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Phase-Difference Ambiguity Resolution for a Single-Frequency Signal

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Abstract

In this letter we propose a novel method to unambiguously estimate the phase-difference of a single-frequency signal measured between a pair of spatially separated sensors. First, we mathematically prove that, in a noiseless system, the phase between a pair of spatially separated sensors with inter-sensor spacing exceeding half wavelength ($\frac{\lambda}{2}$) of the signal of interest, can unambiguously be estimated utilizing a third collinear sensor, provided that the difference of the two smaller inter-sensor spacings does not exceed $\frac{\lambda}{2}$. The performance of the method is characterized by estimating the variance and the probability of failure in noisy cases.

1 Introduction

Time-Delay (TD) is an important signal parameter the accurate estimation of which represents an important problem in different fields including radar, sonar and ultrasonic. The problem generally consists of estimating the time-delay of a signal as observed by two spatially separated sensors. Various methods exist for the purpose of Time-Delay Estimation (TDE). The most well known are

those based on the Generalized Cross-Correlation (GCC) [1]. Generally, these methods provides for accuracy in order of an integral sample. If better accuracy is required, computationally intensive interpolation is needed [4]. Another group of TDE methods are DFT and cross-spectrum based methods (e.g., [2,3,6,7]) that estimate the time-delay based on the phase-difference estimated directly in the frequency domain. These methods provides for sub-sample accuracy and additionally require less computational power. However, a big limitation associated with the frequency domain methods, is that they restrict the delay values that can unambiguously be estimated to a range dictated by the permissible phase-difference range in the frequency domain. In other words, the phase-difference is naturally restricted to the range $\{-\pi, \pi\}$ and consequently, the delay should be in the range $\{\frac{-1}{2f}, \frac{1}{2f}\}$, where f is the signal frequency. This restriction is emphasized by forcing an inter-sensor spacing that complies with the traditional $\frac{\lambda}{2}$ rule in far-field models. If the inter-sensor spacing exceeds $\frac{\lambda}{2}$, the well known phase-difference ambiguity will occur and the estimated delay need to be disambiguated.

In many cases in practise, the satisfaction of the $\frac{\lambda}{2}$ is prohibitive. As an example, the wavelengths of acoustic signals above the audible range in air, decreases to values that makes the placement of the sensors physically unrealizable due to the smallness of the required separation compared to the senors radii. Same phenomenon, occurs in ultra-wideband RF systems. In these systems, phase-difference ambiguity is inherent and the use of frequency domain approaches is prohibitive unless a method for disambiguation exists. Moreover, even if $\frac{\lambda}{2}$ is not too small, a widely separated pair of receiver could be required to provide high Direction Of Arrival (DOA) resolution.

In [3], phase unwrapping was achieved exploiting frequency diversity of the received signal. However, in many systems as in indoor ultrasonic location systems, single-frequency pulses are used. in such cases, another diversity need to be exploited in order to disambiguate the measured phase-differences. In [5], a method that utilizes a third (auxilliary) collinear senor to provide such diversity has been used. However, the approach restricts the inter-sensor spacing, in such a way that the triplet is well-suited only for a single predefined frequency. Due to this restriction, the approach is susceptible to both Doppler frequency shifts as well frequency deviations at the

transmitter. Additionally, the approach can not accommodate for multiple different frequencies in case of a multi-channel impulsive systems.

In this letter, we propose a novel method to disambiguate the phase-difference of a single-frequency signal received by two spatially separated sensors. The propped method is intended to be more robust to frequency offsets. In addition, it can easily be generalized to frequency-diverse signals by successively applying it to individual frequency bins. The method can also be considered as a general theorem, to unambiguously estimate delays from a partial array of three sensors. In this letter, we first explain the theoretical basis of the method with a noiseless assumption, and following that we study the performance in presence of noise. Since the method is devised to work with TDE methods that suffer from the phase wrapping problem, the effect of noise is modeled as an error in the ambiguous delays estimated by the TDE method. The performance is thus characterized by varying the parameters of an assumed probability distribution of the error observed in the ambiguous estimates. we also emphasize that, the method is non-recursive and very computationally simple.

The rest of this document is organized as flows. Section 2 gives a detailed description of the proposed phase/delay disambiguation method. Section 3 presents an error analysis. In Section 4, performance is evaluated using ultrasonic signals. Section 5 is the conclusion of this letter.

2   Delay Disambiguation

This section details the description and the mathematical proof of the proposed method. Since the relation between time-delay and phase-difference is clearly known, we prefer for the sake of simplicity to explain the proposed method in terms of delays rather then phase-differences.

Theorem: In a far-field model, a signal delay measured between a pair of sensors spaced at more than $\frac{\lambda}{2}$ can almost surely be disambiguated utilizing a third (auxiliary) collinear sensor, provided that the absolute difference of the two smaller inter-sensor spacings does not exceed $\frac{\lambda}{2}$, where $\frac{\lambda}{2}$ is the half-wavelength of the impinging sinusoid.

The theorem does not make any assumption about the inter-sensor spacing distances. A maxi-

mum value could be any value that makes the far-filed condition valid.

Proof: Let $[d_{12}, \ d_{23}]$ be the inter-sensor spacing vector of a three-elements sensor array expressed in half-wavelength. Without loss of generality assume that sensor 3 is the auxiliary sensor and that $d_{23} > d_{12}$, and $d_{12} > \frac{\lambda}{2}$ The relation between the true delays and the ambiguous delays observed by the array can be stated as:

$$\delta_{12} = \delta_{12}^a + \frac{n_{12}}{f} \tag{1}$$

$$\delta_{23} = \delta_{23}^a + \frac{n_{23}}{f} \tag{2}$$

where $[\delta_{12}, \ \delta_{23}]$ are the true delays ($\delta_{12} \in \{\frac{-d_{12}}{2f}, \ \frac{d_{12}}{2f}\}$ and $\delta_{23} \in \{\frac{-d_{23}}{2f}, \ \frac{d_{23}}{2f}\}$); $[\delta_{12}^a, \ \delta_{23}^a] \in \{\frac{-1}{2f}, \ \frac{1}{2f}\}$ are the ambiguous delays that are measured between each pair of sensors; $[n_{12}, \ n_{23}] \in \mathbf{Z}$ represent the phase wrapping process; f is the received signal frequency. From the far-field assumption, we get:

$$\delta_{23} = \frac{d_{23}}{d_{12}} \delta_{12} \tag{3}$$

Subtracting (2) from (1), and then substituting (3) yields:

$$\delta_{12} = \mu \left(\delta_{23}^a - \delta_{12}^a + \frac{n_{23} - n_{12}}{f} \right) \tag{4}$$

where

$$\mu = \left(\frac{d_{12}}{d_{23} - d_{12}} \right) \tag{5}$$

Since the difference $n_{23} - n_{12}$ is unknown, (4) can be rewritten as:

$$\delta_{12}^{(k)} = \mu \left(\delta_{23}^a - \delta_{12}^a + \frac{k}{f} \right), \quad k \in \mathbf{Z} \tag{6}$$

where:

$$k = n_{23} - n_{12} \tag{7}$$

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Now, assume that the true delay estimate is $\delta_{12}^{(i)} = \mu\left(\delta_{23}^{a} - \delta_{12}^{a} + \frac{i}{f}\right)$, $i \in \mathbf{Z}$ and it satisfies:

$$\delta_{12}^{(i)} \in \left\{\frac{-d_{12}}{2f}, \frac{d_{12}}{2f}\right\} \tag{8}$$

Now, $\delta_{12}^{(k)}$ can be written as:

$$\delta_{12}^{(k)} = \delta_{12}^{(i)} + \mu\frac{k-i}{f}, \; [i,k] \in \mathbf{Z}, \; k = i, i \pm 1, i \pm 2, ... \tag{9}$$

In order to disambiguate the set in (9), $\delta_{12}^{(i)}$ should be made *distinct* from all the other (theoretically infinite) estimates in the set. Such distinction could be to make $\delta_{12}^{(i)}$ the only estimate in the set that satisfies (8). To achieve that, the following condition should be satisfied by all the other *false* estimates:

$$\mid \delta_{12}^{(k)} \mid > \frac{d_{12}}{2f}, \; [i,k] \in \mathbf{Z}, \; k = i \pm 1, i \pm 2, ... \tag{10}$$

where $\mid . \mid$ denotes the unsigned value. Assuming a positive value for μ, the condition in (10) can be written using (9) as:

$$\mu\frac{k_p - i}{f} > \frac{d_{12}}{2f} - \delta_{12}^{(i)}, \; [i,k_p] \in \mathbf{Z}, \; k_p = i+1, i+2, ... \tag{11}$$

and

$$\mu\frac{k_m - i}{f} < -\frac{d_{12}}{2f} - \delta_{12}^{(i)}, \; [i,k_m] \in \mathbf{Z}, \; k_m = i-1, i-2, ... \tag{12}$$

Now, considering only the minimum and maximum possible values of the LHS of each of (11) and (12) respectively, the necessary condition for getting a unique value for $\delta_{12}^{(i)}$ can be stated as satisfying both:

$$\mu > \frac{d_{12}}{2} - f\delta_{12} \tag{13}$$

and

$$\mu > \frac{d_{12}}{2} + f\delta_{12} \tag{14}$$

5

where $\delta_{12}^{(i)}$ has been replaced by the the true delay symbol δ_{12}. Combining (13) and (14) yields:

$$\mu > \max\left(\frac{d_{12}}{2} - f\delta_{12}, \frac{d_{12}}{2} + f\delta_{12}\right) = \frac{d_{12}}{2} + \mid f\delta_{12} \mid \tag{15}$$

where $\max(.,.)$ is the maximum value and $\mid . \mid$ is unsigned value. Equation (15) represents the necessary condition for $\delta_{12}^{(i)}$ to be a distinct value. It is noticed that satisfying (15) depends on the value $\mid \delta_{12} \mid$ and thus on the direction of arrival (DOA) of the signal. A *sufficient* condition can straightforwardly be derived from (15) by setting $\mid \delta_{12} \mid$ and hence the RHS of (15) to the maximum possible value. Then μ should satisfy:

$$\mu > d_{12} \tag{16}$$

Finally, substituting (5) in (16) and manipulating summarizes the condition as:

$$d_{23} - d_{12} < 1 \tag{17}$$

Since both d_{12} and d_{23} are expressed in half-wavelength, the sufficient condition for the uniqueness of $\delta_{12}^{(i)}$ is that the absolute difference the smaller inter-sensor spacings should be less than one half-wavelength.

Now, a sufficient condition to guarantee that a single value of the infinite set of candidate estimates of the true delay is distinct. Following, we show that exploiting (17), the number of search set is reduced to only 3 values. Getting back to (7), the equation can be rewritten as:

$$k = \Psi\left(f\delta_{23}\right) - \Psi\left(f\delta_{12}\right) \tag{18}$$

where $\Psi(.)$ is a special rounding function that works exactly like a standard rounding function except for that it rounds a real value $x + 0.5$ to x, where $x \in \mathbf{Z}$. This for instance guarantees that a delay value of $\frac{1}{2f}$ will not be ambiguated into a value of $\frac{-1}{2f}$. using (3), (18) can be expressed as:

$$k = \Psi\left(f\delta_{12} + \frac{f\delta_{12}}{d_{12}}(d_{23} - d_{12})\right) - \Psi\left(f\delta_{12}\right)$$

$$= \Psi\left(n_{12} + \gamma + \frac{f\delta_{12}}{d_{12}}(d_{23} - d_{12})\right) - n_{12} \qquad (19)$$

where

$$-0.5 < \gamma = f\delta_{12}^a \leq 0.5 \qquad (20)$$

Now, without loss of generality, assume both δ_{12} and n_{12} are positive. In such a case, the maximum possible value for k is obtained by setting the δ_{12} and the difference $d_{23} - d_{12}$ in (19) to their maximum permissible values, nominally, $\frac{d_{12}}{2f}$ and 1. That yields:

$$k_{max} = \Psi\left(n_{12} + \gamma + 0.5\right) - n_{12} = 1 \qquad (21)$$

The result in (21) stems directly from (19). Similarly, for a negative delay case, we get:

$$k_{min} = -1 \qquad (22)$$

Now, the search is reduced to the subset:

$$K = \{-1, 0, 1\} \qquad (23)$$

Finally, we summarize the method for delay disambiguation for sensor configuration satisfying (17):

1. Calculate $\delta_{12}^{(k)}$ for $\forall k \in K = \{-1, 0, 1-\}$.
2. Estimate δ_{12} as: $\hat{\delta}_{12} = \delta_{12}^{(i)}$, $i \in K$, where $\delta_{12}^{(i)} \in \left\{\frac{-d_{12}}{2f}, \frac{d_{12}}{2f}\right\}$.

In the preceding discussion, a noiseless data model was assumed. However, in presence of noise and estimation error in δ_{12}^a and δ_{23}^a, performance of the proposed method is questionable. In the following section, we show that the proposed method can achieve acceptable performance even in noisy cases.

3 Error Analysis

The aim of this section is to study the effect of error in δ_{12}^a and δ_{23}^a on the performance of the proposed method. Denote the erroneous principal estimates by δ_{12}' and δ_{23}', then:

$$\delta_{12}' = \delta_{12}^a + e_{12}^a, \ e_{12}^a \tag{24}$$

$$\delta_{23}' = \delta_{23}^a + e_{23}^a, \ e_{23}^a \tag{25}$$

where e_{12}^a and e_{12}^a represent estimation errors that depends on the underlying method. Substituting these erroneous estimates in (6) for $k \in K$ yields:

$$\delta_{12}'^{(k)} = \delta_{12}^{(k)} + e_{12} = \mu\left(\delta_{23}^a - \delta_{12}^a + \frac{k}{f}\right) + \mu(e_{23}^a - e_{12}^a), \ k \in K \tag{26}$$

where $\delta_{12}'^{(k)}$ are erroneous estimates for the candidate triplet estimates; e_{12} represents the error in estimating each of the triplet. From (26) it is evident that the estimated unambiguous delay will suffer error that is contribution from both ambiguous estimates. The error also depends on the parameter μ. The mean and variance of this error and its relation to the SNR can be directly identified from the characteristics of the underlying TDE method. The crucialness of this error is that it affects the hard decision that is taken to select among the three candidate estimates. Literally, the error gives rise to a probability of failure in the disambiguation process. Due to the clearness of the relationship between the error and the errors in the principal estimates, in this section, we focus of studying the probability of failure of the proposed method in noisy cases. The probability od failure can be written in light of (9) as:

$$
\begin{aligned}
P_f = {} & P\left(\frac{-d_{12}}{2f} - \delta_{12}^{(i)} > e_{12} > \frac{d_{12}}{2f} - \delta_{12}^{(i)}\right) \\
& + P\left(\frac{-d_{12}}{2f} - \delta_{12}^{(i)} - \mu\frac{l-i}{f} \le e_{12} \le \frac{d_{12}}{2f} - \delta_{12}^{(i)} - \mu\frac{l-i}{f}\right) \\
& + P\left(\frac{-d_{12}}{2f} - \delta_{12}^{(i)} - \mu\frac{m-i}{f} \le e_{12} \le \frac{d_{12}}{2f} - \delta_{12}^{(i)} - \mu\frac{m-i}{f}\right), \\
& [i,l,m] \in K, \ i \ne l \ne m
\end{aligned}
\tag{27}
$$

Now for simplicity assume that e_{12}^a, e_{23}^a are two i.i.d processes with normal distributions, zero means, and variance σ^2. From (26), e_{12} will also be a zero-mean process with variance:

$$\sigma_{12}^2 = 2\mu^2\sigma^2 \tag{28}$$

and the cumulative distribution:

$$\Phi(x) = \frac{1}{2}\left(1 + erf\left(\frac{x}{2\mu\sigma}\right)\right), \ x \in R \tag{29}$$

where $erf(.)$ is the error function. Further, Since $\delta_{12}^{(i)}$ is related to the DOA (θ) by

$$\delta_{12}^{(i)} = \frac{d_{12}}{2f}\sin(\theta) \tag{30}$$

Based on (29) and (30), (27) can be written as:

$$
\begin{aligned}
P_f &= -\Phi\left(\zeta^+\right) - \Phi\left(-\zeta^-\right) + \Phi\left(\zeta^+ - \mu\frac{l-i}{f}\right) \\
&+ \Phi\left(-\zeta^- + \mu\frac{l-i}{f}\right) + \Phi\left(\zeta^+ - \mu\frac{m-i}{f}\right) \\
&+ \Phi\left(-\zeta^- + \mu\frac{m-i}{f}\right), \ [i,l,m] \in K, \ i \neq l \neq m
\end{aligned} \tag{31}
$$

where $\zeta^+ = \frac{d_{12}}{2f}\left(1 - \sin(\theta)\right)$ and $\zeta^- = -\frac{d_{12}}{2f}\left(1 + \sin(\theta)\right)$.

4   Performance Tests

Different tests based on Eq. (31) have been conducted. The purpose was to understand the relationship between the probability of failure of the proposed method one one side; and the error in the principal estimates and different array parameter on the other side. ultrasonic signals were assumed throughout the tests. In all tests the values of $[i,l,m]$ were calculated based on an Assumed DOA, and reasonable values of σ^2 are assumed. First we evaluated P_f for all DOAs in the range

$\{-90^o, 90^o\}$, and for different values of σ^2 as in Fig. 1. Fig. 1 shows the dependence of p_f on both σ^2 and the DOA. The performance degrades as σ^2 increases, and for a fixed σ^2, it is clear that as we move towards the end-fire P_f increases.

In Fig. 2, d_{12} and σ^2 are kept constant while μ is varied by varying the difference $d_{23} - d_{12}$. The figure emphasizes the fact that the proposed method exhibits worst performance near the end-fire when $d_{23} - d_{12} = 1$, but still the variation of p_f with the difference $d_{23} - d_{12}$ need to be clarified. This variation is studied in Fig 3, where P_f is plotted against the difference $d_{23} - d_{12}$ for selected DOAs. From the figure, it is revealed that the difference $d_{23} - d_{12}$ should be carefully selected. Performance seems to peak in the range $\{0.4, 0.9\}\frac{\lambda}{2}$. However, since $\frac{\lambda}{2}$ depends on the frequency in use, a question concerning the usability of the same triplet for multiple frequencies arises. The answer follows immediately from Fig. 4, where $d_{23} - d_{12}$ is selected to be $0.9\frac{\lambda}{2}$ at 160 KHz. Tests involving the frequency range $\{0, 160\}KHz$ for different DOAs are summarized in the figure. Note that, part of the test frequencies are satisfying the $\frac{\lambda}{2}$ condition, and they thus represent ambiguity-free cases. The figure shows stable performance in the whole range except around the higher frequencies, where performance degradation that varies with the DOA in the same perviously noticed fashion is observable. Note that this figure is plotted for a relatively larger value of σ^2, and a difference $d_{23} - d_{12}$ that is close to the critical upper value for the higher frequencies.

5   Conclusion

A method that disambiguates time-delays observed between two widely-spaced sensors utilizing an auxiliary collinear sensor is presented. The performance of the method is studied and an expression for the probability of failure is obtained. Finally, we emphasize that, we have used the proposed method to extend one of the existing TDE methods. Results will appear in the literature soon.
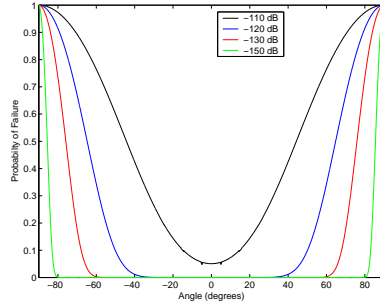
Figure 1: Probability of failure versus DOA for $\sigma^2 = $ -110, -120, -130 and -150 dB; f = 50 KHz; $d_{12} = 11$; $\mu = 11$.
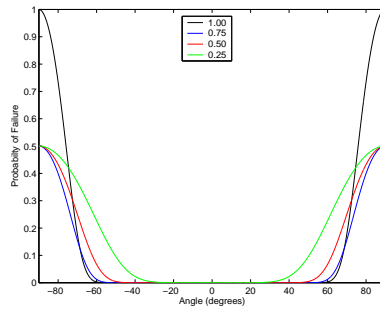

Figure 2: Probability of failure versus DOA for selected values of μ obtained by varying $d_{23} - d_{12}$ to take the values: 0.25, 0.50, 0.75 and 1; f = 50 KHz; $d_{12} = 11$; $\sigma^2 = $ -130 dB
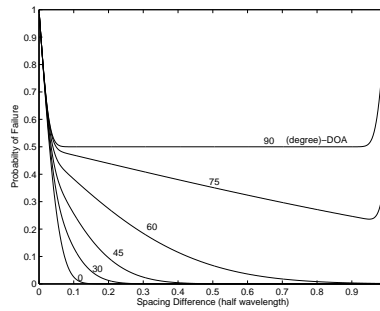

Figure 3: Probability of failure versus the difference $d_{23} - d_{12}$ for selected DOA angles; f = 50 KHz; $d_{12} = 11$; $\sigma^2 = $ -130 dB
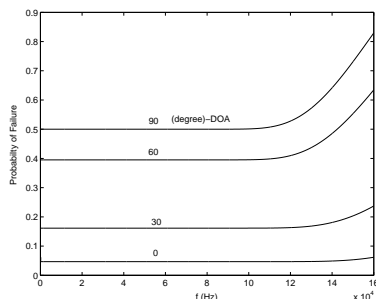
11

Figure 4: Probability of failure versus frequency for selected DOA angles; $d_{12}= 5$; $d_{23} - d_{12} = 0.9$ both at 160 KHz; $\sigma^2 = $ -120 dB

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