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Phase Dynamics of Weakly Unstable Periodic Structures

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Nonlinear phase dynamics of weakly unstable two-dimensional periodic patterns is studied. Four distinct physical situations are specifically considered. They correspond to the Eckhaus instability and zig-zag instability occurring in each of propagating and non-propagating patterns. Consequently, four prototype partial differential equations for phase function are obtained. Their derivation is totally based on symmetry- and scaling arguments. A simple interpretation of the origin of nonlinearity is given. Although the main part of the present theory is phenomenological, a more rigorous asymptotic theory is also developed for reaction-diffusion equations.

§ 1. Introduction

Recent theories on chaos and turbulence in dissipative systems usually take it for granted, explicitly or implicitly, that transitions to chaos are adequately modelled by simple dynamical systems with only a few degrees of freedom. Although this idea proved to be valid in many important applications, there exist equally important realistic cases where the inclusion of infinitely many degrees of freedom is vital even in the immediate vicinity of the onset of weak turbulence. This is peculiar to systems of sufficient spatial extension especially when regular patterns become slowly modulated in space and time as a result of the unstable growth of long-scale phase modes. The resulting turbulence can be very weak in the sense that the maximum Lyapunov number of the corresponding chaotic orbit can be very close to zero, whereas the attractor dimension may possibly be infinitely high. Because of the latter property, one must prepare from the outset an infinitely high dimensional phase space to accommodate the chaotic attractor, hence the breakdown of the usual dynamical-system description. More precisely, the range of its applicability, if expressed in terms of some bifurcation parameter included, would be narrowed down to zero as the system size goes to infinity. This particular type of weak turbulence was previously called phase turbulence.[1]

If the basic pattern which is going to be turbulized is space-periodic, phase turbulence may be visualized as contours varying slowly and irregularly in space and time, whereas the overall topological nature of the pattern is perfectly preserved. Although the experimental situation is not clear, something like this seems to happen in fluid layers heated from below with sufficient horizontal extension and relatively low Prandtl number.[2),3)] The convective rolls may then become weakly deformed through various kinds of instabilities such as the Eckhaus instability,[4)] zig-zag instability and scewed varicose instability,[5)] all these being caused by the unstable growth of long wavelength phase modes. Actually, the resulting dynamical behavior may or may not be chaotic. Still the description in terms of suitably defined phase function $\phi(\boldsymbol{r}, t)$ seems appropriate. For doing this, one has first to contract the original dynamics to the dynamics for ϕ, and this is in fact possible if one exploits the smallness of the bifurcation parameter ε. The equation for ϕ

would take the form of a nonlinear partial differential equation. A number of studies in this direction have been done for diffusion-coupled chemical oscillators,[1),6),7)] chemical wavefronts[8)] and combustion fronts.[9)] Though restricted to the linear regime, the phase dynamics idea was proposed also in the Rayleigh-Bénard convection by Pomeau and Manneville[10)] who derived a phase diffusion equation from the Newell-Whitehead amplitude equation.[11)] A severe limitation inherent in the Newell-Whitehead equation was noticed later by Siggia and Zippelius,[12)] which called for much more elaborate treatment of phase dynamics. This was attempted by Cross[13)] especially in relation to the origin of the scewed varicose instability. As another extension of the Pomeau-Manneville idea, one may mention the multi-phase theory proposed by Brand and Cross[14)] appropriate for describing a slowly modulated wavy vortex state in the Couette-Taylor system. It may be said, however, that any *nonlinear* phase dynamics theory of phase-unstable fluid patterns seems left undeveloped as yet.

Consider a two-dimensional dissipative system extending infinitely in both x and y directions. In order to derive generally the phase dynamics, the first to do is to classify various basic states which are going to be phase-destabilized. The basic states must have at least one phase mode of neutral stability, which implies that the following three cases will be the most important.

1. Uniform states which are oscillating autonomously,
2. Single straight wavefronts or interfaces which may be propagating or non-propagating,
3. Space-periodic patterns which may be propagating or non-propagating .

As noted above, cases 1 and 2 were studied previously at least for some reaction-diffusion and combustion problems. Thus studying the third case would be of interest, and this in fact constitutes the subject of the present paper. It would, however, be difficult to treat periodic patterns in general. Accordingly, the present paper will be restricted to such simple cases that the basic patterns depend only on one space coordinate. In contrast, truly two-dimensional periodic patterns would require multiple phase variables as discussed in Ref. 14).

Consider the dissipative dynamics described by

$$\partial_t \boldsymbol{X} = \boldsymbol{F}(\boldsymbol{X};\ \partial_x,\ \partial_y), \tag{1·1}$$

where $\boldsymbol{X}$ represents a real state vector and $\boldsymbol{F}$ some nonlinear function of $\boldsymbol{X}$ possibly including some spatial derivatives of $\boldsymbol{X}$. The above equation is assumed invariant under coordinate translations, reflections and rotations. Many physical systems share this property, and one obvious example is reaction-diffusion equations

$$\partial_t \boldsymbol{X} = \boldsymbol{F}(\boldsymbol{X}) + D(\partial_x{}^2 + \partial_y{}^2)\boldsymbol{X}\,. \tag{1·2}$$

The present paper will mostly be concerned with the general system (1·1), although certain properties assumed later for system (1·1) will explicitly be demonstrated for a more specific class (1·2). Unfortunately, the present theory does not seem to apply directly to convection problems. This is because of our assumption of smooth expansion of some relevant quantities in various spatial derivatives of phase; in fact, Cross[11)] argued that this assumption is invalid for the Rayleigh-Bénard convection on account of some long-wave singularities originating from the incompressibility condition. Despite this fact, our simple interpretation of the origin of the dominant nonlinear effects in the phase dynamics, which forms one of our main conclusions in the present paper, might enjoy much wider applicability including the Rayleigh-Bénard convection.

Section 2 outlines how to obtain nonlinear evolution equations for phase in various situations. Symmetry- and scaling arguments are all needed for this purpose. Symmetry considerations were also made in the Couette-Taylor system,[14] while scaling idea was used in oscillatory systems[1] and front dynamics.[9] The combination of these two ideas will turn out extremely fruitful. The present paper does not attempt the detailed analysis of the nolinear phase equations obtained, although a brief remark on each equation will be given in §3. The validity of one of our basic assumptions will be checked for reaction-diffusion equations in the Appendix. A rigorous asymptotic theory of phase dynamics based on a precise definition of phase will also be developed there for the same class of systems.

§ 2. Derivation of four prototype equations

2a. *Functional ansatz*

Let Eq. (1·1) admit a continuous family of periodic solutions $\boldsymbol{X}_0(\psi)$, where

$$\boldsymbol{X}_0(\psi+l)=\boldsymbol{X}_0(\psi),$$
$$\psi=x-ct+\phi\,,\qquad l=2\pi/k\,. \tag{2·1}$$

ϕ is an arbitrary constant, and l (or k) is a continuous parameter specifying wavelength (or wavenumber). The functional form of $\boldsymbol{X}_0$ depends generally on k. The propagation velocity c may identically vanish in some finite range of k, which in fact is the case for convective rolls. Such static solutions will be called type B, while propagating solutions type A. The velocity c may depend on k for propagating solutions. Vanishing velocity of type B solutions implies that they preserve the original reflection symmetry of the system. Thus one may require $\boldsymbol{X}_0(\psi)$ to be a symmetric function, or that there exist some ψ_0 such that

$$\boldsymbol{X}_0(-\psi+\psi_0)=\boldsymbol{X}_0(\psi).\quad \text{(for type B only)} \tag{2·2}$$

In contrast, the directions of increasing and decreasing x are not equivalent for type A solutions, and hence the corresponding $\boldsymbol{X}_0(\psi)$ must be asymmetric. A well-known example of type A solutions would be the periodic chemical waves in oscillatory and excitable reaction-diffusion systems,[15] whereas the Turing type solutions[16] belong to type B.

The basic idea of the phase dynamics is now described. Let us deform slightly the periodic pattern $\boldsymbol{X}_0(\psi)$ in such a way that its phase ϕ may be allowed a long-scale dependence on x and y. Such $\boldsymbol{X}_0(\psi)$ would no longer satisfy (1·1) exactly, still it could be "optimized" by allowing ϕ to evolve in time in an appropriate manner. The resulting time-evolution of ϕ is expected slow because ϕ is a long-wave extension of the uniform translational disturbance whose stability is neutral. In general, one may call some field quantity a slow field if its time-dependence vanishes in the long-wave limit. In the present theory ϕ is assumed to be the only slow field involved.*) This leads to the picture that all

*) This is not the case for the Rayleigh-Bénard convection with free top and bottom boundary conditions and infinite horizontal extension;[5),12)] in that case the vertical vorticity forms the second slow field. For rigid boundaries, however, ϕ represents the only slow field.

the other degrees of freedom which relax much faster than ϕ follow adiabatically the motion of ϕ (or ψ), and consequently the instantaneous spatial distribution of ϕ (or ψ) practically specifies the complete state of the system. Because the starting equation of motion (1·1) is first order in time, the above implies that the instantaneous rate of change of ϕ (or ψ) would also be determined by the distribution of ϕ (or ψ) itself, which we express as

$$\partial_t\phi=\Omega[\phi], \tag{2·3a}$$

$$\partial_t\psi=f[\psi]. \tag{2·3b}$$

It is now assumed that the right-hand sides can be represented in terms of local ϕ (or ψ) and its various spatial derivatives, i.e.,

$$\Omega[\phi]=\Omega(\phi,\ \partial_x\phi,\ \partial_y\phi,\ \partial_x{}^2\phi,\ \partial_x\partial_y\phi,\ \partial_y{}^2\phi,\ \cdots), \tag{2·4a}$$

$$f[\psi]=f(\psi,\ \partial_x\psi,\ \partial_y\psi,\ \partial_x{}^2\psi,\ \partial_x\partial_y\psi,\ \partial_y{}^2\psi,\ \cdots), \tag{2·4b}$$

and further that they can be expanded into multiple Taylor series. Although these assumptions would be difficult to justify rigorously, one may obtain explicitly such Taylor expansions to any desired order at least for reaction-diffusion dynamics as will be shown in the Appendix. Note that the derivative expansion of Ω represents itself the phase dynamics in the form of a partial differential equation. Thus the remaining problem is how to reduce such an infinite series to a finite one. As is seen below, symmetry- and scaling considerations are all we need for this purpose.

2b. *Symmetry considerations*

The spatial invariance properties of the original dynamics assumed in §1 are expected to be carried over to the equation for ψ (but not to the equation for ϕ because ϕ represents a deviation from a *symmetry-breaking* pattern). First, the translation $x\rightarrow x+x_0$ must leave f invariant. Since this transformation has no effect on the space derivatives of ψ, only replacing ψ by $\psi+x_0$, the above condition requires that f be independent of ψ-value itself. Equivalently,

[I] Ω is independent of ϕ.

Note that the translation $y\rightarrow y+y_0$ automatically leaves f and Ω invariant. Secondly, the reflection symmetry requires the invariance of f under $y\rightarrow -y$, or

[II] $y\rightarrow -y$ leaves Ω invariant.

The invariance of f under the reflection $x\rightarrow -x$ or under rotations cannot lead to any simple properties of Ω, although they may lead to some relations to be satisfied by some coefficients in the expansion of Ω. Finally, the symmetry property in (2·2) is still to be used. For type B solutions (possibly with space-time dependent ϕ) the change of the sign of ψ has no physical effects within a spatial translation. This requires that f be invariant under $\psi\rightarrow -\psi$ or, equivalently,

[III] Simultaneous transformations $\phi\rightarrow -\phi$ and $x\rightarrow -x$ leave Ω invariant (for type B only).

Various terms in the Taylor expansion of $\Omega[\phi]$ vanish because of their violation of [I] ~[III]. For instance, terms like $\partial_y\phi\partial_y{}^2\phi$ and $\partial_y{}^3\phi$ cannot appear on account of [II].

The assumption that ϕ depends slowly on x and y is yet to be used. Slow variation of ϕ in fact occurs when the basic periodic pattern $\boldsymbol{X}_0$ becomes weakly unstable with respect to long-scale phase disturbances. Our system is now supposed to be in such a weakly unstable state, which enables us to single out most important terms from the expansion of Ω. Before going into term-by-term estimation, however, it is necessary to make clear what specific types of phase instabilities are to be considered.

2c. *Instability types and the corresponding linear dispersions*

There exist at least two types of phase instabilities, namely,
(E) Eckhaus instability
and
(Z) Zig-zag instability.
The above names have been borrowed from fluid systems. They may occur either for type A solutions or for type B solutions. Therefore, four distinct physical situations are possible. They will be abbreviated as A-E, A-Z, B-E and B-Z, the meaning of which should naturally be understood.

Each type of instability is now briefly described. Let the periodic solution $\boldsymbol{X}_0(x-ct)$, which may be A type or B type, be slightly perturbed. Equation (1·1) is then linearized with respect to the deviation $\boldsymbol{\rho}$ defined by

$$\boldsymbol{X}(x, y, t)=\boldsymbol{X}_0(x-ct)+\boldsymbol{\rho}(x-ct, y, t). \tag{2·5}$$

The solution of the linearized equation may be sought in the form

$$\boldsymbol{\rho}(x-ct, y, t)=\boldsymbol{u}(x-ct)\exp\{i(q_x x+q_y y)+\lambda t\},$$

where $\boldsymbol{u}$ is some l-periodic function of $x-ct$ possibly depending on $\boldsymbol{q}\equiv(q_x, q_y)$. The eigenvalue λ as a function of $\boldsymbol{q}$ is now considered. For vanishing $\boldsymbol{q}$, one of the eigenvalues must identically vanish. This is the eigenvalue of the phase mode or translational mode. Its extension to finite $\boldsymbol{q}$ forms a phase branch. Since instabilities different from phase instabilities are not considered here, the real part of λ for the other branches is assumed to be finite negative. In accordance with the Taylor expansion of Ω, let the eigenvalues of long-scale phase modes be expanded in powers of q_x and q_y. This expansion must coincide with the linearlized Ω via transformations $\lambda\rightarrow\partial_t$, $iq_x\rightarrow\partial_x$ and $iq_y\rightarrow\partial_y$. Then, from property [II], odd powers of iq_y must not appear in λ. Further, property [III] implies that odd powers of iq_x must not appear for type B, but may appear for type A. Thus the general form of λ becomes

$$\left.\begin{aligned}
&\lambda=i\lambda_i-\lambda_r,\\
&\lambda_i=(a_1q_x+a_3q_x^3+\cdots)+(c_{12}q_xq_y^2+c_{32}q_x^3q_y^2+\cdots),\\
&\lambda_r=(a_2q_x^2+a_4q_x^4+\cdots)+(b_2q_y^2+b_4q_y^4+\cdots)+(c_{22}q_x^2q_y^2+\cdots),
\end{aligned}\right\} \tag{2·6}$$

where a_i, b_i and c_{ij} are real constants depending generally on the basic wavenumber k, and λ_i is vanishing for type B solutions. By the Eckhaus instability we mean that a_2 changes sign from positive to negative, whereas the zig-zag instability means that the same occurs for b_2. They may also be called longitudinal and transversal instabilities, respectively. In the present paper a_4 and b_4 are always assumed positive so that they may contribute to stabilization.

2d. *Scaling idea and final equation forms*

The derivative expansion of $\Omega[\phi]$ can further be simplified by scaling considerations. Let a_2 or b_2 be a small parameter depending on the type of instability of concern, and put $a_2=-\varepsilon$ or $b_2=-\varepsilon$. It is reasonable to assume that the solutions of the partial differential equation for ϕ have a scaling form for sufficiently small ε, or

$$\phi(x, y, t)=\varepsilon^{\beta}\tilde{\phi}(\varepsilon^{\nu}x, \varepsilon^{\nu'}y, e^{\delta}t), \tag{2·7}$$

where β, ν, ν' and δ are non-negative constants yet to be determined. The above scaling form enables us to express the order of magnitude of various terms in the expansion of Ω. For instance, $\partial_x\phi\sim\varepsilon^{\nu+\beta}$ and $(\partial_y\phi)^2\sim\varepsilon^{2(\nu'+\beta)}$. How to find the exponent values goes as follows. Note first that the determination of δ is automatic if the other three exponents are known. Further, the destabilizing term $-\varepsilon\partial_x{}^2\phi$ in the case of the Eckhaus instability is expected to be balanced with the dissipation term $\partial_x{}^4\phi$ and also with the lateral diffusion $\partial_y{}^2\phi$. Similarly, one may expect the balancing $-\varepsilon\partial_y{}^2\phi\sim\partial_y{}^4\phi\sim\partial_x{}^2\phi$ to hold for zig-zag instability. It is easy to check that, even if $\beta(\geq 0)$ is unknown, knowing the values of ν and ν' in this way determines uniquely which nonlinear terms are most dominant. The balance condition between the dominant nonlinear terms and the above linear terms finally determines β. The values of the exponents in each case are listed in Table I. Note that $\partial_y\phi$ is small in all cases, which means that the contours of constant ϕ remains almost parallel to the y-direction even if the pattern is destabilized. The final equation forms are the following.

(A-E) $\quad \partial_t\phi=L\phi+g(\partial_x\phi)^2,$

$$L=a_1\partial_x+a_2\partial_x{}^2-a_3\partial_x{}^3-a_4\partial_x{}^4+b_2\partial_y{}^2, \qquad a_2=-\varepsilon, \tag{2·8}$$

(A-Z) $\quad \partial_t\phi=L\phi+g(\partial_y\phi)^2,$

$$L=b_2\partial_y{}^2-c_{12}\partial_x\partial_y{}^2-b_4\partial_y{}^4+a_2\partial_x{}^2, \qquad b_2=-\varepsilon, \tag{2·9}$$

(B-E) $\quad \partial_t\phi=L\phi+g\partial_x\phi\cdot\partial_x{}^2\phi,$

$$L=a_2\partial_x{}^2-a_4\partial_x{}^4+b_2\partial_y{}^2, \qquad a_2=-\varepsilon, \tag{2·10}$$

(B-Z) $\quad \partial_t\phi=L\phi+\{g_1\partial_x\phi+g_2(\partial_y\phi)^2\}\partial_y{}^2\phi,$

$$L=b_2\partial_y{}^2-b_4\partial_y{}^4+a_2\partial_x{}^2, \qquad b_2=-\varepsilon. \tag{2·11}$$

The above results must be consistent with the linear dispersion (2·6), which requires that a_i, b_i and c_{ij} above be identical with the respective quantities under the same notations in Eq. (2·6). For simplicity, a common notation g has been used for the nonlinearity parameters. It should be noted that our scaling arguments break down if the above equations turned out to give solutions escaping to infinity.

Table I.

	β	ν	ν'	δ
A-E	1/2	1/2	1	3/2
A-Z	1	1	1/2	2
B-E	1/2	1/2	1	2
B-Z	0	1	1/2	2

A few more remarks should be given regarding Eq. (2·8). Let the term $a_1\partial_x\phi$ be eliminated through the coordinate transformation $x\to x-a_1t$. The scaling exponents in Table I actually refer to this new representation. Note also that the terms $a_2\partial_x{}^2\phi$, $\partial_x{}^4\phi$ and $\partial_y{}^2\phi$, all being $O(\varepsilon^{5/2})$, are smaller than the terms $\partial_x{}^3\phi$ and $(\partial_x\phi)^2$ which are $O(\varepsilon^2)$. The former should still be retained because they represent the first nonvanishing dissipation effects and hence are the very cause of the spontaneous occurrence of the scaling behavior of ϕ. For the other quantities, in consrast, all terms in each case are of the same order in ε.

2e. *Determination of g*

The parameter g can be related to a certain derivative of a_2 or b_2 or velocity c, and this fact leads to a simple physical interpretation of the nonlinearity. Let us begin with the determination of g is each case.

(A-E) As a particular solution of Eq. (2·8), consider

$$\phi=\varkappa x+(a_1\varkappa+g\varkappa^2)t\,, \tag{2·12}$$

where $\varkappa$ must be small by assumption. The corresponding phase-perturbed solution of Eq. (1·1) becomes

$$\boldsymbol{X}_0(\phi)=\boldsymbol{X}_0((1+\varkappa)x-(c(k)-a_1\varkappa-g\varkappa^2)t),$$

which still represents a perfectly periodic pattern though with a modified wavenumber $\tilde{k}$ and modified velocity $\tilde{c}$ given respectively by

$$\tilde{k}=k(1+\varkappa)$$

and

$$\tilde{c}=c(k)-a_1\varkappa-g\varkappa^2\,.$$

Since $\tilde{c}$ must coincide with $c(\tilde{k})$, the Taylor expansion of $c(\tilde{k})$ about $c(k)$ gives

$$a_1=-k\frac{dc(k)}{dk}\,,$$

$$g=-k^2\frac{d^2c(k)}{dk^2}\,. \tag{2·13}$$

(A-Z) Analogously to the above, consider

$$\phi=\varkappa y+g\varkappa^2 t \tag{2·14}$$

as a particular solution of Eq. (2·9). This leads to

$$\boldsymbol{X}_0(\phi)=\boldsymbol{X}_0(x+\varkappa y-(c(k)-g\varkappa^2)t),$$

which represents again a periodic pattern though it is slightly non-parallel to the y-direction. The wavenumber has now been changed to $\tilde{\varkappa}$, where

$$\tilde{\varkappa}=\sqrt{1+\varkappa^2}\,,$$

and we must require

$$\tilde{c} \equiv c(k) - g\varkappa^2 = c(\tilde{\varkappa}),$$

which gives

$$g = \frac{k}{2}\frac{dc(k)}{dk}. \tag{2·15}$$

(B-E) The appropriate form of ϕ in this case is

$$\phi = \varkappa x + \phi_0 \exp(iq_x x + \lambda t), \tag{2·16}$$

where ϕ_0 is supposed to be small. Equation (2·10) is now linearized with respect to ϕ_0. Then

$$\lambda = -\tilde{a}_2 q_x^2 - a_4 q_x^4,$$

where

$$\tilde{a}_2 = a_2(k) + g\varkappa^2.$$

The approximate solution (2·16) of Eq. (2·10) describes a small wavy phase perturbation upon a periodic pattern with wavenumber $\tilde{k}$, where

$$\tilde{k} = k(1 + \varkappa).$$

The equality $\tilde{a}_2 = a_2(\tilde{k})$ must be satisfied, which leads to

$$g = k\frac{da_2(k)}{dk}. \tag{2·17}$$

(B-Z) Let us consider the form

$$\phi = \varkappa_1 x + \varkappa_2 y + \phi_0 \exp(iq_y y + \lambda t), \tag{2·18}$$

and linearize Eq. (2·11) with respect to ϕ_0. Then

$$\lambda = -\tilde{b}_2 q_y^2 - b_4 q_y^4,$$

where

$$\tilde{b}_2 = b_2(k) + g_1\varkappa_1 + g_2\varkappa_2^2.$$

The basic wavenumber has now been changed to

$$\tilde{k} = k\sqrt{(1+\varkappa_1)^2 + \varkappa_2^2}.$$

Requiring $\tilde{b}_2 = b_2(\tilde{k})$, one obtains

$$g_1 = 2g_2 = k\frac{db_2(k)}{dk}. \tag{2·19}$$

From the above arguments it is now understood where the nonlinearity of our phase dynamics comes from. We see that the dominant nonlinearity is caused by the local velocity change in the case of propagating patterns, and the local change in the phase diffusion constant in the case of static patterns, both due to the local periodicity change. The same mechanism is expected to underlie wider classes of systems to which the present theory may not strictly apply.

§ 3. Remarks on the four prototype equations

Detailed analysis of our four prototype equations is not the purpose of the present paper. Instead, a brief remark will be given on each of them.

(A-E) Let us work with the aforementioned moving coordinate to eliminate the $a_1\partial_x\phi$ term in Eq. (2·8). If the basic periodic pattern is stable, or if $a_2>0$, no higher order derivatives than the second will be important. If one neglects the y-dependence, too, one obtains

$$\partial_t\phi=a_2\partial_x{}^2\phi+g(\partial_x\phi)^2\,. \tag{3·1}$$

This is transformed to the Burgers equation through $\partial_x\phi=v$. The original pattern may be a periodic pulse train such as shown by the Hodgkin-Huxley nerve conduction equation. In that case Eq. (3·1) tells us that long wave fluctuations v in the pulse number density obeys the Burgers equation. This fact was used[17] for explaining observed spectral anomaly of $v(x,\ t)$ in nerve axons.[18]

For weakly unstable case, which is of our main concern here, the terms $\partial_x{}^3\phi$ and $(\partial_x\phi)^2$ are the dominant as noted in §2-2d. If the other terms are ignored, the Korteweg-de Vries equation

$$\partial_t v=-a_3\partial_x{}^3 v+2gv\partial_x v \tag{3·2}$$

is obtained. Because the next order terms, which are dissipation terms $\varepsilon\partial_x{}^2\phi$ and $\partial_x{}^4\phi$, represent "energy" source and sink, they seem to be crucial in driving the system to some attractor which may be a chaotic attractor of infinite dimension. The present type of dissipative KdV equation is met in some different physical contexts such as viscous fluids flowing on an inclined plane[19] and dissipative trapped-ion mode in plasmas.[20]

(A-Z) In this case the instability is strongest in the y-direction, and the x-dependence of ϕ may be neglected in the first approximation. Then

$$\partial_t\phi=-\varepsilon\partial_y{}^2\phi-b_4\partial_y{}^4\phi+g(\partial_y\phi)^2\,. \tag{3·3}$$

The same equation was also met in reaction-diffusion turbulence[1),7),9)] and frame-front turbulence.[8] By including y-dependence, a continuously coupled chaotic system is obtained, where each subsystem is described by Eq. (3·3).

(B-E) In contrast to the above case, the y-dependence seems less important. Let us drop it and normalize some coefficients by a suitable rescaling. Then

$$\partial_t\phi=-\alpha\partial_x{}^2\phi-\partial_x{}^4\phi+\partial_x\phi\partial_x{}^2\phi\,, \tag{3·4a}$$

where α is supposed to be positive or the periodic pattern is supposed to be unstable. For infinite system size the above has a potential, or

$$\partial_t\phi=-\delta U/\delta\phi\,,$$

$$U=\frac{1}{2}\int dx\left\{-\alpha(\partial_x\phi)^2+\frac{1}{3}(\partial_x\phi)^3+(\partial_x{}^2\phi)^2\right\}. \tag{3·4b}$$

Thus the system is expected to approach a local minimum of U. In order to get some insight into this final state, Eq. (3·4a) is expressed as

$$\partial_t v = \partial_x(-a\partial_x v - \partial_x{}^3 v + v\partial_x v), \tag{3·5}$$

where $v \equiv \partial_x \phi$. Constant v, or $v = v_0$, is always a solution. This is stable if $v_0 > a$ and unstable otherwise. Note that nonvanishing spatial average of v, denoted as $\bar{v}$, means a nonvanishing change in the average periodicity of the pattern. In general, this may cause a change in stability, too. Note, however, that $\bar{v}$ is a conserved quantity, which means that a phase-destabilized periodic pattern cannot spontaneously develop into a stable periodic pattern by changing its periodicity. Thus the phase instability of a periodic pattern will inevitably produce some nonuniform distribution of v corresponding to a local minimum of U. Among such nonuniform states the most interesting would be the ones satisfying

$$-a\partial_x v - \partial_x{}^3 v + v\partial_x v = 0. \tag{3·6}$$

Its solutions are nothing but the equilibrium solutions of the KdV equation supplemented with a propagation term $\partial_x v$. The unstable case where $\bar{v} < a$ is of the present interest. It is clear that one-soliton solutions of Eq. (3·6), i.e.,

$$v(x) = v_0 + 3\tilde{a}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{-\tilde{a}}\,x\right),$$
$$\tilde{a} = a - v_0 < 0 \tag{3·7}$$

cannot satisfy $\bar{v} < a$ because v_0 must be identical to $\bar{v}$ for infinite system size which one is presently working with. Possible realistic states may then be periodic equilibrium solutions or

$$v(x) = v_0 + 3\tilde{a}k\,\mathrm{cn}^2\left(\frac{1}{2}\sqrt{-\tilde{a}}\,x,\ k\right) \tag{3·8}$$

for which the above inequality could be satisfied.

(B-Z) It seems difficult to see, at least by inspection, any simple properties of the solutions of Eq. (2·10). The associated dynamics may be essentially two-dimensional, and detailed numerical analysis is desired.

§ 4. Summary

The dynamics of some simple two-dimensional periodic patterns has been studied in particular when they become weakly deformed through some phase instabilities such as the Eckhaus and zig-zag instabilities. It has been found that the dynamics is best described in terms of suitably defined phase function ϕ. Depending on the type of instability and also on whether the basic periodic structure is propagating or non-propagating, four distinct physical situations are possible. Four corresponding partial differential equations for ϕ have been obtained by means of symmetry- and scaling arguments. All parameters included have then been related to some coefficients appearing in the linear dispersion of the phase branch, the unperturbed propagation velocity and some suitable derivatives of these quantities with respect to the basic periodicity of the pattern. This finding has enabled us a simple physical interpretation of the nonlinear terms involved. Some of these partial differential equations are expected to show turbulent behavior, although no detailed analysis of them has been attempted in the present paper. The crucial assumption of the present theory is the possibility of smooth expansion of relevant quantities in powers of various spatial derivatives of ϕ. Although this property has been demonstrated for reaction-diffusion systems, some important fluid-dynamical problems such as the Rayleigh-Bénard convection seem to lack in this property, and hence may remain beyond direct applicability of the present theory. In this connection, the present theory would have to be generalized to introduce some auxiliary variables (which are not necessarily slow fields) so that the possibility of smooth expansion may be recovered.

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Appendix

Reaction-diffusion systems (1·2) are considered here in order to demonstrate how $\Omega[\phi]$ is obtained explicitly in the form of derivative expansion. To begin with, we introduce some notions associated with the linear system about the l-periodic solution $\boldsymbol{X}_0(z)$ of Eq. (1·2), where $z=x-ct$ and $\boldsymbol{X}_0(z)$ satisfies

$$\boldsymbol{F}(\boldsymbol{X}_0)+c\partial_z\boldsymbol{X}_0+D\partial_z{}^2\boldsymbol{X}_0=0\,. \tag{A·1}$$

The linearized equation of Eq. (1·2) about $\boldsymbol{X}_0(z)$ is expressed as

$$\partial_t\boldsymbol{\rho}=\tilde{L}\boldsymbol{\rho}\,, \tag{A·2}$$

where $\boldsymbol{\rho}$ is the deviation of $\boldsymbol{X}$ from $\boldsymbol{X}_0(z)$, and

$$\tilde{L}=L(z)+c\partial_z+D(\partial_z{}^2+\partial_y{}^2), \tag{A·3a}$$

$$L_{ij}(z)=\partial F_i(\boldsymbol{X}_0)/\partial X_{0j}\,. \tag{A·3b}$$

Let $\boldsymbol{\rho}$ be restricted to such perturbations that are l-periodic and independent of y. By putting $\boldsymbol{\rho}=\boldsymbol{u}(z)e^{\lambda t}$, where $\boldsymbol{u}(z)=\boldsymbol{u}(z+l)$, Eq. (A·2) becomes

$$\lambda\boldsymbol{u}=\tilde{L}_0\boldsymbol{u}\,, \tag{A·4a}$$

where

$$\tilde{L}_0=L(z)+c\partial_z+D\partial_z{}^2\,. \tag{A·4b}$$

The eigenfunctions and eigenvalues associated with $\tilde{L}_0$ are denoted as $\boldsymbol{u}_n$ and λ_n ($n=0, 1, 2, \cdots$), respectively. Let $\boldsymbol{u}_0$ denote the eigenfunction of the translational mode. One may take

$$\boldsymbol{u}_0(z)=\frac{d\boldsymbol{X}_0(z)}{dz} \tag{A·5}$$

and $\lambda_0=0$. Note that the linear dispersion (2·6) is a long-wave extension of this zero-eigenvalue. It is assumed that all eigenvalues are algebraically simple, that $\mathrm{Re}\lambda_n < 0$ for all $n \neq 0$, and that the eigenfunctions are orthonormalized as

$$(\boldsymbol{u}_m^*, \boldsymbol{u}_n) \equiv \int_0^l \boldsymbol{u}_m^*(z)\boldsymbol{u}_n(z)dz = \delta_{mn}, \tag{A·6}$$

where $\boldsymbol{u}_m^*$ denotes an eigenfunction of the adjoint operator of $\tilde{L}_0$ belonging to the eigenvalue λ_m.

Our systematic method of derivative expansion is now described. Let the periodic pattern $\boldsymbol{X}_0(z)$ be given a long-scale phase perturbation. In order to indicate this slowness explicitly, it is appropriate to introduce scaled coordinates X and Y by

$$X = \varepsilon_1 x, \qquad Y = \varepsilon_2 y. \tag{A·7}$$

As in usual multi-scale asymptotic theories, x, X and Y are treated as independent variables. Note that there is no reason for the unstretched coordinate y to enter the theory. In the above, ε_1 and ε_2 are used as tracers of smallness and will finally be equated to 1. Still expressions like "terms of order ε_1^2" will be used to indicate quantities such as $\partial_x^2\phi$ and $(\partial_x\phi)^2$. With these new coordinates the phase-perturbed state may approximately be expressed as

$$\boldsymbol{X} \simeq \boldsymbol{X}_0(\psi), \tag{A·8a}$$

$$\psi = x - ct + \phi(X, Y, t). \tag{A·8b}$$

How ϕ evolves in time is now considered. Since phase perturbations would inevitably induce deviations in wave profile (seen on the sections locally normal to the contours of constant ψ), one should rather work with a form more general than (A·8a), and put

$$\boldsymbol{X} = \boldsymbol{X}_0(\psi) + \boldsymbol{\rho}(\psi, X, Y, t). \tag{A·9}$$

Note that the X-and Y-dependence of $\boldsymbol{\rho}$ refers to its long-scale variation, while its short-scale variation has separately been taken into account by its l-periodic dependence on ψ or

$$\boldsymbol{\rho}(\psi + l, X, Y, t) = \boldsymbol{\rho}(\psi, X, Y, t). \tag{A·10}$$

The definition of $\boldsymbol{\rho}$ in Eq. (A·9) is not unique because the reference state $\boldsymbol{X}_0(\psi)$ itself depends on yet unspecified ϕ. The separation of the disturbances into $\boldsymbol{\rho}$ and ϕ may be made unique by requiring

$$(\boldsymbol{u}_0^*, \boldsymbol{\rho}) \equiv \int_0^l \boldsymbol{u}_0^*(\psi)\boldsymbol{\rho}(\psi, X, Y, t)d\psi = 0, \tag{A·11}$$

where the integral is taken under fixed X, Y and t. Condition (A·11) simply means that $\boldsymbol{\rho}$ includes no phase disturbances.

Rather than trying to find directly the expressions for the set of the unknowns $(\phi, \boldsymbol{\rho})$, it is more appropriate to do the same for $(\Omega, \boldsymbol{\rho})$, where $\Omega \equiv \partial_t\phi$. This is simply because the phase disturbance need not remain small, while its time derivative is expected to be small. The same spirit commonly underlies all asymptotic theories of nonlinear oscillations including secularity. The unknowns $(\Omega, \boldsymbol{\rho})$ are now sought in the form

$$\boldsymbol{\rho}(\psi, X, Y, t) = \boldsymbol{\rho}(\psi, [\phi]), \tag{A·12a}$$

$$\Omega(X,\ Y,\ t)=\Omega[\phi]. \tag{A·12b}$$

Here $[\phi]$ denotes dependence on all possible derivatives of ϕ with respect to the scaled coordinates X and Y or, symbolically,

$$[\phi]\equiv(\varepsilon_1\partial_X\phi,\ \varepsilon_2\partial_Y\phi,\ \varepsilon_1{}^2\partial_X{}^2\phi,\ \cdots). \tag{A·13}$$

In accordance with the introduction of multiple spatial scales, it is appropriate to convert Eq. (1·2) to

$$\partial_t\boldsymbol{X}=\boldsymbol{F}(\boldsymbol{X})+D\{(\partial_x+\varepsilon_1\partial_X)^2+\varepsilon_2{}^2\partial_Y{}^2\}\boldsymbol{X}. \tag{A·14}$$

Let (A·12a) and (A·12b) be substituted into (A·14). The left-hand side of (A·14) is then expressed as

$$\begin{aligned}&\partial_t\{\boldsymbol{X}_0(\psi)+\boldsymbol{\rho}(\psi,\ [\phi])\}\\&\quad=\Omega[\phi]\boldsymbol{u}_0(\psi)-c\partial_\psi\boldsymbol{X}_0+\partial_\psi\boldsymbol{\rho}\cdot\Omega[\phi]-c\partial_\psi\boldsymbol{\rho}+\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_t[\phi],\end{aligned} \tag{A·15}$$

where the following abbreviation has been used:

$$\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_t[\phi]\equiv\sum_{i+j\geq1}\sum\varepsilon_1{}^i\varepsilon_2{}^j\partial_X{}^i\partial_Y{}^j\Omega[\phi]\cdot\partial\boldsymbol{\rho}/\partial(\varepsilon_1{}^i\varepsilon_2{}^j\partial_X{}^i\partial_Y{}^j\phi). \tag{A·16}$$

Let $\boldsymbol{F}(\boldsymbol{X})$ be expanded as

$$\boldsymbol{F}(\boldsymbol{X})=\boldsymbol{F}(\boldsymbol{X}_0(\psi))+L(\psi)\boldsymbol{\rho}+M(\psi)\boldsymbol{\rho}\boldsymbol{\rho}+\cdots. \tag{A·17}$$

Then the right-hand side of Eq. (A·14) becomes

$$\boldsymbol{F}(\boldsymbol{X}_0(\psi))+L(\psi)\boldsymbol{\rho}+M(\psi)\boldsymbol{\rho}\boldsymbol{\rho}+D\{(\partial_x+\varepsilon_1\partial_X)^2+\varepsilon_2{}^2\partial_Y{}^2\}(\boldsymbol{X}_0(\psi)+\boldsymbol{\rho})+O(\boldsymbol{\rho}^3). \tag{A·18}$$

Equating (A·15) and (A·18), and using Eq. (A·1) with z replaced by ψ, one obtains

$$\Omega[\phi]\boldsymbol{u}_0(\psi)-\tilde{L}_0(\psi)\boldsymbol{\rho}=\boldsymbol{B}(\psi,\ [\phi]), \tag{A·19a}$$

where

$$\begin{aligned}\boldsymbol{B}(\psi,\ [\phi])=&-\partial_\psi\boldsymbol{\rho}\cdot\Omega[\phi]-\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_t[\phi]+M\boldsymbol{\rho}\boldsymbol{\rho}\\&+D\{(\partial_x+\varepsilon_1\partial_X)^2+\varepsilon_2{}^2\partial_Y{}^2-\partial_\psi{}^2\}(\boldsymbol{X}_0(\psi)+\boldsymbol{\rho})+O(\boldsymbol{\rho}^3)\\=&-\partial_\psi\boldsymbol{\rho}\cdot\Omega[\phi]-\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_t[\phi]+M\boldsymbol{\rho}\boldsymbol{\rho}\\&+D[\partial_\psi{}^2\boldsymbol{X}_0\cdot\{2\varepsilon_1\partial_X\phi+\varepsilon_1{}^2(\partial_X\phi)^2+\varepsilon_2{}^2(\partial_Y\phi)^2\}\\&+\partial_\psi\boldsymbol{X}_0\cdot(\varepsilon_1{}^2\partial_X{}^2\phi+\varepsilon_2{}^2\partial_Y{}^2\phi)]+\boldsymbol{B}_1[\phi]+O(\boldsymbol{\rho}^3)\end{aligned} \tag{A·19b}$$

with

$$\begin{aligned}\boldsymbol{B}_1[\phi]&=D\{(\partial_x+\varepsilon_1\partial_X)^2+\varepsilon_2{}^2\partial_Y{}^2-\partial_\psi{}^2\}\boldsymbol{\rho}\\&=D(2\varepsilon_1\partial_x\partial_X+\varepsilon_1{}^2\partial_X{}^2+\varepsilon_2{}^2\partial_Y{}^2)\boldsymbol{\rho}\\&=D[2\varepsilon_1\{\partial_\psi{}^2\boldsymbol{\rho}\cdot\partial_X\phi+\partial_{[\phi]}(\partial_\psi\boldsymbol{\rho})\cdot\partial_X[\phi]\}\\&\quad+\varepsilon_1{}^2\{\partial_\psi{}^2\boldsymbol{\rho}\cdot(\partial_X\phi)^2+\partial_\psi\boldsymbol{\rho}\cdot\partial_X{}^2\phi+2\partial_{[\phi]}(\partial_\psi\boldsymbol{\rho})\cdot\partial_X[\phi]\cdot\partial_X\phi\\&\quad+\partial_{[\phi]}^2\boldsymbol{\rho}\cdot(\partial_X[\phi])^2+\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_X{}^2[\phi]\}\end{aligned}$$

$$+\varepsilon_2{}^2\{\partial_\psi{}^2\boldsymbol{\rho}\cdot(\partial_Y\phi)^2+\partial_\psi\boldsymbol{\rho}\cdot\partial_Y{}^2\phi+2\partial_{[\phi]}(\partial_\psi\boldsymbol{\rho})\cdot\partial_Y[\phi]\cdot\partial_Y\phi$$
$$+\partial_{[\phi]}^2\boldsymbol{\rho}\cdot(\partial_Y[\phi])^2+\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_Y{}^2[\phi]\}]\cdot \tag{A·19c}$$

Here again some abbreviations have been used, i.e.,

$$\varepsilon_1\partial_{[\phi]}(\partial_\psi\boldsymbol{\rho})\cdot\partial_X[\phi]=\sum_{i+j\geq1}\sum\varepsilon_1{}^{i+1}\varepsilon_2{}^j\partial_X{}^{i+1}\partial_Y{}^j\phi\cdot\partial(\partial_\psi\boldsymbol{\rho})/\partial(\varepsilon_1{}^i\varepsilon_2{}^j\partial_X{}^i\partial_Y{}^j\phi), \tag{A·20a}$$

$$\varepsilon_1{}^2\partial_{[\phi]}^2\boldsymbol{\rho}\cdot(\partial_X[\phi])^2=\sum_{i+j\geq1}\sum_{,}\sum_{k+l\geq1}\sum\varepsilon_1{}^{i+k+2}\varepsilon_2{}^{j+l}(\partial_X{}^{i+1}\partial_Y{}^j\phi)(\partial_X{}^{k+1}\partial_Y{}^l\phi)$$
$$\times\partial^2\boldsymbol{\rho}/\partial(\varepsilon_1{}^i\varepsilon_2{}^j\partial_X{}^i\partial_Y{}^j\phi)\partial(\varepsilon_1{}^k\varepsilon_2{}^l\partial_X{}^k\partial_Y{}^l\phi), \tag{A·20b}$$

$$\varepsilon_1{}^2\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_X{}^2[\phi]=\sum_{i+j\geq1}\sum\varepsilon_1{}^{i+2}\varepsilon_2{}^j\partial_X{}^{i+2}\partial_Y{}^j\phi\cdot\partial\boldsymbol{\rho}/\partial(\varepsilon_1{}^i\varepsilon_2{}^j\partial_X{}^i\partial_Y{}^j\phi), \tag{A·20c}$$

and similar definitions apply to $\varepsilon_2\partial_{[\phi]}(\partial_\psi\boldsymbol{\rho})\cdot\partial_Y[\phi]$, $\varepsilon_2{}^2\partial_{[\phi]}^2\boldsymbol{\rho}\cdot(\partial_Y[\phi])^2$ and $\varepsilon_2{}^2\partial_{[\phi]}\boldsymbol{\rho}\cdot\partial_Y{}^2[\phi]$. Note that $\boldsymbol{B}$ contains all perturbation effects and is of the order of ε_1 or ε_2 or higher.

Let $\boldsymbol{\rho}(\psi, [\phi])$ be expanded as

$$\boldsymbol{\rho}(\psi,\ [\phi])=\sum_{m\neq0}c_m[\phi]\boldsymbol{u}_l(\psi). \tag{A·21}$$

A scalar product $(\boldsymbol{u}_{m'}^*\cdot)$ with each side of Eq. (A·19a) is now taken. Then we get

$$\Omega[\phi]=(\boldsymbol{u}_0{}^*,\ \boldsymbol{B}) \tag{A·22a}$$

for $m=0$, and

$$c_m[\phi]=-\lambda_m{}^{-1}(\boldsymbol{u}_m{}^*,\ \boldsymbol{B}) \tag{A·22b}$$

for $m\neq0$. The solutions $\Omega[\phi]$ and $c_m[\phi]$ of the above set of equations can be obtained perturbatively. Let some relevant quantities be cast into power series expansions

$$\begin{bmatrix}\Omega[\phi]\\ \boldsymbol{\rho}(\psi,\ [\phi])\\ c_m[\phi]\\ \boldsymbol{B}(\psi,\ [\phi])\end{bmatrix}=\sum_{\mu+\nu\geq1}\sum\varepsilon_1{}^\mu\varepsilon_2{}^\nu\begin{bmatrix}\Omega_{\mu\nu}[\phi]\\ \boldsymbol{\rho}_{\mu\nu}(\psi,\ [\phi])\\ c_{m,\mu\nu}[\phi]\\ \boldsymbol{B}_{\mu\nu}(\psi,\ [\phi])\end{bmatrix}. \tag{A·23}$$

For given μ and ν, each $[\phi]$-dependent quantity on the right-hand side of (A·23) may further be decomposed into all possible types of derivatives. For example, $\Omega_{22}[\phi]$ is decomposed as

$$\Omega_{22}[\phi]=\Omega_{22}^{(1)}\partial_X{}^2\phi\partial_Y{}^2\phi+\Omega_{22}^{(2)}(\partial_X\phi)^2\partial_Y{}^2\phi+\Omega_{22}^{(3)}\partial_X{}^2\phi(\partial_Y\phi)^2+\Omega_{22}^{(4)}(\partial_X\phi\partial_Y\phi)^2. \tag{A·24}$$

Different types of derivative terms will be indicated by a superscript (σ). Then Eqs. (A·22a) and (A·22b) may be decomposed into a set of finer balance equations

$$\Omega_{\mu\nu}^{(\sigma)}=(\boldsymbol{u}_0{}^*,\ \boldsymbol{B}_{\mu\nu}^{(\sigma)}) \tag{A·25a}$$

and

$$c_{m,\mu\nu}^{(\sigma)}=-\lambda_m{}^{-1}(\boldsymbol{u}_m{}^*,\ \boldsymbol{B}_{\mu\nu}^{(\sigma)}). \tag{A·25b}$$

It is easy to see that the right-hand sides of Eqs. (A·25a) and (A·25b) contain only the lower order unknowns $\Omega_{\mu'\nu'}^{(\sigma')}$ and $c_{m',\mu'\nu'}^{(\sigma')}$ where $\mu'+\nu'<\mu+\nu$. This means that the equations can be solved iteratively for every $\Omega_{\mu\nu}^{(\sigma)}$ and $c_{m,\mu\nu}^{(\sigma)}$. By substituting $\Omega_{\mu\nu}^{(\sigma)}$ thus obtained into the equation

$$\partial_t \phi = \sum_{\mu+\nu\geq 1}\sum \sum_{\sigma} \varepsilon_2{}^{\nu} \epsilon_2{}^{\nu} \Omega^{(\sigma)}_{\mu\nu}[\phi],$$

and equating ε_1 and ε_2 to 1, a nonlinear partial differential equation for ϕ is obtained. How to simplify such an equation follows the theory developed in the text.

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