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Perturbations and Hopf bifurcation of the planar discontinuous dynamical system

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Abstract

The objective of the paper is to obtain results on the behavior of a specific plane discontinuous dynamical system in the neighbourhood of the singular point. A new technique of investigation is presented. Conditions for existence of the foci and centres are proposed. The focus-centre problem and Hopf bifurcation are considered. Appropriate examples are given to ilustrate the bifurcation theorem.

Keywords: Planar dynamical systems; Discontinuous systems; Focus; Centre; Stability; Hopf bifurcation

1. Introduction and Preliminaries

The theory of dynamics with discontinuous trajectories has been developed through applications [3,4,6,15,19,20,22,30,32,34,35] and theoretical challenges [15,16,22,26,36–38]. The present paper can be considered as an attempt to apply ideas of the perturbations theory, which was founded by Poincaré and Lyapunov [28,33], and methods of the bifurcation theory [4,8,15,18,21,29,30,33] to the object which combines features of vector fields and maps. In fact, we consider the problem for equations with variable time of impulses. Effective methods of the investigation of systems with nonfixed moments of impulsive action can be found in [9,12,13,26,27,38]. Theoretical problems of nonsmooth dynamics and

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discontinuous maps [5,10,11,14,23–25] also very close to the subject of our paper. In our view there have been two principal obstacles to a thorough investigation of the subject. While the absence of sufficiently general results on the smoothness of solutions has been the first one, the problem of choice of a nonperturbed system convenient for study has been the second. The present work utilizes extensively the differentiable and analytical dependence discontinuous solutions on parameters [1,2,26,40]. Moreover, the nonperturbed equation is specifically defined. And, while all of its terms are linear, this equation is essentially a nonlinear one. We should remark that in general the perturbed systems with sets of discontinuities of linear nature have been considered previously. One example is the clock model [22,38]. But we investigate systems all of whose terms are nonlinear. This necessitates the use of the standard linearization method and application of the concept of B-equivalent discontinuous systems which has been developed in [1–3] for nonautonomous case. The approach of the paper can be effectively employed for investigation of oscillations in mechanics, electronics, biology and medicine [4,15,30–32,34].

The paper is organized in the following manner. In Section 1, we give the description of the systems under consideration and prove the theorem of existence of foci and centres of the nonperturbed system. The main subject of Section 2 is foci of the perturbed equation. The noncritical case is considered. In Section 3 the problem of distinguishing between the centre and the focus is solved. Bifurcation of a periodic solution is investigated in Section 4. Section 5 consists of examples illustrating the bifurcation theorem.

1.1. *The nonperturbed system*

Let N, R be sets of all natural and real numbers, respectively, R^2 be a real euclidean space. Denote by $\langle x, y\rangle$ the dot-product of vectors $x, y \in R^2$. Let $||x|| = \langle x, x\rangle^{\frac{1}{2}}$ be the norm of a vector $x \in R^2$, $\mathscr{R}$ be the set of all real-valued constant 2×2 matrices, $I \in \mathscr{R}$ be an identity matrix. We shall consider in R^2 the following dynamical system:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = Ax, \quad x \notin \Gamma_0,\\ &\Delta x|_{x\in\Gamma_0} = B_0 x, \end{aligned} \tag{1}$$

where $A, B_0 \in \mathscr{R}$, Γ_0 is a subset of R^2 and will be described below. The phase point of (1) moves between two consecutive intersections with the set Γ_0 along one of the trajectories of the system $x' = Ax$. When the solution meets the set Γ_0 at the moment τ, the point $x(t)$ has an jump $\Delta x|_\tau := x(\tau+) - x(\tau)$. Thus we suppose that the solutions are left continuous functions. The following assumptions will be needed throughout the paper:

(C1) $\Gamma_0 = \bigcup_{i=1}^p s_i$, $p \in N$, where s_i are half-lines starting at the origin and defined by equations $\langle a^i, x\rangle = 0$, $i = \overline{1, p}$, where $a^i = (a_1^i, a_2^i) \in R^2$ are constant vectors;

(C2)
$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

where $\beta \neq 0$;

(C3) there exists a regular matrix $Q \in \mathscr{R}$ and nonnegative real numbers k and θ such that

$$B_0 = kQ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} Q^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also, for the sake of brevity, in what follows every angle for a point or a line is considered with respect to the positive half-line of the first coordinate axis.
Denote $s_i' = (I + B_0)s_i, i = \overline{1, p}$. Let γ_i and ζ_i be angles of s_i and $s_i', i = \overline{1, p}$, respectively,

$$B_0 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

(C4) $0 < \gamma_1 < \zeta_1 < \gamma_2 < \cdots < \gamma_p < \zeta_p < 2\pi$, and $(b_{11} + 1) \cos \gamma_i + b_{12} \sin \gamma_i \neq 0, \ i = \overline{1, p}$.

The system (1) is said to be a D_0-*system* if conditions (C1)–(C4) hold. It is easy to see that the origin is a unique singular point of D_0-*system* and (1) is not linear.

Let us subject (1) to the transformation $x_1 = r \cos(\phi), x_2 = r \sin(\phi)$ and exclude time variable t. Then solution $r(\phi, r_0)$ which starts at the point $(0, r_0)$, satisfies the following system in polar coordinates:

$$\begin{aligned} &\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r, \quad \phi \neq \gamma_i \ (\mathrm{mod}\, 2\pi), \\ &\Delta r|_{\phi=\gamma_i \ (\mathrm{mod}\, 2\pi)} = k_i r, \end{aligned} \tag{2}$$

where $\lambda = \frac{\alpha}{\beta}$, the variable ϕ is ranged over the time-scale

$$R_\phi = R \backslash \bigcup_{i=-\infty}^{\infty} \left[\bigcup_{j=1}^{p-1} (2\pi i + \zeta_j, 2\pi i + \gamma_{j+1}] \cup (2\pi i + \zeta_p, 2\pi(i+1) + \gamma_1] \right]$$

and

$$\begin{aligned} k_i = [&((b_{11} + 1) \cos(\gamma_i) + b_{12} \sin(\gamma_i))^2 \\ &+ (b_{21} \cos(\gamma_i) + (b_{22} + 1) \sin(\gamma_i))^2]^{\frac{1}{2}} - 1. \end{aligned}$$

Eq. (2) is 2π-periodic, so we shall consider just the section $\phi \in [0, 2\pi]$ in what follows. That is the system

$$\begin{aligned} &\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r, \quad \phi \neq \gamma_i, \\ &\Delta r|_{\phi=\gamma_i} = k_i r, \end{aligned} \tag{3}$$

is provided for discussion, where $\phi \in [0, 2\pi]_\phi = [0, 2\pi] \backslash \bigcup_{i=1}^{p} (\gamma_i, \zeta_i]$. System (3) is a sample of time-scale differential equation [7]. We show that one can reduce (3) to an impulsive differential equation for the investigation's needs. Indeed, let us introduce a new variable $\psi = \phi - \sum_{0<\gamma_j<\phi} \theta_j, \theta_j = \zeta_j - \gamma_j$, with the range $[0, 2\pi - \sum_{i=1}^{p} \theta_i]$. We shall call this new variable ψ-*substitution*. It is easy to check that upon ψ-*substitution* the solution $r(\phi, r_0)$ satisfies the following impulsive equation:

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} = \lambda r, \quad \psi \neq \delta_j,$$
$$\Delta r|_{\psi=\delta_j} = k_j r, \tag{4}$$

where $\delta_j = \gamma_j - \sum_{0<\gamma_i<\gamma_j} \theta_i$. Solving (4) as an impulsive system [26,38] and using ψ-*substitution* one can obtain that the solution $r(\phi, r_0)$ of (2) has the form

$$r(\phi, r_0) = \exp\left(\lambda\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right) \prod_{0<\gamma_i<\phi} (1 + k_i) r_0 \tag{5}$$

if $\phi \in [0, 2\pi]_\phi$.

Denote

$$q = \exp\left(\lambda\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right) \prod_{i=1}^{p} (1 + k_i). \tag{6}$$

Construct the Poincaré return map $r(2\pi, r_0)$ on positive half-axis of Ox_1 and compare (5) with (6). Then the following theorem follows:

Theorem 1.1. *If*

(1) $q = 1$, *then the origin is a centre and all solutions are periodic with period* $T = (2\pi - \sum_{i=2}^{p} \theta_i)\beta^{-1}$;
(2) $q < 1$, *then the origin is a stable focus*;
(3) $q > 1$, *then the origin is an unstable focus of* D_0-*system*.

Remark 1.1. Conditions (C1)–(C4) and Theorem 1.1 imply that a trajectory of (1) either spirals to the origin when time increases (decreases) or is a discontinuous cycle. Moreover, if the solution spirals to the origin as time decreases (increases) then it spirals to infinity as time increases (decreases). Thus, the behavior of the trajectory is very similar to the behavior of trajectories of the planar linear system of ordinary differential equations with constant coefficients [17,39]. In what follows, we will consider how a perturbation may change the phase portrait of the system.

1.2. The perturbed system

Consider the following system in the neighbourhood G:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x), \quad x \notin \Gamma,$$
$$\Delta x|_{x\in\Gamma} = B(x)x. \tag{7}$$

Our basic assumptions for system (7) are the following:

(C5) $\Gamma = \bigcup_{i=1}^{p} l_i$ is a set of curves starting at the origin and which are defined by the equations $\langle a^i, x\rangle + \tau_i(x) = 0,\ i = \overline{1, p}$;

(C6)

$$B(x) = (k + \kappa(x))Q \begin{pmatrix} \cos(\theta + \upsilon(x)) & -\sin(\theta + \upsilon(x)) \\ \sin(\theta + \upsilon(x)) & \cos(\theta + \upsilon(x)) \end{pmatrix} Q^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$(I + B(x))x \in G$ for all $x \in G$;

(C7) $\{f, k, \upsilon\} \subset C^{(1)}(G), \{\tau_i, i = \overline{1, p}\} \subset C^{(2)}(G)$;

(C8) $f(x) = o(||x||),\ \kappa(x) = o(||x||),\ \upsilon(x) = o(||x||),\ \tau_i(x) = o(||x||^2),\ i = \overline{1, p}$; Moreover, it is supposed that the matrices A, Q, the vectors a^i, $i = \overline{1, p}$, constants k, θ are the same as for (1), i.e.

(C9) the associated with (7) system (1) is D_0-*system*.

The system (7) is said to be a D-*system* if the conditions (C1)–(C8) hold.

Remark 1.2. Conditions (C5) and (C8) imply that curves l_i do not intersect each other in G, except at the origin, and neither of them has self intersect points in G if G is sufficiently small. The origin is a unique singular point of the D-system.

In what follows we assume without loss of generality that $\gamma_i \neq \frac{\pi}{2} j,\ j = 1, 2, 3$. Then one can transform the equation in (C5) to the polar coordinates so that $l_i : a_i^1 r\ \cos(\phi) + a_i^2 r\ \sin(\phi) + \tau_i(r\ \cos(\phi),\ r\ \sin(\phi)) = 0$ and, hence,

$$\phi = \tan^{-1}\left(\tan\gamma_i - \frac{\tau_i}{a_i^2 r\ \cos(\phi)}\right).$$

Using Taylor expansion gives, the previous equation can be expressed as follows: if r is sufficiently small

$$l_i : \phi = \gamma_i + r\psi_i(r, \phi), \quad i = \overline{1, p}, \tag{8}$$

where functions ψ_i are 2π-periodic in ϕ, continuously differentiable and $\psi_i = O(r)$.

If the phase point $x(t)$ meets the discontinuity line l_i at the angle θ then after the jump the point $x(\theta+)$ will belong to the line $l_i' = \{z \in R^2 \mid z = (I + B(x))x, x \in l_i\}$. For the remaining part of the paper the following assertion is very important.

Lemma 1.1. *If the conditions* (C7) *and* (C8) *are valid then the line l_i' is placed between lines l_i and l_{i+1} for every i if G is sufficiently small.*

Proof. Suppose that lines $s_i,\ s_{i+1},\ l_i,\ l_{i+1}$ are transformed by the map $y = Q^{-1}x$ into lines $s_i'',\ s_{i+1}'',\ l_i'',\ l_{i+1}''$, respectively. Let $L_i = \{z \in R^2 \mid z = Q^{-1}(I + B(Qy))Q,\ y \in l_i''\}$, $\xi_i = Q^{-1}(I + B_0)Qs_i''$ and $\gamma_i',\ \gamma_{i+1}',\ \zeta_i'$ be the angles of straight lines$s_i'',\ s_{i+1}'',\ \xi_i$. Without loss of generality we may assume that $\gamma_i' < \zeta_i' < \gamma_{i+1}'$. Now, to prove the lemma, it is sufficient to check whether L_i lies between lines l_i'', l_{i+1}''. We assume that $0<\gamma_i'<\zeta_i'<\gamma_{i+1}'<\frac{\pi}{2}$, otherwise one can use a linear transformation which will not change the position of the lines with respect to each other. Let $c_1y_1+c_2y_2+l^*(y_1,y_2)=0$ be the equation of the line l_i''. Introducing the polar coordinates $y_1=\rho\cos(\phi)$, $y_2=\rho\sin(\phi)$ one can write this equation as $\phi=\gamma_i'+\rho\psi^*(\rho,\phi)$, where $\psi^*(\rho,\phi)=O(\rho)$ and ψ^* is a 2π-periodic in ϕ function. If $y=(y_1,y_2)\in l_i''$then the point $y^+=(y_1^+,y_2^+)$, where

$$y^+=Q^{-1}(B(Qy)+I)Qy, \tag{9}$$

belongs to L_i. Assume without loss of generality that $y_1^+\neq 0$. Otherwise, we shall use the condition $y_2^+\neq 0$. If we denote $\rho=(y_1^2+y_2^2)^{\frac{1}{2}}$, $\phi=\tan^{-1}(\frac{y_2}{y_1})$, $\rho^+=((y_1^+)^2+(y_2^+)^2)^{\frac{1}{2}}$, $\phi^+=\tan^{-1}(\frac{y_2^+}{y_1^+})$ then (9) implies that

$$\rho^+=k_i\rho+\rho\beta^*(\rho,\phi), \tag{10}$$

$$\phi^+=\phi+\theta+\gamma^*(\rho,\phi), \tag{11}$$

where β^* and γ^* are 2π-periodic in ϕ and $\beta^*=O(\rho)$, $\gamma^*=O(\rho)$. Denote $\sigma(y_1,y_2)=c_1y_1+c_2y_2+l^*(y_1,y_2)$. Then

$$\begin{aligned}\sigma(y_1^+,y_2^+)&=\rho^+(c_1\cos(\phi^+)+c_2\sin(\phi^+))+l^*(\rho^+\cos(\phi^+),\rho^+\sin(\phi^+))\\&=\rho^+\sqrt{c_1^2+c_2^2}\sin(\theta+\upsilon(\rho,\phi)-\rho\psi^*(\rho,\psi))\\&\quad+l^*(\rho^+\cos(\phi^+),\rho^+\sin(\phi^+)),\end{aligned}$$

where $\upsilon(\rho,\phi)=\upsilon(Qy)$. It is readily seen that the sign of $\sigma(\rho^+,\phi^+)$ is the same as that of $\sin(\theta)$ for sufficiently small ρ. Consequently, $\sigma(\rho^+,\phi^+)>0$. Thus the line L_i is placed above the line l_i'' in the first-quarter of the plane Ox_1x_2. Similarly, one can show that it is placed below l_{i+1}''. The lemma is proved. □

Remark 1.3. Notice that Lemma 1.1 guarantees that every nontrivial trajectory meets precisely once each of the lines l_i, $i=\overline{1,p}$, within any time interval of length T if G is sufficiently small.

2. Foci of D-systems

Using the polar coordinates $x_1=r\cos(\phi)$, $x_2=r\sin(\phi)$ one can find that the differential part of (7) has the following form:

$$\frac{\mathrm{d}r}{\mathrm{d}\phi}=\lambda r+P(r,\phi),$$

where, as is known [39], the function $P(r,\phi)$ is 2π-periodic, continuously differentiable and $P=o(r)$. Denote $x^+=(x_1^+,x_2^+)=(I+B(x))x$, $x^+=r^+(\cos\phi^+,\sin\phi^+)$, $\tilde{x}^+=(\tilde{x}_1^+,\tilde{x}_2^+)=(I+B(0))x$, where $x=(x_1,x_2)\in l_i$, $i=\overline{1,p}$. The inequality

$||x^+ - \tilde{x}^+|| \leqslant ||B(x) - B(0)||||x||$ implies that $r^+ = k_i r + \omega(r, \phi)$. Moreover, using the relation between $\frac{x_2^+}{x_1^+}$ and $\frac{\tilde{x}_2^+}{\tilde{x}_1^+}$ and condition (C5) one can conclude that $\phi^+ = \phi + \theta_i + \gamma(r, \phi)$. Functions ω, γ are 2π-periodic in ϕ and $\omega = o(r), \gamma(r, \phi) = o(r)$. Finally, transformed system (7) is of the following form:

$$\begin{aligned} &\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r + P(r, \phi), \quad (\rho, \phi) \notin \Gamma, \\ &\Delta r|_{(\rho,\phi)\in l_i} = k_i r + \omega(r, \phi), \\ &\Delta \phi|_{(\rho,\phi)\in l_i} = \theta_i + \gamma(r, \phi). \end{aligned} \tag{12}$$

Let us introduce the following system besides the system (12):

$$\begin{aligned} &\frac{\mathrm{d}\rho}{\mathrm{d}\phi} = \lambda \rho + P(\rho, \phi), \quad \phi \neq \gamma_i, \\ &\Delta \rho|_{\phi=\gamma_i} = k_i \rho + w_i(\rho), \\ &\Delta \phi|_{\phi=\gamma_i} = \theta_i, \end{aligned} \tag{13}$$

where all elements, except $w_i, \ i = \overline{1, p}$, are the same as in (12) and the domain of (13) is $[0, 2\pi]_\phi$. We shall define functions w_i below.

Let $r(\phi, r_0)$ be a solution of (12) and ϕ_i be the angle where the phase point intersects l_i. Denote also $\chi_i = \phi_i + \theta_i + \gamma(r(\phi_i, r_0), \phi_i)$ the angle, where $r(\phi, r_0)$ has to be after the jump.

Further $(\alpha,\hat{} \beta], \ \{\alpha, \beta\} \subset R$ denotes the oriented interval, that is

$$(\alpha,\hat{} \beta] = \begin{cases} (\alpha, \beta] & \text{if } \alpha \leqslant \beta, \\ (\beta, \alpha] & \text{otherwise.} \end{cases}$$

Definition 2.1. We shall say that systems (12) and (13) are B-equivalent in G if for every solution $r(\phi, r_0)$ of (12) whose trajectory is in G for all $\phi \in [0, 2\pi]_\phi$ there exists a solution $\rho(\phi, r_0)$ of (13) which satisfies the relation

$$r(\phi, r_0) = \rho(\phi, r_0), \phi \in [0, 2\pi]_\phi \setminus \bigcup_{i=1}^{p} \{[\phi_i,\hat{} \gamma_i,] \cup [\zeta_i,\hat{} \chi_i]\}, \tag{14}$$

And, conversely, for every solution $\rho(\phi, r_0)$ of (13) whose trajectory is in G there exists a solution $r(\phi, r_0)$ of (12) which satisfies (14).

Fix $i = \overline{1, p}$. Let $r_1(\phi, \gamma_i, \rho)$, $r_1(\gamma_i, \gamma_i, \rho) = \rho$, be a solution of the equation

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r + P(r, \phi) \tag{15}$$

and $\phi = \eta_i$ be the meeting angle of $r_1(\phi, \gamma_i, \rho)$ with the line l_i. Then

$$r_1(\eta_i, \gamma_i, \rho) = \exp(\lambda(\eta_i - \gamma_i))\rho + \int_{\gamma_i}^{\eta_i} \exp(\lambda(\eta_i - s)) P(r_1(s, \gamma_i, \rho), s) \, \mathrm{d}s.$$

Let now $\eta_i^1 = \eta_i + \theta + \gamma(r_1(\eta_i, \gamma_i, \rho), \eta_i)$ and $\rho^1 = (1+k_i)r_1(\eta_i, \gamma_i, \rho) + \omega(r(\eta_i, \gamma_i, \rho), \eta_i)$. Let $r_2(\phi, \eta_i^1, \rho^1)$ be a solution of system (15),

$$r_2(\zeta_i, \eta_i^1, \rho^1) = \exp(\lambda(\zeta_i - \eta_i^1))\rho^1 + \int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))P(r_2(s, \eta_i^1, \rho^1), s)\,\mathrm{d}s.$$

We define that

$$\begin{aligned} w_i(\rho) = r_2(\zeta_i, \eta_i^1, \rho^1) - (1+k_i)\rho = \exp(\lambda(\zeta_i - \eta_i^1)) \Bigg[(1+k) \Bigg(\exp(\lambda(\eta_i - \gamma_i))\rho \\ + \int_{\gamma_i}^{\eta_i} \exp(\lambda(\eta_i - s))P(r_1(s, \gamma_i, \rho), s)\,\mathrm{d}s \Bigg) + \omega(r_1(\eta_i, \gamma_i, \rho), \eta_i) \Bigg] \\ + \int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))P(r_2(s, \eta_i^1, \rho^1), s)\,\mathrm{d}s - (1+k)\rho \end{aligned}$$

or, if simplified,

$$\begin{aligned} w_i(\rho) &= (1+k)[\exp(-\lambda\gamma(r_1(\eta_i, \gamma_i, \rho), \eta_i)) - 1]\rho \\ &\quad + (1+k)\int_{\gamma_i}^{\eta_i} \exp(\lambda(\zeta_i - \theta - s - \rho\gamma(r_1(\eta_i, \gamma_i, \rho), \eta_i))) \\ &\quad \times P(r_1(s, \gamma_i, \rho), s)\,\mathrm{d}s + \int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))P(r_2(s, \eta_i^1, \rho^1), s)\,\mathrm{d}s \\ &\quad + \exp(\lambda(\zeta_i - \eta_i^1))\omega(r_1(\eta_i, \gamma_i, \rho), \eta_i). \end{aligned} \tag{16}$$

Differentiating (8) and (16) one can find that

$$\frac{\mathrm{d}\eta_i}{\mathrm{d}\rho} = \frac{\frac{\partial r_1}{\partial \rho}[\psi_i + r_1 \frac{\partial \psi_i}{\partial r}]}{1 - (\lambda r_1 + P)[\psi_i + r_1 \frac{\partial \psi_i}{\partial r}] - r_1 \frac{\partial \psi_i}{\partial \phi}}, \qquad \frac{\mathrm{d}\eta_i^1}{\mathrm{d}\rho} = \frac{\mathrm{d}\eta_i}{\mathrm{d}\rho}\left(1 + \frac{\partial \gamma}{\partial \phi}\right) + \frac{\partial \gamma}{\partial r}\,\frac{\partial r_1}{\partial \rho},$$

$$\begin{aligned} \frac{\mathrm{d}w_i}{\mathrm{d}\rho} &= (1+k_i)[\exp(-\lambda\gamma) - 1] - \lambda(1+k_i)\exp(-\lambda\gamma)\left(\frac{\partial \gamma}{\partial r}\,\frac{\partial r_1}{\partial \rho} + \frac{\partial \gamma}{\partial \phi}\,\frac{\mathrm{d}\eta_i}{\mathrm{d}\rho}\right)\rho \\ &\quad + (1+k_i)\exp(\lambda(\zeta_i - \theta_i - \eta_i - \gamma))P\,\frac{\mathrm{d}\eta_i}{\mathrm{d}\rho} \\ &\quad + (1+k_i)\int_{\gamma_i}^{\eta_i} \exp(\lambda(\zeta_i - \theta - s - \gamma))\left\{-\lambda\left(\frac{\partial \gamma}{\partial r}\,\frac{\partial r_1}{\partial \rho} + \frac{\partial \gamma}{\partial \phi}\,\frac{\mathrm{d}\eta_i}{\mathrm{d}\rho}\right]P\right. \\ &\quad \left. - \frac{\partial P}{\partial r}\,\frac{\partial r_1}{\partial \rho} - \frac{\partial P}{\partial \phi}\,\frac{\mathrm{d}\eta_i}{\mathrm{d}\rho}\right\}\mathrm{d}s \\ &\quad + \int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))\,\frac{\partial P(r_2(s, \eta_i^1, \rho^1), s)}{\partial r}\,\frac{\partial r_2}{\partial \rho}\,\mathrm{d}s \\ &\quad - \exp(\lambda(\zeta_i - \eta_i^1))P(\rho^1, \eta_i^1)\,\frac{\partial \eta_i^1}{\partial \rho} \\ &\quad + \exp(\lambda(\zeta_i - \eta_i^1))\left[-\frac{\partial \eta_i^1}{\partial \rho}\,\omega + \frac{\partial \omega}{\partial r}\,\frac{\partial r_1}{\partial \rho} + \frac{\partial \omega}{\partial \phi}\,\frac{\mathrm{d}\eta_i}{\mathrm{d}\rho}\right]. \end{aligned} \tag{17}$$

Analysis of (16) and (17) implies that the following two lemmas are valid.

Lemma 2.1. *The systems* (12) *and* (13) *are B-equivalent if G is sufficiently small.*

Lemma 2.2. *If conditions* (C1)–(C5) *are valid then w_i, $i = \overline{1, p}$, are continuously differentiable functions and $w_i(\rho) = o(\rho)$, $i = \overline{1, p}$.*

Theorem 2.1. *Assume that conditions* (C1)–(C6) *hold and $q < 1$ $(1 < q)$. Then the origin is a stable* (*unstable*) *focus of Eq.* (7).

Proof. Let $r(\phi, r_0)$ be a solution of (12) such that $r(0, r_0) = r_0$ and $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$ be the solution of (13). Using *ψ-substitution* one can obtain that

$$\begin{aligned}\rho(\phi, r_0) = {} & \exp(\lambda\phi) \left\{ \prod_{i=1}^{m} (1 + k_i) \exp\left(-\lambda \sum_{s=1}^{m} \theta_s\right) r_0 \right. \\ & + \prod_{i=1}^{m} (1 + k_i) \exp\left(-\lambda \sum_{s=1}^{m} \theta_s\right) \int_0^{\gamma_1} \exp(-\lambda u) P \,\mathrm{d}u \\ & + \prod_{i=2}^{m} (1 + k_i) \exp\left(-\lambda \sum_{s=2}^{m} \theta_s\right) \int_{\zeta_1}^{\gamma_2} \exp(-\lambda u) P \,\mathrm{d}u \\ & + \cdots \int_{\zeta_m}^{\phi} \exp(-\lambda u) P \,\mathrm{d}u + \prod_{i=2}^{m} (1 + k_i) \exp\left(-\lambda \sum_{s=2}^{m} \theta_s\right) w_1 \\ & \left. + \prod_{i=3}^{m} (1 + k_i) \exp\left(-\lambda \sum_{s=3}^{m} \theta_s\right) w_2 \ldots + \exp(-\lambda\zeta_m) w_m \right\}, \end{aligned} \tag{18}$$

where $\phi \in [0, 2\pi]_\phi$, $P = P(\rho(\phi, r_0), \phi)$, $w_i = w_i(\rho(\gamma_i, r_0), \gamma_i)$. In [1] the differentiable dependence on parameters for solutions of impulsive systems was considered. Using the theorem of the paper, conditions (C4), (C5) and Lemma 2.2 one can find that solution $\rho(\psi, r_0)$ is differentiable in r_0 and its derivative $\frac{\partial\rho(\phi, r_0)}{\partial r_0}$ at the point $\phi = 2\pi$, $r_0 = 0$ is equal to q. Since (12) and (13) are B-equivalent it follows that:

$$\frac{\partial r(2\pi, 0)}{\partial r_0} = q$$

and the proof is complete. □

3. The problem of distinguishing between the center and the focus

Consider the critical case $q = 1$. Throughout this section we assume that functions $f, g, \tau_i, i = \overline{1, p}$, are analytic in G. The condition (C8) implies that the Taylor expansions of f and g start with members of order not less than 2, and the expansions of τ_i, $i = \overline{1, p}$, start with members of order not less than 3. First we investigate the problem for (13) all of whose elements are analytic if ρ is sufficiently small. Theorems from [2] imply that w_i, $i=\overline{1,p}$, are analytic in ρ and the solution $\rho(\phi, r_0)$ of Eq. (13) has the following expansion:

$$\rho(\phi, r_0) = \sum_{i=0}^{\infty} \rho_i(\phi) r_0^i, \tag{19}$$

where $\phi \notin (\gamma_i, \zeta_i]$, $i=\overline{1,p}$, $\rho_0(\phi)=0$, $q=\rho_1(\phi)=1$. One can define a Poincaré return map

$$\rho(2\pi, r_0) = \sum_{i=1}^{\infty} a_i r_0^i, \tag{20}$$

where $a_i = \rho_i(2\pi)$, $i=0,1,\ldots$, $a_1=q=1$. We consider behavior of coefficients of (19) and define what type of a singular point the origin is. The expansions exist [2] such that

$$\begin{aligned} P(\rho, \phi) &= \sum_{i=2}^{\infty} P_i(\phi)\rho^i, \\ w_j(\phi) &= \sum_{j=2}^{\infty} w_{ji}(\phi)\rho^i, \end{aligned} \tag{21}$$

where $P_i(\phi), w_{ji}(\phi)$, $j=2,3,\ldots$, are 2π-periodic functions which can be defined by using elements of (12). The coefficients $\rho_j(\phi)$, $j=2,3,\ldots$, are solutions of the following systems:

$$\begin{aligned} &\frac{d\rho}{d\phi} = P_j(\phi), \quad \phi \neq \gamma_i, \\ &\Delta\rho|_{\phi\neq\gamma_i} = w_{ji}, \\ &\Delta\phi|_{\phi\neq\gamma_i} = \theta_i \end{aligned} \tag{22}$$

with initial values $\rho_j(0)=0$, $j=2,3,\ldots$. Hence, coefficients in (20) are equal to

$$a_j = \int_0^{\gamma_1} P_j(\phi)\,d\phi + \sum_{i=1}^{p-1}\int_{\zeta_i}^{\gamma_{i+1}} P_j(\phi)\,d\phi + \int_{\zeta_p}^{2\pi} P_j(\phi)\,d\phi + \sum_{i=1}^{p} w_{ji}. \tag{23}$$

From (20) and (23) it follows that the following lemma is true.

Lemma 3.1. *Let $q=1$ and the first nonzero element of the sequence a_j, $j=2,3,\ldots$ be negative* (*positive*). *Then the origin is a stable* (*unstable*) *focus of* (13). *If all $a_j=0$, $j=2,3,\ldots$ then the origin is the centre of* (13).

B-equivalence of systems (12) and (13) implies immediately that the following theorem is valid.

Theorem 3.1. *Let $q=1$ and the first nonzero element of the sequence a_j, $j=2,3,\ldots$ be negative* (*positive*). *Then the origin is a stable* (*unstable*) *focus of the Eq.* (7). *If all $a_j=0$, $j=2,3,\ldots$ then the origin is the centre of* (7).

4. Bifurcation of periodic solutions

In this section we prove the bifurcation theorem of a periodic solution from an equilibrium for the discontinuous dynamical system. After the initial impetus of Poincaré [33], Andronov [4] and Hopf [18] this method of research of periodic motions has been used very successfully by many authors for various types of differential see [8,10,15,17,21,29,30] and references cited there). Let us consider the system

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x) + \mu F(x, \mu), \quad x \notin \Gamma(\mu),\\ &\Delta x|_{x \in \Gamma(\mu)} = B(x, \mu)x.\end{aligned} \tag{24}$$

Assume that the following conditions are satisfied:

(A1) the set $\Gamma(\mu) = \bigcup_{i=1}^{p} l_i(\mu)$ is a union of curves in G which start at the origin and are defined by the $l_i : (a^i, x) + \tau_i(x) + \mu v(x, \mu) = 0,\ i = \overline{1, p}$;

(A2) there exists a matrix $Q(\mu) \in \mathscr{R}$, $Q(0) = Q$, analytic in $(-\mu_0, \mu_0)$, and real numbers $\gamma,\ \chi$ such that

$$\begin{aligned}&Q^{-1}(\mu)B(x, \mu)Q(\mu)\\ &\quad = (k + \mu\gamma + \kappa(x))\begin{pmatrix}\cos(\theta + \mu\chi + \upsilon(x)) & -\sin(\theta + \mu\chi + \upsilon(x))\\ \sin(\theta + \mu\chi + \upsilon(x)) & \cos(\theta + \mu\chi + \upsilon(x))\end{pmatrix}\\ &\qquad - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};\end{aligned}$$

(A3) associated with (24) systems

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = Ax, \quad x \notin \Gamma_0,\\ &\Delta x|_{x \in \Gamma(\mu)} = B_0 x,\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x), \quad x \notin \Gamma(0),\\ &\Delta x|_{x \in \Gamma(0)} = B(x, 0)x,\end{aligned} \tag{26}$$

are D_0-*system* and D-*system*, respectively;

(A4) functions $\kappa, \upsilon : G \to R^2$ and $F, v : G \times (-\mu_0, \mu_0) \to R^2$ are analytical in x and μ;

(A5) $F(0, \mu) = 0$, $v(0, \mu) = 0$, uniformly for $\mu \in (-\mu_0, \mu_0)$.

We need also the following system:

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = A(\mu)x, \quad x \notin \Gamma_0(\mu),\\ &\Delta x|_{x \in \Gamma_0(\mu)} = B(0, \mu)x,\end{aligned} \tag{27}$$

where $A(\mu) = A + \mu \frac{\partial F(0,\mu)}{\partial x}$, and $\Gamma_0(\mu) = \bigcup_{i=1}^{p} m_i$ where

$$m_i: \quad \left(a^i + \mu \frac{\partial v(0,\mu)}{\partial x}, x\right) = 0, \quad i = \overline{1,p}.$$

Using polar transformation one can write system (24) in the following form:

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r + P(r,\phi,\mu), \quad (r,\phi) \notin \Gamma(\mu),\\
&\Delta r|_{(r,\phi)\in l_i(\mu)} = k_i r + \omega(r,\phi,\mu),\\
&\Delta \phi|_{(r,\phi)\in l_i(\mu)} = \theta_i + r\gamma(r,\phi,\mu).
\end{aligned} \tag{28}$$

Let the system

$$\begin{aligned}
&\frac{\mathrm{d}\rho}{\mathrm{d}\phi} = \lambda \rho + P(\rho,\phi,\mu), \quad \phi \neq \gamma_i(\mu),\\
&\Delta \rho|_{\phi=\gamma_i(\mu)} = k_i \rho + w_i(\rho,\mu),\\
&\Delta \phi|_{\phi=\gamma_i(\mu)} = \theta_i(\mu),
\end{aligned} \tag{29}$$

where $\gamma_i(\mu)$, $i = \overline{1,p}$, are angles of m_i, be B-equivalent to (28). The functions $w_i(\rho,\mu)$ can be defined in the same manner as in (16). Similar to (6) one can define the function

$$q(\mu) = \exp\left(\lambda(\mu)\left(2\pi - \sum_{j=1}^{p}(\zeta_j(\mu) - \gamma_j(\mu))\right)\right)\prod_{j=p}^{1}(1 + k_j(\mu)) \tag{30}$$

for system (27). The theorem from [2] implies that the function $q(\mu)$ is analytic in μ.

Theorem 4.1. *Assume that* $q(0) = 1$, $q'(0) \neq 0$ *and the origin is a focus for* (26). *Then, for sufficiently small* r_0, *there exists a function* $\mu = \delta(r_0)$ *such that the solution* $r(\phi, r_0, \delta(r_0))$ *of* (28) *is periodic with a period* $T = (2\pi - \sum_{i=1}^{p}\theta_i)\beta^{-1} + o(|\mu|)$. *Furthermore, if the origin is the stable focus of* (26) *then the closed trajectory is the limit cycle.*

Proof. Let $\rho(\phi, r_0, \mu)$ be a solution of (29). By the theorem of analyticity of solutions [2] we have that

$$\rho(2\pi, r_0, \mu) = \sum_{i=1}^{\infty} a_i(\mu) r_0^i,$$

where $a_i(\mu) = \sum_{j=0}^{\infty} a_{ij}\mu^j$, $a_{10} = q(0) = 1$, $a_{11} = q'(0) \neq 0$. We define the displacement function as follows:

$$\begin{aligned}
\mathscr{V}(r_0,\mu) &= \rho(2\pi, r_0, \mu) - r_0\\
&= q'(0)\mu r_0 + \sum_{i=2}^{\infty} a_{i0} r_0^i + r_0\mu^2 G_1(r_0,\mu) + r_0^2 \mu G_2(r_0,\mu),
\end{aligned}$$

where G_1, G_2 are functions analytical in r_0, μ in neighbourhood of the $(0, 0)$. The bifurcation equation is $\mathscr{V}(r_0, \mu) = 0$. Cancelling by r_0 one can write the bifurcation equation as

$$\mathscr{H}(r_0, \mu) = 0, \tag{31}$$

where

$$\mathscr{H}(r_0, \mu) = q'(0)\mu + \sum_{i=2}^{\infty} a_{i0} r_0^{i-1} + \mu^2 G_1(r_0, \mu) + r_0 \mu G_2(r_0, \mu).$$

Since

$$\mathscr{H}(0, 0) = 0, \quad \frac{\partial \mathscr{H}(0, 0)}{\partial \mu} = q'(0) \neq 0$$

for sufficiently small r_0 there exists a function $\mu = \delta(r_0)$ such that $r(\phi, r_0, \delta(r_0))$ is a periodic solution. If we assume that $a_{i0} = 0, i = \overline{2, l-1}$ and $a_{l0} \neq 0$, then one can obtain from (31) that

$$\delta(r_0) = -\frac{a_{l0}}{q'(0)} r_0^{l-1} + \sum_{i=l}^{\infty} \delta_i r_0^i. \tag{32}$$

Analysing the latter expression one can conclude that the bifurcation of periodic solutions exists if a stable for $\mu = 0$ focus is unstable for $\mu \neq 0$ and conversely. Let $\rho(\phi) = \rho(\phi, \bar{r_0}, \bar{\mu})$ be a periodic solution of (29). It is known that the trajectory is limit cycle if

$$\frac{\partial \mathscr{V}(\bar{r}_0, \bar{\mu})}{\partial r_0} < 0. \tag{33}$$

We have that

$$\frac{\partial \mathscr{V}(r_0, \mu)}{\partial r_0} = q'(0)\mu + \sum_{i=2}^{\infty} i a_{i0} r_0^{i-1} + \mu^2 G_1(r_0, \mu) + 2 r_0 \mu G_2(r_0, \mu).$$

Assume that a_{l0} is the first nonzero element among a_{i0} and $a_{l0} < 0$. Then, using (32) one can obtain

$$\frac{\partial \mathscr{V}(\bar{r_0}, \bar{\mu})}{\partial r_0} = (l-1) a_{l0} \bar{r}_0^{l-1} + Q(\bar{r}_0),$$

where Q starts with a member whose order is not less than l. Hence, (33) is valid. From B-equivalence of (28) and (29) one can conclude that the theorem is proved.

Remark 4.1. Notice that the result of the section can be obtained by applying the bifurcation theorem for fixed points of a map [8,21] and theorems of differentiability from [1]. We follow the approach which focused on the expansions of solutions [17,30].

5. Examples

Example 1. Consider the following dynamical system:

$$
\begin{aligned}
&x_1' = (2+\mu)x_1 - x_2 + x_1^2 x_2,\\
&x_2' = x_1 + (2+\mu)x_2 + 3x_1^3 x_2, \quad x \notin l,\\
&\Delta x_1|_{x\in l} = \left((\kappa+\mu^2)\cos\left(\frac{\pi}{6}\right) - 1\right)x_1 - (\kappa+\mu^2)\sin\left(\frac{\pi}{6}\right)x_2,\\
&\Delta x_2|_{x\in l} = (\kappa+\mu^2)\sin\left(\frac{\pi}{6}\right)x_1 + \left((\kappa+\mu^2)\cos\left(\frac{\pi}{6}\right) - 1\right)x_2,
\end{aligned}
\tag{34}
$$

where $x =: (x_1, x_2)$, $\kappa = \exp(-\frac{11\pi}{6})$, the curve l is given by the equation $x_2 = x_1^3, x_1 > 0$. One can define, using (30), that $q(\mu) = (\kappa + \mu^2)\exp((2+\mu)\,\frac{11\pi}{6})$, $q(0) = \kappa\exp(\frac{11\pi}{3}) = 1$, $q'(0) = -\frac{11\pi}{6} \neq 0$. Thus, by Theorem 4.1, system (34) has a periodic solution with period $\approx \frac{11\pi}{12}$ if $|\mu|$ is sufficiently small.

Example 2. Let the following system be given

$$
\begin{aligned}
&x_1' = (\mu-1)x_1 - x_2, \quad x_2' = x_1 + (\mu-1)x_2, \quad x \notin l,\\
&\Delta x_1|_{x\in l} = \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_1 - (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_2,\\
&\Delta x_2|_{x\in l} = (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_1 + \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_2,
\end{aligned}
\tag{35}
$$

where l is a curve given by the equation $x_2 = x_1 + \mu x_1^2$, $x_1 > 0$, $\kappa = \exp(\frac{7\pi}{4})$. Using (30) one can find that $q(\mu) = \kappa\,\exp((\mu-1)\frac{7\pi}{4})$, $q(0) = \kappa\,\exp(-\frac{7\pi}{4}) = 1$, $q'(0) = \frac{7\pi}{4} \neq 0$. Moreover, one can see that for the associated *D-system*

$$
\begin{aligned}
&x_1' = -x_1 - x_2, \quad x_2' = x_1 - x_2, \quad x \notin s,\\
&\Delta x_1|_{x\in s} = \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_1 - (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_2,\\
&\Delta x_2|_{x\in s} = (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_1 + \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_2,
\end{aligned}
\tag{36}
$$

where s is given by the equation $x_2 = x_1, x_1 > 0$, the origin is the stable focus. Indeed, using polar coordinates, denote by $r(\phi, r_0)$ the solution of (36) starting at the angle $\phi = \frac{\pi}{4}$. We can define that $r(\frac{\pi}{4} + 2\pi n, r_0) = (\kappa - r^2(\frac{\pi}{4} + 2\pi(n-1), r_0))\exp(-\frac{7\pi}{4})$. From the last expression it is easily seen that the sequence $r_n = r(\frac{\pi}{4} + 2\pi n, r_0)$ is monotonically decreasing and there exists a limit of r_n. Assume that $r_n \to \sigma \neq 0$. Then it implies that there exists a periodic solution of (36) and $\sigma = (\kappa - \sigma^2)\exp(-\frac{7\pi}{4})\sigma$ which is a contradiction. Thus $\sigma = 0$. Consequently, the origin is the stable focus of (36) and by Theorem 4.1 the system (35) has the limit cycle with period $\approx \frac{7\pi}{4}$ if $\mu > 0$ is sufficiently small.

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