PERTURBATIONS PRESERVING ASYMPTOTICS
OF SPECTRUM WITH REMAINDER[1]

A. G. RAMM

ABSTRACT. Let A be a compact linear operator on a Hilbert space H, $s_n(A) = \lambda_n^{1/2}(A^*A)$, Q be a linear operator, $|Qf| \leq c|Af|^a|f|^{1-a}$, $a > 0$, $c > 0$, $\forall f \in H$. Let

$$s_n(A) = cn^{-r}\{1 + O(n^{-q})\}, \qquad r, q > 0, B = A(I + Q).$$

Then

$$s_n(B) = s_n(A)\{1 + O(n^{-\gamma})\}, \qquad \gamma = \min\left(q, \frac{ra}{1 + ra}\right).$$

Some applications of this result to the spectral theory of elliptic operators are given.

1. Introduction. In many cases it is interesting to know when the perturbed and the unperturbed operators have the same spectral asymptotic behavior. This question was discussed in [1]. Here we give an estimate of the remainder in the asymptotic formulas. This estimate is sharp.

2. Main result.

THEOREM 1. *Let A be a compact linear operator on a Hilbert space H, Q be a linear operator such that $|Qf| \leq c|Af|^a|f|^{1-a}$, $a > 0$, $c > 0$, here and below denotes various constants. Let $B = A(I + Q)$ and $I + Q$ be invertible. If $s_n(A) = cn^{-r}(1 + O(n^{-q}))$, $r, q > 0$, then*

$$(1) \qquad s_n(B) = s_n(A)\{1 + O(n^{-\gamma})\}, \qquad \gamma = \min\left(q, \frac{ra}{1 + ra}\right).$$

PROOF. Let $U = A^*A$, $V \equiv B^*B = (I + Q^*)U(I + Q)$, M_n be the n-dimensional subspace such that the condition $f \perp M_n$ is equivalent to the condition $g \equiv (I + Q)f \perp L_n$, and L_n is the linear span of the first n eigenvectors of U. From the minimax variational principle it follows that

$$s_{n+1+m}^2(B) = \lambda_{n+1+m}(V) \leq \sup_{\substack{f \perp L_m \\ f \perp M_n}} \frac{(Vf, f)}{|f|^2} \leq \sup_{f \perp M_n} \frac{(Ug, g)}{(g, g)} \cdot \sup_{f \perp L_m} \frac{(g, g)}{|f|^2}$$

$$(2) \qquad = s_{n+1}^2(A)\left\{1 + \sup_{f \perp L_m}\left(\frac{2|Qf|}{|f|} + \frac{|Qf|^2}{|f|^2}\right)\right\}$$

$$= s_{n+1}^2(A)\{1 + O(s_m^a(A))\}.$$

Here we used the estimate

$$\sup_{f \perp L_m} \frac{|Qf|}{|f|} \le c \sup_{f \perp L_m} \frac{|Af|^a}{|f|^a} \le c_1 s_m^a(A).$$

Since $U = (I + S^*)V(I + S)$, $I + S \equiv (I + Q)^{-1}$, $S = -(I + Q)^{-1}Q$, similar argument shows that

$$(3) \qquad s_{n+1+2m}^2(A) \le s_{n+1+m}^2(B)\{1 + O(s_m^a(B))\}.$$

One should use in the derivation of (3) M_n' instead of M_n, and the inequality $|Sf| \le c|Qf| \le c_1|Af|^a|f|^{1-a} \le c_2|Bf|^a|f|^{1-a}$. Here M_n' is so chosen that the condition $f \perp M_n'$ is equivalent to the condition $g' \equiv (I + S)f \perp L_n'$ and L_n' is the linear span of the first n eigenvectors of V. Since $s_n(A) \to 0$, and $s_n(B) \to 0$ it follows from (2) and (3) that $s_n(B) \cdot s_n^{-1}(A) \to 1$ as $n \to \infty$ and (1) holds. Indeed, take $mn^{-1} = n^{-x}$. Then $s_m^a(A) = O(n^{-ra(1-x)})$ and $s_{n+m+1}(A) \cdot s_n^{-1}(A) = 1 + O(n^{-x}) + O(n^{-q})$ as $n \to \infty$ because $s_n(A) = cn^{-r}\{1 + O(n^{-q})\}$. Therefore

$$\gamma = \min(q, x, ra(1 - x)) = \min\left(q, \frac{ra}{1 + ra}\right).$$

Below $B \in R_b$ means that the root system of B forms a Riesz basis with brackets of H (see [2, p. 284] for definition).

THEOREM 1'. *Let* $B = A(I+Q)$, $A > 0$, $\lambda_n(A) = cn^{-r}(1+O(n^{-q}))$, $r, q > 0$, $|Qf| \le c|A^b f|$, $b > 0$, $\mathrm{Ker}(I + Q) = \{0\}$. *If* $rb \ge 1$, *then* $B \in R_b$.

PROOF. One has

$$B^{-1} = (I + Q)^{-1}A^{-1} = A^{-1} - (I + Q)^{-1}QA^{-1},$$
$$|(I + Q)^{-1}QA^{-1}f| \le c_1|QA^{-1}f| \le c_2|A^{-(1-b)}f|,$$
$$\lambda_n(A^{-1}) = c^{-1}n^r\{1 + O(n^{-q})\}.$$

From [2, p. 286, Theorem 1] it follows that $rb \ge 1$ implies $B^{-1} \in R_b$. Therefore $B \in R_b$.

Theorem 1 from [2] is stated in §3 below as Proposition 1.

REMARK 1. Estimate (1) is close to sharp: for the elliptic operators in $L^2(D)$ (in §3 below) the remainder is of order given in Theorem 1.

REMARK 2. Let L be a closed densely defined linear operator on H and $A = L^{-1}$ be compact. Let M be a linear operator $D(M) \supset D(L)$ and $T = ML^{-1}$ be compact. Then $B \equiv (L + M)^{-1} = A(I + T)^{-1} = A(I + Q)$, $Q \equiv -(I + T)^{-1}T$. If $s_n(A) = cn^{-r}\{1 + O(n^{-q})\}$, and $|Qf| \le c|Af|^a|f|^{1-a}$, $a > 0$, then Theorem 1 is applicable. If one defines $s_n(L) = s_n^{-1}(L^{-1})$ then $s_n(B) = s_n(A)\{1 + O(n^{-\gamma})\}$, i.e. $s_n(L + M) = s_n(L)\{1 + O(n^{-\gamma})\}$.

REMARK 3. The conclusion of Theorem 1 and its proof are valid also in the case when $B = (I + Q)A$ and $|Q^*f| \le c|Af|^a|f|^{1-a}$. It follows from the equality $s_n(B) = s_n(B^*)$. In applications both estimates $|Qf| \le c|Af|^a|f|^{1-a}$ and $|A^*f| \le c|Af|^a|f|^{1-a}$ are usually available.

3. Applications to differential operators.

1. Let L be an elliptic selfadjoint operator with smooth coefficients on $H = L^2(D)$, where $D \subset \mathbf{R}^d$ is a bounded domain with a smooth boundary. Let $l = \mathrm{ord}\,L$, $m = \mathrm{ord}\,M$, $m < l$, where M is some differential operator with smooth

coefficients. Without loss of generality we assume that L^{-1} and $(L+M)^{-1}$ exist. Let us define $s_n(L) = s_n^{-1}(L^{-1})$. Then $s_n(L) = cn^{l/d}\{1 + O(n^{-d^{-1}})\}$.

$$(4) \qquad s_n(L+M) = s_n(L)\{1 + O(n^{-\gamma})\}, \qquad \gamma = \min\left\{d^{-1}, \frac{l-m}{l-m+d}\right\}$$

as $n \to \infty$. Indeed, $(L+M)^{-1} = (I + L^{-1}M)^{-1}L^{-1} = (I+Q)L^{-1}$, $Q = -(I+L^{-1}M)^{-1}L^{-1}M$ and

$$(5) \qquad |Qf| \le c|L^{-1}f|^{1-a}|f|^a, \qquad a = ml^{-1}.$$

The last inequality is an interpolation one which will be discussed in detail later. If $L^{-1} = A$, then (5) becomes

$$(6) \qquad |Qf| \le c|Af|^{1-a}|f|^a$$

and from Theorem 1 it follows that $s_n\{(L+M)^{-1}\} = s_n(L^{-1})\{1 + O(n^{-\gamma})\}$, $\gamma = \min\{d^{-1}, (l-m)/(l-m+d)\}$. This is equivalent to (4). Let us now explain (6). If $A > 0$ then

$$(7) \qquad |A^b f| \le |Af|^b |f|^{1-b}, \qquad f \in D(A), 0 < b < 1.$$

Indeed, by Hölder inequality

$$|A^b f|^2 = \int_0^\infty t^{2b} d(E_t f, f)$$

$$\le \left(\int_0^\infty t^2 d(E_t f, f)\right)^b \left(\int_0^\infty d(E_t f, f)\right)^{1-b} = |Af|^{2b} \cdot |f|^{2(1-b)}.$$

This is equivalent to (7). Inequality (6) is a particular case of (7) with $L^{-1} = A$, $b = 1 - a$. Note that $|L^{-1}Mf| \le c|L^{-(1-a)}f| = c|A^{1-a}f|$. Let us formulate the result.

THEOREM 2. *Let L be an elliptic selfadjoint operator, $\operatorname{ord} L = l$, on $H = L^2(D)$, where $D \subset \mathbf{R}^d$ is a bounded domain with a smooth boundary. Let M be a differential operator $\operatorname{ord} M = m < l$. Assume that the coefficients of L and M are smooth. Then*

$$s_n(L+M) = s_n(L)\{1 + O(n^{-\gamma})\}, \qquad \gamma = \min\{d^{-1}, (l-m)(l-m+d)^{-1}\}.$$

REMARK 4. Under the assumptions of Theorem 2 the root system of the operator $L + M$ forms a Riesz basis of $H = L^2(D)$ with brackets if $l - m \ge d$.
This follows from Proposition 1.

PROPOSITION 1. *Let $L = L^* > 0$ be an operator with a discrete spectrum $\lambda_j = cj^p + O(j^{p_1})$, $p > 0$, $p_1 < p$, and $|Mu| \le c|L^a u|$, $a < 1 \forall u \in H$. Let $p(1-a) \ge 1$. Then the root system of $L + M$ forms a Riesz basis of H with brackets.*

A proof is given in [2]. The condition $l - m \ge d$ in Remark 1 guarantees that $p(1-a) \ge 1$. Indeed for the elliptic operator L of order l it is known that $p = ld^{-1}$. From the argument given in §3 it follows that $a = ml^{-1}$. Therefore $p(1-a) = d^{-1}(1 - ml^{-1}) = (l-m)d^{-1} \ge 1$ because $l - m \ge d$.

4. Bibliographical remarks.

1. If $|A^{-a}Q| \leq c$, or $|Qf| \leq c|A^a f|$, $a < 1$, then it is often possible to derive the inequality used in Theorem 1: $|Qf| \leq c|Af|^a|f|^{1-a}$. For example this is so if A satisfies the inequality

$$(*) \quad |A^\beta f| \leq c(\alpha, \beta, \gamma)|A^\gamma f|^{(\beta-\alpha)/(\gamma-\alpha)} \cdot |A^\alpha f|^{(\gamma-\beta)/(\gamma-\alpha)},$$
$$\alpha < \beta < \gamma, f \in D(A^\gamma).$$

This inequality is satisfied e.g. if $|(A + sI)^{-1}| \leq c(1 + s)^{-1}$, $s > 0$. If $|(A + sI)^{-1}| \leq cs^{-1}$, $\alpha < \beta < \gamma \leq 1$, and $f \in D(A^\gamma)$ then $(*)$ holds. If $A = A^* > 0$ then $c(\alpha, \beta, \gamma) = 1$. Inequality $(*)$ is closely connected with the theory of fraction powers of linear operators. In [3] one can find the related results and references.

2. In [4] the following theorem was proved: if A and T are compact linear operators on H and for some $r > 0$ $\lim s_n(A)n^r = c$, $\lim s_n(T)n^r = 0$ as $n \to \infty$, then $\lim s_n(A + T)n^r = c$ as $n \to \infty$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109

Current address: Department of Mathematics, Kansas State University, Manhattan, Kansas 66506