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PHASE EQUATIONS FOR RELAXATION OSCILLATORS*

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Abstract. We use the Malkin theorem to derive phase equations for networks of weakly connected relaxation oscillators. We find an explicit formula for the connection functions when the oscillators have one-dimensional slow variables. The functions are discontinuous in the relaxation limit $\mu \to 0$, which provides a simple alternative illustration to the major conclusion of the fast threshold modulation (FTM) theory by Somers and Kopell [*Biological Cybernetics,* 68 (1993), pp. 393–407] that synchronization of relaxation oscillators has properties that are quite different from those of smooth (nonrelaxation) oscillators. We use Bonhoeffer–Van Der Pol relaxation oscillators to illustrate the theory numerically.

Key words. weakly connected oscillators, fast threshold modulation (FTM), synchronization, class 2 excitability, pulse-coupled oscillators

AMS subject classifications. 92B20, 34C, 34D, 58F, 82C32, 92-02, 92C20

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1. Introduction.

Synchronization of coupled oscillators is a ubiquitous phenomenon in many areas of science [24] and engineering [2]. Among many examples we mention synchronization of pacemaker cells of the heart [17], central patter generation [11, 18], chemical waves [13], rhythmic activity in the brain [6, 20], and pattern recognition [23]. Synchronization depends on the intrinsic mechanism of oscillation as well as on the nature of coupling.

Somers and Kopell [12, 21, 22] have proven that synchronization of relaxation oscillators has properties quite different from that of nonrelaxation ones: Relaxation oscillators need just a few or even one cycle to synchronize, and the synchronization is stable in the presence of nonuniformity of natural frequencies. A potential problem with their argument is that they compare moderately or strongly connected relaxation oscillators with phase oscillators that describe *weakly* connected networks. If relaxation oscillators are connected weakly, then they would also need $\mathcal{O}(1/\varepsilon)$ cycles to synchronize, where $\varepsilon \ll 1$ is the strength of connections.

In the present paper we study weakly connected relaxation oscillators. In sections 2 and 3 we present a rigorous and consistent way of their reduction to phase equations. Each phase variable describes the position of the corresponding oscillator along the limit cycle attractor, as we illustrate in Figure 1.1. Resulting phase equations are fundamentally different from those for smooth (nonrelaxation) oscillators because they become discontinuous in the relaxation limit $\mu \to 0$, where $\mu \ll 1$ is the ratio of the fast and slow time scales. We use this fact in section 4 to illustrate the most important conclusions of the fast threshold modulation (FTM) theory [21]. In particular, we show that the rate of in-phase synchronization is indeed *relatively* fast if compared with that for smooth oscillators. Moreover, the rate increases even further when the relaxation oscillators become class 2 excitable, as we show in section 4.3.

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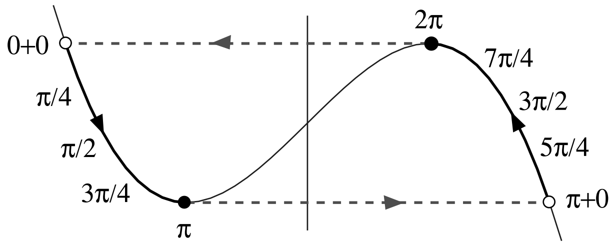

FIG. 1.1. *An example of phase parameterization of a relaxation limit cycle attractor. The cycle is discontinuous in the "relaxation limit" $\mu \to 0$, and so is the parameterization at the "landing points" $\pi + 0$ and $0 + 0$.*

We stress that our results are valid in the relaxation limit $\mu \to 0$, but caution should be used when applying them when μ is not infinitesimal.

1.1. Phase equations. Analysis of locking behavior of a pair of weakly connected oscillators

$$\begin{array}{rcl} \dot{X}_1 & = & F_1(X_1) + \varepsilon G_1(X_1, X_2), \\ \dot{X}_2 & = & F_2(X_2) + \varepsilon G_2(X_1, X_2), \end{array} \qquad X_1, X_2 \in \mathbb{R}^m , \qquad \varepsilon \ll 1 , \tag{1.1}$$

having nearly equal frequencies $\Omega + \varepsilon\omega_i$ can be reduced to the analysis of the phase system [2, 4, 8, 13, 18]

$$\begin{array}{rcl} \varphi_1' & = & \omega_1 + H_1(\varphi_2 - \varphi_1), \\ \varphi_2' & = & \omega_2 + H_2(\varphi_1 - \varphi_2), \end{array} \tag{1.2}$$

where each ω_i denotes small (rescaled) deviation from the common frequency Ω, each φ_i is the phase deviation of the ith oscillator from the natural phase $\vartheta(t) = \Omega t$, and each H_i is some periodic function; see [8, Chapter 9]. Let $\chi = \varphi_2 - \varphi_1$ denote the phase difference, then

$$\chi' = \omega + H(\chi) , \tag{1.3}$$

where $\omega = \omega_2 - \omega_1$ measures the nonuniformity, and

$$H(\chi) = H_2(-\chi) - H_1(\chi) . \tag{1.4}$$

Each root of the equation

$$0 = \omega + H(\chi^\star) \tag{1.5}$$

corresponds to a synchronized solution of (1.2) and hence of (1.1) with the phase difference $\chi^\star$. The solution is stable if $H'(\chi^\star) < 0$.

In this paper we show how to determine the connection functions H_i when (1.1) consists of relaxation oscillators. Many interesting results follow from the fact that H_i may have a discontinuity at the origin as we illustrate at the top of Figure 1.2. One immediate consequence is that the in-phase synchronization ($\chi^\star = 0$) is relatively rapid and persistent under the heterogeneity of natural frequencies ω.

Indeed, if H is smooth, then χ slows down while it approaches $\chi^\star = 0$. As a result, the complete synchronization is an asymptotic process that requires an infinite period of time. In contrast, when H has a discontinuity at the origin, variable χ does not slow down, and it takes a finite period of time to lock.

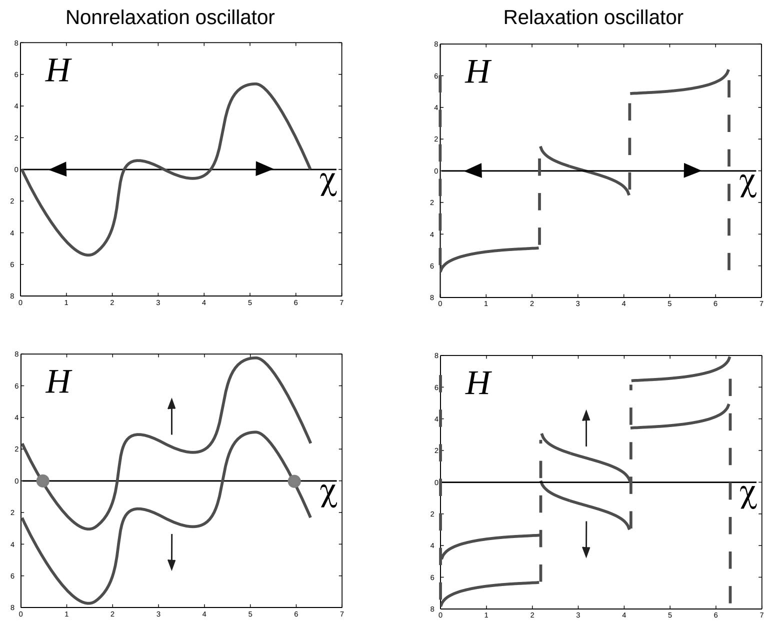

FIG. 1.2. Top: *Function* $H(\chi)$ *defined by* (1.4) *for identical weakly connected oscillators of nonrelaxation and relaxation type.* (*The latter are Bonhoeffer–van der Pol oscillators from Figure* 3.1.) Bottom: *Heterogeneity of natural frequencies shifts the graph of* H *up or down. Nonrelaxation oscillator creates phase differences (depicted by filled circles) while relaxation oscillator does not.*

Changing the heterogeneity parameter ω shifts vertically the graph of H, and hence the root of (1.5) if H is continuous at the origin. In contrast, the root $\chi^\star = 0$ remains for a wide range of ω if H is discontinuous as we illustrate at the bottom of Figure 1.2.

1.2. Malkin theorem. The theorem below is due to Malkin [14, 15]. It has been applied to weakly connected systems, e.g., by Blechman [2] and Ermentrout and Kopell [4]. We state the theorem following Hoppensteadt and Izhikevich [8].

THEOREM 1.1 (see [14]). *Consider a weakly connected system of the form*

$$\dot{X}_i = F_i(X_i) + \varepsilon G_i(X)\ , \qquad X_i \in \mathbb{R}^m\ ,\ \varepsilon \ll 1, \tag{1.6}$$

such that each equation in the uncoupled system

$$\dot{X}_i = F_i(X_i) \tag{1.7}$$

has an exponentially orbitally stable T*-periodic solution* $\gamma_i(t) \in \mathbb{R}^m$. *Let* $\tau = \varepsilon t$ *be slow time and let* $\varphi_i(\tau) \in \mathbb{S}^1$ *be the phase deviation from the natural oscillation* $\gamma_i(t),\ t \geq 0$*; that is,*

$$X_i(t) = \gamma_i(t + \varphi_i(\tau)) + \mathcal{O}(\varepsilon)\ .$$

Then, there is an $\varepsilon_0 > 0$ *such that for all* $\varepsilon \leq \varepsilon_0$ *the vector of phase deviations* $\varphi = (\varphi_1, \ldots, \varphi_n)^\top \in \mathbb{T}^n$ *is a solution to*

$$\varphi_i' = H_i(\varphi - \varphi_i) + \mathcal{O}(\varepsilon), \tag{1.8}$$

where $' = d/d\tau$, *the vector* $\varphi - \varphi_i = (\varphi_1 - \varphi_i, \dots, \varphi_n - \varphi_i)^\top \in \mathbb{T}^{n-1}$, *and the function*

$$H_i(\varphi - \varphi_i) = \frac{1}{T}\int_0^T Q_i(t)^\top G_i(\gamma(t + \varphi - \varphi_i))\,dt,$$

where $Q_i(t) \in \mathbb{R}^m$ *is the unique nontrivial* T*-periodic solution to the linear system*

$$\dot{Q}_i = -\{DF_i(\gamma_i(t))\}^\top Q_i \qquad \text{(adjoint system)} \tag{1.9}$$

satisfying the normalization condition

$$Q_i(t)^\top F_i(\gamma_i(t)) = 1 \tag{1.10}$$

for some (and hence all) t.

It is much easier to study synchronization in the n-dimensional phase model (1.8) than in the original nm-dimensional system (1.6). However, one needs to solve the adjoint problem (1.9), which may pose a major challenge. In the next section we show that the solution can be found analytically (without resort to computers) if (1.7) is a relaxation oscillator with one-dimensional slow variable. A multidimensional case is considered in the appendix.

2. Relaxation oscillators with scalar slow variable.

First, we consider a singularly perturbed system of the form

$$\begin{array}{rcl} \mu\dot{x} &=& f(x,y), \\ \dot{y} &=& g(x,y), \end{array} \qquad \mu \ll 1\ , \tag{2.1}$$

where both x and y are scalars. We impose some technical conditions on f and g, such as smoothness, transversality, etc., that are satisfied for most functions; see the appendix for a detailed list.

We assume that (2.1) has a relaxation limit cycle attractor. A typical example of such a system is the van der Pol oscillator depicted at the top of Figure 2.1. We consider (2.1) in the "relaxation limit" $\mu \to 0$. The limit cycle becomes discontinuous in this case. More precisely, it loses right-hand side continuity at each jump point a_1 and a_2, which we denote by filled circles in Figure 2.1.

THEOREM 2.1. *Suppose* (1.7) *is a planar singularly perturbed system of the form* (2.1) *having a relaxation limit cycle attractor converging to* $\gamma(t)$ *as* $\mu \to 0$. *Suppose* $\gamma(t)$ *has two discontinuities (jumps) at* $t = t_1$ *and at* $t = t_2$*; see Figure* 2.1*, bottom. Then the solution of the corresponding adjoint system* (1.9) *converges as* $\mu \to 0$ *to*

$$Q(t) = \frac{1}{g(\gamma(t))}\left(-g_x'(\gamma(t))\, f_x'(\gamma(t))^{-1}, \quad 1\right)^\top \qquad \textit{when} \quad t \neq t_j, \tag{2.2}$$

and

$$Q(t_j) = \left(c_j\delta(t - t_j), \quad \frac{1}{g(a_j)}\right)^\top , \tag{2.3}$$

where

$$c_j = \frac{1}{f_y'(a_j)}\left(\frac{1}{g(a_j)} - \frac{1}{g(b_j)}\right) \tag{2.4}$$

and a_j *and* b_j *are the end points of the* j*th jump,* $j = 1, 2$.

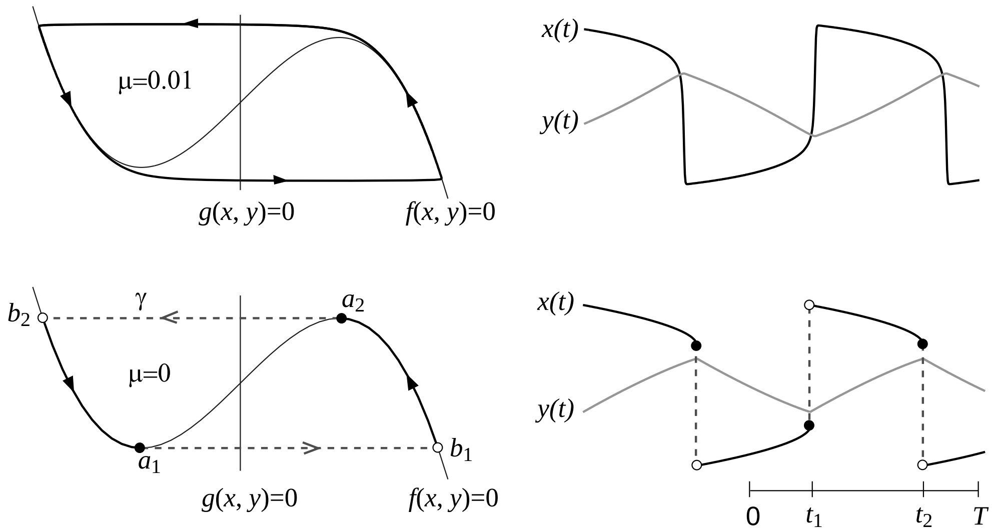

FIG. 2.1. Top: *Nullclines and a periodic solution of the van der Pol relaxation oscillator* $\mu\dot{x} = -y + x - x^3/3$, $\dot{y} = x$ *for* $\mu = 0.01$. Bottom: *The periodic solution becomes discontinuous as* $\mu \to 0$.

Proof. Since the theorem is a special case of a more general theorem for multidimensional relaxation oscillators, one can find a detailed proof in the appendix. Below we only sketch the proof.

We let $\mu \to 0$ and consider an algebraic-differential system of the form

$$\begin{aligned} 0 &= f(x,y), \\ \dot{y} &= g(x,y). \end{aligned} \tag{2.5}$$

Let $Q(t) = (Q_1(t), Q_2(t))^\top$ be the solution of the adjoint to (2.5) problem

$$0 = -f'_x(\gamma(t))Q_1 - g'_x(\gamma(t))Q_2, \tag{2.6}$$

$$\dot{Q}_2 = -f'_y(\gamma(t))Q_1 - g'_y(\gamma(t))Q_2, \tag{2.7}$$

with the normalization condition (1.10) of the form

$$Q_1(t)f(\gamma(t)) + Q_2(t)g(\gamma(t)) = 1 \, .$$

First, consider $t \neq t_1$ and $t \neq t_2$. Since $f(\gamma(t)) = 0$, we have

$$Q_2(t) = \frac{1}{g(\gamma(t))}. \tag{2.8}$$

Since $f'_x(\gamma(t)) \neq 0$, (2.6) results in

$$Q_1(t) = -\frac{g'_x(\gamma(t))}{f'_x(\gamma(t))}Q_2(t) \;=\; -\frac{g'_x(\gamma(t))}{f'_x(\gamma(t))\,g(\gamma(t))} \, . \tag{2.9}$$

Now suppose $t = t_j$. According to (2.8), $Q_2(t)$ jumps from $1/g(a_j)$ to $1/g(b_j)$. Solving (2.7) for Q_1 results in $Q_1(t) = c_j\delta(t - t_j)$, where c_j is defined in (2.4). ∎

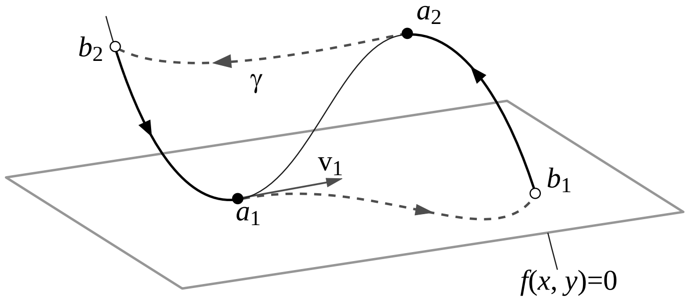

FIG. 2.2. *Fast nullcline and a relaxation limit cycle of* (2.1) *as* $\mu \to 0$ *in the phase space* $\mathbb{R}^2 \times \mathbb{R}$.

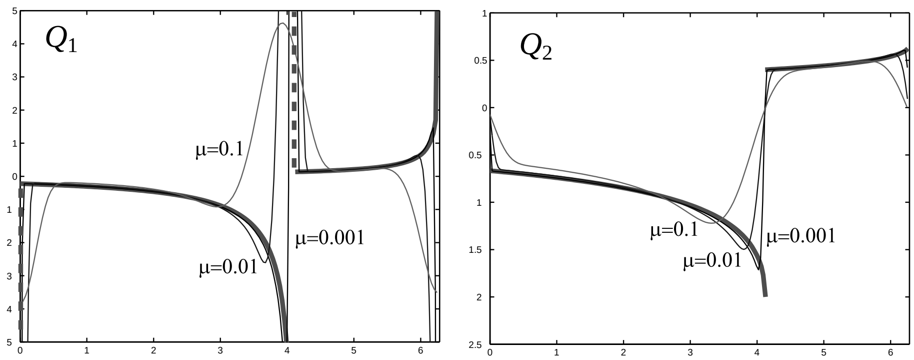

FIG. 2.3. *Illustration to Theorem* 2.1 *using Bonhoeffer–van der Pol oscillator* (2.11) *for* $d = 0.5$ *and* $\mu = 0.1,\ 0.01,$ *and* 0.001. *The asymptotic solution* $Q = (Q_1, Q_2)^\top$ *defined by* (2.2) *is depicted in bold. All periods are normalized to be* 2π.

2.1. Multidimensional fast variable. Now consider (2.1) when $x \in \mathbb{R}^m$ but $y \in \mathbb{R}$. We assume that the fast nullcline is a "cubic curve" in the $m+1$-dimensional phase space; see Figure 2.2. Notice that f'_x, f'_y, and g'_x are not scalars but a matrix, a column-, and a row-vector, respectively. The Jacobian matrix f'_x is stable along the limit cycle γ except the jump points a_1 and a_2 where it has a simple zero eigenvalue. Let $v_j \in \mathbb{R}^m$ be the corresponding eigenvector, $j = 1, 2$. From the central manifold theorem [8] it follows that $x(t)$ jumps along v_j; see Figure 2.2. An analogue of Theorem 2.1 can still be proven, but (2.4) has a new form

$$c_j = \frac{w_j^\top}{w_j f'_y(a_j)} \left(\frac{1}{g(a_j)} - \frac{1}{g(b_j)} \right), \qquad j = 1, 2\ , \tag{2.10}$$

where $w_j \in \mathbb{R}^m$ is dual to v_j row-vector, which is incidentally the left eigenvector of $f'_x(a_j)$.

2.2. Numerical illustration. We illustrate convergence of the solution of the adjoint system (1.9) to $Q(t)$ defined by (2.2) and (2.4) in Figure 2.3. We use Bonhoeffer–van der Pol oscillator

$$\begin{aligned} \mu \dot{x} &= x - x^3/3 - y, \\ \dot{y} &= x + d \end{aligned} \tag{2.11}$$

with $d = 0.5$ and $\mu = 0.1,\ 0.01$, and 0.001.

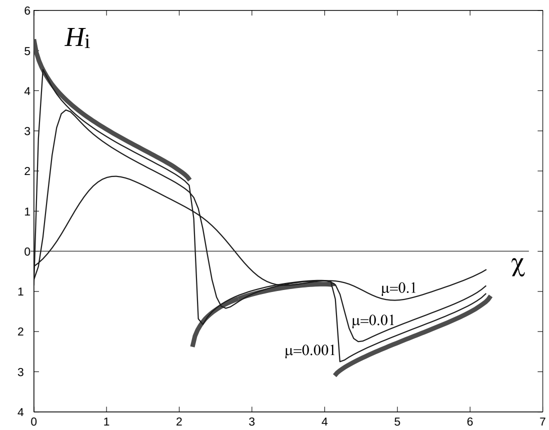

FIG. 3.1. *Illustration to Corollary* 3.1 *using two weakly connected Bonhoeffer–van der Pol oscillators from Figure* 2.3. *Connection functions:* $p_1(x, y) = x_2,\ \ q_1(x, y) = 0$.

3. Weakly connected relaxation oscillators.

Now consider a weakly connected singularly perturbed system of the form

$$\begin{aligned} \mu \dot{x}_i &= f_i(x_i, y_i) + \varepsilon p_i(x, y), \\ \dot{y}_i &= g_i(x_i, y_i) + \varepsilon q_i(x, y), \end{aligned} \qquad i = 1, \dots, n, \qquad \mu,\ \varepsilon \ll 1\ , \tag{3.1}$$

where the connection functions p_i and q_i may also depend on ε and μ. To avoid many technical issues arising when forced relaxation oscillators pass jump points, we require $\varepsilon \ll \mu$ below.

COROLLARY 3.1. *Consider* (3.1) *and suppose that* $\varepsilon \ll \mu$ *and each subsystem*

$$\begin{aligned} \mu \dot{x}_i &= f_i(x_i, y_i), \\ \dot{y}_i &= g_i(x_i, y_i) \end{aligned}$$

is a relaxation oscillator satisfying conditions of the Malkin theorem and Theorem 2.1. *Then as* $\mu \to 0$, *function* $H_i(\chi)$ *converges to*

$$- \frac{1}{T} \int_0^T \frac{g_x'(\gamma(t))\, f_x'(\gamma(t))^{-1}\, p(\gamma(t+\chi)) - q(\gamma(t+\chi))}{g(\gamma(t))}\, dt \tag{3.2}$$

$$+ \frac{1}{T} \sum_{j=1}^{2} c_j^\top p(\gamma(t_j + \chi)), \tag{3.3}$$

where we omitted subscript i *for the sake of simplicity of notation.*

We illustrate numerically the corollary in Figure 3.1. As in the case of Theorem 2.1, the convergence is not uniform when the limit function is discontinuous. In fact, one can prove the following lemma, which we will not need in what follows.

LEMMA 3.2 (discontinuity of H_i). *Consider* (3.1) *having continuous* p_i.

- *Function* (3.2) *is continuous if* $g(a_1) \neq 0$ *and* $g(a_2) \neq 0$.
- *Function* (3.3) *is continuous if and only if* $p(b_j) - p(a_j)$ *is orthogonal to both* c_1 *and* c_2.

In particular, if the oscillators are planar, then (3.3) *is continuous if and only if* $p(a_1) = p(b_1)$ *and* $p(a_2) = p(b_2)$, *which means that the connection function* p *keeps its value during each jump.*

4. Applications. The major purpose of the Malkin theorem is to convert weakly connected networks (3.1) into the corresponding phase models

$$\varphi_i' = H_i(\varphi - \varphi_i) + \mathcal{O}(\varepsilon), \tag{4.1}$$

whose synchronization properties are easier to analyze. It is customary to drop the unknown small order function $\mathcal{O}(\varepsilon)$ and consider the truncated phase model

$$\varphi_i' = H_i(\varphi - \varphi_i), \tag{4.2}$$

which is equivalent to (4.1) in the formal limit $\varepsilon = \mu = 0$. However, one should be aware that (4.2) provides results that are rigorous only in the relaxation limit $\mu \to 0$ and caution should be used when extending them for nonzero μ.

For example, in section 4.1 we study various aspects of the in-phase synchronized solution $\chi = \varphi_2 - \varphi_1 = 0$ of (4.2). If μ is small but nonzero, then we should consider (4.1) instead. All of the results persist, but the in-phase synchronization is no longer perfect, but approximate; i.e., $\chi = \mathcal{O}(\varepsilon)$, where the small shift occurs due to the small-order term $\mathcal{O}(\varepsilon)$ in (4.1).

4.1. In-phase locking. The fact that H_i may be discontinuous reflects the profound difference between behaviors of weakly connected oscillators of relaxation and nonrelaxation type. Revealing such a difference is the major purpose of the FTM theory by Kopell and Somers [12, 21, 22], whose major results are summarized in the following corollaries.

COROLLARY 4.1 (in-phase synchronization). *Consider two weakly connected nearly identical planar relaxation oscillators having "cubic" fast nullclines similar to the one in Figure* 2.1 *and fast* $\to$ *fast excitatory instantaneous connections. The in-phase synchronized solution in the relaxation limit* $\mu \to 0$ *has the following properties:*

- *It is stable.*
- *It is persistent in the presence of nonuniformity of natural frequencies.*
- *It has a rapid rate of convergence.*

Proof. The corollary, and hence the FTM theory, is a direct result of the fact that function H defined by (1.4) and illustrated in Figure 1.2 has a negative discontinuity at $\chi = 0$. To prove this notice that $f_y'(a_1) < 0$ because $f(x, y)$ is positive below the nullcline $f(x, y) = 0$ and negative above it. Since $g(a_1) < 0$ and $g(b_1) > 0$, the constant c_1 defined by (2.10) is positive. Similarly, $c_2 < 0$. We say that the connections are excitatory and instantaneous if $p(a_1) < p(b_1)$ and $p(a_2) > p(b_2)$. Hence (3.3) has a step-like increase when χ crosses 0, and so does $H_i(\chi)$. Therefore, $H(\chi)$ defined by (1.4) has a step-like decrease at the origin.

The stability of the in-phase solution $\chi = 0$ is obvious. Its persistence to heterogeneity of natural frequencies follows from the observation that the equation $0 = \omega + H(\chi)$ has a root $\chi = 0$ for a range of ω due to the discontinuity of H.

Since convergence is an asymptotic process that usually requires an infinite period of time, we ask the following question: "How much time does it take to converge to a small λ-neighborhood of the in-phase solution?" (We consider $\lambda \gg \varepsilon$ to cover nearly in-phase synchronized state $\chi = \mathcal{O}(\varepsilon)$ if $\mu > 0$.) Since $H(\chi)$ is discontinuous and bounded from 0 near $\chi = 0$, the convergence takes a finite $\mathcal{O}(1)$ period of slow time $\tau = \varepsilon t$ for any λ. In contrast, if H were differentiable at $\chi = 0$, then it would take $\mathcal{O}(\ln \lambda^{-1})$, which grows as $\lambda \to 0$. Even though both periods look large on the normal time scale t, the differentiable case takes infinitely longer time to lock. □

COROLLARY 4.2. *The above corollary is also applicable when all x_i are voltage-like variables and the oscillators are connected via gap junctions, i.e., when*

$$p_i(x,y) = \sum_{j=1}^{n} k_{ij}(x_j - x_i)$$

in (3.1)*, where $k_{ij} \geq 0$ are gap junction conductances.*

Due to the apparent linearity it suffices to consider the case $n = 2$. The term $-k_{12}x_1$ does not change the form of the connection function H_1, but shifts it up or down. The remaining term $k_{12}x_2$ can be treated as a fast → fast excitatory instantaneous connection.

COROLLARY 4.3. *Let ω_1 and ω_2 denote the rescaled deviations from the common frequency. The in-phase synchronized solution is persistent to the nonuniformity of frequencies $\omega = \omega_2 - \omega_1$ when*

$$|\omega| \ \leq \ \frac{1}{T}\sum_{j=1}^{2} c_j(p(b_j) - p(a_j)), \tag{4.3}$$

where c_j are defined by (2.4) *and T is the common period.*

Indeed, from (3.3) it follows that the size of discontinuity of H at the origin is described by the right-hand side of (4.3). It is easy to see that Bonhoeffer–van der Pol oscillators from Figure 3.1 synchronize in-phase when $|\omega| \leq 11.2/T \approx 6.2$.

COROLLARY 4.4. *A chain of weakly connected relaxation oscillators does not exhibit propagating waves even in the presence of nonuniformity of natural frequencies.*

Indeed, smooth (nonrelaxation) oscillators compensate for nonuniformities by creating phase differences among adjacent oscillators, which usually leads to propagating waves [22]. Relaxation oscillators do not, as it follows from the previous corollary. That is, they all oscillate with zero phase differences. Notice that the corollary is not applicable to chains of *quiescent but excitable* relaxation systems, which are studied elsewhere.

COROLLARY 4.5. *If relaxation oscillators do not have fast → fast connections, then their locking behavior is similar to that of smooth (nonrelaxation) oscillators.*

Indeed, if p is continuous and $p = p(y)$, then (3.3) is continuous because $y(t)$ is also. That is, a discontinuity may arise only due to a nontrivial dependence of p on $x(t)$.

COROLLARY 4.6. *The above results are also applicable to relaxation oscillators having multidimensional variables unless $p(b) - p(a)$ is orthogonal to c.*

Proof. The proof repeats that of Lemma 3.2. Suppose $c^\top\{p(a) - p(b)\} \neq 0$; then $c^\top p(a) \neq c^\top p(b)$, resulting in a discontinuity of (3.3) when $p(\gamma(t_j + \chi))$ jumps from $p(a)$ to $p(b)$. □

4.2. Antiphase locking. Below we continue to consider two weakly connected planar relaxation oscillators having "cubic" fast nullclines similar to the one in Figure 2.1 and fast → fast excitatory connections. If the oscillators are nearly identical, then the antiphase solution exists and it is exponentially stable if and only if $H'(T/2) < 0$, where $T > 0$ is the period, and H is defined in (1.4). This criterion has a simple interpretation in terms of f and g if we follow Kopell and Somers [12] and make the following assumptions:

- (Duty cycle.) The limit cycle spends more time in the passive phase; that is, on the left branch of the fast nullcline.

- (Heaviside coupling.) The connection function is 0 on the left branch and 1 on the right one.

COROLLARY 4.7 (antiphase synchronization). *The antiphase synchronized solution in the relaxation limit $\mu \to 0$ is exponentially stable if and only if*

$$Q_1(t_1 + T/2) > Q_1(t_2 + T/2), \tag{4.4}$$

where Q_1 is defined by (2.9) *and t_1 and t_2 are jump moments.*

Proof. Consider H_i defined by (3.2, 3.3) for $\chi \approx T/2$. Since the time spent on the right branch is shorter than $T/2$, the points $\gamma(t_j + \chi)$, $j = 1, 2$, are on the left branch, where $p = 0$. Therefore, (3.3) is zero. Since $p(\gamma(t + \chi)) \neq 0$ only for $t \in [t_1 - \chi, t_2 - \chi]$, we have

$$H_i(\chi) = \frac{1}{T} \int_{t_1-\chi}^{t_2-\chi} Q_1(t)\, dt$$

and

$$H_i'(T/2) = \frac{1}{T} \left(Q_1(t_1 - T/2) - Q_1(t_2 - T/2) \right) > 0.$$

Since $H'(T/2) = -2H_i'(T/2) < 0$, we are done. □

For example, the antiphase synchronization for Bonhoeffer–van der Pol oscillators (2.11) is always stable because $g_x' = 1$ and both $|f_x'|$ and $|g|$ decrease along the left branch.

COROLLARY 4.8. *Let $\Delta > 0$ be the difference between the times spent on the left and right branches. When Δ is sufficiently small, the antiphase synchronized solution is exponentially stable if and only if the oscillator slows down before the jump; that is, if and only if $g_x'(a_1) > 0$, where a_1 is the jump point; see Figure* 2.1.

Proof. First, notice that $g < 0$ and $f_x' < 0$ on the left branch. Next, as $\Delta \to 0$, $\gamma(t_2 - T/2) \to a_1$, hence $f_x'(\gamma(t_2 - T/2)) \to 0$, and (4.4) grows. Its sign coincides with that of $g_x'(a_1)$. □

Since the quantity (4.4) is proportional to $H'(T/2)$, the rate of convergence to the antiphase solution increases as $\Delta \to 0$. However, the basin of attraction, which has width Δ because it lies between the two middle positive discontinuities in the Figure 1.2 that are due to the term (3.3), decreases.

If the fast variable is multidimensional, and the connection function is zero on the left branch and some nonzero vector p_0 on the right one, then the equation in the inequality (4.4) must be replaced by $g^{-1} g_x' (f_x')^{-1} p_0$.

COROLLARY 4.9. *If the oscillators have fast $\to$ slow connections, then the antiphase synchronized solution is exponentially stable if and only if*

$$g(\gamma(t_1 - T/2)) < g(\gamma(t_2 - T/2))$$

regardless of the dimension of the fast variable.

The proof is similar to that of the Corollary 4.7.

4.3. Class 2 excitable oscillators. Consider a relaxation oscillator

$$\begin{aligned} \mu \dot{x} &= f(x, y, \lambda), \\ \dot{y} &= g(x, y, \lambda), \end{aligned} \qquad \mu \ll 1\,, \tag{4.5}$$

and suppose that the distance between the slow nullcline and the left jump point a_1 decreases as $\lambda \to 0$; see Figure 4.1. This results in $|g(a_1, \lambda)| \to 0$. A typical example of such a system is the Bonhoeffer–van der Pol oscillator (2.11) with $d = 1 - \lambda$.

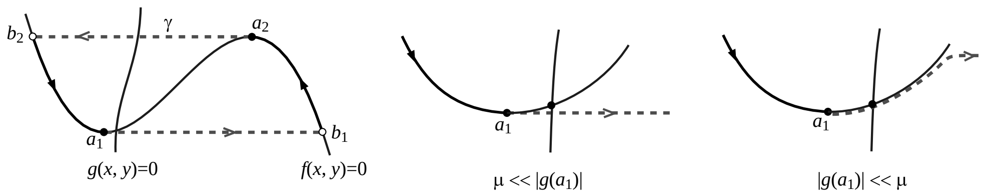

FIG. 4.1. Left: *Nullclines and a periodic solution of a class* 2 *excitable oscillator.* Right: *A magnification of a neighborhood of the jump point a_1 for various $|g(a_1)|$ and μ. "French ducks" can appear when $|g(a_1)| \ll \mu$.*

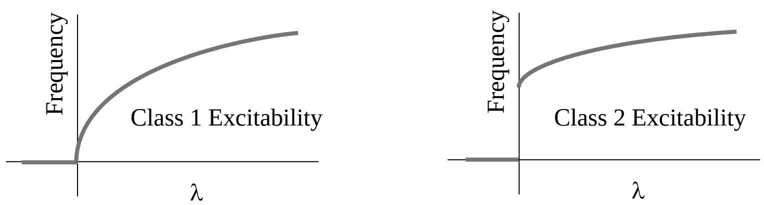

FIG. 4.2. *Class* 1 (*class* 2) *excitability corresponds to zero (nonzero) emerging oscillations.*

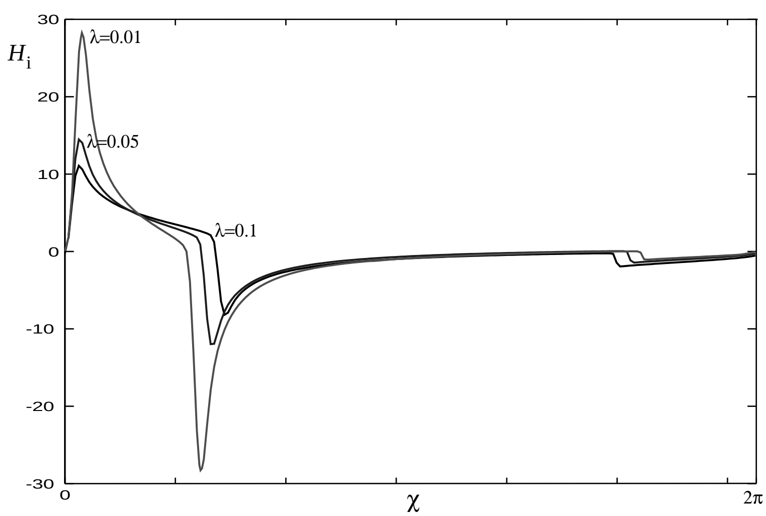

FIG. 4.3. *Illustration to Corollary* 4.10 *using Bonhoeffer–van der Pol oscillator* (2.11) *with connection functions as in Figure* 3.1 *and parameters $\mu = 0.001$ and $d = 1 - \lambda$, where $\lambda = 0.1$,* 0.05, *or* 0.01. *The magnitude of the connection function H_i increases as the distance to the bifurcation, λ, decreases.*

When a stable equilibrium appears, (2.1) ceases to oscillate and becomes excitable. Many interesting phenomena, such as "French duck" (Canard) solutions [1, 5], can occur when $|g(a_1, \lambda)| \ll \mu$. To avoid unnecessary complications we consider a simpler case $\mu \ll |g(a_1, \lambda)|$ below.

Since the oscillations (dis)appear with a nonzero frequency, such a system is referred to as being *class* 2 *excitable*, as opposed to *class* 1 *excitable* systems having zero-frequency emerging oscillations [7, 8, 19]; see Figure 4.2.

COROLLARY 4.10 (locking of class 2 oscillators). *The rate of locking of relaxation*

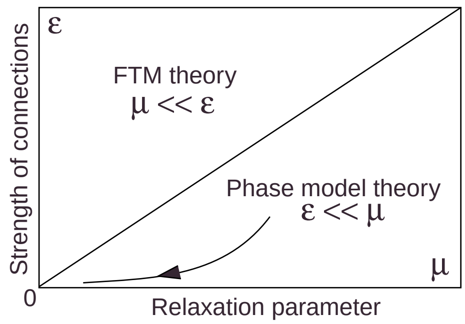

FIG. 5.1. *Partition of the ε, μ-parameter space. FTM theory is applicable when $\mu \ll \varepsilon$. Phase model theory is applicable when $\varepsilon \ll \mu \to 0$, that is, when ε and μ approach the origin along the depicted path.*

oscillators increases significantly when they become class 2 *excitable.*

Indeed, the absolute value of the function H_i from Corollary 3.1 increases as $|g(a_1, \lambda)| \to 0$; see the illustration in Figure 4.3. The larger the function H_i, the faster φ_i moves.

5. Discussion. The major purpose of the paper is derivation of the phase equations for the weakly connected relaxation oscillators. The major mathematical result is the exact solution of the adjoint system (Theorem 2.1) when slow variable y is one-dimensional and $\mu \to 0$. The solution has an especially appealing form, $1/g(\gamma(t))$, when there are only fast $\to$ slow and slow $\to$ slow connections.

Incidentally, this is the third known example of an exact solution of the adjoint system for a general oscillator $\dot{X} = F(X)$. The first one is when the oscillator is near a supercritical Andronov–Hopf bifurcation so that it exhibits small amplitude almost harmonic oscillation [8]. The second example is when the oscillator has class 1 excitability via saddle-node bifurcation on a limit cycle [3, 8, 10, 9].

5.1. Limits $\boldsymbol{\varepsilon \ll \mu \to 0}$. We would like to stress that *we found the solution of the adjoint problem in the relaxation limit* $\mu \to 0$. Since the Malkin theorem may be applied only when $\varepsilon \ll \mu$, the discontinuous phase model describes locking behavior of the relaxation oscillators only for $\varepsilon \to 0$. (If the oscillators were smooth, then ε might be as large as $\mathcal{O}(1)$ when $\varepsilon_0 = \mathcal{O}(1)$; see the Malkin theorem.)

5.2. FTM theory by Kopell and Somers [21]. The fact that the connection function H_i in the phase model (1.8) is discontinuous allows us to provide simple proofs for the major results of the FTM theory [12, 21, 22]. Let us compare and contrast our approaches: We, as well as Kopell and Somers, perform the analysis in the relaxation limit $\mu \to 0$, and then hypothesize that the results will be valid for small but nonzero μ. Kopell and Somers also assume that the oscillators are planar and the connection functions are piecewise constant, i.e., Heaviside coupling. They, however, do not require $\varepsilon \ll 1$. In contrast, our assumptions regarding dynamics of the oscillators seem to be less restrictive: Any oscillators, any dimension, any connection functions, but we do require $\varepsilon \ll \mu \ll 1$. Thus, our approaches do not repeat but complement each other; see Figure 5.1.

One of the most important conditions in the FTM theory is the "compression hypothesis": The rate of change of the slow variable, y, before the jump must be less than that after the jump; that is, $|g(a_j)/g(b_j)| < 1$. This condition is important

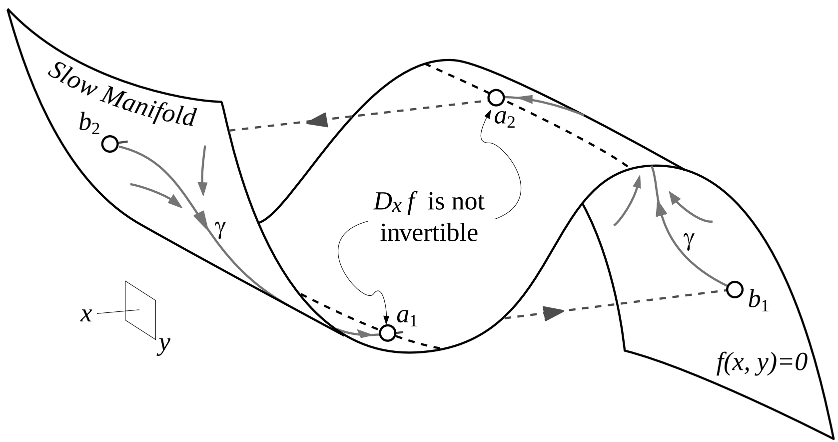

FIG. A.1. *An example of the slow manifold defined by* (A.2).

to prove the stability of the in-phase synchronized solution when $\mu \ll \varepsilon$ because the oscillators can skip certain parts of the limit cycle. As a result, identical oscillators can change order when they jump. Such a behavior is impossible when $\varepsilon \ll \mu$, hence the "compression hypothesis" does not play any role in our analysis.

5.3. Weakly pulse-coupled oscillators. As was mentioned by Kopell and Somers [21], behavior of relaxation oscillators is similar to that of pulse-coupled oscillators, such as the integrate-and-fire model [16], even though their equations are quite different. It turns out that the phase system for weakly pulse-coupled oscillators is discontinuous too [10, 9], which suggests that their locking properties might be indistinguishable. A possible theoretical significance of this observation has yet to be understood.

Appendix A. Multidimensional slow variable. We consider a singularly perturbed system of the form

$$\begin{array}{rcl} \mu\dot{x} &=& f(x,y), \\ \dot{y} &=& g(x,y), \end{array} \qquad x \in \mathbb{R}^m \,,\; y \in \mathbb{R}^k \,, \qquad \mu \ll 1 \,, \tag{A.1}$$

having a relaxation limit cycle attractor; that is, an attractor consisting of slow and fast motions of $x(t)$. We treat (A.1) as a multidimensional generalization of a two-dimensional relaxation oscillator, such as the van der Pol oscillator depicted at the top of Figure 2.1.

If we let $\mu \to 0$, then the singularly perturbed system (A.1) becomes an algebraic-differential system of the form

$$0 = f(x,y), \tag{A.2}$$

$$\dot{y} = g(x,y), \tag{A.3}$$

which is often referred to as the *reduced problem* for (A.1). The condition (A.2) determines the *slow manifold* of the system, which often looks like a "cubic" surface in Figure A.1.

If the Jacobian matrix $f'_x(x,y)$ is invertible, we can solve (A.2) locally for x; that is, we can find $x = s(y)$ such that

$$0 = f(s(y),y) \; .$$

Using $x = s(y)$ in (A.3) results in

$$\dot{y} = g(s(y), y), \tag{A.4}$$

which defines the flow on the slow manifold. Whenever y reaches a value such that the Jacobian matrix $f'_x(x, y)$ is not invertible, an instantaneous jump to another branch of the slow manifold occurs. Thus, a limit cycle attractor of the algebraic-differential system above may be discontinuous, as we illustrate in Figures 2.1 and A.1.

THEOREM A.1. *Consider* (A.1) *satisfying the following assumptions:*

(1) *(Convergence.) It has an exponentially stable limit cycle attractor converging to γ as $\mu \to 0$. The attractor γ has period T.*

(2) *(Jumps.) The attractor γ has two discontinuities (jumps) at $t = t_1$ and $t = t_2$. The corresponding jump points are*

$$a_j = \gamma(t_j) \quad \textit{and} \quad b_j = \lim_{t \to t_j + 0} \gamma(t)$$

for $j = 1$ and 2; see Figure A.1.

(3) *(Stability.) The Jacobian matrix f'_x is stable everywhere on γ except a_j. It has a simple zero eigenvalue at each a_j.*

(4) *(Transversality.) Let $w_j \in \mathbb{R}^m$ be the corresponding left row-eigenvector of $f'_x(a_j)$. We assume that*

$$w_j\, f'_y(a_j)\, g(b_j) \neq 0$$

for all j.

Let $Q_2(t) \in \mathbb{R}^k$ be a (unique) periodic solution of the adjoint to (A.4) *problem*

$$\dot{Q}_2 = -g'(\gamma(t))^\top Q_2 \qquad \textit{(adjoint system)} \tag{A.5}$$

satisfying the normalization condition

$$Q_2^\top(t)\, g(\gamma(t)) = 1\,, \qquad t \in [0, T]\,, \tag{A.6}$$

for some (and hence all) t. When t crosses t_j, $Q_2(t)$ jumps according to

$$Q_2(t_j + 0) = Q_2(t_j) - f'_y(a_j)^\top c_j, \tag{A.7}$$

where

$$c_j = \frac{w_j^\top}{w_j\, f'_y(a_j)\, g(b_j)} \left(Q_2(t_j)^\top g(b_j) - 1\right). \tag{A.8}$$

Then, the solution of the adjoint to (A.1) *problem converges to $Q(t) = (Q_1(t), Q_2(t))$ as $\mu \to 0$, where*

$$Q_1(t) = -\left(g'_x(\gamma(t))\, f'_x(\gamma(t))^{-1}\right)^\top Q_2(t) \quad \textit{when } t \neq t_j$$

and

$$Q_1(t_j) = c_j \delta(t - t_j)\,. \tag{A.9}$$

Proof. Differentiating $f(s(y), y) = 0$ with respect to y results in

$$f'_x(s(y), y)\, s'(y) + f'_y(s(y), y) = 0 \ ,$$

which can be solved for $s'(y)$ when f'_x is invertible,

$$s'(y) = -f'_x(s(y), y)^{-1}\, f'_y(s(y), y) \ .$$

Therefore,

$$\begin{aligned} g'(s(y), y) &= g'_x(s(y), y)\, s'(y) + g'_y(s(y), y) \\ &= -g'_x(s(y), y)\, f'_x(s(y), y)^{-1}\, f'_y(s(y), y) + g'_y(s(y), y) \ . \end{aligned} \tag{A.10}$$

Now consider (A.2), (A.3) and the corresponding adjoint system

$$0 = -f'_x(\gamma(t))^\top Q_1 - g'_x(\gamma(t))^\top Q_2, \tag{A.11}$$

$$\dot{Q}_2 = -f'_y(\gamma(t))^\top Q_1 - g'_y(\gamma(t))^\top Q_2. \tag{A.12}$$

If $t \neq t_j$, then $f'_x(\gamma(t))$ is invertible, and (A.11) results in

$$Q_1(t) = -\left(f'_x(\gamma(t))^\top\right)^{-1} g'_x(\gamma(t))^\top Q_2(t) \ .$$

Using this in (A.12) and taking into account (A.10) yields

$$\begin{aligned} \dot{Q}_2 &= \left(g'_x(\gamma(t)) f'_x(\gamma(t))^{-1} f'_y(\gamma(t)) - g'_y(\gamma(t))\right)^\top Q_2 \\ &= -g'(\gamma(t))^\top Q_2 \ , \end{aligned}$$

which is the adjoint system (A.5). It is easy to check that

$$\begin{aligned} \frac{d}{dt}\left(Q_2(t)^\top g(\gamma(t))\right) &= \dot{Q}_2(t)^\top g(\gamma(t)) + Q_2(t)^\top g'(\gamma(t))\, g(\gamma(t)) \\ &= -Q_2(t)^\top g'(\gamma(t))\, g(\gamma(t)) + Q_2(t)^\top g'(\gamma(t))\, g(\gamma(t)) \\ &= 0 \ , \end{aligned}$$

hence the normalization condition (A.6).

When $\gamma(t)$ jumps, function $g(\gamma(t))$ may undergo a discontinuity, and so may $Q_2(t)$ to respect the normalization condition (A.6). The discontinuity is caused by the δ-behavior of Q_1 which grows along $w_j^\top$, the eigenvector of $f'_x(a_j)^\top$ corresponding to the simple zero eigenvalue. If we represent Q_1 in the form (A.9) for some vector c_j colinear to $w_j^\top$, then

$$Q_2(t_j + 0) - Q_2(t_j) = -f'_y(a_j)^\top c_j \ , \tag{A.13}$$

where $Q_2(t_j + 0)$ is the value after the jump. Multiplying both sides by $g(b_j)$ and solving for c_j yields (A.8). ∎

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