1}{(2\pi)^2} \int z^2 \{\log |a(z')|, \log |a(z)|\}(q(t))\, dz = 0, \end{aligned} \tag{3.21}$$

$$\frac{d}{dt} \arg b(z'; q(t)) = \frac{1}{2\pi} \int z^2 \{\arg b(z'), \log |a(z)|\}(q(t))\, dz = (z')^2. \tag{3.22}$$

Thus $\{-(1/2\pi)\log|a(z)|\}_{z\in\mathbf{R}}$ give the actions and $\{\arg b(z)\}_{z\in\mathbf{R}}$ give the angles for NLS. Of course, (3.21) and (3.22) are nothing more than the familiar fact that $(d/dt)r(z';q(t))=-i(z')^2r(z';q(t))$.

Now as (1.1) and the NLS equation are Hamiltonian, it follows immediately that the maps $q\mapsto U_t^\varepsilon(q)$, $q\mapsto U_t^{\mathrm{NLS}}(q)$ are symplectic for any $t\in\mathbf{R}$. In particular, $q\mapsto U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon(q)$ is symplectic for any t. The nontrivial fact, which can be proved by the methods of this paper, and whose details are left to the (energetic) reader, is that this map remains symplectic in the limit as $t\to\infty$. More precisely, $W^+=\lim_{t\to\infty}U_{-t}^{\mathrm{NLS}}\circ U_t^\varepsilon$ is symplectic on $\mathcal{B}_\varepsilon^+$. Thus

$$\{H\circ W^+, K\circ W^+\}(q)=\{H,K\}(W^+(q)) \quad \text{for } q\in\mathcal{B}_\varepsilon^+. \tag{3.23}$$

It follows immediately that

$$\left\{-\frac{1}{2\pi}\log|a(z;W^+(q)|, z\in\mathbf{R}\right\} \quad \text{and} \quad \{\arg b(z';W^+(q)), z'\in\mathbf{R}\}$$

provide action–angle variables for (1.1) on $\mathcal{B}_\varepsilon^+$. Indeed, the commutation relations (3.20) are preserved by (3.23), and for all $z,z'\in\mathbf{R}$,

$$\begin{aligned}-\frac{1}{2\pi}\log|a(z;W^+(U_t^\varepsilon(q))|&=-\frac{1}{2\pi}\log|a(z;U_t^{\mathrm{NLS}}(W^+(q)))|\\&=-\frac{1}{2\pi}\log|a(z;W^+(q))|,\end{aligned} \tag{3.24}$$

$$\begin{aligned}\arg b(z';W^+(U_t^\varepsilon(q)))&=\arg b(z';U_t^{\mathrm{NLS}}(W^+(q)))\\&=\arg b(z';W^+(q))+(z')^2t.\end{aligned} \tag{3.25}$$

In particular, $\{-(1/2\pi)\log|a(z;W^+(q)|, z\in\mathbf{R}\}$ provide a complete set of integrals for the perturbed NLS equation. This completes the proof of Theorem 1.34.

Observe that if $t\mapsto U_t$ is a Hamiltonian flow on $H^{1,1}$, then the same is true for the flow $t\mapsto V_t\equiv\widehat{W}^+\circ U_t\circ W^+$ on $\mathcal{B}_\varepsilon^+$. Indeed if K is the Hamiltonian for $t\mapsto U_t$, then for $q\in\mathcal{B}_\varepsilon^+$, $(d/dt)H(U_t\circ W^+(q))=\{H,K\}(U_t\circ W^+(q))$, which can be rewritten using (3.23) in the form $(d/dt)H\circ W^+(V_t(q))=\{H\circ W^+,K\circ W^+\}(V_t(q))$, and so $t\mapsto V_t$ is generated by the Hamiltonian $K\circ W^+$. In particular, $t\to U_t^\varepsilon=\widehat{W}^+\circ U_t^{\mathrm{NLS}}\circ W^+$ is generated by the Hamiltonian

$$K^{\mathrm{NLS}}\circ W^+(q)=\frac{1}{2}\int_{\mathbf{R}}(|\partial_x W^+(q)|^2+|W^+(q)|^4)\,dx.$$

But we know that $t\mapsto U_t^\varepsilon$ is generated by the Hamiltonian

$$K^\varepsilon(q)=\frac{1}{2}\int_{\mathbf{R}}\left(|\partial_x q|^2+|q|^4+\frac{2\varepsilon}{l+2}|q|^{l+2}\right)dx$$

so that for $q\in\mathcal{B}_\varepsilon^+$ we must have the interesting identity

$$\int_{\mathbf{R}}\left(|\partial_x q|^2+|q|^4+\frac{2\varepsilon}{l+2}|q|^{l+2}\right)dx=\int_{\mathbf{R}}(|\partial_x W^+(q)|^2+|W^+(q)|^4)\,dx. \tag{3.26}$$

A similar argument shows that

$$\int_{\mathbf{R}}|q|^2\,dx=\int_{\mathbf{R}}|W^+(q)|^2\,dx. \tag{3.27}$$

For any $z\in\mathbf{R}$, let $U_t^{(z)}(q_0)$ denote the flow generated by the Hamiltonian

$$-\frac{1}{2\pi}\log|a(z;q)|$$

(suitably mollified with respect to z). These flows form a commuting family of flows for the NLS equation. But then by the above comments, the flows $U_t^{(\varepsilon,z)}(q_0)\equiv\widehat{W}^+\circ U_t^{(z)}\circ W^+(q_0)$, $z\in\mathbf{R}$, form a commuting family of Hamiltonian flows for the perturbed NLS equation (1.1), with Hamiltonians $-(1/2\pi)\log|a(z;W^+(q))|$, $z\in\mathbf{R}$. Said differently, we see in particular that $\mathcal{B}_\varepsilon^+$ is invariant under the flows generated by all the commuting integrals $-(1/2\pi)\log|a(z;W^+(q))|$, $z\in\mathbf{R}$, for the perturbed equation (1.1).

Observe that if we replace q by $W^+(q)$ in the Lax pair U,W for NLS (see (2.35)), $(U(q),W(q))\to U(W^+(q)),W(W^+(q))$, then the zero curvature condition

$$[\partial_x-U\circ W^+,\partial_t-W\circ W^+]=0 \tag{3.28}$$

is equivalent to the fact that $\tilde{q}(t)=W^+(q(t))$ solves NLS, i.e., $W^+(q(t))=U_t^{\mathrm{NLS}}\circ W^+(q)$, $q(t=0)=q_0$. But then by the intertwining relation, $q(t)=U_t^\varepsilon(q_0)$. Thus $\partial_x-U\circ W^+$, $\partial_t-W\circ W^+$ constitute a Lax pair for the perturbed NLS equation on $\mathcal{B}_\varepsilon^+$. Of course, $U\circ W^+$ and $W\circ W^+$ are highly nonlocal.

Remark 3.29. Keeping careful track of all the orders of decay, the reader may check that the proofs of Theorems 1.29, 1.30, 1.32, 1.34, as well as the proof of the corollary to Theorem 1.29, go through for Λ satisfying the following conditions: (i) $\Lambda\in C^2(\mathbf{R}_+)$, $\Diamond\Lambda''\in\mathrm{Lip}$, (ii) $\Lambda,\Lambda'\geqslant 0$, $\Lambda(0)=\Lambda'(0)=\Lambda''(0)=0$, (iii) $(x\Lambda''(x))'=O(x^s)$ as $x\downarrow 0$, for some $s>\frac{3}{4}$.

4. Smoothing estimates

In this section we will prove various smoothing estimates for the solution m of the normalized RHP $(\mathbf{R},v_\theta)$, where

$$v_\theta=e^{i\theta\,\mathrm{ad}\,\sigma}v=e^{i\theta\,\mathrm{ad}\,\sigma}\begin{pmatrix}1-|r|^2 & r\\ -\bar{r} & 1\end{pmatrix},\quad \theta=xz-tz^2.$$

Fig. 4.4. $\widetilde{\mathbf{R}}_{z_0}$ and Γ_{z_0}.

Our main results are given in Theorem 4.16 below.

Henceforth we will always assume that $r \in H_1^{1,1}$, which corresponds to potentials $q = \mathcal{R}^{-1}(r)$ in $H^{1,1}$ by Proposition 2.27. We will use ϱ, λ and η to denote L^∞-, $H^{1,0}$- and $H^{1,1}$-bounds for r, respectively. Thus $\|r\|_{L^\infty(\mathbf{R})} \leqslant \varrho$, $\|r\|_{H^{1,0}} \leqslant \lambda$, $\|r\|_{H^{1,1}} \leqslant \eta$. Of course we only consider $\varrho < 1$. By virtue of the Sobolev inequality, we can, and will, always assume that $\varrho \leqslant \lambda \leqslant \eta$. For $t \geqslant 0$, $0 < \varrho < 1$, we will also use the notation

$$\begin{bmatrix} k & l \\ i & j \end{bmatrix} = \frac{c\eta^k (1+\eta)^l}{(1+t)^i (1-\varrho)^j} \tag{4.1}$$

for some constant c and nonnegative integers k, l, i and j. Note that

$$\begin{bmatrix} k_1 & l_1 \\ i_1 & j_1 \end{bmatrix} \begin{bmatrix} k_2 & l_2 \\ i_2 & j_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 & l_1+l_2 \\ i_1+i_2 & j_1+j_2 \end{bmatrix}, \tag{4.2}$$

$$\begin{bmatrix} k_1 & l_1 \\ i_1 & j_1 \end{bmatrix} + \begin{bmatrix} k_2 & l_2 \\ i_2 & j_2 \end{bmatrix} \leqslant \begin{bmatrix} \min(k_1,k_2) & \max(k_1+l_1, k_2+l_2) - \min(k_1,k_2) \\ \min(i_1,i_2) & \max(j_1,j_2) \end{bmatrix}. \tag{4.3}$$

Let δ be given as in (2.57). Reverse the orientation of $\mathbf{R}_- + z_0$ to obtain $\widetilde{\mathbf{R}}_{z_0}$,

$$\widetilde{\mathbf{R}}_{z_0} = e^{i\pi}(\mathbf{R}_+ + z_0) \cup (\mathbf{R}_+ + z_0),$$

and extend $\widetilde{\mathbf{R}}_{z_0}$ to a complete(1) contour Γ_{z_0} as shown in Figure 4.4. As Γ_{z_0} is complete, $C^+_{\Gamma_{z_0}} C^-_{\Gamma_{z_0}} = 0$ by Cauchy's theorem.

(4.5) Denote the boundary values of $\delta(z)$ on $\widetilde{\mathbf{R}}_{z_0}$ by $\tilde{\delta}_\pm(z)$. Thus $\tilde{\delta}_\pm(z) = \delta(z)$ for $z > z_0$, and $\tilde{\delta}_\pm(z) = \delta_\mp(z)$ for $z < z_0$.

(1) A contour is *complete* (see e.g. [Z1]) if $\overline{\mathbf{C}} \setminus \Gamma$ is a disjoint union of two possibly disconnected open regions Ω_+ and Ω_-, and Γ may be viewed as the positively oriented boundary of Ω_+ and also as the negatively oriented boundary of Ω_-.

For $z \in \mathbf{C} \backslash \mathbf{R} + z_0$, set

$$\widetilde{m}(z) = m(z)\delta(z)^{-\sigma_3}. \tag{4.6}$$

It is easy to see that $\widetilde{m}$ solves the normalized RH problem $(\widetilde{\mathbf{R}}_{z_0}, \tilde{v}_\theta)$ where

$$\tilde{v}_\theta = e^{i\theta \operatorname{ad}\sigma} \tilde{v}, \quad \tilde{v} = \begin{cases} \tilde{\delta}^{\sigma_3} v \tilde{\delta}^{-\sigma_3}, & z > z_0, \\ \tilde{\delta}_-^{\sigma_3} v^{-1} \tilde{\delta}_+^{-\sigma_3}, & z < z_0. \end{cases} \tag{4.7}$$

Note that in the notation of §2,

$$\tilde{v}_\theta(z) = \breve{v}_\theta(z) \quad \text{for } z > z_0 \quad \text{and} \quad \tilde{v}_\theta(z) = \breve{v}_\theta^{-1}(z) \quad \text{for } z < z_0. \tag{4.8}$$

We have $\tilde{v} = (I - \widetilde{w}^-)^{-1}(I + \widetilde{w}^+) = (\tilde{v}^-)^{-1}\tilde{v}^+$ where

$$\widetilde{w} = (\widetilde{w}^-, \widetilde{w}^+) = \begin{cases} \left(\begin{pmatrix} 0 & r\delta^2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\bar{r}\delta^{-2} & 0 \end{pmatrix} \right), & \text{for } z > z_0, \\ \left(\begin{pmatrix} 0 & -r\delta_+\delta_- \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \bar{r}\delta_+^{-1}\delta_-^{-1} & 0 \end{pmatrix} \right), & \text{for } z < z_0, \end{cases} \tag{4.9}$$

which can also be written as

$$\widetilde{w} = \left(\begin{pmatrix} 0 & -r\tilde{\delta}_-^2/(1-|r|^2) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \bar{r}\tilde{\delta}_+^{-2}/(1-|r|^2) & 0 \end{pmatrix} \right) \tag{4.10}$$

for $z < z_0$. As usual $\widetilde{w}_\theta = e^{i\theta \operatorname{ad}\sigma}\widetilde{w} = (e^{i\theta \operatorname{ad}\sigma}\widetilde{w}^-, e^{i\theta \operatorname{ad}\sigma}\widetilde{w}^+)$. We consider the singular integral equation associated with the normalized RH problem $(\widetilde{\mathbf{R}}_{z_0}, \tilde{v}_\theta)$, as described in §2 (see (2.4)),

$$\tilde{\mu} = I + C_{\widetilde{w}_\theta}\tilde{\mu}. \tag{4.11}$$

We have $\widetilde{m}_\pm = \tilde{\mu}\tilde{v}_{\theta\pm}$ and

$$m_- = \tilde{\mu}\delta^{\sigma_3} \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}_\theta, \quad z > z_0; \quad m_- = \tilde{\mu}\delta_-^{\sigma_3} \begin{pmatrix} 1 & 0 \\ \bar{r}/(1-|r|^2) & 1 \end{pmatrix}_\theta, \quad z < z_0. \tag{4.12}$$

Introduce

$$w = (w^-, w^+) = \left(\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\bar{r} & 0 \end{pmatrix} \right) \tag{4.13}$$

corresponding to the factorization

$$v = (v^-)^{-1} v^+ = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\bar{r} & 1 \end{pmatrix} \quad \text{for all } z \in \mathbf{R}.$$

Again $w_\theta = e^{i\theta \operatorname{ad}\sigma} w$. By the results of §2, both the operators $(1-C_{w_\theta})^{-1}$ and $(1-C_{\widetilde{w}_\theta})^{-1}$ are bounded from L^2 to L^2, and

$$\|(1-C_{w_\theta})^{-1}\|_{L^2\to L^2},\ \|(1-C_{\widetilde{w}_\theta})^{-1}\|_{L^2\to L^2} \leqslant \frac{c}{1-\varrho} \tag{4.14}$$

for all $x, t \in \mathbf{R}$. Similarly for $p>2$, we obtain from (2.68)

$$\|(1-C_{w_\theta})^{-1}\|_{L^p\to L^p},\ \|(1-C_{\widetilde{w}_\theta})^{-1}\|_{L^p\to L^p} \leqslant K_p \tag{4.15}$$

for all $x, t \in \mathbf{R}$. In particular,

$$\tilde{\mu} = (1-C_{\widetilde{w}_\theta})^{-1} I = I + \frac{1}{1-C_{\widetilde{w}_\theta}} (C_{\widetilde{w}_\theta} I)$$

exists in $I + L^p(\mathbf{R})$ for all $p \geqslant 2$.

Notational remark. Observe that if m_- is obtained from (4.12) for given x, t and r, then in the notation of §1, $m_- = m_-(x, z; re^{-it\Diamond^2})$. In particular if $r = r(z)$ is independent of x and t, then m_- is the boundary value $m_-(x, z; q(t))$ of the Beals–Coifman solution of (2.8) with potential $q(t) = \mathcal{R}^{-1}(e^{-it\Diamond^2} r)$ solving NLS. In the calculations that follow, r in (4.12) should be regarded simply as a function in $H_1^{1,0}$ or $H_1^{1,1}$ which may or may not depend on external parameters such as x and t.

The goal of this section is to prove the following smoothing estimates. Recall that $K_p \geqslant 1$ and increases with p.

THEOREM 4.16. *Let $r, r_j \in H^{1,0}$ and set $r(t) = e^{-itz^2} r$, $r_j(t) = e^{-itz^2} r_j$, $j = 1, 2$. Let $\tilde{\mu} = \tilde{\mu}(r(t))$, $\tilde{\mu}_j = \tilde{\mu}(r_j(t))$, $j = 1, 2$, and as in (2.20) set $\widetilde{\mathbf{Q}} = \int \tilde{\mu}(\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-)$, $\widetilde{\mathbf{Q}}_j = \int \tilde{\mu}_j(\widetilde{w}_{j\theta}^+ + \widetilde{w}_{j\theta}^-)$, $j = 1, 2$. Then*

$$\|\tilde{\mu} - I\|_{L^p} \leqslant \frac{c}{(1+t)^{1/2p}} \frac{\lambda K_p}{(1-\varrho)^2} \leqslant \begin{bmatrix} 1 & 0 \\ 1/2p & 2 \end{bmatrix} K_p \quad \textit{for any } p \geqslant 2, \tag{4.17}$$

$$\begin{aligned} \|\tilde{\mu}_2 - \tilde{\mu}_1\|_{L^p} &\leqslant \frac{c}{(1+t)^{1/2p'}} \frac{(1+\lambda)^2 K_{p'} K_p}{(1-\varrho)^4} \|r_2 - r_1\|_{H^{1,0}} \\ &\leqslant \begin{bmatrix} 0 & 2 \\ 1/2p' & 4 \end{bmatrix} K_{p'}^2 \|r_2 - r_1\|_{H^{1,0}} \quad \textit{for any } p' > p \geqslant 2, \end{aligned} \tag{4.18}$$

$$\|\widetilde{\mathbf{Q}}\|_{L^\infty} \leqslant \frac{c}{t^{1/2}} \frac{\lambda(1+\lambda) K_2}{(1-\varrho)^4} \leqslant \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix} \quad \textit{for any } t > 1. \tag{4.19}$$

Moreover, if $r \in H^{1,1}$, $\|r\|_{H^{1,1}} \leqslant \eta$, *then*

$$\|\widetilde{\mathbf{Q}}\|_{L^\infty} \leqslant \frac{c}{(1+t)^{1/2}} \frac{\eta(1+\eta)K_2}{(1-\varrho)^4} \leqslant \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix} \quad \textit{for any } t>0, \tag{4.20}$$

$$\begin{aligned}\|\widetilde{\mathbf{Q}}_2-\widetilde{\mathbf{Q}}_1\|_{L^\infty} &\leqslant \frac{c}{(1+t)^{1/2p+1/4}} \frac{(1+\lambda)^3 K_p}{(1-\varrho)^7} \|r_2-r_1\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p \|r_2-r_1\|_{H^{1,1}}\end{aligned} \tag{4.21}$$

for any $2<p<\infty$ *and for any* $t>0$.

Proof. For $x=0$, (4.17)–(4.21) follow from Corollary 4.58 and Lemmas 4.70, 4.74 and 4.75 below. When $x\neq 0$, a simple translation argument (see [DZW, (4.132)]) shows that

$$\tilde{\mu} = \tilde{\mu}(x,z;e^{-it\Diamond^2} r(\Diamond)) = e^{i\theta(z_0)\,\mathrm{ad}\,\sigma} \tilde{\mu}(0, z-z_0; e^{-it\Diamond^2} r(\Diamond+z_0)), \tag{4.22}$$

where again $z_0=x/2t$. Also,

$$\widetilde{w}_\theta(z) = e^{i\theta(z_0)\,\mathrm{ad}\,\sigma}(e^{-it(z-z_0)^2\,\mathrm{ad}\,\sigma}\widetilde{w}(z)) \tag{4.23}$$

and hence

$$\widetilde{Q}(x,r(t)) = e^{i\theta(z_0)\,\mathrm{ad}\,\sigma}\widetilde{Q}(0;e^{-it\Diamond^2} r(\Diamond+z_0)), \tag{4.24}$$

with similar formulae for $\widetilde{Q}_j$, $j=1,2$. As the L^∞- and $H^{1,0}$-norms of r are independent of translation, the inequalities (4.17)–(4.19) in the case $x\neq 0$ now follow from the case $x=0$ with r replaced by $r(\cdot+z_0)$. However, examining the proof of (4.20) in Lemma 4.74 below, we see that the $H^{1,1}$-norm of r is only needed to control the L^1-norm of $r(\cdot+z_0)$. As the L^1-norm is translation invariant, (4.20) remains true for $x\neq 0$. Similar considerations apply to (4.21). □

As $\|r\|_\infty<1$, we have the estimate,

$$|\log(1-|r(z)|^2)| \leqslant \frac{|r(z)|^2}{1-|r(z)|^2}. \tag{4.25}$$

In particular,

$$\|\log(1-|r|^2)\|_{L^2} \leqslant \frac{\|r\|_{L^\infty}\|r\|_{L^2}}{1-\|r\|^2_{L^\infty}} \leqslant \frac{\|r\|_{L^\infty}\|r\|_{L^2}}{1-\|r\|_{L^\infty}}. \tag{4.26}$$

From (2.60), we have

$$\Delta = \delta_+\delta_- = e^{-H(\chi_{(-\infty,z_0)}\log(1-|r|^2))}, \tag{4.27}$$

where $\chi_{(-\infty,z_0)}$ is the characteristic function of $(-\infty,z_0)$ and

$$Hf(z)=\text{P.V.}\ \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{f(s)}{z-s}\,ds$$

is the Hilbert transform (see e.g. [DZW, Appendix I]). Observe again as in (2.60) that $|\Delta|=1$. For the remainder of this section we will assume that $x=0$. Thus $z_0=0$, $\widetilde{\mathbf{R}}\equiv\widetilde{\mathbf{R}}_{z_0=0}$, $\Gamma\equiv\Gamma_{z_0=0}$ and $\theta=-tz^2$. As noted in §2, the signature table for $\operatorname{Re} i\theta$ (see Figure 2.51) plays a crucial role.

LEMMA 4.29. *Let* $r_1,r_2\in H_1^{1,0}(\mathbf{R})$, $\|r_i\|_\infty\leqslant\varrho<1$, $\|r_i\|_{H^{1,0}(\mathbf{R})}<\lambda$. *Then* $|\Delta_2-\Delta_1|$, $|\Delta_2^{-1}-\Delta_1^{-1}|\leqslant \mathrm{I}+\mathrm{II}$ *where*

$$\|\mathrm{I}\|_{H^{1,0}}\leqslant\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}},\tag{4.30}$$

$$\begin{aligned}\mathrm{II}(z)&\leqslant\frac{c\varrho}{1-\varrho}\left|\frac{1}{\pi}\text{P.V.}\int_{-1}^{0}\frac{1+s}{s-z}\,ds\right|\|r_2-r_1\|_{H^{1,0}}\\&=\frac{c\varrho}{1-\varrho}\,\frac{1}{\pi}\left|1+(z+1)\log\left|\frac{z}{1+z}\right|\right|\|r_2-r_1\|_{H^{1,0}}.\end{aligned}\tag{4.31}$$

Proof of Lemma 4.29. First we consider $\Delta_2-\Delta_1$; the proof of the estimate for $\Delta_2^{-1}-\Delta_1^{-1}$ is the same. Define $\boldsymbol{\chi}(s)=(1+s)\chi_{(-1,0)}(s)$, where $\chi_{(-1,0)}$ denotes the characteristic function of the set $(-1,0)$. We have

$$\begin{aligned}|\Delta_2(z)-\Delta_1(z)|\leqslant&\left|H\left(\left(\log\left(\frac{1-|r_2|^2}{1-|r_1|^2}\right)-\log\left(\frac{1-|r_2(0)|^2}{1-|r_1(0)|^2}\right)\boldsymbol{\chi}\right)\chi_{\mathbf{R}_-}\right)\right|\\&+\left|\log\left(\frac{1-|r_2(0)|^2}{1-|r_1(0)|^2}\right)\right||H(\boldsymbol{\chi})|=\mathrm{I}+\mathrm{II}.\end{aligned}$$

Note that
$$\left(\log\left(\frac{1-|r_2|^2}{1-|r_1|^2}\right)-\log\left(\frac{1-|r_2(0)|^2}{1-|r_1(0)|^2}\right)\boldsymbol{\chi}\right)\chi_{\mathbf{R}_-}\in H^{1,0}(\mathbf{R}).$$

A simple calculation shows that

$$\left|\log\left(\frac{1-|r_2(z)|^2}{1-|r_1(z)|^2}\right)\right|\leqslant\frac{2\varrho}{1-\varrho}|r_2(z)-r_1(z)|$$

and hence

$$\begin{aligned}&\left\|\log\left(\frac{1-|r_2|^2}{1-|r_1|^2}\right)-\log\left(\frac{1-|r_2(0)|^2}{1-|r_1(0)|^2}\right)\boldsymbol{\chi}\right\|_{L^2(\mathbf{R})}\\&\leqslant\frac{2\varrho}{1-\varrho}\|r_2-r_1\|_{L^2}+\frac{2\varrho}{1-\varrho}|r_2(0)-r_1(0)|\,\|\boldsymbol{\chi}\|_{L^2(\mathbf{R})}\leqslant\frac{c\varrho}{1-\varrho}\|r_2-r_1\|_{H^{1,0}}.\end{aligned}$$

Also,

$$\begin{aligned}\Big\| \frac{d}{dz}\Big(\Big(\log\Big(\frac{1-|r_2|^2}{1-|r_1|^2}\Big)-\log\Big(\frac{1-|r_2(0)|^2}{1-|r_1(0)|^2}\Big)\chi\Big)\chi_{\mathbf{R}_-}\Big)\Big\|_{L^2} &\leqslant \frac{1}{1-\varrho}\|r_2\bar r_2'+r_2'\bar r_2-r_1\bar r_1'-r_1'\bar r_1\|_{L^2}\\ &\quad+\frac{4\varrho^2}{(1-\varrho)^2}\|r_1-r_2\|_{L^\infty}\|r_1'\|_{L^2}+\frac{2\varrho}{1-\varrho}\|r_1-r_2\|_{L^\infty}\\ &\leqslant \frac{2\|r_2'\|_{L^2}\|r_2-r_1\|_{L^\infty}}{1-\varrho}+\frac{2\varrho}{1-\varrho}\|r_2'-r_1'\|_{L^2}\\ &\quad+\frac{4\varrho^2}{(1-\varrho)^2}\|r_2-r_1\|_{L^\infty}\|r_1'\|_{L^2}+\frac{2\varrho}{1-\varrho}\|r_1-r_2\|_{L^\infty}\\ &\leqslant \frac{c\lambda\|r_2-r_1\|_{H^{1,0}}}{(1-\varrho)^2}.\end{aligned}$$

As H is bounded from $L^2\to L^2$ and commutes with differentiation, this proves the bound for I. On the other hand,

$$\begin{aligned}\mathrm{II}(z) &\leqslant \frac{c\varrho}{1-\varrho}\|r_2-r_1\|_{H^{1,0}}\Big|\frac{1}{\pi}\,\mathrm{P.V.}\int_{-1}^{0}\frac{(1+s)\,ds}{s-z}\Big|\\ &=\frac{c\varrho}{1-\varrho}\|r_2-r_1\|_{H^{1,0}}\Big|\frac{1}{\pi}(1+(z+1))\log\Big|\frac{z}{1+z}\Big|\Big|. \qquad \square\end{aligned}$$

LEMMA 4.32. *Let r_1, r_2 be as in Lemma 4.29. Then*

$$|\delta_{2\pm}^{\pm 2}-\delta_{1\pm}^{\pm 2}|\leqslant c\Big(\frac{\lambda}{(1-\varrho)^2}+\frac{\varrho}{1-\varrho}\Big|1+(z+1)\log\Big|\frac{z}{1+z}\Big|\Big|\Big)\|r_2-r_1\|_{H^{1,0}}, \tag{4.33}$$

$$|\delta_{2\pm}^{\mp 2}-\delta_{1\pm}^{\mp 2}|\leqslant c\Big(\frac{\lambda}{(1-\varrho)^3}+\frac{\varrho}{(1-\varrho)^2}\Big|1+(z+1)\log\Big|\frac{z}{1+z}\Big|\Big|\Big)\|r_2-r_1\|_{H^{1,0}}. \tag{4.34}$$

Proof. For $\pm\operatorname{Im} z\geqslant 0$, direct calculation shows that

$$\begin{aligned}|\delta_2^{\pm 2}-\delta_1^{\pm 2}| &\leqslant \max_{0\leqslant T\leqslant 1}\big|e^{\pm 2C_{\mathbf{R}_-}(\log(1-|r_2|^2)(1-T)+\log(1-|r_1|^2)T)}\big|\,\Big|2C_{\mathbf{R}_-}\log\frac{1-|r_2|^2}{1-|r_1|^2}\Big|\\ &\leqslant 2\Big|C_{\mathbf{R}_-}\log\frac{1-|r_2|^2}{1-|r_1|^2}\Big|.\end{aligned}$$

Again for $\pm\operatorname{Im} z\leqslant 0$,

$$\begin{aligned}|\delta_2^{\mp 2}-\delta_1^{\mp 2}| &\leqslant \max_{0\leqslant T\leqslant 1}\big|e^{\mp 2C_{\mathbf{R}_-}(\log(1-|r_2|^2)(1-T)+\log(1-|r_1|^2)T)}\big|\,\Big|2C_{\mathbf{R}_-}\log\frac{1-|r_2|^2}{1-|r_1|^2}\Big|\\ &\leqslant \frac{2}{1-\varrho}\Big|C_{\mathbf{R}_-}\log\frac{1-|r_2|^2}{1-|r_1|^2}\Big|.\end{aligned}$$

The lemma now follows from the proof of Lemma 4.29 above, together with the identities $C^{\pm}=\frac{1}{2}(\pm 1-H)$. □

A consequence of the proofs of Lemmas 4.29 and 4.32 is

$$\begin{aligned}|\delta_2^{\pm 2}-\delta_1^{\pm 2}| &\leqslant \frac{c\lambda}{(1-\varrho)^2}(1+|\log z|)\,\|r_2-r_1\|_{H^{1,0}}, \quad \pm\operatorname{Im} z\geqslant 0,\\ |\delta_2^{\mp 2}-\delta_1^{\mp 2}| &\leqslant \frac{c\lambda}{(1-\varrho)^3}(1+|\log z|)\,\|r_2-r_1\|_{H^{1,0}}, \quad \pm\operatorname{Im} z\geqslant 0,\end{aligned} \tag{4.35}$$

where $\log(\cdot)$ denotes the principal branch. These inequalities are useful for $|z|$ small, but when more precise bounds are needed for $|z|$ large, we will use the full inequalities (4.33), (4.34) in Lemma 4.32.

LEMMA 4.36. *Let* $r\in H_1^{1,0}$, $\|r\|_{L^\infty}\leqslant\varrho<1$, $\|r\|_{H^{1,0}}\leqslant\lambda$. *For* $z\in\mathbf{R}\setminus 0$,

$$|\Delta'(z)|\leqslant \mathrm{I}+\mathrm{II}, \tag{4.37}$$

$$\|\mathrm{I}\|_{L^2}\leqslant\frac{c\varrho\lambda}{1-\varrho}, \tag{4.38}$$

$$\mathrm{II}\leqslant\frac{c\varrho^2}{1-\varrho}\,\frac{1}{|z|}. \tag{4.39}$$

Proof. We have

$$\begin{aligned}\Delta'(z) &= -\Delta(z)\frac{d}{dz}H((\log(1-|r|^2))\chi_{\mathbf{R}_-})\\ &= \Delta(z)H\biggl(\frac{|r|^{2\prime}}{1-|r|^2}\chi_{\mathbf{R}_-}\biggr)-\frac{i\Delta}{\pi}\,\frac{\log(1-|r(0)|^2)}{z}.\end{aligned}$$

The result now follows as before. □

LEMMA 4.40. *Let* $r_i\in H_1^{1,0}$, $\|r_i\|_{L^\infty}\leqslant\varrho<1$, $\|r_i\|_{H^{1,0}(\mathbf{R})}\leqslant\lambda$, $i=1,2$. *Then for any* $0<a<1$,

$$\|\Delta_2^{\pm 1\prime}-\Delta_1^{\pm 1\prime}\|_{L^2(|z|>a)}\leqslant\frac{c\lambda(1+\lambda)}{(1-\varrho)^3}\,\frac{1+|\log a|}{\sqrt{a}}\,\|r_2-r_1\|_{H^{1,0}}. \tag{4.41}$$

Proof. As in Lemma 4.36,

$$\Delta_j' = -i\Delta_j\biggl(iH\biggl(\frac{|r_j|^{2\prime}}{1-|r_j|^2}\chi_{\mathbf{R}_-}\biggr)+\frac{\log(1-|r_j(0)|^2)}{z}\biggr).$$

Hence

$$\begin{aligned}\Delta_2'-\Delta_1' &= -i(\Delta_2-\Delta_1)\biggl(iH\biggl(\frac{|r_2|^{2\prime}}{1-|r_2|^2}\chi_{\mathbf{R}_-}\biggr)+\frac{\log(1-|r_2(0)|^2)}{z}\biggr)\\ &\quad +i\Delta_1\biggl(iH\biggl(\biggl(\frac{|r_1|^{2\prime}}{1-|r_1|^2}-\frac{|r_2|^{2\prime}}{1-|r_2|^2}\biggr)\chi_{\mathbf{R}_-}\biggr)+\frac{1}{z}\log\frac{1-|r_1(0)|^2}{1-|r_2(0)|^2}\biggr)\end{aligned}$$

so that

$$|\Delta_2'(z)-\Delta_1'(z)|\leqslant\left[|\Delta_2-\Delta_1|\left|H\left(\frac{|r_2|^{2\prime}}{1-|r_2|^2}\chi_{\mathbf{R}_-}\right)\right|+\left|H\left(\left(\frac{|r_1|^{2\prime}}{1-|r_1|^2}-\frac{|r_2|^{2\prime}}{1-|r_2|^2}\right)\chi_{\mathbf{R}_-}\right)\right|\right]$$
$$+\left[|\Delta_2-\Delta_1|\frac{|\log(1-|r_2(0)|^2)|}{z}+\frac{1}{z}\left|\frac{\log(1-|r_1(0)|^2)}{\log(1-|r_2(0)|^2)}\right|\right].$$

From previous estimates we have

$$\begin{aligned}\|\Delta_2'-\Delta_1'\|_{L^2(|z|>a)}&\leqslant\left(\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}+\frac{\varrho c\|r_2-r_1\|_{H^{1,0}}}{1-\varrho}(1+|\log a|)\right)\varrho\frac{\|r_2'\|_{L^2}}{1-\varrho}\\&\quad+\frac{c\lambda\|r_2-r_1\|_{H^{1,0}}}{(1-\varrho)^2}+\left(\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\right.\\&\quad\left.+\frac{c\varrho\|r_2-r_1\|_{H^{1,0}}}{1-\varrho}(1+|\log a|)\right)\frac{\varrho^2}{(1-\varrho)\sqrt{a}}+\frac{2\varrho}{1-\varrho}\|r_2-r_1\|_{L^\infty}\frac{1}{\sqrt{a}}\\&\leqslant\frac{c\lambda(1+\lambda)}{(1-\varrho)^3}\|r_2-r_1\|_{H^{1,0}}\frac{1+|\log a|}{\sqrt{a}}.\end{aligned}$$

The proof for $(\Delta^{-1})'$ is similar. □

LEMMA 4.42. *Suppose* $r\in H_1^{1,0}$, $\|r\|_{L^\infty}\leqslant\varrho<1$, $\|r\|_{H^{1,0}}\leqslant\lambda$, *and suppose* $f\in H^{1,0}$. *Then for all* $t>1$,

$$\left|\int_{\mathbf{R}}f\Delta^{\pm1}e^{\mp itz^2}dz\right|\leqslant\frac{c}{t^h}\frac{1+\lambda}{1-\varrho}\|f\|_{H^{1,0}},\tag{4.43}$$

where $h=\frac{1}{2}$ *in the general case and* $h=\frac{3}{4}$ *if* $f(0)=0$.

Proof. We only consider the case $\Delta=\Delta^{+1}$ above. The other case is similar. Decompose the integral as

$$\int_{-\infty}^{\infty}f\Delta e^{-itz^2}dz=\int_{|z|<1/\sqrt{t}}f\Delta e^{-itz^2}dz+\int_{|z|>1/\sqrt{t}}f\Delta e^{-itz^2}dz\equiv\mathrm{I}+\mathrm{II}.$$

Changing variables,

$$|\mathrm{I}|=\frac{1}{\sqrt{t}}\left|\int_{-1}^{1}f\left(\frac{z}{\sqrt{t}}\right)\Delta\left(\frac{z}{\sqrt{t}}\right)e^{-iz^2}dz\right|\leqslant\frac{1}{\sqrt{t}}\|f\|_{L^\infty}\leqslant\frac{c}{\sqrt{t}}\|f\|_{H^{1,0}}.$$

We consider $z<-t^{-1/2}$. The case $z>t^{1/2}$ is similar. Integration by parts leads to

$$\begin{aligned}\int_{z<-t^{-1/2}}f\Delta e^{-itz^2}dz&=\frac{e^{-i}}{2i\sqrt{t}}f\left(-\frac{1}{\sqrt{t}}\right)\Delta\left(-\frac{1}{\sqrt{t}}\right)\\&\quad+\frac{1}{2it}\int_{-\infty}^{-t^{-1/2}}e^{-itz^2}\left(\frac{f'\Delta}{z}+\frac{f\Delta'}{z}-\frac{f\Delta}{z^2}\right)dz\\&\equiv\mathrm{I}'+\mathrm{II}'+\mathrm{III}'+\mathrm{IV}'.\end{aligned}$$

Clearly

$$|\mathrm{I}'| \leqslant \frac{c\|f\|_{H^{1,0}}}{t^{1/2}}, \quad |\mathrm{II}'| \leqslant \frac{c}{t^{3/4}}\|f\|_{H^{1,0}} \quad \text{and} \quad |\mathrm{IV}'| \leqslant \frac{c\|f\|_{H^{1,0}}}{t^{1/2}}.$$

Finally using Lemma 4.36,

$$|\mathrm{III}'| \leqslant \frac{\|f\|_{L^\infty}}{2t}\int_{-\infty}^{-t^{-1/2}} \left|\frac{\Delta'}{z}\right| dz \leqslant \frac{\|f\|_{H^{1,0}}}{2t}\left(\frac{\varrho\lambda}{1-\varrho}t^{1/4}+\frac{c\varrho^2}{1-\varrho}t^{1/2}\right),$$

which now leads directly to (4.43) for $h=\frac{1}{2}$. If $f(0)=0$, then the same arguments together with the bound $|f(z)|\leqslant|z|^{1/2}\|f\|_{H^{1,0}}$ for $|z|\leqslant 1$, say, in I, I′, III′ and IV′, yield (4.43) for $h=\frac{3}{4}$. □

The following result is a Lipschitz version of the lemma above.

LEMMA 4.44. *Suppose $f\in H^{1,0}$. Then for all $t\geqslant 0$,*

$$\left|\int_{\mathbf{R}} f(\Delta_2^{\pm 1}-\Delta_1^{\pm 1})e^{\mp itz^2}dz\right| \leqslant \frac{c\log(2+t)}{(1+t)^{1/2}}\,\frac{\lambda(1+\lambda)}{(1-\varrho)^3}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}. \tag{4.45}$$

Proof. Again we only consider the case $\Delta_2-\Delta_1$; the other case is similar. As in the proof of Lemma 4.42 we decompose the integral

$$\begin{aligned}\int_{-\infty}^{\infty} f(\Delta_2-\Delta_1)e^{-itz^2}dz &= \int_{|z|<t^{-1/2}} f(\Delta_2-\Delta_1)e^{-itz^2}dz+\int_{|z|>t^{-1/2}} f(\Delta_2-\Delta_1)e^{-itz^2}dz\\ &\equiv \mathrm{I}+\mathrm{II}.\end{aligned}$$

Using Lemma 4.29,

$$\begin{aligned}|\mathrm{I}| &= \frac{1}{\sqrt{t}}\left|\int_{-1}^{1} f\left(\frac{z}{\sqrt{t}}\right)\left(\Delta_2\left(\frac{z}{\sqrt{t}}\right)-\Delta_1\left(\frac{z}{\sqrt{t}}\right)\right)e^{-iz^2}dz\right|\\ &\leqslant \frac{c}{\sqrt{t}}\|f\|_{H^{1,0}}\left\|\Delta_2\left(\frac{\lozenge}{\sqrt{t}}\right)-\Delta_1\left(\frac{\lozenge}{\sqrt{t}}\right)\right\|_{L^1((-1,1),dz)}\\ &\leqslant \frac{c}{\sqrt{t}}\|f\|_{H^{1,0}}\left(\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}+\frac{c\varrho}{1-\varrho}\|r_2-r_1\|_{H^{1,0}}\right.\\ &\qquad\left.\times\int_{-1}^{1}\left|1+\left(\frac{z}{\sqrt{t}}+1\right)\log\left|\frac{z/\sqrt{t}}{1+z/\sqrt{t}}\right|\right|dz\right)\\ &\leqslant \frac{c(\log t)\lambda}{\sqrt{t}\,(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}} \quad \text{for } t\geqslant 2.\end{aligned}$$

In estimating II, we again only consider $z\leqslant -t^{-1/2}$; the case $z\geqslant t^{-1/2}$ is similar. Integrating by parts, we obtain as before

$$\begin{aligned}\mathrm{II}&=\frac{e^{-i}}{2i\sqrt{t}}f\left(-\frac{1}{\sqrt{t}}\right)(\Delta_2-\Delta_1)(-1/\sqrt{t})\\&\quad+\frac{1}{2it}\int_{-\infty}^{-t^{-1/2}}e^{-itz^2}\left(\frac{f(\Delta_2-\Delta_1)}{z}+\frac{f(\Delta_2-\Delta_1)'}{z}-\frac{f(\Delta_2-\Delta_1)}{z^2}\right)dz\\&\equiv \mathrm{I}'+\mathrm{II}'+\mathrm{III}'+\mathrm{IV}'.\end{aligned}$$

Again by Lemma 4.29,

$$\begin{aligned}|\mathrm{I}'|&\leqslant\frac{c\|f\|_{H^{1,0}}}{\sqrt{t}}\left(\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}+\frac{c\varrho\|r_2-r_1\|_{H^{1,0}}}{1-\varrho}\left|1+\left(1-\frac{1}{\sqrt{t}}\right)\log\left|\frac{1/\sqrt{t}}{1-1/\sqrt{t}}\right|\right|\right)\\&\leqslant\frac{c\log t}{\sqrt{t}}\|f\|_{H^{1,0}}\frac{\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\quad\text{for } t\geqslant 2.\end{aligned}$$

Also,

$$\begin{aligned}|\mathrm{II}'|&\leqslant\frac{c}{t^{3/4}}\left(\int_{-\infty}^{-t^{-1/2}}|f'|^2|\Delta_2-\Delta_1|^2\,dz\right)^{1/2}\\&\leqslant\frac{c}{t^{3/4}}\|f\|_{H^{1,0}}\sup_{z<-t^{-1/2}}|\Delta_2(z)-\Delta_1(z)|\\&\leqslant\frac{c}{t^{3/4}}\|f\|_{H^{1,0}}\left(\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}+\frac{c\varrho}{1-\varrho}\|r_2-r_1\|_{H^{1,0}}(1+\log t)\right)\\&\leqslant c\frac{\log t}{t^{3/4}}\frac{\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}\quad\text{for } t\geqslant 2,\end{aligned}$$

and

$$|\mathrm{IV}'|\leqslant\frac{\|f\|_{H^{1,0}}}{2t^{1/2}}\sup_{z<-t^{-1/2}}|\Delta_2(z)-\Delta_1(z)|\leqslant\frac{c\log t}{t^{1/2}}\frac{\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}},$$

again for $t\geqslant 2$. Finally, using Lemma 4.40, we obtain for $t\geqslant 2$,

$$\begin{aligned}|\mathrm{III}'|&\leqslant\frac{c\|f\|_{H^{1,0}}}{t}t^{1/4}\|\Delta_2'-\Delta_1'\|_{L^2(-\infty,-t^{-1/2})}\leqslant\frac{c\|f\|_{H^{1,0}}}{t^{3/4}}\frac{\lambda(1+\lambda)}{(1-\varrho)^3}\frac{\log t}{t^{-1/4}}\|r_2-r_1\|_{H^{1,0}}\\&=\frac{c\lambda(1+\lambda)}{(1-\varrho)^3}\frac{\log t}{t^{1/2}}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.\end{aligned}$$

Assembling these estimates, we have proved

$$\left|\int_{-\infty}^{\infty}f(\Delta_2-\Delta_1)e^{-itz^2}dz\right|\leqslant c\frac{\log t}{t^{1/2}}\frac{\lambda(1+\lambda)}{(1-\varrho)^3}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}\tag{4.46}$$

for $t \geqslant 2$. On the other hand, by the proof of Lemma 4.29, for all t,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(\Delta_2-\Delta_1)e^{-itz^2}dz\right| &\leqslant \|f\|_{L^2(\mathbf{R})}\|\Delta_1-\Delta_2\|_{L^2}\\ &\leqslant \|f\|_{H^{1,0}}\left(\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}+\frac{c\varrho}{1-\varrho}\|r_2-r_1\|_{H^{1,0}}\|H\chi\|_{L^2}\right)\\ &\leqslant \frac{c\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.\end{aligned}$$

Together with (4.46), this proves (4.45). □

COROLLARY 4.47 (to Lemmas 4.42 and 4.44). *Suppose* $r_i \in H_1^{1,0}$, $\|r_i\|_{L^\infty} \leqslant \varrho < 1$, $\|r_i\|_{H^{1,0}} \leqslant \lambda$, $i=1,2$, *and suppose* $f_j \in H^{1,0}$, $j=1,2$. *Then for all* $t>1$,

$$\begin{aligned}\left|\int_{\mathbf{R}}(f_2\Delta_2^{\pm}-f_1\Delta_1^{\pm})e^{\mp itz^2}dz\right| &\leqslant \frac{c}{t^{1/2}}\,\frac{1+\lambda}{1-\varrho}\|f_2-f_1\|_{H^{1,0}}\\ &\quad+\frac{c\log(2+t)}{(1+t)^{1/2}}\,\frac{\lambda(1+\lambda)}{(1-\varrho)^3}\|f_1\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.\end{aligned}\tag{4.48}$$

Let D_j, $j=1,...,4$, be the jth quadrant in $\mathbf{C}\backslash\Gamma$,

$$\begin{array}{c|c} D_2 & D_1 \\ \hline D_3 & D_4 \end{array}\ \Gamma \tag{4.49}$$

In the lemma below, H^q denotes Hardy space. A general reference for Hardy spaces is, for example, [Du].

LEMMA 4.50. *Suppose* $f \in H^{1,0}$. *Then for* $2 \leqslant p < \infty$ *and for all* $t \geqslant 0$,

$$\begin{aligned}\|C^-_{\mathbf{R}_+\to\Gamma}\delta^{-2}fe^{it\Diamond^2}\|_{L^p} &\leqslant \frac{c}{(1+t)^{1/2p}}\|\delta^{-2}\|_{L^\infty(D_1)}\|f\|_{H^{1,0}} \leqslant \frac{c}{(1+t)^{1/2p}}\,\frac{\|f\|_{H^{1,0}}}{1-\varrho},\\ \|C^-_{e^{i\pi}\mathbf{R}_+\to\Gamma}\tilde{\delta}_+^{-2}fe^{it\Diamond^2}\|_{L^p} &\leqslant \frac{c}{(1+t)^{1/2p}}\|\delta^{-2}\|_{L^\infty(D_3)}\|f\|_{H^{1,0}} \leqslant \frac{c}{(1+t)^{1/2p}}\|f\|_{H^{1,0}},\\ \|C^+_{\mathbf{R}_+\to\Gamma}\delta^{2}fe^{-it\Diamond^2}\|_{L^p} &\leqslant \frac{c}{(1+t)^{1/2p}}\|\delta^{2}\|_{L^\infty(D_4)}\|f\|_{H^{1,0}} \leqslant \frac{c}{(1+t)^{1/2p}}\,\frac{\|f\|_{H^{1,0}}}{1-\varrho},\\ \|C^+_{e^{i\pi}\mathbf{R}_+\to\Gamma}\tilde{\delta}_-^{2}fe^{-it\Diamond^2}\|_{L^p} &\leqslant \frac{c}{(1+t)^{1/2p}}\|\delta^{2}\|_{L^\infty(D_2)}\|f\|_{H^{1,0}} \leqslant \frac{c}{(1+t)^{1/2p}}\|f\|_{H^{1,0}}.\end{aligned}\tag{4.51}$$

Suppose in addition that $f(0)=0$ *and that* g *is a function in the Hardy space* $H^q(\mathbf{C}\backslash\mathbf{R})$ *for some* $2 \leqslant q \leqslant \infty$. *Then for all* $t \geqslant 0$,

$$\left.\begin{array}{r}\|C^-_{\mathbf{R}_+\to\Gamma}g_+fe^{it\Diamond^2}\|_{L^2}\\ \|C^-_{e^{i\pi}\mathbf{R}_+\to\Gamma}\tilde{g}_+fe^{it\Diamond^2}\|_{L^2}\\ \|C^+_{\mathbf{R}_+\to\Gamma}g_-fe^{-it\Diamond^2}\|_{L^2}\\ \|C^+_{e^{i\pi}\mathbf{R}_+\to\Gamma}\tilde{g}_-fe^{-it\Diamond^2}\|_{L^2}\end{array}\right\} \leqslant \frac{c}{(1+t)^{1/2-1/q}}\|g\|_{H^q(\mathbf{C}\backslash\mathbf{R})}\|f\|_{H^{1,0}},\tag{4.52}$$

where $g_\pm$ *are the boundary values of* g *on* $\mathbf{R}$ *and* $\tilde{g}_\pm = g_\mp$ *on* $e^{i\pi}\mathbf{R}_+$.

Proof. Consider the first inequality in (4.51). The other cases in (4.51) are similar. By Fourier theory,

$$f(z)=\frac{1}{\sqrt{2\pi}}\int e^{-iyz}\check{f}(y)\,dy.$$

We have for any $\varepsilon>0$,

$$C^-_{\mathbf{R}_+\to\Gamma}\delta^{-2}e^{-\varepsilon\diamond}fe^{it\diamond^2}=\frac{1}{\sqrt{2\pi}}\int\check{f}(y)e^{-iy^2/4t}F_1\,dy+\frac{1}{\sqrt{2\pi}}\int\check{f}(y)e^{-iy^2/4t}F_2\,dy,$$

where

$$F_1(y)=C^-_{\mathbf{R}_+\to\Gamma}(\delta^{-2}e^{-\varepsilon\diamond}\chi_{(0,a)}e^{it(\diamond-y/2t)^2}),\tag{4.53}$$

$$F_2(y)=C^-_{\mathbf{R}_+\to\Gamma}(\delta^{-2}e^{-\varepsilon\diamond}\chi_{(a,\infty)}e^{it(\diamond-y/2t)^2})\tag{4.54}$$

and $a=\max(0,y/2t)$. (The factor $e^{-\varepsilon z}$ is included just to ensure that $F_2(y)$ exists in L^p.) Clearly $F_1(y)$ is supported on $\mathbf{R}_+$. Assume first that $p>2$. Then for $y>0$,

$$\|F_1\|_{L^p(\Gamma)}\leqslant c\|\delta^{-2}\|_{L^\infty(\mathbf{R}_+)}\|\chi_{(0,a)}\|_{L^p}\leqslant c\|\delta^{-2}\|_{L^\infty(\mathbf{R}_+)}\frac{|y|^{1/p}}{t^{1/p}}$$

and hence

$$\begin{aligned}\left\|\frac{1}{\sqrt{2\pi}}\int dy\,\check{f}(y)e^{-iy^2/4t}F_1\right\|_{L^p(\Gamma)}&\leqslant\frac{c\|\delta^{-2}\|_{L^\infty(\mathbf{R}_+)}}{t^{1/p}}\int_0^\infty|\check{f}(y)|\,y^{1/p}\,dy\\&\leqslant\frac{c\|\delta^{-2}\|_{L^\infty(D_1)}}{t^{1/p}}\|f\|_{H^{1,0}}.\end{aligned}\tag{4.55}$$

For $p=2$, rewrite the integral as

$$\frac{1}{\sqrt{2\pi}}C^-_{\mathbf{R}_+\to\Gamma}\left(\int_{2t\diamond}^\infty\delta^{-2}e^{-\varepsilon\diamond}\check{f}(y)e^{-iy\diamond}e^{it\diamond^2}\,dy\right).$$

Using Hardy's inequality [HLP],

$$\left\|\int_{2t\diamond}^\infty|\check{f}(y)|\,dy\right\|_{L^2(\mathbf{R}_+)}\leqslant\sqrt{\frac{2}{t}}\,\|\diamond\check{f}\|_{L^2(\mathbf{R}_+)}\tag{4.56}$$

and hence

$$\left\|\frac{1}{\sqrt{2\pi}}\int\check{f}e^{-\varepsilon\diamond}e^{-iy^2/4t}F_1\,dy\right\|_{L^2(\Gamma)}\leqslant\frac{c\|\delta^{-2}\|_{L^\infty(D_1)}}{t^{1/2}}\|f\|_{H^{1,0}}.$$

For F_2, first consider the case when $y<0$, and hence $a=0$. Then for $p\geqslant 2$ and by Cauchy's theorem,

$$\begin{aligned}\|C^-_{\mathbf{R}_+\to\Gamma}(\delta^{-2}e^{-\varepsilon\lozenge}e^{it(\lozenge-y/2t)^2})\|_{L^p} &= \|C_{e^{i\pi/4}\mathbf{R}_+\to\Gamma}(\delta^{-2}e^{-\varepsilon\lozenge}e^{it(\lozenge-y/2t)^2})\|_{L^p}\\ &\leqslant c\|\delta^{-2}\|_{L^\infty(D_1)}\|e^{it(\lozenge-y/2t)^2})\|_{L^p(e^{i\pi/4}\mathbf{R}_+)}\\ &\leqslant c\|\delta^{-2}\|_{L^\infty(D_1)}\|e^{it\lozenge^2}e^{-iy\lozenge}\|_{L^p(e^{i\pi/4}\mathbf{R}_+)}\\ &\leqslant c\|\delta^{-2}\|_{L^\infty(D_1)}\|e^{it\lozenge^2}\|_{L^p(e^{i\pi/4}\mathbf{R}_+)}\\ &\leqslant (c/t^{1/2p})\|\delta^{-2}\|_{L^\infty(D_1)},\end{aligned}$$

by scaling. The fact that $y<0$ is used to obtain the third inequality. If $y>0$ then $a=y/2t$, and for $p\geqslant 2$, again by Cauchy's theorem,

$$\begin{aligned}&\|C^-_{\mathbf{R}_+\to\Gamma}\delta^{-2}e^{-\varepsilon\lozenge}\chi_{(a,\infty)}e^{it(\lozenge-y/2t)^2}\|_{L^p(\Gamma)}\\ &\quad= \|C^-_{(y/2t,\infty)\to\Gamma}\delta^{-2}e^{-\varepsilon\lozenge}e^{it(\lozenge-y/2t)^2}\|_{L^p(\Gamma)}\\ &\quad= \|C_{(y/2t+e^{i\pi/4}\mathbf{R}_+)\to\Gamma}\delta^{-2}e^{-\varepsilon\lozenge}e^{it(\lozenge-y/2t)^2}\|_{L^p(\Gamma)}\\ &\quad\leqslant c\|\delta^{-2}\|_{L^\infty(D_1)}\|e^{it(\lozenge-y/2t)^2}\|_{L^p(y/2t+e^{i\pi/4}\mathbf{R}_+)}\leqslant \frac{c\|\delta^{-2}\|_{L^\infty(D_1)}}{t^{1/2p}},\end{aligned}$$

again by scaling. Thus for $p\geqslant 2$,

$$\left\|\frac{1}{\sqrt{2\pi}}\int dy\,\check{f}(y)e^{-iy^2/4t}F_2\right\|_{L^p(\Gamma)}\leqslant\frac{c\|\delta^{-2}\|_{L^\infty(D_1)}}{t^{1/2p}}\|f\|_{H^{1,0}}. \tag{4.57}$$

Letting $\varepsilon\downarrow 0$ in (4.55)–(4.57), we obtain for $p\geqslant 2$ and $t\geqslant 1$,

$$\|C^-_{\mathbf{R}_+\to\Gamma}(\delta^{-2}fe^{it\lozenge^2})\|_{L^p(\Gamma)}<\frac{c}{t^{1/2p}}\|\delta^{-2}\|_{L^\infty(D_1)}\|f\|_{H^{1,0}}.$$

As

$$\|C^-_{\mathbf{R}_+\to\Gamma}(\delta^{-2}fe^{it\lozenge^2})\|_{L^p(\Gamma)}\leqslant c\|\delta^{-2}\|_{L^\infty(D_1)}\|f\|_{H^{1,0}}\quad\text{for all } t>0,$$

we obtain (4.51).

Now we prove the first inequality in (4.52). Again, the remaining inequalities are similar. As before, we have the representation

$$C^-_{\mathbf{R}_+\to\Gamma}g_+e^{-\varepsilon\lozenge}fe^{it\lozenge^2}=\frac{1}{\sqrt{2\pi}}\int\check{f}(y)e^{-iy^2/4t}F_1\,dy+\frac{1}{\sqrt{2\pi}}\int\check{f}(y)e^{-iy^2/4t}F_2\,dy,$$

but now as $\int\check{f}(y)\,dy=\sqrt{2\pi}\,f(0)=0$,

$$\begin{aligned}F_1(y)&=C^-_{\mathbf{R}_+\to\Gamma}(g_+e^{-\varepsilon\lozenge}\chi_{(0,a)}e^{it(\lozenge-y/2t)^2}(1-e^{iy\lozenge})),\\ F_2(y)&=C^-_{\mathbf{R}_+\to\Gamma}(g_+e^{-\varepsilon\lozenge}\chi_{(a,\infty)}e^{it(\lozenge-y/2t)^2}(1-e^{iy\lozenge})).\end{aligned}$$

As before, for $y>0$, the integral involving F_1 can be rewritten as

$$\frac{1}{\sqrt{2\pi}} C^{-}_{\mathbf{R}_+\to\Gamma}\left(\int_{2t\diamond}^{\infty} g_+ e^{-\varepsilon\diamond}\check{f}(y)(e^{-iy\diamond}-1)e^{it\diamond^2}\,dy\right).$$

For $q'\geqslant 2$, using the inequality

$$\left(\int_{2tz}^{\infty}|\check{f}(y)|\,dy\right)^{q'} \leqslant \left(\int_{0}^{\infty}|\check{f}(y)|\,dy\right)^{q'-2}\left(\int_{2tz}^{\infty}|\check{f}(y)|\,dy\right)^{2}$$

together with the above Hardy inequality, we obtain for $t>0$,

$$\left\|\int_{2t\diamond}^{\infty}|\check{f}(y)|\,dy\right\|_{L^{q'}(\mathbf{R}_+)} \leqslant \frac{c}{t^{1/q'}}\|f\|_{H^{1,0}}.$$

Hence for $1/q'+1/q=1/2$,

$$\left\|\frac{1}{\sqrt{2\pi}}\int \check{f}e^{-\varepsilon\diamond}e^{-iy^2/4t}F_1\,dy\right\|_{L^2(\Gamma)} \leqslant \frac{c\|g_+\|_{L^q(\mathbf{R}_+)}}{t^{1/q'}}\|f\|_{H^{1,0}} \leqslant \frac{c\|g\|_{H^q(\mathbf{C}\setminus\mathbf{R})}}{t^{1/2-1/q}}\|f\|_{H^{1,0}}.$$

Now for $y<0$,

$$\|F_2(y)\|_{L^2} \leqslant c\|g e^{it(\diamond-y/2t)^2}(1-e^{iy\diamond})\|_{L^2(e^{i\pi/4}\mathbf{R}_+)} = c\|g e^{it\diamond^2}(e^{-iy\diamond}-1)\|_{L^2(e^{i\pi/4}\mathbf{R}_+)}.$$

Hence for $\alpha>0$ and $t>0$,

$$\begin{aligned}
&\left\|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\check{f}e^{-\varepsilon\diamond}e^{-iy^2/4t}F_2\,dy\right\|_{L^2(\Gamma)}\\
&\qquad\leqslant c\int_{-\infty}^{0}dy\,|\check{f}(y)|\left(\int_0^{\infty}d\gamma\,|g(e^{i\pi/4}\gamma)|^2e^{-2t\gamma^2}|e^{-ie^{i\pi/4}\gamma y}-1|^2\right)^{1/2}\\
&\qquad\leqslant c\int_{-t^{\alpha}}^{0}dy\,|y\check{f}(y)|\left(\int_0^{\infty}d\gamma\,\gamma^2|g(e^{i\pi/4}\gamma)|^2e^{-2t\gamma^2}\right)^{1/2}\\
&\qquad\qquad+\int_{-\infty}^{-t^{\alpha}}dy\,|\check{f}(y)|\left(\int_0^{\infty}d\gamma\,|g(e^{i\pi/4}\gamma)|^2e^{-2t\gamma^2}\right)^{1/2}\\
&\qquad\leqslant c(t^{-3/4+\alpha/2}+t^{-\alpha/2-1/4})\|f\|_{H^{1,0}}\|g(\diamond/\sqrt{t})\|_{H^q(\mathbf{C}\setminus\mathbf{R})}.
\end{aligned}$$

Taking $\alpha=\frac{1}{2}$, we obtain

$$\left\|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\check{f}e^{-\varepsilon\diamond}e^{-iy^2/4t}F_2\,dy\right\|_{L^2(\Gamma)} \leqslant \frac{c}{t^{1/2-1/2q}}\|f\|_{H^{1,0}}\|g\|_{H^q(\mathbf{C}\setminus\mathbf{R})}.$$

Finally for $y>0$,

$$\|F_2(y)\|_{L^2} \leqslant c\|g e^{it(\Diamond - y/2t)^2}(1-e^{iy\Diamond})\|_{L^2(y/2t+e^{i\pi/4}\mathbf{R}_+)},$$

and hence for $\alpha>0$ and $t>0$,

$$\begin{aligned}
&\left\|\frac{1}{\sqrt{2\pi}}\int_0^\infty \check{f} e^{-\varepsilon\Diamond} e^{-iy^2/4t} F_2\,dy\right\|_{L^2(\Gamma)} \\
&\quad\leqslant c\int_0^\infty dy\,|\check{f}(y)|\left(\int_0^\infty d\gamma\,|g(y/2t+e^{i\pi/4}\gamma)|^2 e^{-2t\gamma^2}|1-e^{iy(y/2t+e^{i\pi/4}\gamma)}|^2\right)^{1/2} \\
&\quad\leqslant c\int_0^{t^\alpha} dy\,|\check{f}(y)|\left(\int_0^\infty d\gamma\,|g(y/2t+e^{i\pi/4}\gamma)|^2 e^{-2t\gamma^2}(|e^{-iy^2/2t}-1|^2+|1-e^{ie^{i\pi/4}\gamma y}|^2)\right)^{1/2} \\
&\qquad + c\int_{t^\alpha}^\infty dy\,|\check{f}(y)|\left(\int_0^\infty d\gamma\,|g(y/2t+e^{i\pi/4}\gamma)|^2 e^{-2t\gamma^2}\right)^{1/2} \\
&\quad\leqslant c(t^{-5/4+3\alpha/2}+t^{-3/4+\alpha/2}+t^{-\alpha/2-1/4})\|f\|_{H^{1,0}}\|g(\Diamond/\sqrt{t})\|_{H^q(\mathbf{C}\setminus\mathbf{R})}.
\end{aligned}$$

Again, setting $\alpha=\frac{1}{2}$, we find for $t>0$,

$$\left\|\frac{1}{\sqrt{2\pi}}\int_0^\infty \check{f} e^{-\varepsilon\Diamond} e^{-iy^2/4t} F_2\,dy\right\|_{L^2(\Gamma)} \leqslant \frac{c}{t^{1/2-1/2q}}\|f\|_{H^{1,0}}\|g\|_{H^q(\mathbf{C}\setminus\mathbf{R})}.$$

On the other hand, as before,

$$\|C^-_{\mathbf{R}_+\to\Gamma}(g_+ f e^{it\Diamond^2})\|_{L^2(\Gamma)} \leqslant c\|g_+\|_{L^q(\mathbf{R}_+)}\|f\|_{H^{1,0}} \leqslant c\|g\|_{H^q(\mathbf{C}\setminus\mathbf{R})}\|f\|_{H^{1,0}} \quad \text{for all } t>0,$$

and (4.52) follows. □

Applying Lemma 4.50 to appropriate choices of f (see $\widetilde{w}$ in (4.9)), we obtain the first part of the following corollary. The second part follows from the formula $\tilde{\mu}-I=(1-C_{\widetilde{w}_\theta})^{-1}(C_{\widetilde{w}_\theta}I)\in L^p(\widetilde{\mathbf{R}})$, $p\geqslant 2$. Of course, in order to prove (4.60) below, all we need are the mapping properties of $C^{\pm}_{\widetilde{\mathbf{R}}\to\widetilde{\mathbf{R}}}$, etc. The full estimates in (4.59) for $C^{\pm}_{\widetilde{\mathbf{R}}\to\Gamma}$ will be needed later.

COROLLARY 4.58 (to Lemma 4.50). *For any $2\leqslant p<\infty$, and for all $t\geqslant 0$,*

$$\|C^{\pm}_{\widetilde{\mathbf{R}}\to\Gamma}\widetilde{w}^{\mp}_\theta\|_{L^p} \leqslant \frac{c}{(1+t)^{1/2p}}\frac{\lambda}{(1-\varrho)^2}, \tag{4.59}$$

$$\|\tilde{\mu}-I\|_{L^p} \leqslant \frac{c}{(1+t)^{1/2p}}\frac{\lambda K_p}{(1-\varrho)^2}. \tag{4.60}$$

Remark 4.61. The proof of (4.60) clearly also shows that

$$\|(1-C_{\widetilde{w}_{2\theta}})^{-1}C_{\widetilde{w}_{1\theta}}I\|_{L^p} \leqslant \frac{c}{(1+t)^{1/2p}}\frac{\lambda K_p}{(1-\varrho)^2}$$

as long as $r_1, r_2\in H^{1,0}$, $\|r_j\|_{L^\infty}<\varrho$, $\|r_j\|_{H^{1,0}}<\lambda$, $j=1,2$.

We now prove a Lipschitz version of Lemma 4.50.

LEMMA 4.62. *Suppose* $f\in H^{1,0}$. *Then for* $2\leqslant p<\infty$,

$$\|C^{\pm}_{\mathbf{R}_+\to\Gamma}(\delta_2^{\pm 2}-\delta_1^{\pm 2})fe^{\mp it\lozenge^2}\|_{L^p}\leqslant \frac{c}{(1+t)^{1/2p}}\,\frac{\lambda}{(1-\varrho)^3}\,\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}, \tag{4.63}$$

$$\|C^{\pm}_{e^{i\pi}\mathbf{R}_+\to\Gamma}(\tilde\delta_{2\mp}^{\pm 2}-\tilde\delta_{1\mp}^{\pm 2})fe^{\mp it\lozenge^2}\|_{L^p}\leqslant \frac{c}{(1+t)^{1/2p}}\,\frac{\lambda}{(1-\varrho)^2}\,\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}. \tag{4.64}$$

Proof. We only prove the bound for $C^{-}_{\mathbf{R}_+\to\Gamma}((\delta_2^{-2}-\delta_1^{-2})fe^{it\lozenge^2})$. Again the other cases are similar.

As in the proof of Lemma 4.50, we write for any $\varepsilon>0$,

$$C^{-}_{\mathbf{R}_+\to\Gamma}((\delta_2^{-2}-\delta_1^{-2})fe^{-\varepsilon\lozenge}e^{it\lozenge^2})=\frac{1}{\sqrt{2\pi}}\int dy\,\check f(y)e^{-iy^2/4t}F_1+\frac{1}{\sqrt{2\pi}}\int dy\,\check f(y)e^{-iy^2/4t}F_2,$$

where now

$$F_1(y)=C^{-}_{\mathbf{R}_+\to\Gamma}((\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\lozenge}\chi_{(0,a)}e^{it(\lozenge-y/2t)^2}),$$
$$F_2(y)=C^{-}_{\mathbf{R}_+\to\Gamma}((\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\lozenge}\chi_{(a,\infty)}e^{it(\lozenge-y/2t)^2})$$

and again $a=\max(0,y/2t)$.

Assume first that $p>2$. For F_1, again we only need to consider $y>0$. By (4.35),

$$\begin{aligned}\|F_1\|_{L^p(\Gamma)}&\leqslant c\|(\delta_2^{-2}-\delta_1^{-2})\chi_{(0,a)}\|_{L^p}\leqslant c\frac{\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\left(\int_0^a(1+|\log z|)^p\,dz\right)^{1/p}\\&\leqslant c\frac{\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}a^{1/p}\left(\int_0^1(1+|\log s|+|\log a|)^p\,ds\right)^{1/p}\\&\leqslant c\frac{\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}a^{1/p}(1+|\log a|).\end{aligned}$$

Hence for $t\geqslant 2$,

$$\begin{aligned}&\left\|\frac{1}{\sqrt{2\pi}}\int\check f(y)e^{-iy^2/4t}F_1\right\|_{L^p(\Gamma)}\\&\qquad\leqslant\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\int_0^\infty|\check f(y)|\left|\frac{y}{t}\right|^{1/p}(1+|\log y|+|\log t|)\,dy\\&\qquad\leqslant\frac{c\lambda}{(1-\varrho)^2}\,\frac{\log t}{t^{1/p}}\,\|f\|_{H^{1,0}}.\end{aligned}\tag{4.65}$$

For $p=2$, we can rewrite the integral as

$$\begin{aligned}F_1&=\frac{1}{\sqrt{2\pi}}C^{-}_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\lozenge}\int_0^1 dy\,\check f(y)e^{-iy\lozenge+it\lozenge^2}\chi_{(0,y/2t)}\\&\quad+\frac{1}{\sqrt{2\pi}}C^{-}_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\lozenge}\int_{\max(1,2tz)}^\infty dy\,\check f(y)e^{-iy\lozenge+it\lozenge^2}\equiv\mathrm{I}+\mathrm{II}.\end{aligned}$$

For $t\geqslant 2$, by (4.35) and some elementary calculus,

$$\begin{aligned}\|\mathrm{I}\|_{L^2(\Gamma)} &\leqslant \int_0^1 dy\,|\check{f}(y)|\,\|\delta_2^{-2}-\delta_1^{-2}\|_{L^2(0,y/2t)}\\ &\leqslant \frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\frac{1+\log t}{t^{1/2}}\int_0^1 dy\,(1+|\log y|)\check{f}(y)\\ &\leqslant \frac{1+\log t}{t^{1/2}}\,\frac{c\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.\end{aligned}$$

Also,

$$\begin{aligned}|\mathrm{II}| \leqslant &\left|\frac{1}{\sqrt{2\pi}}C^-_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\Diamond}\chi_{(0,1/2t)}\int_1^\infty dy\,\check{f}(y)\,e^{-iy\Diamond+it\Diamond^2}\right|\\ &+\left|\frac{1}{\sqrt{2\pi}}C^-_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\Diamond}\chi_{(1/2t,\infty)}\int_{2tz}^\infty dy\,\check{f}(y)\,e^{-iy\Diamond+it\Diamond^2}\right| \equiv \mathrm{II}_a+\mathrm{II}_b.\end{aligned}$$

Again for $t\geqslant 2$,

$$\|\mathrm{II}_a\|_{L^2} \leqslant c\|(\delta_2^{-2}-\delta_1^{-2})\chi_{(0,1/2t)}\|_{L^2}\|f\|_{H^{1,0}} \leqslant \frac{c(1+\log t)}{\sqrt{t}}\,\frac{\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.$$

Set $g(\zeta)=\int_\zeta^\infty dy\,|\check{f}(y)|$. By Hardy's inequality, $\|g\|_{L^2}\leqslant 2\|f\|_{H^{1,0}}$. For $t\geqslant 2$,

$$\begin{aligned}\|\mathrm{II}_b\|_{L^2} &\leqslant c\left\|(\delta_2^{-2}-\delta_1^{-2})\int_{2t\Diamond}^\infty dy\,|\check{f}(y)|\right\|_{L^2(1/2t,\infty)}\\ &= \frac{c}{\sqrt{t}}\left\|\left(\delta_2^{-2}\left(\frac{\Diamond}{2t}\right)-\delta_1^{-2}\left(\frac{\Diamond}{2t}\right)\right)g(\Diamond)\right\|_{L^2(1,\infty)}\\ &\leqslant \frac{c(1+\log t)}{\sqrt{t}}\,\frac{c\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}},\end{aligned}$$

by Lemma 4.32. For F_2, again we first consider the case where $y<0$, and hence $a=0$. Then for $p\geqslant 2$ we deform the contour as in the proof of Lemma 4.50, to obtain for $t\geqslant 2$, as $y<0$,

$$\begin{aligned}\|C^-_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\Diamond}e^{it(\Diamond-y/2t)^2}\|_{L^p} &\leqslant c\|(\delta_2^{-2}-\delta_1^{-2})e^{it\Diamond^2}\|_{L^p(e^{i\pi/4}\mathbf{R}_+)}\\ &\leqslant \frac{c(1+\log t)}{t^{1/2p}}\,\frac{\lambda}{(1-\varrho)^3},\end{aligned}$$

by scaling and Lemma 4.32. For $y>0$, we obtain similarly for $t\geqslant 2$,

$$\begin{aligned}&\|C^-_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})e^{-\varepsilon\Diamond}\chi_{(a,\infty)}e^{it(\Diamond-y/2t)^2}\|_{L^p(\Gamma)}\\ &\qquad\leqslant c\|(\delta_2^{-2}-\delta_1^{-2})e^{it(\Diamond-y/2t)^2}\|_{L^p(y/2t+e^{i\pi/4}\mathbf{R}_+)} \leqslant c\frac{1+\log t}{t^{1/2p}}\,\frac{\lambda}{(1-\varrho)^3},\end{aligned}$$

by scaling and Lemma 4.32. Thus for all $p\geqslant 2$, and for $t\geqslant 2$,

$$\left\|\frac{1}{\sqrt{2\pi}}\int dy\, f(y)e^{-iy^2/4t}F_2\right\|_{L^p(\Gamma)}\leqslant\frac{c(1+\log t)}{t^{1/2p}}\frac{\lambda}{(1-\varrho)^3}\|f\|_{H^{1,0}}.$$

On the other hand, for all t, and for all $p\geqslant 2$, we have

$$\begin{aligned}\|C^-_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})fe^{it\Diamond^2}\|_{L^p}&\leqslant c\|(\delta_2^{-2}-\delta_1^{-2})f\|_{L^p(\mathbf{R}_+)}\\&\leqslant\frac{c\lambda}{(1-\varrho)^2}\left(\|f\|_{L^p}+\|f\|_{H^{1,0}}\left\|\int_{-1}^0\frac{1+s}{s-\Diamond}\,ds\right\|_{L^p(\mathbf{R}_+)}\right)\|r_2-r_1\|_{H^{1,0}}\\&\leqslant\frac{c\lambda}{(1-\varrho)^2}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.\end{aligned}$$

We conclude that for all $t\geqslant 0$, and for all $p\geqslant 2$,

$$\|C^-_{\mathbf{R}_+\to\Gamma}(\delta_2^{-2}-\delta_1^{-2})fe^{it\Diamond^2}\|_{L^p}\leqslant\frac{c\lambda}{(1-\varrho)^3(1+t)^{1/2p}}\|f\|_{H^{1,0}}\|r_2-r_1\|_{H^{1,0}}.\qquad\square$$

Using the fact that

$$\left|\left(\frac{r}{1-|r|^2}\right)'\right|\leqslant\frac{c|r'|}{(1-|r|^2)^2}$$

we obtain the following Lipschitz estimate:

COROLLARY 4.66 (to Lemmas 4.50 and 4.62).

$$\|C^\pm_{\mathbf{R}_+\to\Gamma}(\widetilde{w}^\mp_{2\theta}-\widetilde{w}^\mp_{1\theta})\|_{L^p}\leqslant\frac{c}{(1+t)^{1/2p}}\frac{(1+\lambda)^2}{(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}}.$$

LEMMA 4.67. *Let $f\in H^{1,0}$. Then the multipliers $\Diamond\mapsto\Diamond f(\Delta_1-\Delta_2)$ are bounded from $L^q\to L^p$, $q>p\geqslant 2$:*

$$\begin{aligned}\|\Diamond f(\Delta_2^{\pm1}-\Delta_1^{\pm1})\|_{L^q\to L^p}&\leqslant\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\|f\|_{L^\infty}\\&\leqslant\frac{c\lambda}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\|f\|_{H^{1,0}}.\end{aligned}\tag{4.68}$$

Also, the multipliers $\Diamond\to\Diamond(\widetilde{w}_2^\pm-\widetilde{w}_1^\pm)$ are bounded and

$$\|\Diamond(\widetilde{w}_2^\pm-\widetilde{w}_1^\pm)\|_{L^q\to L^p}\leqslant\frac{c(1+\lambda)}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}.\tag{4.69}$$

Proof. For $g\in L^q(\mathbf{R})$,

$$\begin{aligned}\|gf(\Delta_2^{\pm1}-\Delta_1^{\pm1})\|_{L^p}&\leqslant\|f\|_{L^\infty}\|g\|_{L^q}\|(\Delta_2^{\pm1}-\Delta_1^{\pm1})\|_{L^{pq/(q-p)}}\\&\leqslant\frac{c\lambda}{(1-\varrho)^2}\|f\|_{L^\infty}\|g\|_{L^q}\|r_2-r_1\|_{H^{1,0}},\end{aligned}$$

by Lemma 4.29, and this proves (4.68). Setting $f=r_j$ or $\bar{r}_j$ in (4.68), and using $|\Delta_j^{\pm1}|=1$, we obtain (4.69). $\square$

LEMMA 4.70. *For* $q>p\geqslant 2$,

$$\|\tilde{\mu}_2-\tilde{\mu}_1\|_{L^p(\tilde{\mathbf{R}})}\leqslant\frac{c}{(1+t)^{1/2q}}\frac{K_qK_p(1+\lambda)^2}{(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}}\leqslant\begin{bmatrix}0 & 2\\ 1/2q & 4\end{bmatrix}K_q^2\|r_2-r_1\|_{H^{1,0}}.$$

Proof. $\tilde{\mu}_2-\tilde{\mu}_1=m_1+m_2$, where $m_1=(1-C_{\tilde{w}_{2\theta}})^{-1}C_{\tilde{w}_{2\theta}-\tilde{w}_{1\theta}}(1-C_{\tilde{w}_{1\theta}})^{-1}C_{\tilde{w}_{2\theta}}I$ and $m_2=(1-C_{\tilde{w}_{1\theta}})^{-1}C_{\tilde{w}_{2\theta}-\tilde{w}_{1\theta}}I$. By Remark 4.61 and inequality (4.69), we obtain

$$\|m_1\|_{L^p}\leqslant\frac{cK_qK_p\lambda(1+\lambda)}{(1+t)^{1/2q}(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}}.$$

From Corollary 4.66 to Lemmas 4.50 and 4.62, we have

$$\|m_2\|_{L^p}\leqslant\frac{cK_p(1+\lambda)^2}{(1+t)^{1/2p}(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}}.$$

This proves the lemma. □

Recall from §2 that if $\mu=\mu(x,t,z)$ solves $(1-C_{w_\theta})\mu=I$, then

$$m=m(x,t,z)=I+C(\mu(w_\theta^++w_\theta^-))(z),\quad z\notin\mathbf{R},$$

and $m\sim I+\mathbf{Q}/(-2\pi iz)+\ldots$ as $z\to\infty$, where $\mathbf{Q}=\int_{\mathbf{R}}\mu(w_\theta^++w_\theta^-)$ as in (2.20). Similarly $\tilde{m}=I+\tilde{\mathbf{Q}}/(-2\pi iz)+\ldots$ as $z\to\infty$, where $\tilde{\mathbf{Q}}=\int_{\tilde{\mathbf{R}}}\tilde{\mu}(\tilde{w}_\theta^++\tilde{w}_\theta^-)$ (cf. Theorem 4.16). But $\delta=e^{C_{\mathbf{R}_-}\log(1-|r|^2)}=1+(\int_{\mathbf{R}_-}\log(1-|r|^2))/(-2\pi iz)+\ldots$, and hence from (4.6),

$$\mathbf{Q}=\tilde{\mathbf{Q}}+\left(\int_{\mathbf{R}_-}\log(1-|r|^2)\right)\sigma_3.\tag{4.71}$$

Now if $q(x,t)$ solves NLS with $q(x,0)=(\mathcal{R}^{-1}(r))(x)$, then by (2.45),

$$Q=\begin{pmatrix}0 & q(x,t)\\ \overline{q(x,t)} & 0\end{pmatrix}=\frac{\operatorname{ad}\sigma}{2\pi}\mathbf{Q}.$$

But then by (4.71),

$$Q=\frac{\operatorname{ad}\sigma}{2\pi}\tilde{\mathbf{Q}}.\tag{4.72}$$

For later reference, we note that expanding to order $1/z^2$, we obtain

$$\operatorname{ad}\sigma\left(\int_{\mathbf{R}}\mu s(w_\theta^++w_\theta^-)\,ds\right)=\operatorname{ad}\sigma\left(\int_{\tilde{\mathbf{R}}}\tilde{\mu}s(\tilde{w}_\theta^++\tilde{w}_\theta^-)\,ds\right)+iQ\sigma_3\int_{-\infty}^0\log(1-|r|^2)\,ds.\tag{4.73}$$

LEMMA 4.74 (smoothing estimate). *If* $r \in H^{1,0}$, $\|r\|_{L^\infty} \leqslant \varrho < 1$, $\|r\|_{H^{1,0}} \leqslant \lambda$, *then for all* $t \geqslant 1$,

$$|\widetilde{\mathbf{Q}}| \leqslant \frac{c\lambda(1+\lambda)}{t^{1/2}(1-\varrho)^5}.$$

If in addition $r \in H_1^{1,1}$, $\|r\|_{H^{1,1}} \leqslant \eta$, *then*

$$|\widetilde{\mathbf{Q}}| \leqslant \frac{c\eta(1+\eta)}{(1+t)^{1/2}(1-\varrho)^5} \leqslant \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix} \quad \textit{for all } t \geqslant 0.$$

Proof.

$$\begin{aligned}
\widetilde{\mathbf{Q}} &= \int_{\widetilde{\mathbf{R}}} (\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-) + \int_{\widetilde{\mathbf{R}}} (\tilde{\mu} - I)(\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-) \\
&= \int_{\widetilde{\mathbf{R}}} (\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-) + \int_{\widetilde{\mathbf{R}}} (C_{\widetilde{w}_\theta} I)(\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-) + \int_{\widetilde{\mathbf{R}}} (C_{\widetilde{w}_\theta}(\tilde{\mu} - I))(\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-) \\
&\equiv Q^{(0)} + Q^{(1)} + Q^{(2)}.
\end{aligned}$$

From Lemma 4.42 and (4.9), for $t > 1$, $|Q^{(0)}| \leqslant c(1+\lambda)\lambda / t^{1/2}(1-\varrho)$. If $r \in H^{1,1}$, then clearly $|Q^{(0)}| \leqslant c\|r\|_{H^{1,1}} < c\eta$.

Now, by triangularity,

$$\begin{aligned}
Q^{(1)} &= \int_{\widetilde{\mathbf{R}}} (C_{\widetilde{\mathbf{R}}}^+ \widetilde{w}_\theta^- + C_{\widetilde{\mathbf{R}}}^- \widetilde{w}_\theta^+)(\widetilde{w}_\theta^+ + \widetilde{w}_\theta^-) = \int_{\widetilde{\mathbf{R}}} (C_{\widetilde{\mathbf{R}}}^+ \widetilde{w}_\theta^-) \widetilde{w}_\theta^+ + \int_{\widetilde{\mathbf{R}}} (C_{\widetilde{\mathbf{R}}}^- \widetilde{w}_\theta^+) \widetilde{w}_\theta^- \\
&= \int_\Gamma (C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^-)(C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^+ - C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^+) + \int_\Gamma (C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^+)(C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^- - C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^+),
\end{aligned}$$

where we have used the fact that $C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^\pm - C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^\pm = \widetilde{w}_\theta^\pm$ on $\widetilde{\mathbf{R}}$, and equals 0 on $\Gamma \setminus \widetilde{\mathbf{R}}$. Thus, by Cauchy's theorem,

$$\begin{aligned}
|Q^{(1)}| &\leqslant \left| \int_\Gamma (C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^-)(C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^+) \right| + \left| \int_\Gamma (C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^+)(C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^-) \right| \\
&\leqslant \frac{c}{(1+t)^{1/2}} \frac{\lambda^2}{(1-\varrho)^4},
\end{aligned}$$

by (4.59). Similarly, using (4.59) and (4.60),

$$\begin{aligned}
|Q^{(2)}| &\leqslant \left| \int_\Gamma (C_{\widetilde{\mathbf{R}} \to \Gamma}^+ (\tilde{\mu} - I) \widetilde{w}_\theta^-) C_{\widetilde{\mathbf{R}} \to \Gamma}^- \widetilde{w}_\theta^+ \right| + \left| \int_\Gamma (C_{\widetilde{\mathbf{R}} \to \Gamma}^- (\tilde{\mu} - I) \widetilde{w}_\theta^+) C_{\widetilde{\mathbf{R}} \to \Gamma}^+ \widetilde{w}_\theta^- \right| \\
&\leqslant \frac{c\varrho\lambda K_2}{(1+t)^{1/4}(1-\varrho)^2} \frac{\lambda}{(1+t)^{1/4}(1-\varrho)^2} \leqslant \frac{c\varrho\lambda^2}{(1+t)^{1/2}(1-\varrho)^5}.
\end{aligned}$$

The above inequalities prove the lemma. □

LEMMA 4.75 (Lipschitz smoothing estimate). *Suppose* $r_1, r_2 \in H^{1,1}$ *with* $\|r_i\|_{H^{1,1}} \leqslant \eta$, $\|r_i\|_{L^\infty} \leqslant \varrho < 1$, $i=1,2$. *Let* $2<q<\infty$. *For all* $t \geqslant 0$,

$$|\widetilde{\mathbf{Q}}_2 - \widetilde{\mathbf{Q}}_1| \leqslant \frac{c(1+\lambda)^3 K_q}{(1+t)^{1/2q+1/4}(1-\varrho)^7}\|r_2-r_1\|_{H^{1,1}} \leqslant \begin{bmatrix} 0 & 3 \\ 1/2q+1/4 & 7 \end{bmatrix} K_q \|r_2 - r_1\|_{H^{1,1}}.$$

Proof. In the notation of the previous lemma, set $Q_j^{(k)} = Q^{(k)}(r_j)$, $k=0,1,2$, $j=1,2$. Then $|Q_2 - Q_1| \leqslant |Q_2^{(0)} - Q_1^{(0)}| + |Q_2^{(1)} - Q_1^{(1)}| + |Q_2^{(2)} - Q_1^{(2)}|$. By Corollary 4.47, and the fact that $\left|\int_{\mathbf{R}} f \Delta^{\pm} e^{\mp itz^2}\right| \leqslant c\|f\|_{H^{1,1}}$,

$$|Q_2^{(0)} - Q_1^{(0)}| \leqslant \frac{c \log(2+t)}{(1+t)^{1/2}} \frac{(1+\lambda)^3}{(1-\varrho)^3}\|r_2 - r_1\|_{H^{1,1}}.$$

We find

$$\begin{aligned} |Q_2^{(1)} - Q_1^{(1)}| &\leqslant \left|\int (C_{\widetilde{w}_{2\theta} - \widetilde{w}_{1\theta}} I)(\widetilde{w}_{2\theta}^+ + \widetilde{w}_{2\theta}^-)\right| + \left|\int (C_{\widetilde{w}_{1\theta}} I)(\widetilde{w}_{2\theta}^+ + \widetilde{w}_{2\theta}^- - \widetilde{w}_{1\theta}^+ - \widetilde{w}_{1\theta}^-)\right| \\ &= \left|\int_1\right| + \left|\int_2\right|. \end{aligned}$$

Again by triangularity, extension to Γ and Cauchy's theorem,

$$\begin{aligned} \left|\int_1\right| &\leqslant \left|\int_\Gamma (C^+_{\widetilde{\mathbf{R}}\to\Gamma}(\widetilde{w}_{2\theta}^- - \widetilde{w}_{1\theta}^-) C^-_{\widetilde{\mathbf{R}}\to\Gamma}\widetilde{w}_{2\theta}^+\right| + \left|\int_\Gamma C^-_{\widetilde{\mathbf{R}}\to\Gamma}(\widetilde{w}_{2\theta}^+ - \widetilde{w}_{1\theta}^+) C^+_{\widetilde{\mathbf{R}}\to\Gamma}\widetilde{w}_{2\theta}^-\right| \\ &\leqslant \frac{c}{(1+t)^{1/2}} \frac{\lambda(1+\lambda)^2}{(1-\varrho)^6}\|r_2 - r_1\|_{H^{1,0}}, \end{aligned}$$

where we have used Corollaries 4.58 and 4.66. The estimate for $\int_2$ is similar. Thus

$$|Q_2^{(1)} - Q_1^{(1)}| \leqslant \frac{c\lambda(1+\lambda)^2}{(1-\varrho)^6} \frac{1}{(1+t)^{1/2}}\|r_2 - r_1\|_{H^{1,0}}.$$

Expanding, we obtain

$$\begin{aligned} |Q_2^{(2)} - Q_1^{(2)}| &\leqslant \left|\int_{\widetilde{\mathbf{R}}} (C_{\widetilde{w}_{2\theta} - \widetilde{w}_{1\theta}}(\tilde{\mu}_2 - I))(\widetilde{w}_{2\theta}^+ + \widetilde{w}_{2\theta}^-)\right| + \left|\int_{\widetilde{\mathbf{R}}} (C_{\widetilde{w}_{1\theta}}(\tilde{\mu}_2 - \tilde{\mu}_1))(\widetilde{w}_{2\theta}^+ + \widetilde{w}_{2\theta}^-)\right| \\ &\quad + \left|\int_{\widetilde{\mathbf{R}}} (C_{\widetilde{w}_{1\theta}}(\tilde{\mu}_1 - I))(\widetilde{w}_{2\theta}^+ + \widetilde{w}_{2\theta}^- - \widetilde{w}_{1\theta}^+ - \widetilde{w}_{1\theta}^-)\right| \\ &\equiv \left|\int_{1'}\right| + \left|\int_{2'}\right| + \left|\int_{3'}\right|. \end{aligned}$$

As before

$$\left|\int_{1'}\right| \leqslant \left|\int_\Gamma (C^+_{\tilde{\mathbf{R}}\to\Gamma}(\tilde{\mu}_2-I)(\widetilde{w}^-_{2\theta}-\widetilde{w}^-_{1\theta}))C^-_{\tilde{\mathbf{R}}\to\Gamma}\widetilde{w}^+_{2\theta}\right| + \left|\int_\Gamma (C^-_{\tilde{\mathbf{R}}\to\Gamma}(\tilde{\mu}_2-I)(\widetilde{w}^+_{2\theta}-\widetilde{w}^+_{1\theta}))C^+_{\tilde{\mathbf{R}}\to\Gamma}\widetilde{w}^-_{2\theta}\right|.$$

Now by (4.69) for $q>p=2$,

$$\|(\tilde{\mu}_2-I)(\widetilde{w}^-_{2\theta}-\widetilde{w}^-_{1\theta})\|_{L^2} \leqslant \frac{c(1+\lambda)}{(1-\varrho)^2}\|r_2-r_1\|_{H^{1,0}}\|\tilde{\mu}_2-I\|_{L^q} \leqslant \frac{c}{(1+t)^{1/2q}}\frac{\lambda(1+\lambda)K_q}{(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}},$$

by Corollary 4.58. Then, again by the corollary,

$$\left|\int_{1'}\right| \leqslant \frac{c\lambda^2(1+\lambda)K_q\|r_2-r_1\|_{H^{1,0}}}{(1+t)^{1/4+1/2q}(1-\varrho)^6}.$$

Now for $q>p=2$,

$$\begin{aligned}\left|\int_{3'}\right| &\leqslant \left|\int_\Gamma (C^+_{\tilde{\mathbf{R}}\to\Gamma}(\tilde{\mu}_1-I)\widetilde{w}^-_{1\theta})C^-_{\tilde{\mathbf{R}}\to\Gamma}(\widetilde{w}^+_{2\theta}-\widetilde{w}^+_{1\theta})\right| + \left|\int_\Gamma (C^-_{\tilde{\mathbf{R}}\to\Gamma}(\tilde{\mu}_1-I)\widetilde{w}^+_{1\theta})C^+_{\tilde{\mathbf{R}}\to\Gamma}(\widetilde{w}^-_{2\theta}-\widetilde{w}^-_{1\theta})\right| \\ &\leqslant \frac{c\varrho}{(1+t)^{1/4}}\frac{\lambda K_2}{(1-\varrho)^2}\frac{1}{(1+t)^{1/4}}\frac{(1+\lambda)^2}{(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}} = \frac{c\varrho\lambda(1+\lambda)^2K_2}{(1+t)^{1/2}(1-\varrho)^6}\|r_2-r_1\|_{H^{1,0}},\end{aligned}$$

again by Corollaries 4.58 and 4.66. Finally, by Lemma 4.70 and Corollary 4.58,

$$\begin{aligned}\left|\int_{2'}\right| &\leqslant \left|\int_\Gamma (C^+_{\tilde{\mathbf{R}}\to\Gamma}(\tilde{\mu}_2-\tilde{\mu}_1)\widetilde{w}^-_{1\theta})C^-_{\tilde{\mathbf{R}}\to\Gamma}\widetilde{w}^+_{2\theta}\right| + \left|\int_\Gamma (C^-_{\tilde{\mathbf{R}}\to\Gamma}(\tilde{\mu}_2-\tilde{\mu}_1)\widetilde{w}^+_{1\theta})C^+_{\tilde{\mathbf{R}}\to\Gamma}\widetilde{w}^-_{2\theta}\right| \\ &\leqslant \frac{c\varrho K_2K_q(1+\lambda)^2}{(1+t)^{1/2q}(1-\varrho)^4}\|r_2-r_1\|_{H^{1,0}}\frac{\lambda}{(1+t)^{1/4}(1-\varrho)^2} \\ &= \frac{c\varrho\lambda(1+\lambda)^2K_2K_q\|r_2-r_1\|_{H^{1,0}}}{(1+t)^{1/4+1/2q}(1-\varrho)^6}.\end{aligned}$$

Thus for any $q>2$,

$$|Q^{(2)}_2-Q^{(2)}_1| \leqslant \frac{c\lambda(1+\lambda)^2K_q\|r_2-r_1\|_{H^{1,0}}}{(1+t)^{1/4+1/2q}(1-\varrho)^7}.$$

Assembling the above estimates, the lemma is proved. □

5. Supplementary estimates

This section plays an intermediary role. Our goal here is to supplement the estimates in §4, and place them in a form that is directly applicable to the analysis of the evolution equation (2.46). We continue to use the notation of §4 without further comment. Thus m solves the normalized RHP $(\mathbf{R}, v_\theta)$, etc. Throughout this section we consider reflection coefficients r, r_1, r_2 in $H_1^{1,1}$, and as before, we assume that their L^∞-, $H^{1,0}$- and $H^{1,1}$-norms are bounded by $\varrho<1$, λ and η respectively. If h, say, is a quantity which depends on the reflection coefficient, $h=h(r)$, and $h_j=h(r_j)$, we write $\Delta h=h_2-h_1$. In order to simplify the writing of Lipschitz estimates we will replace quantities h_j simply by h. With this notation Δ, in particular, operates formally like a derivation: $\Delta hg=(\Delta h)g+h(\Delta g)$. As r_1, r_2 have the same L^∞-, $H^{1,0}$- and $H^{1,1}$-bounds, the Lipschitz estimates one obtains are not affected by this lack of precision. In addition, to further simplify notation, we occasionally use w_θ, which is defined in §2 as (w_θ^-, w_θ^+), also to denote $w_\theta^+ + w_\theta^-$ (see e.g. Lemma 5.1 below). These abuses of notation should not lead to confusion. Note finally, once again, that if

$$X=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

is a (2×2)-matrix with $\det X=1$, then

$$X^{-1}=\begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix},$$

and so estimates for X immediately imply the same estimates for X^{-1}, for example, estimates on $\|m_\pm - I\|_{L^p}$ imply the same estimates on $\|m_\pm^{-1}-I\|_{L^p}=\|m_\pm - I\|_{L^p}$, etc.

LEMMA 5.1. *Let $2\leqslant p<\infty$. Then we have*

$$\|\mu-I\|_{L^2(dz)},\ \|m_\pm-I\|_{L^2(dz)} \leqslant \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\|\mu-I\|_{L^p(dz)},\ \|m_\pm-I\|_{L^p(dz)} \leqslant \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} K_p,$$

$$\|\Delta\mu\|_{L^p(dz)},\ \|\Delta m_\pm\|_{L^p(dz)} \leqslant \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} K_p^2 \|\Delta r\|_{H^{1,1}}.$$

Moreover, the same estimates are true if we replace $\mu, m_\pm$ by $\mu^{-1}, m_\pm^{-1}$.

Proof. Use the equation $\mu-I=(1-C_{w_\theta})^{-1}C_{w_\theta}I$, the relation $m_\pm=\mu v^\pm=\mu(I\pm w^\pm)$ and the second resolvent identity $\Delta(1-C_{w_\theta})^{-1}=(1-C_{w_\theta})^{-1}(\Delta w_\theta)(1-C_{w_\theta})^{-1}$, as expressed in the above simplified notation. □

LEMMA 5.2. *$z(\mu-I)=\mu_1+\mu_2$, $z(m_\pm-I)=(m_\pm)_1+(m_\pm)_2$, where*

$$\|\mu_1\|_{L^2(dz)},\ \|m_{\pm 1}\|_{L^2(dz)} \leqslant \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \textit{and} \quad \|\mu_2\|_{L^\infty(dz)},\ \|m_{\pm 2}\|_{L^\infty(dz)} \leqslant \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Proof. Using $\mu - I = C_{w_\theta}\mu$, we have

$$\langle z\rangle(\mu-I)=\langle z\rangle C_{w_\theta}(\mu-I)+\langle z\rangle C_{w_\theta}I=C_{w_\theta}\langle z\rangle(\mu-I)+C_{w_\theta}\langle z\rangle-\frac{1}{2\pi i}\mathbf{Q}.$$

Using the fact that $C_{w_\theta}\mathbf{Q}=\mathbf{Q}C_{w_\theta}I$, we obtain

$$\begin{aligned}\langle z\rangle(\mu-I)&=(1-C_{w_\theta})^{-1}\left[C_{w_\theta}\langle z\rangle-\frac{1}{2\pi i}\mathbf{Q}\right]\left[(1-C_{w_\theta})^{-1}C_{w_\theta}\langle z\rangle-\frac{1}{2\pi i}\mathbf{Q}(1-C_{w_\theta})^{-1}(C_{w_\theta}I)\right]\\ &\equiv \mu_1+\mu_2.\end{aligned}$$

Now

$$|\mathbf{Q}|=\left|\int(\mu-I)w_\theta+\int w_\theta\right|\leqslant\frac{c\lambda^2}{1-\varrho}+\eta.$$

Hence

$$\|\mu_1\|_{L^2}\leqslant\frac{c\eta}{1-\varrho}+\frac{c\lambda}{1-\varrho}\left(\frac{c\lambda^2}{1-\varrho}+\eta\right)\leqslant\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \|\mu_2\|_{L^\infty}\leqslant\frac{c\lambda^2}{1-\varrho}+\eta\leqslant\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The proof for $m_\pm$ now follows from the relations $m_\pm=\mu v_\theta^\pm$. □

LEMMA 5.3. *In the notation of Lemma 5.2,*

$$\|\Delta\mu_1\|_{L^2(dz)},\ \|\Delta m_{\pm 1}\|_{L^2(dz)}\leqslant\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}\|\Delta r\|_{H^{1,1}}, \tag{5.4}$$

$$\|\Delta\mu_2\|_{L^\infty(dz)},\ \|\Delta m_{\pm 2}\|_{L^\infty(dz)}\leqslant\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\|\Delta r\|_{H^{1,1}}. \tag{5.5}$$

Proof. Let w be the data corresponding to m_-. Again write

$$\langle z\rangle(\mu_- - I)=\left[(1-C_{w_\theta})^{-1}C_{w_\theta}\langle z\rangle-\frac{1}{2\pi i}\mathbf{Q}(1-C_{w_\theta})^{-1}C_{w_\theta}I\right]-\frac{1}{2\pi i}\mathbf{Q}=\mu_1+\mu_2.$$

Now

$$\|(1-C_{w_\theta})^{-1}C_{w_\theta}\|_{L^2(dz)}\leqslant\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned}\|\Delta(1-C_w)^{-1}C_w\langle z\rangle\|_{L^2(dz)}&\leqslant\|(\Delta(1-C_w)^{-1})C_w\langle z\rangle\|_{L^2(dz)}+\|(1-C_w)^{-1}\Delta C_w\langle z\rangle\|_{L^2(dz)}\\ &\leqslant\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\|\Delta r\|_{H^{1,1}}+\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\|\Delta r\|_{H^{1,1}}\leqslant\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Similarly

$$\|\Delta(1-C_{w_\theta})^{-1}C_{w_\theta}I\|_{L^2(dz)} \leqslant \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \|\Delta r\|_{H^{1,1}}.$$

Also,

$$\begin{aligned} |\Delta \mathbf{Q}| &\leqslant \left|\int (\Delta\mu) w\right| + \left|\int (\mu - I)\Delta w\right| + \left|\int \Delta w\right| \\ &\leqslant \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \|\Delta r\|_{H^{1,1}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \|\Delta r\|_{H^{1,1}} + \|\Delta r\|_{H^{1,1}} \\ &\leqslant \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c \right) \|\Delta r\|_{H^{1,1}} \leqslant \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \|\Delta r\|_{H^{1,1}}. \end{aligned}$$

The above inequalities, together with the inequality

$$|\mathbf{Q}| \leqslant \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

from the proof of Lemma 5.2, prove (5.4), (5.5) for μ. Again the proof for $m_\pm$ now follows from the relations $m_\pm = \mu v_\theta^\pm$. □

LEMMA 5.6.

$$\|(1+\Diamond^2)^{1/4}(\mu - I)\|_{L^4(dz)},\ \|(1+\Diamond^2)^{1/4}(m_\pm - I)\|_{L^4(dz)} \leqslant \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} K_4,$$

$$\|(1+\Diamond^2)^{1/4}(\Delta\mu)\|_{L^4(dz)},\ \|(1+\Diamond^2)^{1/4}(\Delta m_\pm)\|_{L^4(dz)} \leqslant \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} K_4^2 \|\Delta r\|_{H^{1,1}}.$$

Proof.

$$\begin{aligned} (1+z^2)^{1/4}(\mu - I) &= C_{w_\theta}(1+\Diamond^2)^{1/4}\mu + \frac{1}{2\pi i}\int \frac{(1+z^2)^{1/4} - (1+\zeta^2)^{1/4}}{\zeta - z} \mu(\zeta) w_\theta(\zeta)\, d\zeta \\ &= C_{w_\theta}(1+\Diamond^2)^{1/4} + C_{w_\theta}(1+\Diamond^2)^{1/4}(\mu - I) \\ &\quad - \frac{1}{2\pi i}\int \frac{(\zeta + z)\mu(\zeta) w_\theta(\zeta)\, d\zeta}{((1+z^2)^{1/4} + (1+\zeta^2)^{1/4})((1+z^2)^{1/2} + (1+\zeta^2)^{1/2})}. \end{aligned}$$

Thus

$$\begin{aligned} (1+z^2)^{1/4}(\mu - I) &= (1 - C_{w_\theta})^{-1} C_{w_\theta}(1+\Diamond^2)^{1/4} \\ &\quad - (1 - C_{w_\theta})^{-1} \frac{1}{2\pi i}\int \frac{(\zeta + \Diamond)\mu(\zeta) w_\theta(\zeta)\, d\zeta}{((1+\Diamond^2)^{1/4} + (1+\zeta^2)^{1/4})((1+\Diamond^2)^{1/2} + (1+\zeta^2)^{1/2})}. \end{aligned} \tag{5.7}$$

The L^4-norm of the first term on the right-hand side of (5.7) is bounded by

$$cK_4\|\Diamond r\|_{L^2}^{1/2}\|r\|_{L^\infty}^{1/2} \leqslant cK_4\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

The integral in the second term on the right-hand side of (5.7) is bounded by

$$\frac{c}{(1+z^2)^{1/4}}\|\mu w_\theta\|_{L^1} \leqslant \frac{c}{2\pi(1+z^2)^{1/4}}\left(\frac{c\lambda^2}{1-\varrho}+\eta\right) \leqslant \frac{1}{(1+z^2)^{1/4}}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}. \tag{5.8}$$

Thus

$$\|(1+\Diamond^2)^{1/4}(\mu-I)\|_{L^4} \leqslant K_4\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

Now using (5.7), we have

$$\begin{aligned}(1+z^2)^{1/4}\Delta\mu &= \Delta(1-C_{w_\theta})^{-1}C_{w_\theta}(1+\Diamond^2)^{1/4}\\ &\quad -(\Delta(1-C_{w_\theta})^{-1})\frac{1}{2\pi i}\int\frac{(\zeta+\Diamond)\mu(\zeta)w_\theta(\zeta)\,d\zeta}{((1+\Diamond^2)^{1/4}+(1+\zeta^2)^{1/4})((1+\Diamond^2)^{1/2}+(1+\zeta^2)^{1/2})}\\ &\quad -(1-C_{w_\theta})^{-1}\frac{1}{2\pi i}\int\frac{(\zeta+\Diamond)\Delta(\mu(\zeta)w_\theta(\zeta))\,d\zeta}{((1+\Diamond^2)^{1/4}+(1+\zeta^2)^{1/4})((1+\Diamond^2)^{1/2}+(1+\zeta^2)^{1/2})}\\ &\equiv \mathrm{I}+\mathrm{II}+\mathrm{III},\end{aligned}$$

where

$$\begin{aligned}\|\mathrm{I}\|_{L^4(dz)} &\leqslant \|(\Delta(1-C_{w_\theta})^{-1})C_{w_\theta}(1+\Diamond^2)^{1/4}\|_{L^4(dz)}\\ &\quad +\|(1-C_{w_\theta})^{-1}C_{\Delta w_\theta}(1+\Diamond^2)^{1/4}\|_{L^4(dz)}\\ &\leqslant cK_4^2\|\Delta r\|_{H^{1,1}}\eta+K_4\|\Delta r\|_{H^{1,1}} \leqslant \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}K_4^2\|\Delta r\|_{H^{1,1}},\end{aligned}$$

$$\|\mathrm{II}\|_{L^4(dz)} \leqslant K_4^2\|\Delta r\|_{H^{1,1}}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},$$

$$\begin{aligned}\|\mathrm{III}\|_{L^4(dz)} &\leqslant cK_4\|\Delta\mu w_\theta\|_{L^1(dz)}\\ &\leqslant cK_4(\|\Delta\mu\|_{L^2(dz)}\|r\|_{L^2(dz)}+\|\mu-I\|_{L^2(dz)}\|\Delta r\|_{L^2(dz)}+\|\Delta r\|_{L^1(dz)})\\ &\leqslant cK_4\left(\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+1\right)\|\Delta r\|_{H^{1,1}} \leqslant \begin{bmatrix}0 & 2\\ 0 & 2\end{bmatrix}K_4\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Finally,

$$\begin{aligned}\|(1+\Diamond^2)^{1/4}\Delta\mu\|_{L^4(dz)} &\leqslant \|\mathrm{I}\|_{L^4(dz)}+\|\mathrm{II}\|_{L^4(dz)}+\|\mathrm{III}\|_{L^4(dz)}\\ &\leqslant \left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}K_4^2+\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}K_4^2+\begin{bmatrix}0 & 2\\ 0 & 2\end{bmatrix}K_4\right)\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}K_4^2\|\Delta r\|_{H^{1,1}},\end{aligned}$$

as $K_4 \geqslant K_2 = c/(1-\varrho)$. The estimates for $\Delta m_\pm$ are similar. □

Let

$$L = \partial_z - i(x-2zt)\,\mathrm{ad}\,\sigma. \tag{5.9}$$

Clearly $Lf_\theta = f'_\theta$, where again $f_\theta = e^{i\theta\,\mathrm{ad}\,\sigma} f$. Also $(Lf)_{-\theta} = \partial_z(f_{-\theta})$.

The operator L arises naturally as follows (see the discussion on L^p-bounds, $p>2$, for $m_-^\pm - I$ in [DZW, §4 following Corollary 4.5]). Differentiation of the jump relation $m_+ = m_- v_\theta$ leads to

$$\partial_z m_+ = (\partial_z m_-) v_\theta + m_-(i(x-2tz)\,\mathrm{ad}\, v_\theta + (\partial_z v)_\theta), \tag{5.10}$$

which implies bounds for $\|\partial_z m_-\|_{L^2}$ that grow quadratically in x and t. The point, however, is that one can rewrite (5.10) in the form

$$(L+2tQ)m_+ = ((L+2tQ)m_-)v_\theta + m_-(\partial_z v)_\theta. \tag{5.11}$$

The term $2tQ$ is added in to ensure that $(L+2tQ)m_\pm \in L^2$. Then (5.11) is an inhomogeneous RHP (see IRHP2, [DZW, §2]) with an inhomogeneous term that does not involve x, t explicitly apart from θ. Using the associated singular integral operator $1 - C_{w_\theta}$, one can estimate $(L+2tQ)m_\pm$ in terms of $\|m_\pm\|_{L^\infty}$ (and $\|r\|_{H^{1,0}}$), as in (5.22) et seq. below. But then one can hope to estimate $\|m_-\|_{L^\infty}$ in turn in terms of $(L+2tQ)m_\pm$ by a Sobolev-type estimate, and hence obtain a priori bounds which grow at moderate rates. The lemmas that follow show how this scheme can be carried through.

LEMMA 5.12. *Suppose that $f \in L^p(dz)$ for all $2 \leqslant p \leqslant \infty$, and that $(L+2tQ)(f+I) \in L^2(dz)$. Then for any $n \geqslant 3$,*

$$\begin{aligned}\|f\|_{L^\infty(dz)} &\leqslant 2n^{1/n}\|(L+2tQ)(f+I)\|_{L^2}^{1/n}\|f\|_{L^{2(n-1)}(dz)}^{(n-1)/n}\\ &\quad + 2n^{1/n}|2tQ|^{1/n}(\|f\|_{L^n} + \|f\|_{L^{n-1}(dz)}^{(n-1)/n}),\end{aligned}$$

$$\begin{aligned}\|\Delta f\|_{L^\infty(dz)} &\leqslant 2n^{1/n}\|\Delta(L+2tQ)(f+I)\|_{L^2(dz)}^{1/n}\|\Delta f\|_{L^{2(n-1)}(dz)}^{(n-1)/n}\\ &\quad + 2n^{1/n}|2t\Delta Q|^{1/n}\|f\|_{L^n(dz)}^{1/n}\|\Delta f\|_{L^n(dz)}^{(n-1)/n}\\ &\quad + 2n^{1/n}|2t\Delta Q|^{1/n}\|\Delta f\|_{L^{n-1}(dz)}^{(n-1)/n} + 2n^{1/n}|2tQ|^{1/n}\|\Delta f\|_{L^n(dz)}.\end{aligned}$$

Proof. Let $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$ denote the standard basis vectors in $\mathbf{C}^2$. Then for $1 \leqslant i, j \leqslant 2$,

$$\begin{aligned}(e_i^t f_{-\theta}(z) e_j)^n &= -n\int_z^\infty (e_i^t \partial_\zeta f_{-\theta} e_j)(e_i^t f_{-\theta} e_j)^{n-1}\, d\zeta\\ &= -n\int_z^\infty e_i^t((L+2tQ)(f+I))_{-\theta} e_j (e_i^t f_{-\theta} e_j)^{n-1}\, d\zeta\\ &\quad + n\int_z^\infty e_i^t(2tQ(f+I))_{-\theta} e_j (e_i^t f_{-\theta} e_j)^{n-1}\, d\zeta.\end{aligned}$$

It follows that

$$|e_i^t f(z)e_j|^n \leqslant n\|(L+2tQ)(f+I)\|_{L^2(dz)}\|f\|^{n-1}_{L^{2(n-1)}(dz)}+n|2tQ|(\|f\|^n_{L^n(dz)}+\|f\|^{n-1}_{L^{n-1}(dz)})$$

and

$$\begin{aligned}\|f\|_{L^\infty(dz)} \leqslant\ & 2n^{1/n}\|(L+2tQ)(f+I)\|^{1/n}_{L^2(dz)}\|f\|^{(n-1)/n}_{L^{2(n-1)}(dz)}\\ &+2n^{1/n}|2tQ|^{1/n}(\|f\|_{L^n(dz)}+\|f\|^{(n-1)/n}_{L^{n-1}(dz)}).\end{aligned}$$

Here we have used the fact that for a (2×2)-matrix A,

$$|A|=\bigg(\sum_{i,j=1}^{2}|A_{ij}|^2\bigg)^{1/2}\leqslant 2\max_{1\leqslant i,j\leqslant 2}|A_{ij}|.$$

Similarly,

$$\begin{aligned}(e_i^t\Delta f_{-\theta}(z)e_j)^n &= -n\int_z^\infty (e_i^t\partial_\zeta\Delta f_{-\theta}e_j)(e_i^t\Delta f_{-\theta}e_j)^{n-1}\,d\zeta\\ &= -n\int_z^\infty (e_i^t(\Delta(L+2tQ)(f+I))_{-\theta}e_j)(e_i^t\Delta f_{-\theta}e_j)^{n-1}\,d\zeta\\ &\quad +n\int_z^\infty e_i^t(2t\Delta Q(f+I))_{-\theta}e_j(e_i^t\Delta f_{-\theta}e_j)^{n-1}\,d\zeta.\end{aligned}$$

It follows from the equality $\Delta Q(f+I)=(\Delta Q)f+\Delta Q+Q\Delta f$ that

$$\begin{aligned}|e_i^t\Delta f(z)e_j|^n \leqslant\ & n\|\Delta(L+2tQ)(f+I)\|_{L^2(dz)}\|\Delta f\|^{n-1}_{L^{2(n-1)}}+n|2t\Delta Q|\,\|f\|_{L^n(dz)}\|\Delta f\|^{n-1}_{L^n(dz)}\\ &+n2t|\Delta Q|\,\|\Delta f\|^{n-1}_{L^{n-1}(dz)}+n2t|Q|\,\|\Delta f\|^n_{L^n(dz)}\end{aligned}$$

and

$$\begin{aligned}\|\Delta f\|_{L^\infty(dz)} \leqslant\ & 2n^{1/n}\|\Delta(L+2tQ)(f+I)\|^{1/n}_{L^2(dz)}\|\Delta f\|^{(n-1)/n}_{L^{2(n-1)}(dz)}\\ &+2n^{1/n}|2t\Delta Q|^{1/n}\|f\|^{1/n}_{L^n(dz)}\|\Delta f\|^{(n-1)/n}_{L^n(dz)}\\ &+2n^{1/n}|2t\Delta Q|^{1/n}\|\Delta f\|^{(n-1)/n}_{L^{n-1}(dz)}+2n^{1/n}|2tQ|^{1/n}\|\Delta f\|_{L^n(dz)}.\end{aligned}$$ □

Introduce the operator

$$\tilde{L}=ix\,\mathrm{ad}\,\sigma-2t\partial_x. \tag{5.13}$$

Note that $\tilde{L}$ is very close to the operator $L_{\mathrm{MSh}}=x-2it\partial_x$, considered by McKean and Shatah [MKS]. This operator commutes with $i\partial_t-\partial_x^2$ and plays a central role in their analysis of nonlinear Schrödinger flows. We may think of $\tilde{L}$ as a matrix version of L_{MSh}. As we now see, $\tilde{L}$ is also closely related to the operator L introduced above.

LEMMA 5.14.

$$(L+2tQ)\mu = (\partial_z - \tilde{L})\mu, \tag{5.15}$$

$$(L+2tQ)m_\pm = (\partial_z - \tilde{L})m_\pm. \tag{5.16}$$

Proof. Equations (5.15) and (5.16) follow directly from the fact that μ and $m_\pm$ are solutions of the equation $\partial_x m = iz\,\mathrm{ad}\,\sigma\, m + Qm$. □

LEMMA 5.17.

$$\|(L+2tQ)\mu^{\pm 1}\|_{L^2},\ \|(L+2tQ)m_\pm\|_{L^2},\ \|(L+2tQ)m_\pm^{-1}\|_{L^2} \leqslant \begin{bmatrix} 1 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)}, \tag{5.18}$$

$$\|\mu^{\pm 1} - I\|_{L^\infty(dz)},\ \|m_\pm - I\|_{L^\infty(dz)},\ \|m_\pm^{-1} - I\|_{L^\infty(dz)} \leqslant \begin{bmatrix} 1 & 2/n \\ -1/2n & 5/n \end{bmatrix} K_{2(n-1)}, \tag{5.19}$$

$$\|\mu^{\pm 1}\|_{L^\infty(dz)},\ \|m_\pm\|_{L^\infty(dz)},\ \|m_\pm^{-1}\|_{L^\infty(dz)} \leqslant \begin{bmatrix} 0 & 1+2/n \\ -1/2n & 5/n \end{bmatrix} K_{2(n-1)}. \tag{5.20}$$

Proof. It follows from the equation $\mu = I + C_w\mu$ and the commutation relation

$$\lozenge C^\pm - C^\pm \lozenge = -\frac{1}{2\pi i}\int \tag{5.21}$$

that

$$\begin{aligned} (L+2tQ)\mu &= 2tQ - 2tQ + C^+(L+2tQ)\mu w_\theta^- + C^-(L+2tQ)\mu w_\theta^+ \\ &= C_{w_\theta}(L+2tQ)\mu + C_{w'_\theta}\mu. \end{aligned} \tag{5.22}$$

Thus $(L+2tQ)\mu = (1-C_{w_\theta})^{-1}C_{w'_\theta}\mu$, and by Lemma 5.12,

$$\begin{aligned} \|(L+2tQ)\mu\|_{L^2} &\leqslant \frac{c}{1-\varrho}\|r'\|_{L^2}\|\mu\|_{L^\infty} \\ &\leqslant \frac{c\eta}{1-\varrho}\big(1 + 2n^{1/n}\|(L+2tQ)\mu\|_{L^2}^{1/n}\|\mu - I\|_{L^{2(n-1)}}^{(n-1)/n} \\ &\qquad + 2n^{1/n}|2tQ|^{1/n}(\|\mu - I\|_{L^n} + \|\mu - I\|_{L^{n-1}}^{(n-1)/n})\big). \end{aligned}$$

Since for $a, b, y \geqslant 0$, $y \leqslant a y^{1/n} + b$ implies $y \leqslant a^{n/(n-1)} + nb/(n-1)$, it follows then from

Lemma 5.1, (4.20) and the relation $Q=(1/2\pi)\,\mathrm{ad}\,\sigma\,\widetilde{\mathbf{Q}}$ that

$$\begin{aligned}
\|(L+2tQ)\mu\|_{L^2} &\leqslant \frac{c\eta^{n/(n-1)}}{(1-\varrho)^{n/(n-1)}}\|\mu-I\|_{L^{2(n-1)}} \\
&\quad+\frac{c\eta}{1-\varrho}(1+|2tQ|^{1/n}(\|\mu-I\|_{L^n}+\|\mu-I\|_{L^{n-1}}^{(n-1)/n})) \\
&\leqslant \begin{bmatrix} n/(n-1) & 0 \\ 0 & n/(n-1)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_{2(n-1)} \\
&\quad+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(1+\begin{bmatrix}1/n & 1/n\\ -1/2n & 5/n\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_n+\begin{bmatrix}(n-1)/n & 0\\ 0 & 0\end{bmatrix}K_{n-1}^{(n-1)/n}\right)\right) \\
&\leqslant \left(\begin{bmatrix}(2n-1)/(n-1) & 0\\ 0 & n/(n-1)\end{bmatrix}+\begin{bmatrix}1 & 1+2/n\\ -1/2n & 1+5/n\end{bmatrix}\right)K_{2(n-1)} \\
&\leqslant \begin{bmatrix}1 & 2\\ -1/2n & 3\end{bmatrix}K_{2(n-1)},
\end{aligned}$$

as $n\geqslant 3$. The estimate regarding μ^{-1} follows from the facts that

$$\mu^{-1}=\begin{pmatrix}\mu_{22} & -\mu_{12}\\ -\mu_{21} & \mu_{11}\end{pmatrix}$$

and that $\partial_z-\tilde{L}$ is an entrywise operation, i.e. the entries of $(\partial_z-\tilde{L})\mu^{-1}$ are the same as the entries of $(\partial_z-\tilde{L})\mu$ apart from some signs and a rearrangement. Now by Lemmas 5.12, 5.1 and 4.74,

$$\begin{aligned}
\|\mu-I\|_{L^\infty} &\leqslant 2n^{1/n}\|(L+2tQ)\mu\|_{L^2}^{1/n}\|\mu-I\|_{L^{2(n-1)}}^{(n-1)/n} \\
&\quad+2n^{1/n}|2tQ|^{1/n}(\|\mu-I\|_{L^n}+\|\mu-I\|_{L^{n-1}}^{(n-1)/n}) \\
&\leqslant \begin{bmatrix}1/n & 2/n\\ -1/2n^2 & 3/n\end{bmatrix}\begin{bmatrix}(n-1)/n & 0\\ 0 & 0\end{bmatrix}K_{2(n-1)}^{(n-1)/n} \\
&\quad+\begin{bmatrix}1/n & 1/n\\ -1/2n & 5/n\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_n+\begin{bmatrix}(n-1)/n & 0\\ 0 & 0\end{bmatrix}K_{n-1}^{(n-1)/n}\right) \\
&\leqslant \begin{bmatrix}1 & 2/n\\ -1/2n^2 & 3/n\end{bmatrix}K_{2(n-1)}+\begin{bmatrix}1 & 2/n\\ -1/2n & 5/n\end{bmatrix}K_n\leqslant\begin{bmatrix}1 & 2/n\\ -1/2n & 5/n\end{bmatrix}K_{2(n-1)}.
\end{aligned}$$

The estimate regarding μ^{-1} is the same. We have proved the first part of (5.18) and the first part of (5.19). The second part of (5.19) follows from the first part using the relation $m_\pm=\mu v_\theta^\pm$, and then the third part of (5.19) follows as before using $\det m_\pm=1$.

Since $\partial_z - \tilde{L}$ is a derivation, by Lemma 5.14 and the fact that $(\partial_z - \tilde{L})f_\theta = f'_\theta$, we obtain $(L+2tQ)m_\pm = (\partial_z - \tilde{L})\mu v_\theta^\pm = ((\partial_z - \tilde{L})\mu)v_\theta^\pm + \mu(v^\pm)'_\theta$. Thus

$$\begin{aligned}\|(L+2tQ)m_\pm\|_{L^2(dz)} &\leqslant c\|(\partial_z - \tilde{L})\mu\|_{L^2(dz)}\|r\|_{L^\infty(dz)} + \|\mu\|_{L^\infty(dz)}\|r'\|_{L^2(dz)} \\ &\leqslant \begin{bmatrix} 1 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)} + \begin{bmatrix} 0 & 1+2/n \\ -1/2n & 5/n \end{bmatrix} K_{2(n-1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\leqslant \begin{bmatrix} 1 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)}.\end{aligned}$$

As $\partial_z - \tilde{L}$ is an entrywise operation, the estimate regarding $m_\pm^{-1}$ is again the same. This proves the second and third parts of (5.18). Finally, (5.20) follows trivially from (5.19). □

LEMMA 5.23. *For $n \geqslant 3$ and $2(n-1) \geqslant p > 2$, we have*

$$\begin{aligned}\|\Delta(L+2tQ)\mu\|_{L^2(dz)} &\leqslant \begin{bmatrix} 0 & 2+3/n \\ 1/2pn - 3/4n & 1+7/n \end{bmatrix} K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 0 & 3 \\ -1/4 & 10/3 \end{bmatrix} K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}}\end{aligned}$$

and

$$\begin{aligned}\|\Delta\mu\|_{L^\infty(dz)} &\leqslant \begin{bmatrix} 0 & 1+3/n \\ 1/2pn - 3/4n & 7/n \end{bmatrix} K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 0 & 2 \\ -1/4 & 7/3 \end{bmatrix} K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Remark. The condition $2(n-1) \geqslant p$ is only for convenience in order to write the final estimates in the lemma in a compact form.

Proof. Using $(L+2tQ)\mu = (1-C_{w_\theta})^{-1} C_{w'_\theta}\mu$, we compute

$$\begin{aligned}\Delta(L+2tQ)\mu &= \Delta(1-C_{w_\theta})^{-1} C_{w'_\theta}\mu \\ &= (\Delta(1-C_{w_\theta})^{-1}) C_{w'_\theta}\mu + (1-C_{w_\theta})^{-1} C_{\Delta w'_\theta}\mu + (1-C_{w_\theta})^{-1} C_{w'_\theta}\Delta\mu.\end{aligned}$$

Thus by Lemmas 5.17 and 5.12,

$$\begin{aligned}\|\Delta(L+2tQ)\mu\|_{L^2} \leqslant{}& \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \|\Delta r\|_{H^{1,1}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \|\mu\|_{L^\infty} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \|\Delta r\|_{H^{1,1}} \|\mu\|_{L^\infty} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \|\Delta\mu\|_{L^\infty}\end{aligned}$$

$$\begin{aligned}
&\leqslant \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\|\Delta r\|_{H^{1,1}}\begin{bmatrix}0 & 1+2/n\\ -1/2n & 5/n\end{bmatrix}K_{2(n-1)}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\|\Delta\mu\|_{L^\infty(dz)}\\
&\leqslant \begin{bmatrix}0 & 2+2/n\\ -1/2n & 2+5/n\end{bmatrix}K_{2(n-1)}\|\Delta r\|_{H^{1,1}}\\
&\quad+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\big(\|\Delta(L+2tQ)\mu\|_{L^2}^{1/n}\|\Delta\mu\|_{L^{2(n-1)}}^{(n-1)/n}\\
&\quad+|2tQ|^{1/n}\|\mu-I\|_{L^n}^{1/n}\|\Delta\mu\|^{(n-1)/n}\\
&\quad+|2t\Delta Q|^{1/n}\|\mu-I\|_{L^n}^{1/n}\|\Delta\mu\|_{L^n}^{(n-1)/n}+|2t\Delta Q|^{1/n}\|\Delta\mu\|_{L^n}\big).
\end{aligned}$$

Therefore, as in the proof of (5.18),

$$\begin{aligned}
\|\Delta(L+2tQ)\mu\|_{L^2}&\leqslant \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^{n/(n-1)}\|\Delta\mu\|_{L^{2(n-1)}}+\|\Delta r\|_{H^{1,1}}\begin{bmatrix}0 & 2+2/n\\ -1/2n & 2+5/n\end{bmatrix}K_{2(n-1)}\\
&\quad+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}|2t\Delta Q|^{1/n}\|\mu-I\|_{L^n}^{1/n}\|\Delta\mu\|_{L^n}^{(n-1)/n}\\
&\quad+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}|2t\Delta Q|^{1/n}(\|\Delta\mu\|_{L^{n-1}}^{(n-1)/n}+\|\Delta\mu\|_{L^n})\\
&\leqslant \begin{bmatrix}n/(n-1) & 0\\ 0 & n/(n-1)\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}K_{2(n-1)}^2\|\Delta r\|_{H^{1,1}}\\
&\quad+\begin{bmatrix}0 & 2+2/n\\ -1/2n & 2+5/n\end{bmatrix}K_{2(n-1)}\|\Delta r\|_{H^{1,1}}\\
&\quad+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 3/n\\ 1/2pn-3/4n & 7/n\end{bmatrix}K_p^{1/n}\begin{bmatrix}1/n & 0\\ 0 & 0\end{bmatrix}K_n^{1/n}\\
&\quad\times\begin{bmatrix}0 & (n-1)/n\\ 0 & 0\end{bmatrix}K_n^{(2n-2)/n}\|\Delta r\|_{H^{1,1}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
&\quad\times\begin{bmatrix}0 & 3/n\\ 1/2pn-3/4n & 7/n\end{bmatrix}K_p^{1/n}\begin{bmatrix}0 & (n-1)/n\\ 0 & 0\end{bmatrix}K_{n-1}^{(2n-2)/n}\|\Delta r\|_{H^{1,1}}\\
&\quad+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1/n & 1/n\\ -1/2n & 5/n\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}K_n^2\|\Delta r\|_{H^{1,1}}\\
&\leqslant \begin{bmatrix}0 & 2+3/n\\ 1/2pn-3/4n & 1+7/n\end{bmatrix}K_{2(n-1)}^2\|\Delta r\|_{H^{1,1}},
\end{aligned}$$

as $n\geqslant 3$, $2(n-1)\geqslant p>2$. The second inequality follows by Theorem 4.16 and Lemma 5.1.

Hence

$$\begin{aligned}
\|\Delta\mu\|_{L^\infty} &\leqslant c\big(\|\Delta(L+2tQ)\mu\|_{L^2}^{1/n}\|\Delta\mu\|_{L^{2(n-1)}}^{(n-1)/n}+|2tQ|^{1/n}\|\Delta\mu\|_{L^n}\\
&\qquad+|2t\Delta Q|^{1/n}\|\Delta\mu\|_{L^{n-1}}^{(n-1)/n}+|2t\Delta Q|^{1/n}\|\mu-I\|_{L^n}^{1/n}\|\Delta\mu\|_{L^n}^{(n-1)/n}\big)\\
&\leqslant \begin{bmatrix}0 & 3/n\\ -1/4n & 10/3n\end{bmatrix} K_{2(n-1)}^{2/n}\|\Delta r\|_{H^{1,1}}^{1/n}\begin{bmatrix}0 & (n-1)/n\\ 0 & 0\end{bmatrix} K_{2(n-1)}^{2(n-1)/n}\|\Delta r\|_{H^{1,1}}^{(n-1)/n}\\
&\qquad+\begin{bmatrix}1 & 1\\ -1/2 & 5\end{bmatrix}^{1/n}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} K_n^2\|\Delta r\|_{H^{1,1}}\\
&\qquad+\begin{bmatrix}0 & 3\\ 1/2p-3/4 & 7\end{bmatrix}^{1/n} K_p^{1/n}\|\Delta r\|_{H^{1,1}}^{1/n}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}^{(n-1)/n} K_{n-1}^{(2n-2)/n}\|\Delta r\|_{H^{1,1}}^{(n-1)/n}\\
&\qquad+\begin{bmatrix}0 & 3\\ 1/2p-3/4 & 7\end{bmatrix}^{1/n} K_p^{1/n}\|\Delta r\|_{H^{1,1}}^{1/n}\\
&\qquad\times\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}^{1/n} K_n^{1/n}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}^{(n-1)/n} K_n^{(2n-2)/n}\|\Delta r\|_{H^{1,1}}^{(n-1)/n}\\
&\leqslant \begin{bmatrix}0 & 1+3/n\\ 1/2pn-3/4n & 7/n\end{bmatrix} K_{2(n-1)}^2\|\Delta r\|_{H^{1,1}}\\
&\leqslant \begin{bmatrix}0 & 2\\ -1/4 & 7/3\end{bmatrix} K_{2(n-1)}^2\|\Delta r\|_{H^{1,1}}.
\end{aligned}$$

□

LEMMA 5.24. *For $q\in H^{1,1}$, and hence for $r\in H_1^{1,1}$,*

$$\|q\|_{L^2}^2=-\frac{1}{2\pi}\int\log(1-|r|^2)\leqslant\frac{1}{2\pi}\frac{\|r\|_{L^2}^2}{1-\varrho},$$

$$\|q\|_{L^2}\leqslant\frac{1}{\sqrt{2\pi}}\frac{\|r\|_{L^2}}{(1-\varrho)^{1/2}}\leqslant\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix},$$

$$\|\partial_x q\|_{L^2}\leqslant\frac{1}{\sqrt{2\pi}}\frac{\|\Diamond r\|_{L^2}}{(1-\varrho)^{1/2}}\leqslant\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}.$$

Proof. The inequalities follow from the identities

$$\int|q|^2\,dx=-\frac{1}{2\pi}\int\log(1-|r|^2)\,dz, \tag{5.25}$$

$$\int(|\partial_x q|^2+|q|^4)\,dx=-\frac{1}{2\pi}\int z^2\log(1-|r|^2)\,dz, \tag{5.26}$$

for the basic NLS-conserved quantities, probability and energy, respectively (see (2.32), (2.33) with $V(y)=y^2$). These identities are proved by expanding $\log a(z)$ in powers of $1/z$ and using (2.10c), as in [FaT], for example. □

For the perturbed NLS equation (1.1), we must set $\Lambda(s)=2(l+2)^{-1}s^{(l+2)/2}$ in (2.35), and we obtain

$$G(q)=-i|q|^l\begin{pmatrix}0 & q\\ -\bar{q} & 0\end{pmatrix}.$$

LEMMA 5.27.

$$\|G\|_{L^1},\ \|\partial_x G\|_{L^1}\leqslant\begin{bmatrix}l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}.$$

Proof. By Lemma 5.24 and (4.20),

$$\|G\|_{L^1}\leqslant c\|q\|_{L^2}^2\|q\|_{L^\infty}^{l-1}\leqslant\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}l-1 & l-1\\ (l-1)/2 & 5l-5\end{bmatrix}=\begin{bmatrix}l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix},$$

$$\begin{aligned}\|\partial_x G\|_{L^1}&\leqslant c\|q\|_{L^2}\|\partial_x q\|_{L^2}\|q\|_{L^\infty}^{l-1}\\ &\leqslant\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}l-1 & l-1\\ (l-1)/2 & 5l-5\end{bmatrix}=\begin{bmatrix}l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}.\end{aligned}$$ □

We compute, using Lemma 5.14,

$$\begin{aligned}\tilde{L}\mathbf{Q}&=\int((\tilde{L}\mu)w_\theta+\mu\tilde{L}w_\theta)=\int((\tilde{L}\mu)w_\theta+\mu(\partial_z e^{i\theta\,\mathrm{ad}\,\sigma})w)=\int(((-\partial_z+\tilde{L})\mu)w_\theta-\mu w_\theta')\\ &=-\int(((L+2tQ)\mu)w_\theta-(\mu-I)w_\theta')+\int w_\theta'\\ &\equiv L_Q+L_Q'.\end{aligned}$$

By Lemmas 5.17 and 5.1,

$$\begin{aligned}\|L_Q\|_{L^\infty(dx)}&\leqslant\|(L+2tQ)\mu)\|_{L^2(dz)}\|r\|_{L^2(dz)}+\|\mu-I\|_{L^2(dz)}\|r'\|_{L^2(dz)}\\ &\leqslant\begin{bmatrix}2 & 2\\ -1/2n & 3\end{bmatrix}K_{2(n-1)}+\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\leqslant\begin{bmatrix}2 & 2\\ -1/2n & 3\end{bmatrix}K_{2(n-1)}.\end{aligned}$$

Together with the fact that w, and hence w', is off-diagonal, this proves the following result.

LEMMA 5.28.

$$\|L_Q\|_{L^\infty(dx)}\leqslant\begin{bmatrix}2 & 2\\ -1/2n & 3\end{bmatrix}K_{2(n-1)}\quad\textit{and}\quad\|L_Q'\|_{L^2(dx)}\leqslant c\eta\leqslant\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

LEMMA 5.29.

$$\|\tilde{L}G\|_{L^2(dx)} \leqslant \begin{bmatrix} l+1 & l+2 \\ (l-1)/2-1/2n & 5l-1 \end{bmatrix} K_{2(n-1)}, \tag{5.30}$$

$$\|\tilde{L}G\|_{L^1(dx)} \leqslant \begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2 \end{bmatrix} K_{2(n-1)}, \tag{5.31}$$

$$\|\tilde{L}G\|_{L^p(dx)} \leqslant \begin{bmatrix} l+1 & l+3-2/p \\ l/2-1/p-1/2n & 5l+7/2-9/p \end{bmatrix} K_{2(n-1)}, \quad 1\leqslant p\leqslant 2. \tag{5.32}$$

Proof. Estimates (5.30) and (5.31) are just special cases of (5.32), while (5.32) follows from (5.30), (5.31) and the interpolation inequality

$$\|f\|_{L^p} \leqslant \|f\|_{L^2}^{2-2/p} \|f\|_{L^1}^{2/p-1}. \tag{5.33}$$

Using the identity $-2t\partial_x|q|^l = \frac{1}{2}l|q|^{l-2}((ixq-2t\partial_x q)\bar{q}+\overline{(ixq-2t\partial_x q)}q)$, we obtain

$$\tilde{L}G = -i(ix \operatorname{ad}\sigma - 2t\partial_x)|q|^l \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{pmatrix}, \tag{5.34}$$

where $\beta = |q|^l(ixq-2t\partial_x q)+l|q|^{l-2}q\operatorname{Re}(\bar{q}(ixq-2t\partial_x q))$. Thus by Lemmas 5.28, 5.24, and (4.20), together with the fact that the off-diagonal part of $\tilde{L}\mathbf{Q}$ has the form

$$\begin{pmatrix} 0 & (ix-2t\partial_x)q \\ -\overline{(ix-2t\partial_x)q} & 0 \end{pmatrix},$$

we obtain

$$\begin{aligned}
\|\tilde{L}G\|_{L^2(dx)} &\leqslant c\| |q|^l(ix-2t\partial_x)q + l|q|^{l-2}q\operatorname{Re}(\bar{q}(ix-2t\partial_x)q)\|_{L^2(dx)} \\
&\leqslant c\|q\|_{L^2}\|q\|_{L^\infty}^{l-1}\|L_Q\|_{L^\infty} + c\|q\|_{L^\infty(dx)}^l \|L_Q'\|_{L^2(dx)} \\
&\leqslant \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} l-1 & l-1 \\ (l-1)/2 & 5l-5 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)} + \begin{bmatrix} l & l \\ l/2 & 5l \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} l+2 & l+1 \\ (l-1)/2-1/2n & 5l-3/2 \end{bmatrix} K_{2(n-1)} + \begin{bmatrix} l+1 & l \\ l/2 & 5l \end{bmatrix} \\
&\leqslant \begin{bmatrix} l+1 & l+2 \\ (l-1)/2-1/2n & 5l-1 \end{bmatrix} K_{2(n-1)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\|\tilde{L}G\|_{L^1(dx)} &\leqslant c\| |q|^l(ix-2t\partial_x)q + l|q|^{l-2}q\,\mathrm{Re}(\bar{q}(ix-2t\partial_x)q)\|_{L^1(dx)} \\ &\leqslant c\|q\|_{L^2}^2\|q\|_{L^\infty}^{l-2}\|L_Q\|_{L^\infty} + c\|q\|_{L^2(dx)}\|q\|_{L^\infty(dx)}^{l-1}\|L'_Q\|_{L^2(dx)} \\ &\leqslant \begin{bmatrix} 2 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} l-2 & l-2 \\ (l-2)/2 & 5l-10\end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1/2n & 3\end{bmatrix}K_{2(n-1)} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1/2\end{bmatrix}\begin{bmatrix} l-1 & l-1 \\ (l-1)/2 & 5l-5\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} \\ &= \begin{bmatrix} l+2 & l \\ (l-2)/2-1/2n & 5l-6\end{bmatrix}K_{2(n-1)} + \begin{bmatrix} l+1 & l-1 \\ (l-1)/2 & 5l-q/2\end{bmatrix} \\ &\leqslant \begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}.\end{aligned}$$

Here we have used twice the fact that $(1-\varrho)^{-1}\leqslant cK_{2n(n-1)}$. □

LEMMA 5.35. *Let $n\geqslant 3$ and $2<p\leqslant 2(n-1)$. Then for L_Q and L'_Q as in Lemma* 5.28, $\|\Delta L'_Q\|_{L^2(dx)}\leqslant c\|\Delta r\|_{H^{1,1}}$ *and*

$$\begin{aligned}|\Delta L_Q| &\leqslant \begin{bmatrix} 1 & 2+3/n \\ 1/2pn-3/4n & \max(1+7/n,2)\end{bmatrix}K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 1 & 3 \\ -1/4 & 10/3\end{bmatrix}K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Proof. The first part is trivial. For the second part, we compute

$$\begin{aligned}|\Delta\tilde{L}_Q| &= \left|\int[\Delta(L+2tQ)\mu)]w_\theta + \int(L+2tQ)\mu\Delta w_\theta - \int(\Delta\mu)w'_\theta - \int(\mu-I)\Delta w'_\theta\right| \\ &\leqslant \|\Delta(L+2tQ)\mu\|_{L^2(dz)}\|r\|_{L^2(dz)} + \|(L+2tQ)\mu\|_{L^2(dz)}\|\Delta r\|_{L^2(dz)} \\ &\quad + \|\Delta\mu\|_{L^2(dz)}\|r'\|_{L^2(dz)} + \|\mu-I\|_{L^2(dz)}\|\Delta r'\|_{L^2(dz)} \\ &\leqslant \begin{bmatrix} 0 & 2+3/n \\ 1/2pn-3/4n & 1+7/n\end{bmatrix}K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 2 \\ -1/2n & 3\end{bmatrix}K_{2(n-1)}\|\Delta r\|_{H^{1,1}} \\ &\quad + \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}K_2^2\|\Delta r\|_{H^{1,1}}\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}K_2\|\Delta r\|_{H^{1,1}}\end{aligned}$$

(by Lemmas 5.23, 5.17 and 5.1)

$$\leqslant \left(\begin{bmatrix} 1 & 2+3/n \\ 1/2pn-3/4n & 1+7/n\end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1/2n & 2\end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0\end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}\right)K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}$$

(as $K_{2(n-1)}>K_2=c/(1-\varrho)$)

$$\leqslant \begin{bmatrix} 1 & 2+3/n \\ 1/2pn-3/4n & \max(1+7/n,2) \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}$$

$$\leqslant \begin{bmatrix} 1 & 3 \\ -1/4 & 10/3 \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}. \qquad \square$$

LEMMA 5.36. *For L_Q and L'_Q as in Lemma* 5.28, $2<p<\infty$, $n\geqslant 3$, $2\geqslant p'\geqslant 1$,

$$\|\Delta\tilde{L}G\|_{L^2(dx)} \leqslant \begin{bmatrix} l & l+4 \\ l/2+1/2p-1/2n-3/4 & 5l+1 \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}},$$

$$\|\Delta\tilde{L}G\|_{L^1(dx)} \leqslant \begin{bmatrix} l & l+3 \\ l/2+1/2p-1/2n-5/4 & 5l-7/2 \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}$$

and

$$\|\Delta\tilde{L}_G\|_{L^{p'}(dx)} \leqslant \begin{bmatrix} l & l+5-2/p' \\ l/2+1/2p-1/2n-1/4-1/p' & 5l+11/2-9/p' \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}.$$

Proof. Let $2<p\leqslant 2(n-1)$. Using Lemmas 5.28, 5.35, 5.24, Theorem 4.16, the form (5.34) of $\tilde{L}G$, and the form of the off-diagonal part of $\tilde{L}\mathbf{Q}$, we compute

$$\begin{aligned}
\|\Delta\tilde{L}G\|_{L^2(dx)} &\leqslant c(\|\Delta(|q|^l)\tilde{L}\mathbf{Q}\|_{L^2(dx)}+\|\Delta(|q|^{l-2}q^2)\tilde{L}\mathbf{Q}\|_{L^2(dx)}+\|\,|q|^l\Delta\tilde{L}\mathbf{Q}\|_{L^2(dx)})\\
&\leqslant c\|q\|^{l-2}_{L^\infty(dx)}\|q\|_{L^2(dx)}\|\Delta q\|_{L^\infty(dx)}\|L_Q\|_{L^\infty(dx)}\\
&\quad +c\|q\|^{l-1}_{L^\infty(dx)}\|\Delta q\|_{L^\infty(dx)}\|L'_Q\|_{L^2(dx)}\\
&\quad +c\|q\|^{l-1}_{L^\infty(dx)}\|q\|_{L^2(dx)}\|\Delta L_Q\|_{L^\infty(dx)}+c\|q\|^{l}_{L^\infty(dx)}\|\Delta L'_Q\|_{L^2(dx)}\\
&\leqslant \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-2}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\begin{bmatrix} 2 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)}\\
&\quad +\begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-1}\begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\\
&\quad +\begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-1}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 2+3/n \\ 1/2pn-3/4n & \max(1+7/n,2) \end{bmatrix}\\
&\qquad \times K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}\\
&\quad +\begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l}\|\Delta r\|_{H^{1,1}}
\end{aligned}$$

$$\leqslant \left(\begin{bmatrix} l+1 & l+3 \\ l/2+1/2p-1/2n-3/4 & 5l+1/2 \end{bmatrix} + \begin{bmatrix} l & l+2 \\ l/2+1/2p-1/4 & 5l+1 \end{bmatrix} \right.$$

$$+ \begin{bmatrix} l & l+1+3/n \\ l/2+1/2pn-3/4n-1/2 & \max(5l+7/n-7/2, 5l-5/2) \end{bmatrix}$$

$$\left. + \begin{bmatrix} l & l \\ l/2 & 5l \end{bmatrix} \right) K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}} \quad (\text{as } p \leqslant 2(n-1))$$

$$\leqslant \begin{bmatrix} l & l+4 \\ l/2+1/2p-1/2n-3/4 & 5l+1 \end{bmatrix} K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}}.$$

Similarly,

$$\begin{aligned}
\|\Delta \tilde{L} G\|_{L^1(dx)} &\leqslant c(\|\Delta(|q|^l)\tilde{L}\mathbf{Q}\|_{L^1(dx)} + \|\Delta(|q|^{l-2}q^2)\tilde{L}\mathbf{Q}\|_{L^1(dx)} + \| |q|^l \Delta \tilde{L}\mathbf{Q}\|_{L^1(dx)} \\
&\leqslant c\|q\|_{L^\infty(dx)}^{l-3} \|q\|_{L^2(dx)}^2 \|\Delta q\|_{L^\infty(dx)} \|L_Q\|_{L^\infty(dx)} \\
&\quad + c\|q\|_{L^\infty(dx)}^{l-2} \|q\|_{L^2(dx)} \|\Delta q\|_{L^\infty(dx)} \|L_Q'\|_{L^2(dx)} \\
&\quad + c\|q\|_{L^\infty(dx)}^{l-2} \|q\|_{L^2(dx)}^2 \|\Delta L_Q\|_{L^\infty(dx)} \\
&\quad + c\|q\|_{L^\infty(dx)}^{l-1} \|q\|_{L^2(dx)} \|\Delta L_Q'\|_{L^2(dx)} \\
&\leqslant \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-3} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^2 \begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} \\
&\qquad \times K_p \|\Delta r\|_{H^{1,1}} \begin{bmatrix} 2 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)} \\
&\quad + \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-2} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p \|\Delta r\|_{H^{1,1}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-2} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^2 \begin{bmatrix} 0 & 2+3/n \\ 1/2pn-3/4n & \max(1+7/n, 2) \end{bmatrix} \\
&\qquad \times K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}} \\
&\quad + \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}^{l-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \|\Delta r\|_{H^{1,1}} \\
&\leqslant \left(\begin{bmatrix} l+1 & l+2 \\ l/2+1/2p-1/2n-5/4 & 5l-4 \end{bmatrix} + \begin{bmatrix} l & l+1 \\ l/2+1/2p-3/4 & 5l-7/2 \end{bmatrix} \right. \\
&\quad + \begin{bmatrix} l & l-3/n \\ l/2+1/2pn-3/4n-1 & \max(5l+7/n-8, 5l-7) \end{bmatrix} \\
&\quad \left. + \begin{bmatrix} l & l-1 \\ l/2-1/2 & 5l-9/2 \end{bmatrix} \right) K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}}
\end{aligned}$$

$$\leqslant \begin{bmatrix} l & l+3 \\ l/2+1/2p-1/2n-5/4 & 5l-7/2 \end{bmatrix} K_{2(n-1)}^2 \|\Delta r\|_{H^{1,1}}.$$

Finally the $L_{p'}$-results follows as before from (5.33). □

LEMMA 5.37. *For any* $2<p<\infty$, $\Delta\partial_x Q = \mathrm{I}+\mathrm{II}$, *where*

$$\|\mathrm{I}\|_{L^\infty(dx)} \leqslant \begin{bmatrix} 1 & 4 \\ 1/2p & 8 \end{bmatrix} K_p \|\Delta r\|_{H^{1,1}}, \quad \|\mathrm{II}\|_{L^2(dx)} \leqslant \frac{c}{(1-\varrho)^2} \|\Delta r\|_{H^{1,1}}.$$

Proof. For simplicity, we write $w = w_\theta$, $f' = \partial_x f$, and therefore $w' = iz\,\mathrm{ad}\,\sigma\, w$. Noticing that $[\sigma,\mu]w$ is diagonal, we compute

$$\begin{aligned} \Delta\partial_x Q &= \Delta\frac{\mathrm{ad}\,\sigma}{2\pi}\int(\mu' w+\mu w') = \frac{\mathrm{ad}\,\sigma}{2\pi}\int((\Delta\mu')w+\mu'\Delta w+\Delta(\mu w')) \\ &= \frac{\mathrm{ad}\,\sigma}{2\pi}\int((\Delta(iz[\sigma,\mu]+Q\mu))w+(iz[\sigma,\mu]+Q\mu)\Delta w+(\mu w')) \\ &= \frac{\mathrm{ad}\,\sigma}{2\pi}\int((\Delta Q)\mu w+Q(\Delta\mu)w+Q\mu\Delta w)+\frac{\mathrm{ad}\,\sigma}{2\pi}\int\Delta(\mu w') \equiv \mathrm{I}_1+\mathrm{I}_2. \end{aligned}$$

For any $p>2$,

$$\begin{aligned} |\mathrm{I}_1| &\leqslant c\|\Delta Q\|_{L^\infty(dx)}(\|\mu-I\|_{L^2(dz)}\|r\|_{L^2(dz)}+\|r\|_{L^1(dz)}) \\ &\quad +\|Q\|_{L^\infty(dx)}\|\Delta\mu\|_{L^2(dz)}\|r\|_{L^2(dz)} \\ &\quad +\|Q\|_{L^\infty(dx)}(\|\mu-I\|_{L^2(dz)}\|\Delta r\|_{L^2(dz)}+\|\Delta r\|_{L^1(dz)}) \\ &\leqslant \begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda+\eta\right)+\begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\|\Delta r\|_{H^{1,1}}\lambda \\ &\quad +\begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\|\Delta r\|_{H^{1,1}}+\|\Delta r\|_{H^{1,1}}\right) \end{aligned}$$

(by Theorem 4.16 and Lemma 5.1)

$$\begin{aligned} &\leqslant \left(\begin{bmatrix} 1 & 4 \\ 1/2p+1/4 & 8 \end{bmatrix}+\begin{bmatrix} 2 & 2 \\ 1/2 & 7 \end{bmatrix}+\begin{bmatrix} 1 & 2 \\ 1/2 & 6 \end{bmatrix}\right)K_p\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 1 & 4 \\ 1/2p+1/4 & 8 \end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}. \end{aligned}$$

A simple computation shows that $\mathrm{ad}\,\sigma\int_{\mathbf{R}}\mu z\,\mathrm{ad}\,\sigma(w_\theta)\,dz = 2\sigma\,\mathrm{ad}\,\sigma\int\mu z w_\theta\,dz$, and hence by the analog of (4.73) for general z_0,

$$\mathrm{I}_2 = i\frac{\sigma}{\pi}\,\mathrm{ad}\,\sigma\int_{\tilde{\mathbf{R}}_{z_0}} z\Delta(\tilde{\mu}\tilde{w}_\theta)\,dz+\frac{\Delta}{2\pi}\left(Q\int_{-\infty}^{z_0}\log(1-|r(z)|^2)\,dz\right).$$

Set $W=\int_{\tilde{\mathbf{R}}_{z_0}} z\tilde{\mu}\tilde{w}_\theta\, dz$. Then

$$\Delta W=\int_{\tilde{\mathbf{R}}_{z_0}} z(\Delta\tilde{\mu})\tilde{w}_\theta\, dz+\int_{\tilde{\mathbf{R}}_{z_0}} z(\tilde{\mu}-I)\Delta\tilde{w}_\theta\, dz+\int_{\tilde{\mathbf{R}}_{z_0}} z\Delta\tilde{w}_\theta\, dz=W_1+W_2+W_3.$$

By Theorem 4.16, for any $p>2$,

$$|W_1|\leqslant\begin{bmatrix}0 & 2\\ 1/2p & 5\end{bmatrix}K_p\|\Delta r\|_{H^{1,0}}\eta \quad\text{and}\quad |W_2|\leqslant\begin{bmatrix}1 & 0\\ 1/4 & 3\end{bmatrix}\|\Delta r\|_{H^{1,1}}.$$

Reversing the orientation on $\mathbf{R}_-+z_0$, we see from (4.9), (4.10) that

$$W_3=\Delta\int_{\mathbf{R}} z\begin{pmatrix}0 & r^{\#}\delta_+^2e^{i\theta}\\ -\overline{r^{\#}}\delta_-^{-2}e^{-i\theta} & 0\end{pmatrix}dz,$$

where $r^{\#}(z)=r(z)$ for $z>z_0$, $r^{\#}(z)=r(z)/(1-|r(z)|^2)$ for $z<z_0$. Of course, if $\delta_\pm$ were independent of x, then we would obtain immediately a bound for $\|W_3\|_{L^2(dx)}$ in terms of the $H^{0,1}$-norm of Δr. But $\delta_\pm=\delta_\pm(z,z_0)$ depend on x through z_0, and this complicates the estimation of W_3.

We proceed as follows. Consider first

$$\begin{aligned}\int_{\mathbf{R}} z\Delta(r^{\#}\delta_+^2)e^{i\theta}\, dz&=\int z(\Delta r^{\#})e^{i\theta}\, dz+\int z(\Delta r^{\#})(\delta_+^2-1)e^{i\theta}\, dz+\int zr^{\#}(\Delta\delta_+^2)e^{i\theta}\, dz\\ &\equiv \mathrm{I}'+\mathrm{II}'+\mathrm{III}'.\end{aligned}$$

Clearly

$$\|\mathrm{I}'\|_{L^2(dz)}\leqslant c\|\Diamond e^{-it\Diamond^2}\Delta r^{\#}\|_{L^2(dz)}\leqslant c\|\Delta r^{\#}\|_{H^{0,1}}\leqslant\frac{c}{(1-\varrho)^2}\|\Delta r\|_{H^{0,1}}\leqslant\frac{c}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}.$$

Using the identity $xz-tz^2=-t((z-z_0)^2-z_0^2)$, we have

$$\begin{aligned}\mathrm{II}'&=e^{itz_0^2}\left(\int_{z_0}^{\infty} z\Delta r(\delta^2-1)e^{-it(z-z_0)^2}dz+\int_{-\infty}^{z_0} z\Delta r^{\#}(\delta_+^2-1)e^{-it(z-z_0)^2}dz\right)\\ &\equiv e^{itz_0^2}(\mathrm{II}''+\mathrm{III}'').\end{aligned}$$

Now $\delta_\pm^2$ solve the normalized RHP $(\mathbf{R}_-+z_0,(1-|r|^2)^2)$, and hence by the methods of §2,

$$\delta^2(z)=1+\int_{-\infty}^{z_0}\frac{\delta_-^2(s)((1-|r(s)|^2)^2-1)}{s-z}\frac{ds}{2\pi i}$$

for $z\in\mathbf{C}\setminus(-\infty,z_0]$. Thus

$$z(\delta^2(z)-1)=-\int_{-\infty}^{z_0}\delta_-^2(s)((1-|r(s)|^2)^2-1)\frac{ds}{2\pi i}+\int_{-\infty}^{z_0}\delta_-^2(s)\frac{s((1-|r(s)|^2)^2-1)}{s-z}\frac{ds}{2\pi i},$$

and inserting this relation into II″ we obtain

$$\mathrm{II}' = -\left(\int_0^\infty \Delta r(u+z_0)e^{-itu^2}du\right)\left(\int_{-\infty}^{z_0} \delta_-^2((1-|r|^2)^2-1)\,\frac{dz}{2\pi i}\right) + \int_{-\infty}^{0} \delta_-^2(\gamma+z_0)[(\gamma+z_0)((1-|r(\gamma+z_0)|^2)^2-1)]\left[\int_0^\infty e^{-itu^2}\frac{\Delta r(u+z_0)}{\gamma-u}\,du\right]\frac{d\gamma}{2\pi i}.$$

By Proposition 2.55,

$$\left|\int_{-\infty}^{z_0} \delta_-^2((1-|r|^2)^2-1)\,\frac{dz}{2\pi i}\right| \leqslant \frac{c\lambda^2}{1-\varrho}. \tag{5.38}$$

Set

$$G(u) = \int_{e^{-i\pi/4}\infty}^{u} e^{-it\alpha^2}d\alpha, \quad u\geqslant 0, \tag{5.39}$$

where the integration takes place along the contour $\alpha = u+e^{-i\pi/4}\beta$, $\beta\geqslant 0$. Note that (i) $G'(u)=e^{-itu^2}$ and (ii) $G(u)=-e^{-i\pi/4}e^{-itu^2}\int_0^\infty e^{-\beta^2t-2e^{i\pi/4}u\beta t}\,d\beta$, so that $|G(u)|\leqslant c\int_0^\infty e^{-\beta^2 t}e^{-\sqrt{2}u\beta t}\,d\beta$. Integrating by parts,

$$\left|\int_0^\infty \Delta r(u+z_0)e^{-itu^2}du\right| \leqslant |\Delta r(z_0)G(0)| + \left|\int_0^\infty (\partial_u \Delta r(u+z_0))G(u)\,du\right| \leqslant \frac{c\|r_2-r_1\|_{H^{1,0}}}{\sqrt{t}} + c\|r_2-r_1\|_{H^{1,0}}\|G\|_{L^2(0,\infty)}.$$

But by Minkowski's inequality,

$$\|G\|_{L^2(0,\infty)} \leqslant c\int_0^\infty e^{-\beta^2 t}\|e^{-\sqrt{2}\,\Diamond\beta t}\|_{L^2}\,d\beta = c\int_0^\infty \frac{e^{-\beta^2 t}}{(\beta t)^{1/2}}\,d\beta = \frac{c}{t^{3/4}},$$

and hence

$$\left|\int_0^\infty \Delta r(u+z_0)e^{-itu^2}du\right| \leqslant c\|r_2-r_1\|_{H^{1,0}}\left(\frac{1}{t^{1/2}}+\frac{1}{t^{3/4}}\right), \quad t>0. \tag{5.40}$$

On the other hand, for all $t\in\mathbf{R}$, $\left|\int_0^\infty \Delta r(u+z_0)e^{-itu^2}du\right| \leqslant c\|\Delta r\|_{H^{0,1}}$, and it follows that

$$\left|\int_0^\infty \Delta r(u+z_0)e^{-itu^2}du\right| \leqslant \frac{c\|r_2-r_1\|_{H^{1,1}}}{(1+t)^{1/2}}, \quad t\geqslant 0. \tag{5.41}$$

Now consider the second term in II″. Noting that the integration variable γ is negative, this term takes the form

$$-\int_{-\infty}^{0} \delta_-^2(\gamma+z_0)(\gamma+z_0)((1-|r(\gamma+z_0)|^2)^2-1)(C^+_{\mathbf{R}_+\to\Gamma}\Delta r(\Diamond+z_0)e^{-it\Diamond^2})(\gamma)\,d\gamma.$$

By the proof of (4.51) (here the δ^2-term is absent),

$$\|C^+_{\mathbf{R}_+\to\Gamma}\Delta r(\lozenge+z_0)e^{-it\lozenge^2}\|_{L^2} < \frac{c}{(1+t)^{1/4}}\|\Delta r\|_{H^{1,0}}, \quad t\geqslant 0,$$

and hence the term is bounded by

$$\frac{c}{1-\varrho}\,\frac{\|\lozenge((1-|r(\lozenge)|^2)^2-1)\|_{L^2}}{(1+t)^{1/4}}\|\Delta r\|_{H^{1,0}},$$

which is bounded in turn by $c\eta\|\Delta r\|_{H^{1,0}}/(1-\varrho)(1+t)^{1/4}$.

Combining the above results, we conclude that

$$|\mathrm{II}'| \leqslant \frac{c\lambda^2\|\Delta r\|_{H^{1,1}}}{(1-\varrho)(1+t)^{1/2}} + \frac{c\eta\|\Delta r\|_{H^{1,0}}}{(1-\varrho)(1+t)^{1/4}} \leqslant \begin{bmatrix} 1 & 1 \\ 1/4 & 1 \end{bmatrix}\|\Delta r\|_{H^{1,1}}. \tag{5.42}$$

Similarly we rewrite II''' in the form

$$\begin{aligned}\mathrm{II}''' = &-\left(\int_{-\infty}^{0}\Delta r^{\#}(u+z_0)e^{-itu^2}du\right)\left(\int_{-\infty}^{z_0}\delta_-^2((1-|r|^2)^2-1)\frac{dz}{2\pi i}\right)\\ &+\int_{-\infty}^{0}\delta_-^2(\gamma+z_0)(\gamma+z_0)[(1-|r(\gamma+z_0)|^2)^2-1](C^+_{e^{i\pi}\mathbf{R}_+\to\Gamma}\Delta r^{\#}(\lozenge+z_0)e^{-it\lozenge^2})(\gamma)\,d\gamma,\end{aligned}$$

which leads to a similar bound as in (5.42). Here the contour of integration in (5.39) for $G(u)$, $u\leqslant 0$, must be replaced by $\alpha=u+e^{3\pi i/4}\beta$, $\beta\geqslant 0$, and we must again use (4.51). We obtain

$$|\mathrm{II}'''| \leqslant \begin{bmatrix} 1 & 1 \\ 1/4 & 1 \end{bmatrix}\|\Delta r^{\#}\|_{H^{1,1}} \leqslant \begin{bmatrix} 2 & 1 \\ 1/4 & 4 \end{bmatrix}\|\Delta r\|_{H^{1,1}}, \tag{5.43}$$

$$|\mathrm{II}'| \leqslant \begin{bmatrix} 2 & 1 \\ 1/4 & 4 \end{bmatrix}\|\Delta r\|_{H^{1,1}}. \tag{5.44}$$

Finally we consider III', which we again write in the form

$$\begin{aligned}\mathrm{III}' &= e^{itz_0^2}\left(\int_{z_0}^{\infty} zr(\Delta\delta^2)e^{-it(z-z_0)^2}dz + \int_{-\infty}^{z_0} zr^{\#}(\Delta\delta_+^2)e^{-it(z-z_0)^2}dz\right)\\ &\equiv e^{itz_0^2}(\mathrm{III}''+\mathrm{III}''').\end{aligned}$$

As before,

$$\begin{aligned}\mathrm{III}'' = &-\left(\int_0^{\infty} r(u+z_0)e^{-itu^2}du\right)\left(\int_{-\infty}^{z_0}\Delta(\delta_-^2((1-|r|^2)^2-1))\frac{dz}{2\pi i}\right)\\ &-\int_{-\infty}^{0}\Delta\{\delta_-^2(\gamma+z_0)[(\gamma+z_0)((1-|r(\gamma+z_0)|^2)^2-1)]\}(C^+_{\mathbf{R}_+\to\Gamma}r(\lozenge+z_0)e^{-it\lozenge^2})(\gamma)\,d\gamma.\end{aligned}$$

By (2.56), for $z\in\mathbf{C}\setminus(-\infty,z_0]$, $\delta^2(z)=e^{2\int_{-\infty}^{z_0}\frac{\log(1-|r(s)|^2)}{s-z}\frac{ds}{2\pi i}}$, and so

$$\begin{aligned}\Delta\delta^2 &= \int_0^1 \frac{d}{dy} e^{2\int_{-\infty}^{z_0}\frac{\log(1-|r_1+y(r_2-r_1)|^2)}{s-z}\frac{ds}{2\pi i}}\,dy \\ &= -4\int_0^1 \big[e^{2\int_{-\infty}^{z_0}\frac{\log(1-|r_1+y(r_2-r_1)|^2)}{s-z}\frac{ds}{2\pi i}}\big] \\ &\quad\times\left[\int_{-\infty}^{z_0}\frac{\operatorname{Re}[(r_2-r_1)(\overline{r_1+y(r_2-r_1)})]}{1-|r_1+y(r_2-r_1)|^2}\,\frac{ds'}{2\pi i(s'-z)}\right]dy,\end{aligned} \tag{5.45}$$

which implies the estimate

$$\|\Delta\delta_-^2\|_{L^2}\leqslant\frac{c}{(1-\varrho)^2}\|\Delta r\|_{L^2}. \tag{5.46}$$

Hence

$$\left|\int_{-\infty}^{z_0}\Delta\delta_-^2((1-|r|^2)^2-1)\,dz\right|\leqslant\frac{c\lambda}{(1-\varrho)^2}\|\Delta r\|_{L^2}+\frac{c\lambda}{1-\varrho}\|\Delta r\|_{L^2}\leqslant\frac{c\eta}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}},$$

and using Lemma 2.48,

$$\begin{aligned}\|\Delta\{\delta_-^2(\Diamond)\Diamond((1-|r(\Diamond)|^2)^2-1)\}\|_{L^2} &\leqslant \|\Delta\delta_-^2\|_{L^2}\|\Diamond((1-|r(\Diamond)|^2)^2-1)\|_{L^\infty} \\ &\quad+\|\delta_-^2\|_{L^\infty}\|\Delta\Diamond((1-|r(\Diamond)|^2)^2-1)\|_{L^2} \\ &\leqslant \frac{c\eta^2}{(1-\varrho)^2}\|\Delta r\|_{L^2}+\frac{c}{1-\varrho}\|\Delta r\|_{H^{0,1}} \\ &\leqslant c\frac{(1+\eta)^2}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

The estimates for II″ now imply

$$\begin{aligned}|\mathrm{III}''| &\leqslant \frac{c\eta}{(1+t)^{1/2}}\cdot\frac{\eta}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}+\frac{c(1+\eta)^2}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}\cdot\frac{\eta}{(1+t)^{1/4}} \\ &\leqslant \begin{bmatrix}1 & 2\\ 1/4 & 2\end{bmatrix}\|\Delta r\|_{H^{1,1}},\end{aligned} \tag{5.47}$$

and similarly

$$\begin{aligned}|\mathrm{III}'''| &\leqslant \frac{c\|r^\#\|_{H^{1,1}(-\infty,z_0)}}{(1+t)^{1/2}}\cdot\frac{\eta}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}+c\frac{(1+\eta)^2}{(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}\|r^\#\|_{H^{1,1}(-\infty,z_0)} \\ &\leqslant \begin{bmatrix}1 & 2\\ 1/4 & 4\end{bmatrix}\|\Delta r\|_{H^{1,1}}.\end{aligned} \tag{5.48}$$

Thus

$$|\mathrm{III}'| \leqslant \begin{bmatrix} 1 & 2 \\ 1/4 & 4 \end{bmatrix} \|\Delta r\|_{H^{1,1}}, \tag{5.49}$$

and combining (5.44) and (5.49),

$$|\mathrm{II}'+\mathrm{III}'| \leqslant \begin{bmatrix} 1 & 2 \\ 1/4 & 4 \end{bmatrix} \|\Delta r\|_{H^{1,1}}. \tag{5.50}$$

As the $(2,1)$-entry of W_3 is the negative conjugate of the $(1,2)$-entry, it follows that $\Delta W = W_4 + W_5$ where

$$\|W_4\|_{L^2(dx)} \leqslant \frac{c}{(1-\varrho)^2} \|\Delta r\|_{H^{1,1}}$$

and for any $p>2$,

$$\begin{aligned} |W_5| &\leqslant \begin{bmatrix} 1 & 2 \\ 1/2p & 5 \end{bmatrix} K_p \|\Delta r\|_{H^{1,0}} + \begin{bmatrix} 1 & 0 \\ 1/4 & 3 \end{bmatrix} \|\Delta r\|_{H^{1,1}} + \begin{bmatrix} 1 & 2 \\ 1/4 & 4 \end{bmatrix} \|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 1 & 2 \\ 1/2p & 5 \end{bmatrix} K_p \|\Delta\|_{H^{1,1}}. \end{aligned}$$

Finally,

$$\left| \int_{-\infty}^{z_0} \log(1-|r(z)|^2)\, dz \right| \leqslant \frac{\eta^2}{1-\varrho}$$

and

$$\begin{aligned} \left| \Delta \int_{-\infty}^{z_0} \log(1-|r(z)|^2)\, dz \right| &= \left| \int_0^1 \left[\int_{-\infty}^{z_0} \frac{d}{dy} \log(1-|r_1+y(r_2-r_1)|^2)\, dz \right] dy \right| \\ &\leqslant \frac{c\eta}{1-\varrho} \|\Delta r\|_{L^2}. \end{aligned}$$

Hence by Theorem 4.16, for any $p>2$,

$$\begin{aligned} \left| \Delta \left(Q \int_{-\infty}^{z_0} \log(1-|r|^2)\, dz \right) \right| &\leqslant \begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p \|\Delta r\|_{H^{1,0}} \cdot \frac{\eta^2}{1-\varrho} \\ &\quad + \begin{bmatrix} 1 & 1 \\ 1/2 & 5 \end{bmatrix} \cdot \frac{\eta}{1-\varrho} \|\Delta r\|_{L^2} \\ &\leqslant \begin{bmatrix} 2 & 3 \\ 1/2p+1/4 & 8 \end{bmatrix} K_p \|\Delta r\|_{H^{1,1}}. \end{aligned}$$

We conclude that $\mathrm{I}_2 = \mathrm{I}_{21} + \mathrm{I}_{22}$ where

$$\|\mathrm{I}_{21}\|_{L^2(dx)} \leqslant \frac{c}{(1-\varrho)^2} \|\Delta r\|_{H^{1,1}}$$

and for any $p>2$,

$$\begin{aligned}\|\mathrm{I}_{22}\|_{L^\infty(dx)} &\leqslant \begin{bmatrix} 1 & 2 \\ 1/2p & 5 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}} + \begin{bmatrix} 2 & 3 \\ 1/2p+1/4 & 8 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} 1 & 4 \\ 1/2p & 8 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Together with the estimate on I_1, this proves the lemma. □

Remark. Using the fact that $z|r(z)|^2\in L^p$ for all $p\geqslant 1$, more careful estimates show that W_5 in fact falls off like $(1+t)^{-(1/2-\varepsilon)}$ for any $\varepsilon>0$ as $t\to\infty$, but this extra decay is clearly only of academic interest as the leading order of decay in I is governed by other terms.

LEMMA 5.51. *For* $2<p<\infty$,

$$\begin{aligned}\|\Delta G\|_{L^2} &\leqslant \begin{bmatrix} l & l+2 \\ l/2+1/2p-1/4 & 5l+5/2 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}, \\ \|\Delta G\|_{L^1} &\leqslant \begin{bmatrix} l & l+1 \\ l/2+1/2p-3/4 & 5l-2 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}};\end{aligned} \tag{5.52}$$

$$\begin{aligned}\|\Delta GQ\|_{L^2} &\leqslant \begin{bmatrix} l+1 & l+3 \\ l/2+1/2p+1/4 & 5l+15/2 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}, \\ \|\Delta GQ\|_{L^1} &\leqslant \begin{bmatrix} l+1 & l+2 \\ l/2+1/2p-1/4 & 5l+3 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}};\end{aligned} \tag{5.53}$$

$$\begin{aligned}\|\Delta\partial_x G\|_{L^2} &\leqslant \begin{bmatrix} l & l+4 \\ l/2+1/2p-1/2 & 5l+7/2 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}, \\ \|\Delta\partial_x G\|_{L^1} &\leqslant \begin{bmatrix} l & l+3 \\ l/2+1/2p-1 & 5l-1 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}.\end{aligned} \tag{5.54}$$

Proof. For $p>2$, by Theorem 4.16 and Lemma 5.24,

$$\begin{aligned}\|\Delta G\|_{L^2} &\leqslant c\|q\|_{L^\infty}^{l-1}\|q\|_{L^2}\|\Delta q\|_{L^\infty} \\ &\leqslant \begin{bmatrix} l-1 & l-1 \\ (l-1)/2 & 5(l-1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1/2p+1/4 & 7 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l & l+2 \\ l/2+1/2p-1/4 & 5l+5/2 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\end{aligned}$$

and

$$\begin{aligned}\|\Delta G\|_{L^1} &\leqslant \|q\|_{L^\infty}^{l-2}\|q\|_{L^2}^2\|\Delta q\|_{L^\infty}\\ &\leqslant \begin{bmatrix} l-2 & l-2\\ (l-2)/2 & 5(l-2)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}^2\begin{bmatrix}0 & 3\\ 1/2p+1/4 & 7\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix} l & l+1\\ l/2+1/2p-3/4 & 5l-2\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

The proof for (5.53) is similar, replacing l by $l+1$.

For (5.54), let $\Delta\partial_x Q=\mathrm{I}+\mathrm{II}$ as in Lemma 5.37 and let $p>2$. Then

$$\begin{aligned}\|\Delta\partial_x G\|_{L^2} &\leqslant \|q\|_{L^\infty}^{l-1}\|\partial_x q\|_{L^2}\|\Delta q\|_{L^\infty}+\|q\|_{L^\infty}^{l-1}\|q\|_{L^2}\|\mathrm{I}\|_{L^\infty}+\|q\|_{L^\infty}^{l}\|\mathrm{II}\|_{L^2}\\ &\leqslant \begin{bmatrix} l-1 & l-1\\ (l-1)/2 & 5(l-1)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}0 & 3\\ 1/2p+1/4 & 7\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\\ &\quad+\begin{bmatrix} l-1 & l-1\\ (l-1)/2 & 5(l-1)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}1 & 4\\ 1/2p & 8\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\\ &\quad+\begin{bmatrix} l & l\\ l/2 & 5l\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix} l & l+4\\ l/2+1/2p-1/2 & 5l+7/2\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\end{aligned}$$

and

$$\begin{aligned}\|\Delta\partial_x G\|_{L^1} &\leqslant \|q\|_{L^\infty}^{l-2}\|q\|_{L^2}\|q'\|_{L^2}\|\Delta q\|_{L^\infty}+\|q\|_{L^\infty}^{l-2}\|q\|_{L^2}^2\|\mathrm{I}\|_{L^\infty}+\|q\|_{L^\infty}^{l-1}\|q\|_{L^2}\|\mathrm{II}\|_{L^2}\\ &\leqslant \begin{bmatrix} l-2 & l-2\\ (l-2)/2 & 5(l-2)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}^2\begin{bmatrix}0 & 3\\ 1/2p+1/4 & 7\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\\ &\quad+\begin{bmatrix} l-2 & l-2\\ (l-2)/2 & 5(l-2)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}^2\begin{bmatrix}1 & 4\\ 1/2p & 8\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\\ &\quad+\begin{bmatrix} l-1 & l-1\\ (l-1)/2 & 5(l-1)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix} l & l+3\\ l/2+1/2p-1 & 5l-1\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

6. A priori estimates

Set

$$F = P_{12}\int e^{-i\theta} m_-^{-1} G m_-\, dy, \tag{6.1}$$

where P_{12} denotes the projection of (2×2)-matrices onto their $(1,2)$-entries. More precisely,

$$\begin{aligned} F &= F(z,t;r) \qquad (6.2)\\ &= P_{12}\int_{-\infty}^{\infty} e^{-i(yz-tz^2)} m_-^{-1}(y,z;e^{-i\Diamond^2 t}r)\, G(\mathcal{R}^{-1}(e^{-i\Diamond^2 t}r))(y)\, m_-(y,z;e^{-i\Diamond^2 t}r)\, dy. \end{aligned}$$

Note that in terms of F, (2.46) takes the form

$$r(t)(z) = r_0(z) + \varepsilon\int_0^t F(z,s,r(s))\, ds. \tag{6.3}$$

Also note that the term $e^{-i\theta}$ in (6.1) and (6.2) can be replaced by $e^{-i\theta\,\mathrm{ad}\,\sigma}$. The goal of this section is to obtain the estimates in the following theorem, which is a combination of Lemmas 6.27 and 6.51 below.

LEMMA 6.4. *For* $n\geqslant 3$,

$$\|F\|_{H^{1,1}} \leqslant \begin{bmatrix} l+1 & l+5 \\ l/2-1/2n-3/4 & 5l \end{bmatrix} K_{2(n-1)}^3. \tag{6.5}$$

For $n\geqslant 3$, $2<p\leqslant 4$ *and* $p''>4$,

$$\|\Delta F\|_{H^{1,1}} \leqslant \begin{bmatrix} l & l+6 \\ l/2-1/2p-1/2n-1 & 5l+8/3 \end{bmatrix} \max(K_{2(n-1)}^4, K_{p''}^4)\|\Delta r\|_{H^{1,1}}. \tag{6.6}$$

Note that for p close to 2, and n sufficiently large,

$$\|F\|_{H^{1,1}},\ \|\Delta F\|_{H^{1,1}} \leqslant \frac{\text{const}}{(1+t)^{\alpha}} \tag{6.7}$$

for some $\alpha>1$, as long as $l>\frac{7}{2}$ (cf. (3.1)).

Remark 6.8. As noted in §2, uniqueness for solutions of (2.46) follows from the Lipschitz estimate (6.6). Much of the analysis of this paper is concerned with ensuring that this estimate has explicit time decay in order to control the long-time behavior of solutions of (2.46). However, to prove uniqueness, this time decay is clearly not necessary, and it is possible to give a rather short proof of a version of (6.6) without explicit time decay that is sufficient for the purposes. We leave the details to the interested reader.

We decompose F into four terms:

$$\begin{aligned} F &= P_{12}\int e^{-i\theta}G\,dy + P_{12}\int e^{-i\theta}(m_-^{-1}-I)G\,dy \\ &\quad + P_{12}\int e^{-i\theta}G(m_--I)\,dy + P_{12}\int e^{-i\theta}(m_-^{-1}-I)G(m_--I)\,dy \\ &\equiv F^{(1)}+F^{(2)}+F^{(3)}+F^{(4)}. \end{aligned} \tag{6.9}$$

Notation. In the following, for functions $h=h(x,z)$, we use the norm

$$\|h\|_{L^p(dz)\otimes L^q(dx)} \equiv \|\,\|h\|_{L^p(dz)}\|_{L^q(dx)}, \quad 1\leqslant p,q\leqslant\infty.$$

LEMMA 6.10.

$$\|F\|_{L^2} \leqslant \begin{bmatrix} l+1 & l+1 \\ (l-1)/2 & 5l-3/2 \end{bmatrix} K_4^2.$$

Proof. By the L^2-unitarity of the Fourier transform, Theorem 4.16 and Lemma 5.24,

$$\|F^{(1)}\|_{L^2(dz)} = c\|\,|q|^l q\|_{L^2(dx)} \leqslant c\|q\|_{L^\infty}^l\|q\|_{L^2} \leqslant \begin{bmatrix} l & l \\ l/2 & 5l \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \leqslant \begin{bmatrix} l+1 & l \\ l/2 & 5l+1/2 \end{bmatrix}.$$

By Minkowski's inequality, Theorem 4.16, Lemmas 5.24 and 5.1,

$$\begin{aligned} \|F^{(3)}\|_{L^2(dz)} &\leqslant c\|\,|q|^l q\|_{L^1(dx)}\|m_--I\|_{L^2(dz)\otimes L^\infty(dx)} \\ &\leqslant \begin{bmatrix} l-1 & l-1 \\ (l-1)/2 & 5(l-1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leqslant \begin{bmatrix} l+2 & l-1 \\ (l-1)/2 & 5l-3 \end{bmatrix}. \end{aligned}$$

Similarly,

$$\|F^{(2)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+2 & l-1 \\ (l-1)/2 & 5l-3 \end{bmatrix}.$$

Finally,

$$\begin{aligned} \|F^{(4)}\|_{L^2(dz)} &\leqslant c\|\,|q|^l q\|_{L^1(dx)}\|m_--I\|^2_{L^4(dz)\otimes L^\infty(dx)} \\ &\leqslant \begin{bmatrix} l-1 & l-1 \\ (l-1)/2 & 5(l-1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^2 K_4^2\lambda^2 \leqslant \begin{bmatrix} l+3 & l-1 \\ (l-1)/2 & 5l-4 \end{bmatrix} K_4^2, \end{aligned}$$

where the L^4-estimate on m_--I is given by Lemma 5.24. Thus

$$\begin{aligned} \|F\|_{L^2(dz)} &\leqslant \begin{bmatrix} l+1 & l \\ l/2 & 5l+1/2 \end{bmatrix} + \begin{bmatrix} l+2 & l-1 \\ (l-1)/2 & 5l-3 \end{bmatrix} + \begin{bmatrix} l+3 & l-1 \\ (l-1)/2 & 5l-4 \end{bmatrix} K_4^2 \\ &\leqslant \begin{bmatrix} l+1 & l+1 \\ (l-1)/2 & 5l-3/2 \end{bmatrix} K_4^2, \end{aligned}$$

where we have again used the fact that $(1-\varrho)^{-1}\leqslant cK_2\leqslant cK_4$. $\square$

LEMMA 6.11.

$$\|\Diamond F\|_{L^2(dz)} \leqslant \begin{bmatrix} l+1 & l+3 \\ (l-1)/2 & 5l \end{bmatrix} K_4^2.$$

Proof. Again by the unitarity of the Fourier transform,

$$\begin{aligned}\|\Diamond F^{(1)}\|_{L^2(dz)} &\leqslant c\|(|q|^l q)_x\|_{L^2(dx)} \leqslant c\|q\|_{L^\infty}^l \|q_x\|_{L^2} \\ &\leqslant \begin{bmatrix} l & l \\ l/2 & 5l \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \leqslant \begin{bmatrix} l+1 & l \\ l/2 & 5l+1/2 \end{bmatrix}.\end{aligned}$$

Integration by parts, and using (2.8), we obtain

$$\begin{aligned} zF^{(3)} &= -iP_{12}\int e^{-i\theta}G_y(m_- - I)\,dy - iP_{12}\int e^{-i\theta}Gm_{-y}\,dy \\ &= -iP_{12}\int e^{-i\theta}G_y(m_- - I)\,dy - iP_{12}\int e^{-i\theta}G(i[\sigma, z(m_- - I)] + Qm_-)\,dy \qquad (6.12) \\ &= -iP_{12}\int e^{-i\theta}G_y(m_- - I)\,dy - iP_{12}\int e^{-i\theta}GQ(m_- - I) \equiv \mathrm{I} + \mathrm{II}, \end{aligned}$$

as $G[\sigma, z(m_- - I)]$ and GQ are diagonal. Again by Theorem 4.16, Lemma 5.24 and Lemma 5.1,

$$\begin{aligned}\|\mathrm{I}\|_{L^2(dz)} &\leqslant \|G_y\|_{L^1(dx)}\|m_- - I\|_{L^2(dz)\otimes L^\infty(dx)} \\ &= c\|\,|q|^{l-1}\|_{L^\infty}\|q\|_{L^2}\|q_x\|_{L^2}\|m_- - I\|_{L^2(dz)\otimes L^\infty(dx)} \\ &\leqslant \begin{bmatrix} l-1 & l-1 \\ (l-1)/2 & 5(l-1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leqslant \begin{bmatrix} l+2 & l-1 \\ (l-1)/2 & 5l-3 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{II}\|_{L^2(dz)} &\leqslant c\|GQ\|_{L^1(dx)}\|m_- - I\|_{L^2(dz)\otimes L^\infty(dx)} \\ &\leqslant \begin{bmatrix} l & l \\ l/2 & 5l \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leqslant \begin{bmatrix} l+3 & l \\ l/2 & 5l+2 \end{bmatrix}.\end{aligned}$$

Thus

$$\|\Diamond F^{(3)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+2 & l+1 \\ (l-1)/2 & 5l+2 \end{bmatrix}.$$

Using $(d/dy)m_-^{-1} = iz[\sigma, m^{-1}] - m_-^{-1}Q$, we obtain similarly

$$\|\Diamond F^{(2)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+2 & l+1 \\ (l-1)/2 & 5l+2 \end{bmatrix}.$$

By Lemmas 5.27 and 5.6,

$$\begin{aligned}\|\Diamond F^{(4)}\|_{L^2(dz)} &\leqslant c\|G\|_{L^1}\,\||\Diamond|^{1/2}(m_-^{-1}-I)\|_{L^4(dz)\otimes L^\infty(dx)}\,\||\Diamond|^{1/2}(m_--I)\|_{L^4(dz)\otimes L^\infty(dx)}\\ &\leqslant \begin{bmatrix} l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}K_4\right)^2 \leqslant \begin{bmatrix} l+3 & l+1\\ (l-1)/2 & 5l-2\end{bmatrix}K_4^2.\end{aligned}$$

Hence

$$\begin{aligned}\|\Diamond F\|_{L^2(dz)} &\leqslant \begin{bmatrix} l+1 & l\\ l/2 & 5l+1/2\end{bmatrix} + \begin{bmatrix} l+2 & l+1\\ (l-1)/2 & 5l+2\end{bmatrix} + \begin{bmatrix} l+3 & l+1\\ (l-1)/2 & 5l-2\end{bmatrix}K_4^2\\ &\leqslant \begin{bmatrix} l+1 & l+3\\ (l-1)/2 & 5l\end{bmatrix}K_4^2,\end{aligned}$$

again as $(1-\varrho)^{-1}\leqslant cK_4$. □

LEMMA 6.13.

$$\|\partial_z F\|_{L^2(dz)} \leqslant \begin{bmatrix} l+1 & l+5\\ l/2-1/2n-3/4 & 5l\end{bmatrix}K^3_{2(n-1)}.$$

Proof. Set $\mathbf{A}=\partial_z-\tilde{L}$, where $\tilde{L}=ix\,\mathrm{ad}\,\sigma-2t\partial_x$ (see (5.13)). Using the derivation property of $\mathrm{ad}\,\sigma$, and integration once by parts, we obtain

$$\begin{aligned}\partial_z F &= P_{12}\partial_z\int e^{-i\theta\,\mathrm{ad}\,\sigma}m_-^{-1}Gm_-\,dx\\ &= P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}(\mathbf{A}m_-^{-1})Gm_-\,dx\\ &\qquad + P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}m_-^{-1}G(\mathbf{A}m_-)\,dx - P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}m_-^{-1}(\tilde{L}G)m_-\,dx\\ &\equiv F^{(5)}+F^{(6)}+F^{(7)}.\end{aligned}\tag{6.14}$$

Recall from §4 that $m_-=\tilde{\mu}\delta_-^{\sigma_3}v_\theta^{\#}$ where

$$v^{\#}(z)=\begin{pmatrix}1 & -r\\ 0 & 1\end{pmatrix}\quad\text{for } z>z_0$$

and

$$v^{\#}(z)=\begin{pmatrix}1 & 0\\ \bar{r}/(1-|r|^2) & 1\end{pmatrix}\quad\text{for } z<z_0.$$

Substituting in $F^{(7)}$ we obtain

$$\begin{aligned}F^{(7)} &= -P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}[(v_\theta^{\#})^{-1}\delta_-^{-\sigma_3}\tilde{\mu}^{-1}(\tilde{L}G)\tilde{\mu}\delta_-^{\sigma_3}v_\theta^{\#}]\,dx \\ &= -P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}[(v_\theta^{\#})^{-1}\delta_-^{-\sigma_3}(\tilde{L}G)\delta_-^{\sigma_3}v_\theta^{\#}] \\ &\quad -P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}[(v_\theta^{\#})^{-1}\delta_-^{-\sigma_3}(\tilde{\mu}^{-1}-I)(\tilde{L}G)\delta_-^{\sigma_3}v_\theta^{\#}]\,dx \\ &\quad -P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}[(v_\theta^{\#})^{-1}\delta_-^{-\sigma_3}(\tilde{L}G)(\tilde{\mu}-I)\delta_-^{\sigma_3}v_\theta^{\#}]\,dx \\ &\quad -P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}[(v_\theta^{\#})^{-1}\delta_-^{\sigma_3}(\tilde{\mu}^{-1}-I)(\tilde{L}G)(\tilde{\mu}-I)\delta_-^{\sigma_3}v_\theta^{\#}]\,dx \\ &\equiv -F^{(71)}-F^{(72)}-F^{(73)}-F^{(74)}.\end{aligned} \tag{6.15}$$

The factors δ_- and $v_\theta^{\#}$ depend on x through z_0, and hence they cannot simply be removed from the above integrals. As in the proof of Lemma 5.37, this complicates the estimation of $F^{(71)}$ in particular.

After some elementary algebra we find

$$F^{(71)} = \int (\tilde{L}G)_{12}e^{-i\theta}\delta_-^{-2}\,dx - \int_{x<2tz}(\tilde{L}G)_{21}r^2e^{i\theta}\delta^2\,dx. \tag{6.16}$$

Consider functions $H(z)\equiv\int h(x)e^{-izx}\delta_-^{-2}(z;z_0)\,dx$, where $h\in L^1\cap L^2(\mathbf{R})$. Clearly

$$\left\|\int h(x)e^{-i\Diamond x}\,dx\right\|_{L^2(dz)} \leqslant c\|h\|_{L^2(dx)}. \tag{6.17}$$

On the other hand,

$$\int e^{iyz}\left(\int h(x)e^{-izx}(\delta_-^{-2}(z,z_0)-1)\,dx\right)dz = \int h(x)\left(\int e^{i(y-x)z}(\delta_-^{-2}(z,z_0)-1)\,dz\right)dx.$$

By the analyticity properties of $\delta^{-2}(z;z_0)$, $\int e^{isz}(\delta_-^{-2}(z,z_0)-1)\,dz=0$ for $s<0$. For $s>0$, use the fact that $\delta_\pm$ solve the normalized RHP (2.54) to write

$$\int e^{isz}(\delta_-^{-2}-1)\,dz = \int e^{isz}(\delta_+^{-2}(1-\chi_{z<z_0}|r|^2)^2-1)\,dz,$$

where $\chi_{z<z_0}$ denote the characteristic function of the set $\{z<z_0\}$. Again by the analyticity of $\delta^{-2}(z,z_0)$, we have $\int e^{isz}(\delta_+^{-2}-1)\,dz=0$ for $s>0$, and so

$$\int e^{isz}(\delta_-^{-2}-1)\,dz = \int_{-\infty}^{z_0} e^{isz}\delta_+^{-2}(-2|r|^2+|r|^4)\,dz.$$

Introduce the following auxiliary function (cf. (5.39))

$$G(s,z,z_0)=\int_{+i\infty}^{z} e^{is\zeta}\delta^{-2}(\zeta,z_0)\,d\zeta,\quad s>0,\ z<z_0, \tag{6.18}$$

with integration along a contour from $+i\infty$ to z in the upper half-plane. Clearly (i) $\partial_z G(s,z,z_0)=e^{isz}\delta^{-2}(z,z_0)$, $z<z_0$, and by (2.58), (ii) $|G(s,z,z_0)|\leqslant c/s(1-\varrho)$, $s>0$, $z<z_0$. Here for $s>0$,

$$\begin{aligned}\left|\int e^{isz}(\delta_-^{-2}-1)\,dz\right| &\leqslant |G(s,z_0,z_0)(-2|r(z_0)|^2+|r(z_0)|^4)| \\ &\qquad +\left|\int_{-\infty}^{z_0} G(s,z,z_0)(-2\partial_z|r|^2+\partial_z|r|^4)\,dz\right| \\ &\leqslant \frac{c\eta^2}{s(1-\varrho)}.\end{aligned}$$

On the other hand, for all $s\in\mathbf{R}$,

$$\left|\int_{-\infty}^{z_0} e^{isz}\delta_+^{-2}(-2|r|^2+|r|^4)\,dz\right|\leqslant\frac{c\eta^2}{1-\varrho},$$

and we conclude that

$$\left|\int e^{isz}(\delta_-^{-2}-1)\,dz\right|\leqslant\frac{c\eta^2}{1-\varrho}\,\frac{1}{1+|s|} \tag{6.19}$$

for all $s\in\mathbf{R}$. Thus

$$\left|\int e^{iyz}\left(\int h(x)e^{-izx}(\delta_-^{-2}(z,z_0)-1)\,dx\right)dz\right|\leqslant\frac{c\eta^2}{1-\varrho}\int\frac{|h(x)|}{1+|y-x|}\,dx,$$

from which it follows that

$$\begin{aligned}\left\|\int h(x)e^{-i\Diamond x}(\delta_-^{-2}(\Diamond,z_0)-1)\,dx\right\|_{L^2(dz)} &\leqslant\frac{c\eta^2}{1-\varrho}\left\|\int\frac{|h(x)|}{1+|\Diamond-x|}\,dx\right\|_{L^2} \\ &\leqslant\frac{c\eta^2}{1-\varrho}\|h\|_{L^p}\left\|\frac{1}{1+|\Diamond|}\right\|_{L^q},\end{aligned}$$

$1/q+1/p=3/2$, $1<p,q<2$, by Young's inequality. Collecting terms, we find

$$\|H\|_{L^2(dz)}\leqslant c\|h\|_{L^2}+\frac{c\eta^2}{1-\varrho}\|h\|_{L^p},\quad 1<p<2. \tag{6.20}$$

Now consider functions $J(z)=r(z)^2\int_{x<2zt}h(x)e^{izx}\delta^2(z,z_0)\,dz$, where $h\in L^1\cap L^2(\mathbf{R})$ as before. We have

$$\int e^{-iyz}J(z)\,dz=\int h(x)\left(\int_{z_0}^{\infty}e^{iz(x-y)}r(z)^2\delta(z,z_0)\,dz\right)dx.$$

Here we need the auxiliary function for $s>0$,

$$G(s,z,z_0)=\int_{+i\infty}^{z} e^{is\zeta}\delta^2(\zeta,z_0)\,d\zeta,\quad z>z_0, \tag{6.21}$$

with integration again along a contour from $+i\infty$ to z in the upper half-plane. Now (i) $\partial_z G(s,z,z_0)=e^{isz}\delta^2(z,z_0)$, $z>z_0$, and (ii) $|G(s,z,z_0)|\leqslant c/s$, $s>0$, $z>z_0$. For $s<0$, define

$$G(s,z,z_0)=\int_{-i\infty}^{z} e^{is\zeta}\delta^2(\zeta,z_0)\,d\zeta,\quad z>z_0, \tag{6.22}$$

with integration now along a contour from $-i\infty$ to z in the lower half-plane. Again (i) $\partial_z G(s,z,z_0)=e^{isz}\delta^2(z,z_0)$, $z>z_0$, and (ii) $|G(s,z,z_0)\leqslant c/|s|(1-\varrho)$, $s<0$, $z>z_0$. Thus for all $s\in\mathbf{R}\backslash 0$,

$$\left|\int_{z_0}^{\infty} e^{izs}r(z)^2\delta^2(z,z_0)\,dz\right|\leqslant|r(z_0)|^2|G(s,z,z_0)|+2\left|\int_{z_0}^{\infty}G(s,z,z_0)r\partial_z r\,dz\right|\leqslant\frac{c\eta^2}{(1-\varrho)|s|}.$$

But also for all $s\in\mathbf{R}$,

$$\left|\int_{z_0}^{\infty} e^{izs}r(z)^2\delta^2(z,z_0)\,dz\right|\leqslant\frac{c\eta^2}{1-\varrho},$$

and so

$$\left|\int_{z_0}^{\infty} e^{izs}r(z)^2\delta^2(z,z_0)\,dz\right|\leqslant\frac{c\eta^2}{1-\varrho}\frac{1}{1+|s|}$$

for all $s\in\mathbf{R}$, and we obtain as before

$$\|J\|_{L^2}=\frac{c\eta^2}{1-\varrho}\|h\|_{L^p}\quad\text{for any } 1<p<2. \tag{6.23}$$

Finally, applying these estimates to $F^{(71)}$, we conclude that for any $1<p<2$,

$$\begin{aligned}\|F^{(71)}\|_{L^2(dz)}&\leqslant\frac{c\eta^2}{1-\varrho}\|\tilde{L}G\|_{L^p}+c\|\tilde{L}G\|_{L^2}\\&\leqslant\frac{c\eta^2}{1-\varrho}\begin{bmatrix}l+1 & l+3-2/p\\ l/2-1/p-1/2n & 5l+7/2-9/p\end{bmatrix}K_{2(n-1)}\\&\quad+\begin{bmatrix}l+1 & l+2\\ (l-1)/2-1/2n & 5l-1\end{bmatrix}K_{2(n-1)}\\&\leqslant\begin{bmatrix}l+1 & l+5-2/p\\ l/2-1/p-1/2n & \max(5l+9/2-9/p,5l-1)\end{bmatrix}K_{2(n-2)}.\end{aligned} \tag{6.24}$$

The estimates for the remaining terms in $F^{(6)}$ are straightforward. Using once again the fact that the entries of $\tilde{\mu}^{-1}-I$ are simply a rearrangement of the entries of $\tilde{\mu}-I$, Lemma 5.29 and (4.17), we obtain

$$\begin{aligned}
&\|F^{(72)}\|_{L^2}+\|F^{(73)}\|_{L^2}+\|F^{(74)}\|_{L^2}\\
&\quad\leqslant \frac{c}{1-\varrho}\big(\|\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|_{L^2(dz)\otimes L^\infty(dx)}+\|\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|^2_{L^4(dz)\otimes L^\infty(dx)}\big)\\
&\quad\leqslant \frac{c}{1-\varrho}\left(\begin{bmatrix} l+1 & l+1\\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\begin{bmatrix}1 & 0\\ 1/4 & 3\end{bmatrix}\right.\\
&\qquad\left.+\begin{bmatrix} l+1 & l+1\\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\begin{bmatrix}1 & 0\\ 1/8 & 2\end{bmatrix}^2 K_4^2\right)\\
&\quad\leqslant \begin{bmatrix} l+2 & l+2\\ l/2-1/2n-3/4 & 5l-1/2\end{bmatrix}K_{2(n-1)}K_4^2.
\end{aligned}\tag{6.25}$$

Combining this estimate with (6.24) and choosing $2>p>\frac{4}{3}$, we obtain for $n\geqslant 3$,

$$\begin{aligned}
\|F^{(7)}\|_{L^2(dz)}&\leqslant \begin{bmatrix} l+1 & l+5-2/p\\ l/2-3/4-1/2n & 5l-1/2\end{bmatrix}K_{2(n-1)}K_4^2\\
&\leqslant \begin{bmatrix} l+1 & l+4\\ l/2-3/4-1/2n & 5l-1/2\end{bmatrix}K^3_{2(n-1)}.
\end{aligned}\tag{6.26}$$

By Lemmas 5.27, 5.17 and 5.14,

$$\begin{aligned}
\|F^{(6)}\|_{L^2(dz)}&\leqslant c\|G\|_{L^1(dx)}\|m_-^{-1}\|_{L^\infty(dz)\otimes L^\infty(dx)}\|\mathbf{A}m_-\|_{L^2(dz)\otimes L^\infty(dx)}\\
&\leqslant \begin{bmatrix} l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}\begin{bmatrix} 0 & 1+2/n\\ -1/2n & 5/n\end{bmatrix}K_{2(n-1)}\begin{bmatrix} 1 & 2\\ -1/2n & 3\end{bmatrix}K_{2(n-1)}\\
&\leqslant \begin{bmatrix} l+2 & l+2+2/n\\ (l-1)/2-1/n & 5l+1\end{bmatrix}K^2_{2(n-1)}
\end{aligned}$$

for $n\geqslant 3$. As $A=\partial_z-\tilde{L}$ is an entrywise operation (cf. Lemma 5.17), $F^{(5)}$ satisfies the same estimate as $F^{(6)}$.

Combining this estimate with (6.26), we obtain finally for $n\geqslant 3$,

$$\begin{aligned}
\|\partial_z F\|_{L^2(dz)}&\leqslant \|F^{(5)}\|_{L^2(dz)}+\|F^{(6)}\|_{L^2(dz)}+\|F^{(7)}\|_{L^2(dz)}\\
&\leqslant \begin{bmatrix} l+2 & l+2+2/n\\ (l-1)/2-1/n & 5l+1\end{bmatrix}K^2_{2(n-1)}\\
&\qquad+\begin{bmatrix} l+1 & l+4\\ l/2-1/2n-3/4 & 5l-1/2\end{bmatrix}K_{2(n-1)}K_4^2\\
&\leqslant \begin{bmatrix} l+1 & l+5\\ l/2-1/2n-3/4 & 5l\end{bmatrix}K^3_{2(n-1)}.
\end{aligned}$$

This completes the proof of Lemma 6.13. □

From Lemmas 6.10, 6.11 and 6.13, we obtain for $n\geqslant 3$,

$$\begin{aligned}\|F\|_{H^{1,1}} &\leqslant \begin{bmatrix} l+1 & l+1 \\ (l-1)/2 & 5l-3/2 \end{bmatrix} K_4^2 + \begin{bmatrix} l+1 & l+3 \\ (l-1)/2 & 5l \end{bmatrix} K_4^2 \\ &\quad + \begin{bmatrix} l+1 & l+5 \\ l/2-1/2n-3/4 & 5l \end{bmatrix} K_{2(n-1)}^3 \\ &\leqslant \begin{bmatrix} l+1 & l+5 \\ l/2-1/2n-3/4 & 5l \end{bmatrix} K_{2(n-1)}^3.\end{aligned}$$

We have proved the following basic result.

LEMMA 6.27. *For $n\geqslant 3$,*

$$\|F\|_{H^{1,1}} \leqslant \begin{bmatrix} l+1 & l+5 \\ l/2-1/2n-3/4 & 5l \end{bmatrix} K_{2(n-1)}^3.$$

We now begin the derivation of a priori estimates for ΔF.

LEMMA 6.28. *For $2<p\leqslant 4$,*

$$\|\Delta F\|_{L^2(dz)} \leqslant \begin{bmatrix} l & l+3 \\ l/2+1/2p-3/4 & 5l+1/2 \end{bmatrix} K_4^3 \|\Delta r\|_{H^{1,1}}.$$

Proof. In the notation of (6.2), by Lemma 5.51, for $p>2$,

$$\|\Delta F^{(1)}\|_{L^2(dz)} = c\|\Delta G\|_{L^2(dx)} \leqslant \begin{bmatrix} l & l+2 \\ l/2+1/2p-1/4 & 5l+5/2 \end{bmatrix} K_p \|\Delta r\|_{H^{1,1}}.$$

By Lemmas 5.51, 5.27 and 5.1, for $p>2$,

$$\begin{aligned}\|\Delta F^{(3)}\|_{L^2(dz)} &\leqslant c\|\Delta G\|_{L^1(dx)} \|m_- - I\|_{L^2(dz)\otimes L^\infty(dx)} + c\|G\|_{L^1(dx)} \|\Delta m_-\|_{L^2(dz)\otimes L^\infty(dx)} \\ &\leqslant \begin{bmatrix} l & l+1 \\ l/2+1/2p-3/4 & 5l-2 \end{bmatrix} K_p \|\Delta r\|_{H^{1,1}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} l+1 & l-1 \\ (l-1)/2 & 5l-4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l+1 & l+1 \\ l/2+1/2p-3/4 & 5l-1 \end{bmatrix} K_p \|\Delta r\|_{H^{1,1}}.\end{aligned}$$

The estimate for $\Delta F^{(2)}$ is the same.

For $2<p\leqslant 4$, by Lemmas 5.1, 5.27 and 5.51,

$$\begin{aligned}\|\Delta F^{(4)}\|_{L^2(dz)} &\leqslant c\|\Delta m_-^{-1}\|_{L^4(dz)\otimes L^\infty(dx)}\|G\|_{L^1(dx)}\|m_- - I\|_{L^4(dz)\otimes L^\infty(dx)} \\ &\quad + \|m_-^{-1}-I\|_{L^4(dz)\otimes L^\infty(dx)}\|\Delta G\|_{L^1(dx)}\|m_- - I\|_{L^4(dz)\otimes L^\infty(dx)} \\ &\quad + \|m_-^{-1}-I\|_{L^4(dz)\otimes L^\infty(dx)}\|G\|_{L^1(dx)}\|\Delta m_-\|_{L^4(dz)\otimes L^\infty(dx)} \\ &\leqslant \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} K_4^2\|\Delta r\|_{H^{1,1}}\begin{bmatrix} l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_4 \\ &\quad + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_4\begin{bmatrix} l & l+1\\ l/2+1/2p-3/4 & 5l-2\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_4 \\ &\quad + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}K_4\begin{bmatrix} l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}K_4^2\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l+2 & l+1\\ l/2+1/2p-3/4 & 5l-2\end{bmatrix}K_4^3\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Thus using $(1-\varrho)^{-1}\leqslant cK_4$, we get

$$\begin{aligned}\|\Delta F\|_{L^2(dz)} &\leqslant \left(\begin{bmatrix} l & l+2\\ l/2+1/2p-1/4 & 5l+1/2\end{bmatrix} + \begin{bmatrix} l+1 & l+1\\ l/2+1/2p-3/4 & 5l-3\end{bmatrix}\right. \\ &\qquad \left. + \begin{bmatrix} l+1 & l+2\\ l/2+1/2p-3/4 & 5l-2\end{bmatrix}\right)K_4^3\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l & l+3\\ l/2+1/2p-3/4 & 5l+1/2\end{bmatrix}K_4^3\|\Delta r\|_{H^{1,1}}. \qquad \square\end{aligned}$$

LEMMA 6.29. *For $p>2$,*

$$\|\Delta\Diamond F^{(1)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l & l+4\\ l/2+1/2p-1/2 & 5l+7/2\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}.$$

Proof. This lemma follows directly from Lemma 5.51 and the fact that

$$\|\Delta\Diamond F^{(1)}\|_{L^2(dz)} = c\|\Delta\partial_x G\|_{L^2(dx)}. \qquad \square$$

LEMMA 6.30. *For $2<p<\infty$,*

$$\|\Delta\Diamond F^{(2)}\|_{L^2(dz)},\ \|\Delta\Diamond F^{(3)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+1 & l+3\\ l/2+1/2p-1 & 5l+4\end{bmatrix}K_p\|\Delta r\|_{H^{1,1}}.$$

Proof. We will only prove the lemma for $F^{(3)}$. Again the estimate for $F^{(2)}$ is similar. Using the expression (6.12), we have

$$\Delta z F^{(3)} = -iP_{12}\int e^{-i\theta}(\Delta\partial_y G)(m_- - I)\,dy - iP_{12}\int e^{-i\theta}\partial_y G\Delta m_-$$
$$-iP_{12}\int e^{-i\theta}(\Delta(GQ))(m_- - I)\,dy - iP_{12}\int e^{-i\theta}(GQ)\Delta m_-.$$

Thus by Lemmas 5.51, 5.1 and 5.27, for any $p>2$,

$$\begin{aligned}
\|\Delta\Diamond F^{(3)}\|_{L^2(dz)} &\leqslant \|\Delta\partial G\|_{L^1(dx)}\|m_- - I\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\quad + \|\partial G\|_{L^1(dx)}\|\Delta m_-\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\quad + \|\Delta GQ\|_{L^1(dx)}\|m_- - I\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\quad + \|GQ\|_{L^1(dx)}\|\Delta m_-\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\leqslant \begin{bmatrix} l & l+3 \\ l/2+1/2p-1 & 5l-1 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} l+1 & l-1 \\ (l-1)/2 & 5l-4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} K_2^2\|\Delta r\|_{H^{1,1}} \\
&\quad + \begin{bmatrix} l+1 & l+2 \\ l/2+1/2p-1/4 & 5l+3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}} \\
&\quad + \begin{bmatrix} l+2 & l \\ l/2 & 5l+1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} K_2^2\|\Delta r\|_{H^{1,1}} \\
&\leqslant \begin{bmatrix} l+1 & l+3 \\ l/2+1/2p-1 & 5l+4 \end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}. \qquad \square
\end{aligned}$$

LEMMA 6.31. *For* $2<p\leqslant 4$,

$$\|\Delta\Diamond F^{(4)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+2 & l+3 \\ l/2+1/2p-3/4 & 5l \end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}.$$

Proof.

$$\begin{aligned}
\Delta F^{(4)} &= -iP_{12}\int e^{-iyz}(m_-^{-1}-I)G\Delta m_-\,dy \\
&\quad -iP_{12}\int e^{-iyz}(\Delta m_-^{-1})G(m_- - I) \\
&\quad -iP_{12}\int e^{-iyz}(m_-^{-1}-I)\Delta G(m_- - I)\,dy \\
&\equiv \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

Using Lemmas 5.27 and 5.6, we have

$$\begin{aligned}\|\Delta\Diamond \mathrm{I}\|_{L^2(dz)} &\leqslant \|G\|_{L^1(dx)}\| |\Diamond|^{1/2}(m_-^{-1}-I)\|_{L^4(dz)\otimes L^\infty(dx)}\| |\Diamond|^{1/2}\Delta m_-\|_{L^4(dz)\otimes L^\infty(dx)}\\ &\leqslant \begin{bmatrix} l+1 & l-1\\ (l-1)/2 & 5l-4\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} K_4 \begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} K_4^2\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix} l+2 & l+2\\ (l-1)/2 & 5l-2\end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Similarly,

$$\|\Delta\Diamond \mathrm{II}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+2 & l+2\\ (l-1)/2 & 5l-2\end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}.$$

Using Lemma 5.51, we obtain for $2<p\leqslant 4$,

$$\begin{aligned}\|\Delta\Diamond \mathrm{III}\|_{L^2(dz)} &\leqslant \|\Delta G\|_{L^1(dx)}\| |\Diamond|^{1/2}(m_-^{-1}-I)\|_{L^4(dz)\otimes L^\infty(dx)}\\ &\quad\times \| |\Diamond|^{1/2}(m_- - I)\|_{L^4(dz)\otimes L^\infty(dx)}\\ &\leqslant \begin{bmatrix} l & l+1\\ l/2+1/2p-3/4 & 5l-2\end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^2 K_4^2\\ &= \begin{bmatrix} l+2 & l+3\\ l/2+1/2p-3/4 & 5l\end{bmatrix} K_pK_4^2\|\Delta r\|_{H^{1,1}}.\end{aligned}$$

Finally,

$$\begin{aligned}\|\Delta\Diamond F^{(4)}\|_{L^2(dz)} &\leqslant \left(\begin{bmatrix} l+2 & l+2\\ (l-1)/2 & 5l-2\end{bmatrix} + \begin{bmatrix} l+2 & l+3\\ l/2+1/2p-3/4 & 5l\end{bmatrix}\right) K_4^3\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix} l+2 & l+3\\ l/2+1/2p-3/4 & 5l\end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}. \qquad \square\end{aligned}$$

LEMMA 6.32. *For* $2<p\leqslant 4$,

$$\|\Delta\Diamond F\|_{L^2(dz)} \leqslant \begin{bmatrix} l & l+5\\ l/2+1/2p-1 & 5l+2\end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}.$$

Proof.

$$\begin{aligned}\|\Delta\Diamond F\|_{L^2(dz)} &\leqslant \begin{bmatrix} l & l+4\\ l/2+1/2p-1/2 & 5l+7/2\end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\\ &\quad+\begin{bmatrix} l+1 & l+3\\ l/2+1/2p-1 & 5l+4\end{bmatrix} K_p\|\Delta r\|_{H^{1,1}}\\ &\quad+\begin{bmatrix} l+2 & l+3\\ l/2+1/2p-3/4 & 5l\end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}\\ &\leqslant \begin{bmatrix} l & l+5\\ l/2+1/2p-1 & 5l+2\end{bmatrix} K_4^3\|\Delta r\|_{H^{1,1}}. \qquad \square\end{aligned}$$

We now use (6.14) to estimate $\|\partial_z \Delta F\|_{L^2(dz)}$, $\partial_z \Delta F = \Delta F^{(5)} + \Delta F^{(6)} + \Delta F^{(7)}$. In turn

$$\Delta F^{(7)} = -\Delta F^{(71)} - \Delta F^{(72)} - \Delta F^{(73)} - \Delta F^{(74)}, \tag{6.33}$$

and from (6.16)

$$\begin{aligned}\Delta F^{(71)} &= \int (\Delta(\tilde{L}G)_{12}) e^{-i\theta} \delta_-^{-2}\, dx + \int (\tilde{L}G)_{12} e^{-i\theta} \Delta \delta_-^{-2}\, dx \\ &\quad - \int_{x<2tz} (\Delta(\tilde{L}G)_{21}) r^2 e^{i\theta} \delta_-^{-2}\, dx - \int_{x<2tz} (\tilde{L}G)_{21} (\Delta r^2(z)) e^{i\theta} \delta_-^{-2}\, dx \\ &\quad - \int_{x<2tz} (\tilde{L}G)_{21} r^2 e^{i\theta} (\Delta \delta_-^{-2})\, dx \\ &= \Delta F^{(711)} + \Delta F^{(712)} + \Delta F^{(713)} + \Delta F^{(714)} + \Delta F^{(715)}.\end{aligned}$$

By (6.20), (6.22) and Lemma 5.36, for any $1<p'<2$, $p>2$, $n\geqslant 3$,

$$\begin{aligned}\|\Delta F^{(711)}\|_{L^2(dz)} + \|\Delta F^{(713)}\|_{L^2(dz)} &= \frac{c\eta^2}{1-\varrho} \|\Delta \tilde{L}G\|_{L^{p'}} + c\|\Delta \tilde{L}G\|_{L^2} \\ &\leqslant \frac{c\eta^2}{1-\varrho} \begin{bmatrix} l & l+5-2/p' \\ l/2+1/2p-1/2n-1/4-1/p' & 5l+11/2-9/p' \end{bmatrix} K^2_{2(n-1)} \|\Delta r\|_{H^{1,1}} \\ &\quad + \begin{bmatrix} l & l+4 \\ l/2+1/2p-1/2n-3/4 & 5l+1 \end{bmatrix} K^2_{2(n-1)} \|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l & l+7-2/p' \\ l/2+1/2p-1/2n-1/4-1/p' & \max(5l+13/2-9/p', 5l+1) \end{bmatrix} K^2_{2(n-1)} \|\Delta r\|_{H^{1,1}}.\end{aligned} \tag{6.34}$$

Also from the proof of (6.23), for any $1<p<2$, $n\geqslant 3$,

$$\begin{aligned}\|\Delta F^{(714)}\|_{L^2(dz)} &\leqslant \frac{c\eta \|\Delta r\|_{H^{1,1}}}{1-\varrho} \|\tilde{L}G\|_{L^p} \\ &\leqslant \frac{\eta \|\Delta r\|_{H^{1,1}}}{1-\varrho} \begin{bmatrix} l+1 & l+3-2/p \\ l/2-1/p-1/2n & 5l+7/2-9/p \end{bmatrix} K_{2(n-1)} \\ &= \begin{bmatrix} l+2 & l+3-2/p \\ l/2-1/p-1/2n & 5l+9/2-9/p \end{bmatrix} K_{2(n-1)} \|\Delta r\|_{H^{1,1}},\end{aligned} \tag{6.35}$$

where we have again used Lemma 5.29.

Now let $\Delta H(z) = \int h(x) e^{-izx} \Delta \delta_-^{-2}(z; z_0)\, dx$, where $h \in L^1 \cap L^2(\mathbf{R})$, as before, and we have

$$\int e^{iyz} \Delta H(z)\, dz = \int h(x) \left(\int e^{i(y-x)z} \Delta \delta_-^{-2}(z, z_0)\, dz \right) dx.$$

By analyticity, $\int e^{isz}\Delta\delta_-^{-2}\,dz=0$ for $s<0$. For $s>0$, $\int e^{isz}\Delta\delta_+^{-2}\,dz=0$, and so

$$\int e^{isz}\Delta\delta_-^{-2}\,dz=\int_{-\infty}^{z_0} e^{isz}\Delta\delta_+^{-2}(-2|r|^2+|r|^4)\,dz+\int_{-\infty}^{z_0} e^{isz}\delta_+^{-2}(-2\Delta|r|^2+\Delta|r|^4)\,dz.$$

As in the proof of (6.19), we conclude that for all $s\geqslant 0$,

$$\left|\int_{-\infty}^{z_0} e^{isz}\delta_+^{-2}(-2\Delta|r|^2+\Delta|r|^4)\,dz\right|\leqslant\frac{c\eta\|\Delta r\|_{H^{1,1}}}{1-\varrho}\,\frac{1}{1+|s|}. \tag{6.36}$$

On the other hand, from (6.18), we have

$$\Delta G(s,z,z_0)=\int_{+i\infty}^{z} e^{is\zeta}\Delta\delta^{-1}(\zeta,z_0)\,d\zeta,\quad s>0,\ z<z_0. \tag{6.37}$$

Hence

(i) $\partial_z\Delta G(s,z,z_0)=e^{isz}\Delta\delta^{-2}(z,z_0)$, $z<z_0$;

(ii) For any $1<\alpha,\beta<\infty$, $1/\alpha+1/\beta=1$, $s>0$, $z<z_0$,

$$\begin{aligned}|\Delta G(s,z,z_0)|&\leqslant\|e^{-s\Diamond}\|_{L^\alpha(0,\infty)}\|\Delta\delta^{-2}(z+i\Diamond,z_0)\|_{L^\beta(0,\infty)}\\&\leqslant\frac{c}{s^{1/\alpha}}\|\Delta\delta^{-2}(z+i\Diamond,z_0)\|_{L^\beta(0,\infty)}.\end{aligned}$$

But from (5.45) (or alternatively, from (the proof of) Lemma 4.32), for $1<\beta<\infty$,

$$\|\Delta\delta^{-2}(z+i\Diamond,z_0)\|_{L^\beta(0,\infty)}\leqslant\frac{c}{(1-\varrho)^2}\|\Delta r\|_{L^\beta}\leqslant\frac{c\|\Delta r\|_{H^{1,1}}}{(1-\varrho)^2}. \tag{6.38}$$

Hence

$$|\Delta G(s,z,z_0)|\leqslant\frac{c}{s^{1/\alpha}(1-\varrho)^2}\|\Delta r\|_{H^{1,1}}. \tag{6.39}$$

Previous calculations now show that for all $s\geqslant 0$,

$$\int_{-\infty}^{z_0} e^{isz}(\Delta\delta_+^{-2})(-2|r|^2+|r|^4)\,dz\leqslant\frac{c\eta^2}{(1-\varrho)^2}\,\frac{\|\Delta r\|_{H^{1,1}}}{(1+|s|)^{1/\alpha}},$$

which implies in turn for all $s\in\mathbf{R}$ and any $1<\alpha<\infty$,

$$\left|\int e^{isz}\Delta\delta_-^{-2}\,dz\right|\leqslant\frac{c\eta(1+\eta)}{(1-\varrho)^2}\,\frac{\|\Delta r\|_{H^{1,1}}}{(1+|s|)^{1/\alpha}}. \tag{6.40}$$

Hence for $1/q+1/p=3/2$, $1<p<2$, $1<\alpha<q<2$,

$$\begin{aligned}\|\Delta H\|_{L^2(dz)}&\leqslant\frac{c\eta(1+\eta)}{(1-\varrho)^2}\|h\|_{L^p}\left\|\frac{1}{(1+|\Diamond|)^\alpha}\right\|_{L^q}\|\Delta r\|_{H^{1,1}}\\&\leqslant\frac{c\eta(1+\eta)}{(1-\varrho)^2}\|h\|_{L^p}\|\Delta r\|_{H^{1,1}}.\end{aligned} \tag{6.41}$$

Applying this estimate to $(\tilde{L}G)_{12}$, and choosing α and q appropriately, we obtain for any $1<p<2$, $n\geqslant 3$,

$$\begin{aligned}\|\Delta F^{(712)}\|_{L^2(dz)} &\leqslant \frac{\eta(1+\eta)}{(1-\varrho)^2}\begin{bmatrix} l+1 & l+3-2/p \\ l/2-1/p-1/2n & 5l+7/2-9/p\end{bmatrix} K_{2(n-1)}\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l+2 & l+4-3/p \\ l/2-1/p-1/2n & 5l+11/2-9/q\end{bmatrix} K_{2(n-1)}\|\Delta r\|_{H^{1,1}}.\end{aligned} \tag{6.42}$$

Similarly, replacing δ^2 in $J(z)$ by $\Delta\delta^2$, and using ΔG as defined in (6.37) in place of G, we find for any $1<p<2$, $n\geqslant 3$,

$$\begin{aligned}\|\Delta F^{(715)}\|_{L^2(dz)} &\leqslant \frac{c\eta^2}{(1-\varrho)^2}\|\tilde{L}G\|_{L^p}\|\Delta r\|_{H^{1,1}} \\ &\leqslant \begin{bmatrix} l+3 & l+3-2/p \\ l/p-1/p-1/2n & 5l+7/2-9/p\end{bmatrix} K_{2(n-1)}\|\Delta r\|_{H^{1,1}}.\end{aligned} \tag{6.43}$$

Adding up the contributions, we obtain finally for any $1<p'<2$, $p>2$, $n\geqslant 3$,

$$\begin{aligned}\|\Delta F^{(71)}\|_{L^2} \leqslant & \Bigg(\begin{bmatrix} l & l+7-2/p' \\ l/2+1/2p-1/2n-1/4-1/p' & \max(5l+13/2-9/p', 5l+1)\end{bmatrix} K^2_{2(n-1)} \\ &+\begin{bmatrix} l+2 & l+3-2/p \\ l/p-1/p-1/2n & 5l+9/2-9/p\end{bmatrix} K_{2(n-1)} \\ &+\begin{bmatrix} l+2 & l+4-2/p \\ l/2-1/p-1/2n & 5l+11/2-9/p\end{bmatrix} K_{2(n-1)} \\ &+\begin{bmatrix} l+3 & l+3-2/p \\ l/2-1/p-1/2n & 5l+7/2-9/p\end{bmatrix} K_{2(n-1)}\Bigg)\|\Delta r\|_{H^{1,1}} \\ \leqslant & \begin{bmatrix} l & \max(l+7-2/p', l+6-2/p) \\ l/2+1/2p-1/2n-1/4-1/p' & \max(5l+13/2-9/p', 5l+9/2-9/p, 5l+1)\end{bmatrix} \\ & \times K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}},\end{aligned} \tag{6.44}$$

where again we have used $(1-\varrho)^{-1}\leqslant cK_{2(n-1)}$.

Now consider $\Delta F^{(73)}$. As

$$v^{\#}(z)e_2 = \begin{pmatrix} -r \\ 1\end{pmatrix}, \quad e_1(v^{\#}(z))^{-1} = (1, r) \quad \text{for } z > z_0$$

and

$$v^{\#}(z)e_2 = \begin{pmatrix} 0 \\ 1\end{pmatrix}, \quad e_1(v^{\#}(z))^{-1} = (1, 0) \quad \text{for } z < z_0,$$

we see that the factor $\bar{r}/(1-|r|^2)$ in $v^{\#}(z)$ never appears in $\Delta F^{(7)}$. Thus for $2<p\leqslant 4$, $n\geqslant 3$, using (6.38) and its analog for $\Delta\delta^{+2}$, we obtain

$$\begin{aligned}
\|\Delta F^{(73)}\|_{L^2(dz)} &\leqslant \frac{c}{1-\varrho}\|\Delta\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\quad+\frac{c}{1-\varrho}\|\tilde{L}G\|_{L^1(dx)}\|\Delta\tilde{\mu}\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\quad+\frac{c}{1-\varrho}\|\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|_{L^2(dz)\otimes L^\infty(dx)}\|\Delta r\|_{H^{1,1}} \qquad (6.45)\\
&\quad+c\|\tilde{L}G\|_{L^1(dx)}\|(|\Delta\delta_-^2|+|\Delta\delta_-^{-2}|)(\tilde{\mu}-I)\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\leqslant \frac{1}{1-\varrho}\begin{bmatrix} l & l+3 \\ l/2+1/2p-1/2n-5/4 & 5l-7/2\end{bmatrix}K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}\begin{bmatrix}1 & 0\\ 1/4 & 3\end{bmatrix} \\
&\quad+\frac{1}{1-\varrho}\begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\begin{bmatrix}0 & 2\\ 1/2p & 4\end{bmatrix}K_p^2\|\Delta r\|_{H^{1,1}} \\
&\quad+\frac{1}{1-\varrho}\begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\begin{bmatrix}1 & 0\\ 1/4 & 3\end{bmatrix}\|\Delta r\|_{H^{1,1}} \\
&\quad+\begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\frac{\|\Delta r\|_{H^{1,1}}}{(1-\varrho)^2}\begin{bmatrix}1 & 0\\ 1/2p & 2\end{bmatrix}K_p \\
&\leqslant \begin{bmatrix} l+1 & l+3 \\ l/2+1/2p-1/2n-1 & 5l-1/2\end{bmatrix}K^3_{2(n-1)}\|\Delta r\|_{H^{1,1}}.
\end{aligned}$$

The same estimate is clearly true for $\|\Delta F^{(72)}\|_{L^2(dz)}$.

Again for $p''>4$, $n\geqslant 3$, $2<p\leqslant 4$,

$$\begin{aligned}
&\|\Delta F^{(74)}\|_{L^2(dz)} \leqslant \frac{c}{1-\varrho}\|\Delta\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|^2_{L^4(dz)\otimes L^\infty(dx)} \\
&+\frac{c}{1-\varrho}\|\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|_{L^4(dz)\otimes L^\infty(dx)}\|\Delta\tilde{\mu}\|_{L^4(dz)\otimes L^\infty(dx)} \\
&+\frac{c}{1-\varrho}\|\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|^2_{L^4(dz)\otimes L^\infty(dx)}\|\Delta r\|_{H^{1,1}} \qquad (6.46)\\
&+c\|\tilde{L}G\|_{L^1(dx)}\|\tilde{\mu}-I\|_{L^4(dz)\otimes L^\infty(dx)}\|(|\Delta\delta_-^2|+|\Delta\delta_-^{-2}|)(\tilde{\mu}-I)\|_{L^4(dz)\otimes L^\infty(dx)} \\
&\qquad\leqslant \frac{1}{1-\varrho}\begin{bmatrix} l & l+3 \\ l/2+1/2p-1/2n-5/4 & 5l-7/2\end{bmatrix}K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}}\begin{bmatrix}1 & 0\\ 1/8 & 2\end{bmatrix}^2K_4^2 \\
&+\frac{1}{1-\varrho}\begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\begin{bmatrix}1 & 0\\ 1/8 & 2\end{bmatrix}K_4\begin{bmatrix}0 & 2\\ 1/2p'' & 4\end{bmatrix}K^2_{p''}\|\Delta r\|_{H^{1,1}} \\
&+\frac{1}{1-\varrho}\begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2\end{bmatrix}K_{2(n-1)}\begin{bmatrix}1 & 0\\ 1/8 & 2\end{bmatrix}^2K_4^2\|\Delta r\|_{H^{1,1}}
\end{aligned}$$

$$+\frac{1}{1-\varrho}\begin{bmatrix} l+1 & l+1 \\ (l-2)/2-1/2n & 5l-11/2 \end{bmatrix} K_{2(n-1)} \begin{bmatrix} 1 & 0 \\ 1/8 & 2 \end{bmatrix} K_4 \frac{\|\Delta r\|_{H^{1,1}}}{(1-\varrho)^2} \begin{bmatrix} 1 & 0 \\ 1/2p'' & 2 \end{bmatrix} K_{p''}$$

$$\leqslant \begin{bmatrix} l+2 & l+3 \\ l/2+1/2p-1/2n-1 & 5l+3/2 \end{bmatrix} \max(K^4_{2(n-1)}, K^4_{p''}) \|\Delta r\|_{H^{1,1}}.$$

Assembling the above estimates, we have shown that for $2<p\leqslant 4$, $p''>4$, $n\geqslant 3$, $1<p'<2$,

$$\begin{aligned} \|\Delta F^{(7)}\|_{L^2} \leqslant & \begin{bmatrix} l & \max(l+7-2/p', l+6-2/p) \\ l/2+1/2p-1/2n-1/4-1/p' & \max(5l+13/2-9/p', 5l+9/2-9/p, 5l+1) \end{bmatrix} \\ & \times K^2_{2(n-1)} \|\Delta r\|_{H^{1,1}} \\ & + \begin{bmatrix} l+1 & l+3 \\ l/2+1/2p-1/2n-1 & 5l-1/2 \end{bmatrix} K^3_{2(n-1)} \|\Delta r\|_{H^{1,1}} \qquad (6.47) \\ & + \begin{bmatrix} l+2 & l+3 \\ l/2+1/2p-1/2n-1 & 5l+3/2 \end{bmatrix} \max(K^4_{2(n-1)}, K^4_{p''}) \|\Delta r\|_{H^{1,1}} \\ \leqslant & \begin{bmatrix} l & \max\left(l+7-\frac{2}{p'}, l+6-\frac{2}{p}\right) \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-\frac{1}{4}-\frac{1}{p'}, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{3}{2} \end{bmatrix} \\ & \times \max(K^4_{2(n-1)}, K^4_{p''}) \|\Delta r\|_{H^{1,1}}. \end{aligned}$$

In other words, we have proved the following lemma.

LEMMA 6.48. *For $2<p\leqslant 4$, $p''>4$, $n\geqslant 3$ and $1<p'<2$,*

$$\begin{aligned} & \|\Delta F^{(7)}\|_{L^2(dx)} \\ & \leqslant \begin{bmatrix} l & \max\left(l+7-\frac{2}{p'}, l+6-\frac{2}{p}\right) \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-\frac{1}{4}-\frac{1}{p'}, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{3}{2} \end{bmatrix} \\ & \quad \times \max(K^4_{2(n-1)}, K^4_{p''}) \|\Delta r\|_{H^{1,1}}. \end{aligned}$$

LEMMA 6.49. *For $n\geqslant 3$,*

$$\|\Delta F^{(5)}\|_{L^2(dz)}, \|\Delta F^{(6)}\|_{L^2(dz)} \leqslant \begin{bmatrix} l+1 & l+2/n+4 \\ l/2-1/2n-3/4 & 5l+8/3 \end{bmatrix} K^3_{2(n-1)} \|\Delta r\|_{H^{1,1}}.$$

Proof. For $n\geqslant 3$ and $2<p\leqslant 4$, we compute using Lemmas 5.17, 5.23, 5.27 and 5.51,

$$\begin{aligned}
\|\Delta F^{(6)}\|_{L^2(dz)} &= \Big\| P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma}(\Delta m_-^{-1})G\mathbf{A}m_- + P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma} m_-^{-1}G\Delta\mathbf{A}m_- \\
&\qquad + P_{12}\int e^{-i\theta\,\mathrm{ad}\,\sigma} m_-^{-1}(\Delta G)\mathbf{A}m_-\Big\|_{L^2(dz)} \\
&\leqslant \|\Delta m_-^{-1}\|_{L^\infty(dz)\otimes L^\infty(dx)}\|G\|_{L^1(dx)}\|\mathbf{A}m_-\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\qquad + \|m_-^{-1}\|_{L^\infty(dz)\otimes L^\infty(dx)}\|G\|_{L^1(dx)}\|\Delta\mathbf{A}m_-\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\qquad + \|m_-^{-1}\|_{L^\infty(dz)\otimes L^\infty(dx)}\|\Delta G\|_{L^1(dx)}\|\mathbf{A}m_-\|_{L^2(dz)\otimes L^\infty(dx)} \\
&\leqslant \begin{bmatrix} 0 & 2 \\ -1/4 & 7/3 \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}} \begin{bmatrix} l+1 & l-1 \\ l/2-1/2 & 5l-4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)} \\
&\qquad + \begin{bmatrix} 0 & 1+2/n \\ -1/2n & 5/n \end{bmatrix} K_{2(n-1)}\begin{bmatrix} l+1 & l-1 \\ l/2-1/2 & 5l-4 \end{bmatrix} \\
&\qquad\quad \times \begin{bmatrix} 0 & 3 \\ -1/4 & 10/3 \end{bmatrix} K^2_{2(n-1)}\|\Delta r\|_{H^{1,1}} \\
&\qquad + \begin{bmatrix} 0 & 1+2/n \\ -1/2n & 5/n \end{bmatrix} K_{2(n-1)}\begin{bmatrix} l & l+1 \\ l/2+1/2p-3/4 & 5l-2 \end{bmatrix} \\
&\qquad\quad \times K_p\|\Delta r\|_{H^{1,1}}\begin{bmatrix} 1 & 2 \\ -1/2n & 3 \end{bmatrix} K_{2(n-1)} \\
&\leqslant \left(\begin{bmatrix} l+2 & l+3 \\ l/2-1/2n-3/4 & 5l+4/3 \end{bmatrix} + \begin{bmatrix} l+1 & l+2/n+3 \\ l/2-1/2n-3/4 & 5l+5/n-2/3 \end{bmatrix}\right. \\
&\qquad \left. + \begin{bmatrix} l+1 & l+2/n+4 \\ l/2+1/2p-1/n-3/4 & 5l+5/n+1 \end{bmatrix}\right) K^3_{2(n-1)}\|\Delta r\|_{H^{1,1}} \\
&\leqslant \begin{bmatrix} l+1 & l+2/n+4 \\ l/2-1/2n-3/4 & \max(5l+4/3, 5l+5/n+1) \end{bmatrix} K^3_{2(n-1)}\|\Delta r\|_{H^{1,1}}.
\end{aligned}$$

The estimate for $\Delta F^{(5)}$ is the same. □

LEMMA 6.50. *For $n\geqslant 3$, $2<p\leqslant 4$, $p''>4$, $n\geqslant 3$ and $1<p'<2$,*

$$\begin{aligned}
&\|\Delta\partial_z F\|_{L^2(dz)} \\
&\leqslant \begin{bmatrix} l & \max\left(l+7-\frac{2}{p'}, l+6-\frac{2}{p}, l+\frac{2}{n}+5\right) \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-\frac{1}{4}-\frac{1}{p'}, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{8}{3} \end{bmatrix} \\
&\qquad \times\max(K^4_{2(n-1)}, K^4_{p''})\|\Delta r\|_{H^{1,1}}.
\end{aligned}$$

Proof. Combining Lemmas 6.48 and 6.49, we obtain

$$\begin{aligned}
&\|\Delta\partial_z F\|_{L^2(dz)}\\
&\quad\leqslant \|\Delta F^{(5)}\|_{L^2(dz)}+\|\Delta F^{(6)}\|_{L^2(dz)}+\|\Delta F^{(7)}\|_{L^2(dz)}\\
&\quad\leqslant \begin{bmatrix} l+1 & l+2/n+4 \\ l/2-1/2n-3/4 & 5l+8/3 \end{bmatrix} K^4_{2(n-1)}\|\Delta r\|_{H^{1,1}}\\
&\qquad+\begin{bmatrix} l & \max\left(l+7-\frac{2}{p'},\, l+6-\frac{2}{p}\right) \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-\frac{1}{4}-\frac{1}{p'},\, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{3}{2} \end{bmatrix}\\
&\qquad\quad\times\max(K^4_{2(n-1)}, K^4_{p''})\|\Delta r\|_{H^{1,1}}\\
&\quad\leqslant \begin{bmatrix} l & \max\left(l+7-\frac{2}{p'},\, l+6-\frac{2}{p},\, l+\frac{2}{n}+5\right) \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-\frac{1}{4}-\frac{1}{p'},\, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{8}{3} \end{bmatrix}\\
&\qquad\quad\times\max(K^4_{2(n-1)}, K^4_{p''})\|\Delta r\|_{H^{1,1}}.
\end{aligned}$$

□

LEMMA 6.51. *For* $n\geqslant 3$, $2<p\leqslant 4$ *and* $p''>4$,

$$\|\Delta F\|_{H^{1,1}}\leqslant \begin{bmatrix} l & l+6 \\ l/2+1/2p-1/2n-1 & 5l+8/3 \end{bmatrix}\max(K^4_{2(n-1)}, K^4_{p''})\|\Delta r\|_{H^{1,1}}.$$

Proof. Combining Lemmas 6.28, 6.32 and 6.50, we obtain for $1<p'<2$,

$$\begin{aligned}
&\|\Delta F\|_{H^{1,1}}\leqslant \|\Delta F\|_{L^2(dz)}+\|\Diamond F\|_{L^2(dz)}+\|\Delta\partial_z F\|_{L^2(dz)}\\
&\quad\leqslant\left(\begin{bmatrix} l & l+3 \\ l/2+1/2p-3/4 & 5l+1/2 \end{bmatrix}+\begin{bmatrix} l & l+5 \\ l/2+1/2p-1 & 5l+2 \end{bmatrix}\right)K^4_{2(n-1)}\|\Delta r\|_{H^{1,1}}\\
&\qquad+\begin{bmatrix} l & \max\left(l+2-\frac{2}{p'},\, l+6-\frac{2}{p'},\, l+\frac{2}{n}+5\right) \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-\frac{1}{4}-\frac{1}{p'},\, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{8}{3} \end{bmatrix}\\
&\qquad\quad\times\max(K^4_{2(n-1)}, K^4_{p''})\|\Delta r\|_{H^{1,1}}\\
&\quad\leqslant \begin{bmatrix} l & l+6 \\ \min\left(\frac{l}{2}+\frac{1}{2p}-\frac{1}{4}-\frac{1}{p'},\, \frac{l}{2}+\frac{1}{2p}-\frac{1}{2n}-1\right) & 5l+\frac{8}{3} \end{bmatrix}\max(K^4_{2(n-1)}, K^4_{p''})\|\Delta r\|_{H^{1,1}}.
\end{aligned}$$

Taking $\frac{4}{3}<p'<1$, the desired estimate follows. □

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