Phase-Field Relaxation of Topology Optimization with Local Stress Constraints

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Abstract

We introduce a new relaxation scheme for structural topology optimization problems with local stress constraints based on a phase-field method. The starting point of the relaxation is a reformulation of the material problem involving linear and 0–1 constraints only. The 0–1 constraints are then relaxed and approximated by a Cahn-Hilliard type penalty in the objective functional, which yields convergence of minimizers to 0–1 designs as the penalty parameter decreases to zero. A major advantage of this kind of relaxation opposed to standard approaches is a uniform constraint qualification that is satisfied for any positive value of the penalization parameter.

The relaxation scheme yields a large-scale optimization problem with a high number of linear inequality constraints. We discretize the problem by finite elements and solve the arising finite-dimensional programming problems by a primal-dual interior point method. Numerical experiments for problems with stress constraints based on different criteria indicate the success and robustness of the new approach.

Keywords: Topology Optimization, Local Stress Constraints, Phase-Field Methods, Interior-Point Methods.

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1 Introduction

Topology optimization denotes problems of finding optimal material distributions in given design domains subject to certain criteria and, possibly, satisfying several additional constraints. In the last two decades, advances in homogenization, optimization theory, and numerical analysis, as well as

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new engineering approaches caused topology optimization techniques to become a standard tool of engineering design (cf. [5, 17] for an overview), in particular in structural mechanics.

In structural optimization there are two design-constraint combinations of particular importance, namely the maximization of material stiffness at given mass and the minimization of mass while keeping a certain stiffness. The formulation of the first combination as the so-called minimal compliance problem has become standard, and seems to be well-understood with respect to its mathematical properties (cf. e.g. [2, 8, 25]), and various successful numerical techniques have been proposed (cf. e.g. [3, 7, 19, 26, 31]). The treatment of the second problem is by far less understood and until now there seems to be no approach that is capable of computing reliable (global) optima within reasonable computational effort. The main source of difficulties in this problem is a lack of constraint qualification in the feasible set defined by the local stress constraints, which already appear for simple truss structures (cf. [23, 28]). Moreover, there are several complications for specific methods, e.g. convergence issues of homogenized stress criteria for material interpolation schemes (cf. [5, 16]).

We start by describing the main mathematical setup used in the sequel. By $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) we denote the design domain, which we assume to be sufficiently regular. The function $\mathbf{u} : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ denotes the elastic displacement, the strain

$$\mathbf{e}_{ij} := \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i}\right)$$

and the stress σ, determined from the strain via a standard linearly elastic relation

$$\sigma_{ij} = \sum_{k,\ell} C_{ijk\ell}\mathbf{e}_{k\ell}$$

with a suitable symmetric positive definite elasticity tensor $(C_{ijk\ell})$, i.e.,

$$\sum_{i,j,k,\ell} C_{ijk\ell}\mathbf{e}_{ij}\mathbf{e}_{k\ell} \geq \eta \sum_{ij} \mathbf{e}_{ij}^2$$

for some $\eta > 0$. Below we shall abbreviate the stress-strain relation as $\sigma = \mathbf{C} : \mathbf{e}$. The scaled density of the material is denoted by $\chi : \Omega \to \{0, 1\}$, which we normalize to $\chi(\mathbf{x}) = 1$ if there is material at the point $\mathbf{x}$ and $\chi(\mathbf{x}) = 0$ otherwise. Then the stress constrained topology optimization problem is given by

$$
\begin{aligned}
\int_\Omega \chi \,\mathrm{d}\mathbf{x} \quad &\rightarrow \quad \min_{\chi\in\{0,1\}\text{ a.e.}}, \\
\operatorname{div}\sigma = 0 \quad & \quad \text{in } \{\chi = 1\}, \\
\sigma = \mathbf{C} : \mathbf{e} \quad & \quad \text{in } \Omega, \\
\mathbf{u} = \mathbf{0} \quad & \quad \text{on } \Gamma_D \subset \partial\Omega, \\
\sigma \cdot \mathbf{n} = \mathbf{t} \quad & \quad \text{on } \Gamma_N \subset \partial\Omega, \\
\sigma \cdot \mathbf{n} = \mathbf{0} \quad & \quad \text{on } \partial\{\chi = 1\} \cup (\partial\Omega - \Gamma_D - \Gamma_N), \\
\sigma^{\min} \le \sigma \le \sigma^{\max} \quad & \quad \text{in } \{\chi = 1\}, \\
\mathbf{u}^{\min} \le \mathbf{u} \le \mathbf{u}^{\max} \quad & \quad \text{in } \Omega,
\end{aligned}
$$

where $\mathbf{n}$ denotes the outward unit normal and $\mathbf{t}$ is a traction force applied on a part Γ_N of the boundary $\partial\Omega$. In the current geometrical setup, the material is kept fixed on Γ_D, while the remaining part $\partial\Omega - \Gamma_D - \Gamma_N$ needs not to represent material boundaries. Moreover, we have ignored body forces for simplicity, but they could be incorporated by adding a right-hand side to the divergence equation.

The matrices $\sigma^{\min}$ and $\sigma^{\max}$ denote the allowed minimal and maximal value for the local stresses, respectively. We shall call this criterion total stress constraints. Alternatively, the case of von Mises stress constraints is of interest, where the local constraints on σ are replaced by

$$\Phi(\sigma) \le \Phi^{\max}, \tag{1.1}$$

where the von Mises stress is denoted via the functional $\Phi : \mathbb{R}^{d\times d} \to \mathbb{R}$ given by

$$\Phi(\sigma) = \sqrt{\frac{\sum_{i,j}(\lambda_i - \lambda_j)^2}{2}}$$

where $\lambda_j, j = 1, \ldots, d$ are the principal stresses (the eigenvalues of σ). Note that for $d = 2$, the case we are focussing on, we simply have

$$\Phi(\sigma) = \frac{|\lambda_2 - \lambda_1|}{\sqrt{2}} = \sqrt{\frac{(\sigma_{11} + \sigma_{22})^2 + 4\sigma_{12}^2}{2}}.$$

A frequently used approach to overcome the difficulties with the missing constraint qualification is the so-called epsilon-relaxation approach (cf. [15, 23]), originally used in the optimization of truss structures, which perturbs the stress criterion by some small parameter ϵ. In the topology optimization of continuum structures, the ϵ-relaxation can be combined with standard material interpolation schemes and finite element discretization in order to compute approximations of solutions, an approach carried out by Duysinx and Bendsoe [16]. A drawback of the ϵ-relaxation approach is the fact that the constraint qualification is not uniform for positive ϵ and there could still be points at which e.g. the Mangasarian-Fromovitz qualification is violated.

Recently, it has been shown for such examples by Stolpe and Svanberg [28] that the trajectories of minimizers of the ϵ-relaxed approach can have a discontinuity for arbitrarily small ϵ and as a consequence it is difficult (or even impossible) to compute reliable minimizers of topology optimization problems with local stress constraints using this approach.

Due to the well-known ill-posedness of topology optimization problems (cf. [25]), we add a perimeter penalization to the objective functional, i.e., we minimize

$$J^\gamma(\chi) = \gamma \int_\Omega \chi \, \mathrm{d}\mathbf{x} + \sup_{\substack{\mathbf{g} \in C_0^\infty(\Omega;\mathbb{R}^d) \\ \|\mathbf{g}\|_\infty \leq 1}} \int_\Omega \operatorname{div} \mathbf{g} \, \chi \, \mathrm{d}\mathbf{x} \tag{1.2}$$

for a (large) parameter $\gamma > 0$. The additional perimeter term equals the length of the curve $\partial\{\chi = 1\}$ for $d = 2$, and the area of the surface $\partial\{\chi = 1\}$ for $d = 3$. The boundedness of the perimeter regularizes the topology optimization problem, in particular it excludes checkerboard effects as the discretization size decreases to zero (cf. [18, 22]).

In this paper we use a different approach to the relaxation of the local constraints. Starting point of our analysis is a reformulation of the equality constraints describing the elastic equilibrium and the local inequality constraints for stresses and displacements into a system of linear inequality constraints as recently proposed by Stolpe and Svanberg [29, 30]. This reformulation is approximate at the continuum level, but exact for finite element discretizations with suitable parameter choice. The main difficulty is that the arising problem also involves 0–1 constraints in addition to the linear inequalities. The computational effort of methods for the global minimization of these mixed linear programming problems grows fast with the number of degrees of freedom in the discretization, so that the problem could be solved only for very coarse discretizations so far (cf. [27, 30]). Instead of solving mixed linear programming problems, we propose to use a phase-field relaxation of the reformulated problem. The phase-field relaxation consists in using an interpolated material density ρ, similar to material interpolation schemes. In addition, a Cahn-Hilliard type penalization functional (cf. [14]) of the form

$$P^\epsilon(\rho) = \frac{\epsilon}{2} \int_\Omega |\nabla\rho|^2 \, \mathrm{d}\mathbf{x} + \frac{1}{\epsilon} \int_\Omega W(\rho) \, \mathrm{d}\mathbf{x} \tag{1.3}$$

is used to approximate the perimeter, where $W : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a scalar function with exactly two minimizers at 0 and 1 satisfying $W(0) = W(1) = 0$. The second term of the penalty functional ensures that the values of the material density ρ converge to 0 or 1 as $\epsilon \to 0$, while the first term controls the perimeter of level sets of ρ. Due to a famous result by Modica and Mortola [21] (cf. also [1, 20]), minimizers of P^ϵ with fixed volume $\int_\Omega \rho \, \mathrm{d}\mathbf{x}$ converge to minimizers χ of the perimeter at fixed volume over functions satisfying $\chi \in \{0,1\}$ almost everywhere in Ω. This convergence arises in the framework of Γ-convergence (cf. [10] and the references therein), which ensures in particular convergence of minimizers. Together with finite element discretization, we shall use a continuation $\epsilon \to 0$ and solve the arising discretized problems by an interior-point method. Obviously, the phase-field approach is not the only possibility to relax the reformulated constraints, one could e.g. use a direct relaxation without further approximations, standard material interpolation schemes, or level set methods, which are closely related to phase-field methods (cf. [4]). However, the phase-field relaxation incorporates a variety of advantages with respect to such approaches:

- In contrast to a direct relaxation to a continuous density variable (and also in contrast to material interpolation schemes), the phase-field method still provides geometric information. In particular, one can expect $\{\rho > \eta\}$ (with $\eta << \frac{1}{2}$ small) to be a superset of the limit $\{\chi = 1\}$ and $\{\rho < 1 - \eta\}$ to be a subset (for ϵ sufficiently small). Moreover, geometric quantities such as mean curvature can be approximated in terms of derivatives of ρ for small ϵ.

- With the phase-field relaxation one can still use the density linearly in the constraints, which is not true for material interpolation (e.g. in SIMP one has ρ^p, $p > 1$) or for level set methods (where the unknown is a signed distance function to some boundary, and in the relaxation one usually takes an application of a smoothed Heaviside function). The additional nonlinearity does not only complicate the constraints, but might also destroy constraint qualifications.

- The parameter ϵ can be used for continuation. For ϵ being large, the functional P^ϵ is strictly convex, so that one can compute global optima for arbitrary initial values. When decreasing ϵ, the minimizer of the previous step can be expected to provide a good initial guess for the next step carried out with a smaller ϵ.

To our knowledge, the phase-field approach in topology optimization was first introduced by Bourdin and Chambolle [9] for a design problem with design-dependent loads, another type of problem where standard material interpolation schemes encounter diffculties. The approach has recently been applied to minimal compliance type problems by Wang and Zhou [33].

The remainder of this paper is organized as follows: in Section 2 we review the constraint reformulation due to [30] and extend the approach to an approximate reformulation of the continuous problem. In Section 3 we introduce the phase-field relaxation and analyze its basic properties. The finite element discretization yielding linearly constrained programming problems is discussed in Section 4, and the solution of these programming problems by interior-point methods in Section 5. Finally, we present numerical results obtained for local as well as for von Mises stress constraints.

2 Reformulation of Constraints

In the following we consider a reformulation of constraints on subsets of locally bounded stresses, i.e.,

$$\beta|\sigma_{ij}| \leq 1, \qquad \text{in } \Omega,\ i,j = 1,\dots,d, \tag{2.1}$$

for some (small) $\beta > 0$ such that $\beta \max_{i,j} \sigma_{ij}^{\max} < 1$ and $\beta \min_{i,j} \sigma_{ij}^{\min} > -1$ (i.e., the original total stress constraints are more restrictive for $\chi = 1$). The total constraints for the displacement $\mathbf{u}$, the stress σ, and the density χ are given by

$$\begin{aligned}
\operatorname{div} \sigma = 0 &\quad \text{in } \{\chi = 1\}, \\
\sigma = \mathbf{C} : \mathbf{e}(\mathbf{u}) &\quad \text{in } \Omega, \\
\mathbf{u} = \mathbf{0} &\quad \text{on } \Gamma_D \subset \partial\Omega, \\
\sigma \cdot \mathbf{n} = \mathbf{t} &\quad \text{on } \Gamma_N \subset \partial\Omega, \\
\sigma \cdot \mathbf{n} = \mathbf{0} &\quad \text{on } \partial\{\chi = 1\} \cup (\partial\Omega - \Gamma_D - \Gamma_N), \\
\chi \in \{0,1\} &\quad \text{a.e. in } \Omega, \\
\Phi^{\min} \leq \Phi(\sigma) \leq \Phi^{\max} &\quad \text{in } \{\chi = 1\}, \\
\mathbf{u}^{\min} \leq \mathbf{u} \leq \mathbf{u}^{\max} &\quad \text{in } \Omega.
\end{aligned} \tag{2.2}$$

We shall reformulate the constraints (2.1), (2.2) as linear inequality constraints for the case of total stress and von Mises stress constraints.

2.1 Reformulation of Total Stress Constraints

We start with the reformulation in the case of total stress constraints, i.e., $\Phi(\sigma) = \sigma$. For this sake we introduce the approximate constraint sets

$$\mathcal{C}^\beta := \{(\chi, \mathbf{u}, \sigma) \in BV(\Omega; [0,1]) \times L^\infty(\Omega; \mathbb{R}^d) \times L^\infty(\Omega; \mathbb{R}^{d\times d}) \mid (\chi, \mathbf{u}, \sigma) \text{ satisfies } (2.1), (2.2)\}.$$

and an additional artificial stress variable $\mathbf{s} \in L^\infty(\Omega; \mathbb{R}^{d\times d})$.

Let $(\chi, \mathbf{u}, \sigma) \in \mathcal{C}^\beta$ and let $\mathbf{s} = \sigma$ if $\chi = 1$ and $\mathbf{s} = \mathbf{0}$ if $\chi = 0$, i.e., $\mathbf{s} = \chi\sigma$. Then the constraints

$$-(1-\chi)\mathbf{1} \leq \beta(\sigma - \mathbf{s}) \leq (1-\chi)\mathbf{1} \tag{2.3}$$

with the matrix $\mathbf{1} = (1)_{ij}$ and

$$\sigma^{\min}\chi \leq \mathbf{s} \leq \sigma^{\max}\chi \tag{2.4}$$

are satisfied. Vice versa, assume that

$$(\chi, \mathbf{u}, \sigma, \mathbf{s}) \in BV(\Omega; [0,1]) \times L^\infty(\Omega; \mathbb{R}^d) \times L^\infty(\Omega; \mathbb{R}^{d\times d}) \times L^\infty(\Omega; \mathbb{R}^{d\times d})$$

fulfills (2.3), (2.4) and $\chi \in \{0,1\}$ almost everywhere. Then, for $\chi = 0$ (2.4) implies $\mathbf{s} = 0$ and (2.3) gives $-1 \leq \beta\sigma_{ij} \leq 1$. For $\chi = 1$ (2.3) implies $\mathbf{s} = \sigma$ and (2.4) becomes

$$\sigma^{\min}\chi \leq \sigma \leq \sigma^{\max}\chi.$$

Moreover, since either $\mathbf{s} = \sigma$ or $\mathbf{s} = \mathbf{0}$, we obtain that div $\mathbf{s} = 0$ almost everywhere in Ω.

Due to the above arguments we conclude $(\chi, \mathbf{u}, \sigma) \in \mathcal{C}^\beta$ if and only if there exists $\mathbf{s}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{s} &= 0 && \text{in } \Omega, \\ \sigma &= \mathbf{C} : \mathbf{e}(\mathbf{u}) && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D \subset \partial\Omega, \\ \mathbf{s} \cdot \mathbf{n} &= \mathbf{t} && \text{on } \Gamma_N \subset \partial\Omega, \\ \mathbf{s} \cdot \mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega - \Gamma_D - \Gamma_N, \\ -(1-\chi)\mathbf{1} \leq \beta(\sigma - \mathbf{s}) &\leq (1-\chi)\mathbf{1} && \text{in } \Omega, \\ \chi &\in \{0,1\} && \text{a.e. in } \Omega, \\ \sigma^{\min}\chi \leq \mathbf{s} &\leq \sigma^{\max}\chi && \text{in } \Omega, \\ \mathbf{u}^{\min} \leq \mathbf{u} &\leq \mathbf{u}^{\max} && \text{in } \Omega. \end{aligned} \tag{2.5}$$

Note that (except $\chi \in \{0,1\}$) the constraints (2.5) are linear with respect to the new vector of unknowns $(\chi, \mathbf{u}, \sigma, \mathbf{s})$, in particular all constraints are formulated on Ω and not on the unknown set $\{\chi = 1\}$. We would like to mention that the drawback of the reformulation is an increase in the number of unknowns and a high number of inequality constraints. On the other hand, this higher number of unknowns and constraints seems to be a reasonable price for the linear reformulation of the complicated original constraints.

2.2 Reformulation of Von Mises Stress Constraints

In the following we discuss the reformulation of the inequalities in the case of von Mises stress constraints for dimension $d = 2$ (similar computations are possible for $d = 3$). Since both sides of the constraint (1.1) are positive, we can square them and since the constraint must hold only for $\chi = 1$, it can be written equivalently as

$$\chi(\sigma_{11} + \sigma_{22})^2 + 4\chi\sigma_{12}^2 \leq 2\chi(\Phi^{\max})^2.$$

A more conservative version of the von Mises criterion (cf. [30]) is given by

$$\chi|\sigma_{11} + \sigma_{22}| + 2\chi|\sigma_{12}| \leq \sqrt{2}\chi\Phi^{\max}. \tag{2.6}$$

As above we introduce an artificial stress variable $\mathbf{s} = \chi\sigma$, and since $\chi^2 = \chi$, we can reformulate the von Mises constraint as

$$(\mathbf{s}_{11} + \mathbf{s}_{22})^2 + 4(\mathbf{s}_{12})^2 \leq 2\chi(\Phi^{\max})^2.$$

The reformulation of the von Mises stress constraints on the constraint set $\mathcal{C}^\beta$ is given by

$$\begin{aligned}
\operatorname{div} \mathbf{s} = 0 &\quad \text{in } \Omega, \\
\sigma = \mathbf{C} : \mathbf{e}(\mathbf{u}) &\quad \text{in } \Omega, \\
\mathbf{u} = \mathbf{0} &\quad \text{on } \Gamma_D \subset \partial\Omega, \\
\mathbf{s} \cdot \mathbf{n} = \mathbf{t} &\quad \text{on } \Gamma_N \subset \partial\Omega, \\
\mathbf{s} \cdot \mathbf{n} = \mathbf{0} &\quad \text{on } \partial\Omega - \Gamma_D - \Gamma_N, \\
-(1-\chi)\mathbf{1} \leq \beta(\sigma - \mathbf{s}) \leq (1-\chi)\mathbf{1} &\quad \text{in } \Omega, \\
(\mathbf{s}_{11} + \mathbf{s}_{22})^2 + 4(\mathbf{s}_{12})^2 \leq 2\chi(\Phi^{\max})^2 &\quad \text{in } \Omega, \\
\chi \in \{0,1\} &\quad \text{a.e. in } \Omega, \\
\mathbf{u}^{\min} \leq \mathbf{u} \leq \mathbf{u}^{\max} &\quad \text{in } \Omega.
\end{aligned} \tag{2.7}$$

Note that in this case, $\chi = 0$ in the von Mises stress constraint does not imply directly $\mathbf{s} = \mathbf{0}$, but only $\mathbf{s}_{11} + \mathbf{s}_{22} = 0$ and $\mathbf{s}_{12} = 0$. However, since $\mathbf{s}$ is divergence free in addition, we may conclude for those special stresses that $\nabla \mathbf{s}_{11} = \mathbf{0}$. Hence, $\mathbf{s}$ is of the form

$$\mathbf{s} = \chi\sigma + (1-\chi)\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

for any constant $C \in \mathbb{R}$. Since such artificial stresses will not change the von Mises stress, and since $\mathbf{s}$ does not a have a physical meaning for $\chi = 0$, the additional terms will not play a major role (we are basically free to define $\mathbf{s}$ arbitrarily in $\{\chi = 0\}$). The reformulation (2.7) involves convex quadratic constraints, but only with respect to the stress variable $\mathbf{s}$, which still yields constraint qualification after relaxation and discretization.

In this paper we rather use the conservative von Mises stress criterion (2.6), which can actually be reformulated into linear inequalities. Again we use $\mathbf{s} = \chi\sigma$ to obtain

$$|\mathbf{s}_{11} + \mathbf{s}_{22}| + 2|\mathbf{s}_{12}| \leq \sqrt{2}\chi\Phi^{\max}.$$

Moreover, we introduce functions p_1, p_2, q_1, and q_2 such that

$$p_i \geq 0, \quad q_i \geq 0, \qquad i = 1, 2$$

and

$$-p_1 \leq \mathbf{s}_{11} + \mathbf{s}_{22} \leq p_2, \quad -q_1 \leq \mathbf{s}_{12} \leq q_2.$$

From these constraints we obtain that

$$|\mathbf{s}_{11} + \mathbf{s}_{22}| \leq \max\{p_1, p_2\}, \qquad |\mathbf{s}_{12}| \leq \max\{q_1, q_2\}.$$

Finally, we impose the constraints

$$p_i + q_j \leq \sqrt{2}\chi\Phi^{\max}, \qquad i, j = 1, 2.$$

As a consequence of the latter we obtain

$$\max\{p_1, p_2\} + \max\{q_1, q_2\} \leq \sqrt{2}\chi\Phi^{\max},$$

which implies the conservative von Mises constraints for $\mathbf{s}$. On the other hand, if the constraint is satisfied by $\mathbf{s}$, the variables

$$\begin{aligned} &p_1 = -\min\{\mathbf{s}_{11} + \mathbf{s}_{22}, 0\}, \quad p_2 = \max\{\mathbf{s}_{11} + \mathbf{s}_{22}, 0\}, \\ &q_1 = -\min\{\mathbf{s}_{12}, 0\}, \quad q_2 = \max\{\mathbf{s}_{12}, 0\} \end{aligned}$$

satisfy the new linear constraints and thus, we conclude equivalence to the original conservative von Mises constraints. The reformulation of the conservative von Mises stress constraints on the constraint set $\mathcal{C}^\beta$ is given by

$$\begin{aligned} \operatorname{div} \mathbf{s} = 0 \quad & \text{in } \Omega, \\ \sigma = \mathbf{C} : \mathbf{e}(\mathbf{u}) \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \Gamma_D \subset \partial\Omega, \\ \mathbf{s} \cdot \mathbf{n} = \mathbf{t} \quad & \text{on } \Gamma_N \subset \partial\Omega, \\ \mathbf{s} \cdot \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\Omega - \Gamma_D - \Gamma_N, \\ -(1-\chi)\mathbf{1} \leq \beta(\sigma - \mathbf{s}) \leq (1-\chi)\mathbf{1} \quad & \text{in } \Omega, \\ p_i \geq 0, \quad q_i \geq 0 \quad & \text{in } \Omega, \quad i = 1, 2, \\ -p_1 \leq \mathbf{s}_{11} + \mathbf{s}_{22} \leq p_2 \quad & \text{in } \Omega, \\ -q_1 \leq \mathbf{s}_{12} \leq q_2 \quad & \text{in } \Omega, \\ p_i + q_j \leq \sqrt{2}\chi\Phi^{\max} \quad & \text{in } \Omega, \quad i, j = 1, 2, \\ \chi \in \{0, 1\} \quad & \text{a.e. in } \Omega, \\ \mathbf{u}^{\min} \leq \mathbf{u} \leq \mathbf{u}^{\max} \quad & \text{in } \Omega. \end{aligned}$$

3 Phase-Field Relaxation

We now turn our attention to the relaxation of the stress constrained topology optimization problem. For this sake we replace the indicator function χ by a density $\rho : \Omega \to [0, 1]$ and approximate the perimeter term in the regularized objective functional J^γ (1.2) by the Cahn-Hilliard term P^ϵ (1.3). The resulting relaxation in the case of total stress constraints is given by

$$\begin{aligned} M(\rho) = \gamma \int_\Omega \rho \, \mathrm{d}\mathbf{x} + P^\epsilon(\rho) \quad \to \quad & \min, \\ \operatorname{div} \mathbf{s} = 0 \quad & \text{in } \Omega, \\ \sigma = \mathbf{C} : \mathbf{e}(\mathbf{u}) \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \Gamma_D, \\ \mathbf{s} \cdot \mathbf{n} = \mathbf{t} \quad & \text{on } \Gamma_N, \\ \mathbf{s} \cdot \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\Omega - \Gamma_D - \Gamma_N, \\ \rho = 1 \quad & \text{on } \Gamma_N, \\ -(1-\rho)\mathbf{1} \leq \beta(\sigma - \mathbf{s}) \leq (1-\rho)\mathbf{1} \quad & \text{in } \Omega, \\ \sigma^{\min}\rho \leq \mathbf{s} \leq \sigma^{\max}\rho \quad & \text{in } \Omega, \\ 0 \leq \rho \leq 1 \quad & \text{a.e. in } \Omega, \\ \mathbf{u}^{\min} \leq \mathbf{u} \leq \mathbf{u}^{\max} \quad & \text{in } \Omega. \end{aligned} \tag{3.1}$$

The function space setting for this relaxation is given by

$$(\rho, \mathbf{u}, \sigma, \mathbf{s}) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times L^\infty(\Omega; \mathbb{R}^d) \times L^\infty(\Omega; \mathbb{R}^{d\times d}) \times L^\infty(\Omega; \mathbb{R}^{d\times d}).$$

Note that, in addition to the original constraint reformulation (2.5), we have added a Dirichlet boundary condition for ρ on Γ_N, which is well-defined in the sense of traces of functions in $H^1(\Omega)$. The reasoning for adding this condition is as follows: in the original constraint we have $\mathbf{s} \cdot \mathbf{n} = \mathbf{t} \neq \mathbf{0}$ on Γ_N. Hence, there exists a small open neighbourhood of Γ_N in Ω, where $\chi = 1$ (since otherwise $\chi = 0$ would imply $\mathbf{s} = \mathbf{0}$ and thus $\mathbf{s} \cdot \mathbf{n} = \mathbf{0}$ on Γ_N). In the relaxed formulation, the trace of ρ is positive by analogous arguments, but not necessarily equal to one. Thus, the additional constraint will not change the limit of the constraint set, but on the other hand it restricts the relaxation and simplifies the analysis of the relaxed problem.

In a similar way we can give a relaxed formulation of the problem with conservative von Mises stress constraints ($d = 2$) as

$$\begin{array}{rcll}
M(\rho) = \gamma \int_\Omega \rho \, \mathrm{d}\mathbf{x} + P^\epsilon(\rho) & \rightarrow & \min, & \\
\operatorname{div} \mathbf{s} = 0 & & \text{in } \Omega, & \\
\sigma = \mathbf{C} : \mathbf{e}(\mathbf{u}) & & \text{in } \Omega, & \\
\mathbf{u} = \mathbf{0} & & \text{on } \Gamma_D, & \\
\mathbf{s} \cdot \mathbf{n} = \mathbf{t} & & \text{on } \Gamma_N, & \\
\mathbf{s} \cdot \mathbf{n} = \mathbf{0} & & \text{on } \partial\Omega - \Gamma_D - \Gamma_N, & \\
\rho = 1 & & \text{on } \Gamma_N, & \\
-(1-\rho)\mathbf{1} \leq \beta(\sigma - \mathbf{s}) \leq (1-\rho)\mathbf{1} & & \text{in } \Omega, & \\
p_i \geq 0, \quad q_i \geq 0 & & \text{in } \Omega, \quad i = 1, 2, & \\
-p_1 \leq \mathbf{s}_{11} + \mathbf{s}_{22} \leq p_2 & & \text{in } \Omega, & \\
-q_1 \leq \mathbf{s}_{12} \leq q_2 & & \text{in } \Omega, & \\
p_i + q_j \leq \sqrt{2}\chi\Phi^{\max} & & \text{in } \Omega, \quad i, j = 1, 2 & \\
0 \leq \rho \leq 1 & & \text{a.e. in } \Omega, & \\
\mathbf{u}^{\min} \leq \mathbf{u} \leq \mathbf{u}^{\max} & & \text{in } \Omega, &
\end{array} \tag{3.2}$$

with variables

$$\begin{array}{rcl}
(\rho, \mathbf{u}, \sigma, \mathbf{s}, \mathbf{p}, \mathbf{q}) & \in & (H^1(\Omega) \cap L^\infty(\Omega)) \times L^\infty(\Omega; \mathbb{R}^d) \times \\
& & L^\infty(\Omega; \mathbb{R}^{d\times d})^2 \times L^\infty(\Omega; \mathbb{R}^d)^2,
\end{array}$$

with the notation $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$. In our discussion below we will focus on total stress constraints, i.e., the relaxed optimization problem (3.1), but analogous reasoning is possible for von Mises stress constraints.

3.1 Structure of the Relaxed Problem

In the following we further examine the structure of the relaxed problem (3.1). Due to the term $W(\rho)$ in the Cahn-Hilliard penalty, we have to expect the objective functional to be nonconvex, in particular for small ϵ, when minimizers are forced to take values close to 0 or 1. For large ϵ, the first term of the Cahn-Hilliard energy dominates and thus, the optimization problem is convex:

Theorem 3.1. *Let $W \in C^2([0,1])$. Then there exists $\epsilon_0 > 0$ dependent on Ω only, such that the objective functional $\rho \mapsto M(\rho) = \gamma \int_\Omega \rho \; d\mathbf{x} + P^\epsilon(\rho)$ is convex for all $\epsilon > \epsilon_0$.*

Proof. The objective functional is twice continuously differentiable with derivatives

$$M'(\rho)\psi = \gamma \int_\Omega \psi \; \mathrm{d}\mathbf{x} + \epsilon \int_\Omega \nabla\rho \cdot \nabla\psi \; \mathrm{d}\mathbf{x} + \frac{1}{\epsilon}\int_\Omega W'(\rho)\psi \; \mathrm{d}\mathbf{x}$$

and

$$M''(\rho)(\psi_1, \psi_2) = \epsilon \int_\Omega \nabla\psi_1 \cdot \nabla\psi_2 \; \mathrm{d}\mathbf{x} + \frac{1}{\epsilon}\int_\Omega W''(\rho)\psi_1\psi_2 \; \mathrm{d}\mathbf{x}.$$

Due the constraint $0 \leq \rho \leq 1$, we obtain $|W''(\rho)| \leq W_0$, where $W_0 \in \mathbb{R}$ is the maximum of W'' in the interval $[0,1]$. Moreover, since $\rho = 1$ on Γ_N, admissible variations satisfy $\psi = 0$ on Γ_N and due to a Poincaré-type inequality, there exists a constant $C_P > 0$ such that

$$\int_\Omega \psi^2 \; \mathrm{d}\mathbf{x} \leq C_P \int_\Omega |\nabla\psi|^2 \; \mathrm{d}\mathbf{x}$$

for all admissible variations $\psi \in H^1(\Omega)$ with $\psi = 0$ on Γ_N. Hence,

$$M''(\rho)(\psi,\psi) \geq (\epsilon - \frac{W_0 C_P}{\epsilon}) \int_\Omega |\nabla\psi|^2 \; \mathrm{d}\mathbf{x}.$$

Consequently, for $\epsilon \geq \epsilon_0 := \sqrt{W_0 C_P}$, M is convex. Since all constraints are linear, the relaxed optimization problem (3.1) is convex. □

So far, we have not discussed possible choices for the function W. Commonly used in phase-field simulations of phase-transition problems (e.g. in the Allen-Cahn and Cahn-Hilliard equation, cf. e.g. [4, 13]) is the *double-well potential*

$$W(r) = r^2(1-r)^2, \qquad r \in \mathbb{R}.$$

Recently, the so-called *double-obstacle potential*

$$W(r) = r(1-r), \qquad r \in [0,1] \tag{3.3}$$

has received further attention (cf. [6]). In the case of evolutions like the Allen-Cahn equation, the use of the double-obstacle potential is rather a computational complication, since $0 \leq \rho \leq 1$ has to be enforced (in contrast to the evolution with the double-well potential), and the partial differential equation has to be reformulated as a variational inequality. In our case, the choice of the double-obstacle potential (3.3) seems more attractive, since we enforce the bound constraints on ρ anyway (within our system of inequality constraints) and the double-obstacle problem causes a polynomial nonlinearity of lower degree than the double-well potential. Using (3.3) in P^ϵ, we observe that (3.1) and (3.2) are quadratic optimization problems subject to linear constraints.

3.2 Existence of Solutions

We now investigate the existence of solutions of the relaxed problem (3.1). For this sake we introduce the Banach spaces of functions with essentially bounded strain

$$BS^\infty(\Omega) := \{\ \mathbf{u} \in L^\infty(\Omega;\mathbb{R}^d) \mid \mathbf{e}(\mathbf{u}) \in L^\infty(\Omega;\mathbb{R}^{d\times d})\ \}$$

and with square-integrable strain

$$BS^2(\Omega) := \{\ \mathbf{u} \in L^2(\Omega;\mathbb{R}^d) \mid \mathbf{e}(\mathbf{u}) \in L^2(\Omega;\mathbb{R}^{d\times d})\ \},$$

with norms

$$\|\mathbf{u}\|_{BS^\infty} := \max\{\|\mathbf{u}\|_\infty, \|\mathbf{e}(\mathbf{u})\|_\infty\}$$

and

$$\|\mathbf{u}\|_{BS^2} := \sqrt{\|\mathbf{u}\|_2^2 + \|\mathbf{e}(\mathbf{u})\|_2^2}.$$

One can verify by standard arguments that $BS^\infty(\Omega)$ is a Banach space (including all elements of the Sobolev space $W^{1,\infty}(\Omega;\mathbb{R}^d)$) and that $BS^2(\Omega)$ is a Hilbert space with scalar product

$$\langle \mathbf{u}, \mathbf{v}\rangle_{BS^2} := \langle \mathbf{u}, \mathbf{v}\rangle_{L^2} + \langle \mathbf{e}(\mathbf{u}), \mathbf{e}(\mathbf{v})\rangle_{L^2}.$$

As usual for weak solutions of partial differential equations, we understand the equality constraints on $\mathbf{s}$ in a standard weak sense, i.e.,

$$\int_\Omega \mathbf{s} : \nabla\Psi \ \mathrm{d}\mathbf{x} = \int_{\Gamma_N} (\mathbf{t}\cdot\mathbf{n})\Psi \ \mathrm{d}a, \qquad \forall\ \Psi \in W^{1,2}(\Omega;\mathbb{R}^d), \Psi|_{\Gamma_D} = 0.$$

Similarly, we interpret the stress-strain relation in an L^2-sense, i.e.,

$$\int_\Omega [\sigma : \Psi - \Psi : \mathbf{C} : \mathbf{e}(\mathbf{u})] \ \mathrm{d}\mathbf{x} = 0 \qquad \forall\ \Psi \in L^2(\Omega;\mathbb{R}^{d\times d}).$$

We start the analysis with a lower semicontinuity property:

Lemma 3.2. *Let W be defined by* (3.3)*. Then the functional $M : H^1(\Omega) \to \mathbb{R}$ is sequentially weakly lower semicontinuous.*

Proof. Due to the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, the linear functional $\rho \mapsto \gamma \int_\Omega \rho \, d\mathbf{x}$ and the quadratic functional $\rho \mapsto \frac{1}{\epsilon} \int_\Omega W(\rho) \, d\mathbf{x}$ are weakly continuous. Together with the sequential weak lower semicontinuity of the square of the norm in Hilbert spaces applied to the third term in M, we obtain the assertion. □

Besides lower semicontinuity, a fundamental ingredient for the existence of solutions is compactness in appropriate topologies. In order to obtain some weak compactness, we examine the boundedness of the constraint set:

Lemma 3.3. *Let* $\epsilon > 0$ *and let*

$$(\rho, \mathbf{u}, \sigma, \mathbf{s}) \in L^\infty(\Omega) \times BS^\infty(\Omega) \times L^\infty(\Omega; \mathbb{R}^{d\times d})^2$$

satisfy the constraints in (3.1). *Then,* $(\rho, \mathbf{u}, \sigma, \mathbf{s})$ *lies in a bounded set with respect to the corresponding norms.*

Proof. From the bound constraints $0 \le \rho \le 1$ we immediately conclude that ρ lies in the unit ball of $L^\infty(\Omega)$. Consequently, we deduce

$$\min\left\{\mathbf{0}, \sigma^{\min}\right\} \le \mathbf{s} \le \max\left\{\sigma^{\max}, \mathbf{0}\right\}$$

and hence, $\mathbf{s}$ is bounded in the norm of $L^\infty(\Omega; \mathbb{R}^{d\times d})$. Due to

$$\mathbf{s} - \frac{1-\rho}{\beta}\mathbf{1} \le \sigma \le \mathbf{s} + \frac{1-\rho}{\beta}\mathbf{1}$$

we further conclude the boundedness of σ in the norm of $L^\infty(\Omega; \mathbb{R}^{d\times d})$. Finally, the bound constraints on $\mathbf{u}$ imply its boundedness in the norm of $L^\infty(\Omega; \mathbb{R}^d)$ and together with the stress-strain relation and the positive definiteness we may conclude the boundedness of $\mathbf{u}$ in the norm of $BS^\infty(\Omega)$. □

With these preliminary results we can provide an existence result for the relaxed topology optimization problem for arbitrary positive ϵ:

Theorem 3.4. *Let* $\epsilon > 0$, $\beta > 0$, *and let* W *be defined by* (3.3). *Moreover, let the admissible set defined by the constraints in* (3.1) *be nonempty. Then there exists a solution*

$$(\rho, \mathbf{u}, \sigma, \mathbf{s}) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times BS^\infty(\Omega) \times L^\infty(\Omega; \mathbb{R}^{d\times d})^2$$

of the constrained optimization problem (3.1).

Proof. For admissible densities $\rho \ge 0$, the objective functional M is bounded below by zero and hence, the infimum m_0 of M on the admissible set is finite. Hence, we can find a minimizing sequence

$$(\rho^n, \mathbf{u}^n, \sigma^n, \mathbf{s}^n) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times BS^\infty(\Omega) \times L^\infty(\Omega; \mathbb{R}^{d\times d})^2$$

such that $M(\rho^n) \to m_0$. Since $M(\rho^n)$ converges, the sequence is bounded in particular and since

$$\frac{2}{\epsilon} M(\rho^n) \geq \int_\Omega |\nabla \rho^n|^2 \mathrm{d}\mathbf{x},$$

we obtain boundedness of ρ^n in $H^1(\Omega)$. Due to lemma 3.3 and standard precompactness results for bounded sets in weak or weak-* topologies, we can extract a subsequence (again denoted by the superscript n) such that

$$\begin{aligned}
\rho^n \to \hat{\rho} \qquad & \text{weak in } H^1(\Omega), \text{ and weak-* in } L^\infty(\Omega), \\
\mathbf{u}^n \to \hat{\mathbf{u}} \qquad & \text{weak in } BS^2(\Omega), \text{ and weak-* in } L^\infty(\Omega; \mathbb{R}^d), \\
\sigma^n \to \hat{\sigma} \qquad & \text{weak-* in } L^\infty(\Omega; \mathbb{R}^{d \times d}), \\
\mathbf{s}^n \to \hat{\mathbf{s}} \qquad & \text{weak-* in } L^\infty(\Omega; \mathbb{R}^{d \times d}).
\end{aligned}$$

Due to closedness of simple bounds with respect to weak-* convergence in L^∞, we can conclude that the limit $(\hat{\rho}, \hat{\mathbf{u}}, \hat{\sigma}, \hat{\mathbf{s}})$ satisfies all the inequalities in (3.1). Moreover, since for $\Psi \in W^{1,2}(\Omega; \mathbb{R}^d)$ we have in particular $\nabla \Psi \in L^1(\Omega)$, we may conclude that

$$\int_{\Gamma_N} (\mathbf{t} \cdot \mathbf{n}) \Psi \ \mathrm{d}a = \int_\Omega \mathbf{s}^n : \nabla \Psi \ \mathrm{d}\mathbf{x} \to \int_\Omega \mathbf{s} : \nabla \Psi \ \mathrm{d}\mathbf{x}$$

due to weak-* convergence in L^∞. Hence, $\hat{\mathbf{s}}$ satisfies the associated equality constraints. From the weak convergence of $\mathbf{u}^n$ in $BS^2(\Omega)$ we conclude that $\hat{\mathbf{u}}$ satisfies the stress-strain relation and hence, $(\rho, \mathbf{u}, \sigma, \mathbf{s})$ is in the admissible set. With the sequential lower semicontinuity from Lemma 3.2 we finally obtain that thus, $(\rho, \mathbf{u}, \sigma, \mathbf{s})$ is a solution of the optimization problem (3.1). □

4 Discretization

In the following we consider the discretization of the relaxed problems for $\Omega \subset \mathbb{R}^2$, detailing the analysis again for the case of (3.1). For simplicity (and motivated by the typical choices of design domains), we assume that Ω is of polygonal shape.

Our aim is to construct a finite element approximation on a triangular grid, i.e., we decompose $\Omega = \bigcup_{T \in \mathcal{T}^h} \overline{T}$ for a suitable family $\mathcal{T}^h$ of triangles satisfying standard regularity conditions (cf. [11]). The parameter $h > 0$ denotes the grid size (equal to the maximal diameter of triangles in $\mathcal{T}^h$). We shall use two different discrete subspaces, namely the H^1-subspace of linear elements (for the density ρ and displacement components $\mathbf{u}_i$)

$$\mathcal{V}^h := \{\varphi \in C(\Omega) \mid \varphi \text{ is affinely linear in } T, \ \forall \ T \in \mathcal{T}^h\},$$

and the L^∞-subspace of constant elements (for the stress components σ_{ij} and $\mathbf{s}_{ij}$)

$$\mathcal{W}^h := \{\varphi \in L^\infty(\Omega) \mid \varphi \text{ is constant in } T,\ \forall\, T \in \mathcal{T}^h\}.$$

Note that for $\mathbf{u} \in \mathcal{V}^h \times \mathcal{V}^h \subset BS^\infty(\Omega)$, we obtain $\frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \in \mathcal{W}^h$.

The equality constraints are discretized by a standard finite element approach, i.e., we look for $\mathbf{s} \in (\mathcal{W}^h)^{2\times 2}$, $\sigma \in (\mathcal{W}^h)^{2\times 2}$, and $\mathbf{u} \in \mathcal{V}^h \times \mathcal{V}^h$ satisfying

$$\int_\Omega \mathbf{s} : \nabla\Psi \ \mathrm{d}\mathbf{x} = \int_{\Gamma_N} (\mathbf{t}\cdot\mathbf{n})\Psi \ \mathrm{d}a, \qquad \forall\ \Psi \in \mathcal{V}^h \times \mathcal{V}^h, \Psi|_{\Gamma_D} = 0$$

and

$$\int_\Omega [\sigma : \Psi - \Psi : \mathbf{C} : \mathbf{e}(\mathbf{u})] \ \mathrm{d}\mathbf{x} = 0 \qquad \forall\ \Psi \in (\mathcal{W}^h)^{2\times 2}.$$

The bound constraints on the displacement and density $\rho \in \mathcal{V}^h$ can be enforced directly, for piecewise linear functions, the constraints hold if and only if they hold in all nodes of the grid.

Finally, we need to discretize the inequality constraints involving both stress variables and the density. Since the components of σ and $\mathbf{s}$ are in a different subspace than ρ, the discretization is not straight-forward. In particular, we cannot pose local constraints in the grid nodes or on edges, since functions in $\mathcal{W}^h$ are discontinuous over the edges. Consequently, the more promising approach is to interpret the inequality constraints as constraints in $\mathcal{W}^h$. For this sake we introduce the discrete projection operator (with respect to the L^2-norm) $\mathcal{P}^h : \mathcal{V}^h \to \mathcal{W}^h$,

$$(\mathcal{P}^h v)|_T := \frac{1}{|T|}\int_T v \ \mathrm{d}\mathbf{x}, \qquad \forall\ T \in \mathcal{T}^h, \forall\ v \in \mathcal{V}^h.$$

The discretized formulation of constraints in $\mathcal{W}^h$ can then be formulated as

$$\begin{aligned} -(1-\mathcal{P}^h\rho)\mathbf{1} \le \beta(\sigma - \mathbf{s}) &\le (1-\mathcal{P}^h\rho)\mathbf{1} && \text{in } \Omega,\\ \sigma^{\min}\mathcal{P}^h\rho \le \mathbf{s} &\le \sigma^{\max}\mathcal{P}^h\rho && \text{in } \Omega. \end{aligned}$$

In order to obtain a linearly constrained quadratic programming problem, we represent the functions in subspaces by standard basis functions, namely a set of nodal basis functions $\{\phi_j\}_{j=1,\dots,N}$ for $\mathcal{V}^h$ and a set of basis functions $\{\psi_j\}_{j=1,\dots,M}$ with support in single triangles for $\mathcal{W}^h$. We can write

$$\rho = \sum_{j=1}^N R_j\phi_j, \qquad \mathbf{u}_i = \sum_{j=1}^N U_j^i\phi_j$$

and

$$\mathbf{s}_{ij} = \sum_{k=1}^M S_k^{ij}\psi_k, \qquad \sigma_{ij} = \sum_{k=1}^M \Sigma_k^{ij}.$$

The coefficients R_j, U_j^i, S_k^{ij}, and Σ_k^{ij} can be collected into vectors $\mathbf{R} \in \mathbb{R}^N$, $\mathbf{U} \in \mathbb{R}^{2N}$, $\mathbf{S} \in \mathbb{R}^{3M}$, and $\Sigma \in \mathbb{R}^{3M}$ (using the symmetry $S_k^{ij} = S_k^{ji}$ and $\Sigma_k^{ij} = \Sigma_k^{ji}$). The discretized problem can now be written equivalently as a quadratic programming problem with linear constraints for the unknown $(\mathbf{R}, \mathbf{U}, \boldsymbol{\Sigma}, \mathbf{S}) \in \mathbb{R}^{3N+6M}$:

$$
\begin{aligned}
\gamma \mathbf{E}^T \mathbf{R} + \frac{\epsilon}{2} \mathbf{R}^T \mathbf{K} \mathbf{R} + \frac{1}{\epsilon} \mathbf{R}^T (\mathbf{E} - \mathbf{M}\mathbf{R}) \quad &\rightarrow \quad \min \\
\left(\mathbf{I} - \mathbf{L}_1^T \mathbf{L}_1\right) \mathbf{B}^T \mathbf{S} = \mathbf{T}&, \\
\mathbf{D}\boldsymbol{\Sigma} - \mathbf{C}\mathbf{B}\mathbf{U} = 0&, \\
\mathbf{L}_1 \mathbf{U} = 0&, \\
\mathbf{L}_2 \mathbf{R} = 1&, \\
-\mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R}) \leq \beta(\Sigma - \mathbf{S}) \leq \mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R})&, \\
\boldsymbol{\sigma}^{\min} \mathbf{Q}\mathbf{P}\mathbf{R} \leq \mathbf{S} \leq \boldsymbol{\sigma}^{\max} \mathbf{Q}\mathbf{P}\mathbf{R}&, \\
0 \leq \mathbf{R} \leq 1&, \\
\mathbf{u}^{\min} \leq \mathbf{U} \leq \mathbf{u}^{\max}&.
\end{aligned}
\tag{4.1}
$$

In the objective functional, $\mathbf{E} \in \mathbb{R}^N$ is a vector representing the coefficients of the constant function 1 with respect to the basis functions ϕ_j. $\mathbf{K} \in \mathbb{R}^{N \times N}$ is a stiffness matrix arising from the finite element discretization of the negative Laplacian in $\mathcal{V}^h$, and $\mathbf{M} \in \mathbb{R}^{N \times N}$ is a mass matrix for the identity in $\mathcal{V}^h$. In the discretized formulation of the constraints, the matrix $\mathbf{B} \in \mathbb{R}^{3M \times 2N}$ is the discretization of the divergence operator (restricted to symmetric stress tensors), $\mathbf{C} \in \mathbb{R}^{3M \times 3M}$ represents the stress-strain relation, $\mathbf{D}$ is a (diagonal) mass matrix for the identity in $\mathcal{W}^h$, and $\mathbf{T} \in \mathbb{R}^{2N}$ is a discrete representation of the traction force. The matrices $\mathbf{L}_1 \in \mathbb{R}^{2N_1 \times 2N}$ and $\mathbf{L}_2 \in \mathbb{R}^{N_2 \times N}$ with entries 0 or 1 realize the boundary conditions, where N_1 is the number of node points on Γ_D and N_2 is the number of node points on Γ_N. $\boldsymbol{\sigma}^{\min}$ and $\boldsymbol{\sigma}^{\max}$ are diagonal matrices, representing the corresponding entries of $\sigma^{\min}$ and $\sigma^{\max}$. Finally, $\mathbf{Q} \in \mathbb{R}^{3M \times M}$ is an extension matrix and $\mathbf{P} \in \mathbb{R}^{M \times N}$ is the matrix representation of the projection operator $\mathcal{P}^h$.

The above reasoning shows that after discretization we end up with a linearly constrained quadratic programming problem for the variable

$$\mathbf{X} = (\mathbf{R}, \mathbf{U}, \boldsymbol{\Sigma}, \mathbf{S}) \in \mathbb{R}^{3N+6M}$$

with $2N + 3M + 2N_1 + N_2$ equality, $12M$ inequality constraints and $6N$ bound constraints. Note that $2N_1$ equalities corresponding to the divergence constraint for nodal points in Γ_D are actually in the form $0 = 0$ and can be eliminated. In addition, we can eliminate the components of $\mathbf{U}$ corresponding to nodal points on Γ_D and the corresponding bound constraints (we have to assume $\mathbf{u}^{\min} \leq \mathbf{0} \leq \mathbf{u}^{\max}$ in order to obtain feasible points anyway). Moreover, we can eliminate the components of $\mathbf{R}$ corresponding to nodal values on Γ_N and the corresponding bound constraints. Since all values of $\boldsymbol{\Sigma}$ are determined by the corresponding equality constraints, we remove these constraints and replace $\mathbf{\Sigma}$ by $\mathbf{CBU}$ in the inequality constraints. We consequently end up with a smaller programming problem with $3N + 3M - 2N_1 - N_2$ unknowns, $2N - 2N_1 - N_2$ equality constraints, $12M$ inequality constraints, and $6N - 4N_1 - 2N_2$ bound constraints.

The existence of solutions for the reduced programming problem can be verified in an analogous way to the infinite-dimensional situation under the assumption that there exists a feasible point.

4.1 Constraint Qualification

For linear constraints, the common notions of constraint qualification such as *linear independence*, *Mangasarian-Fromovitz*, or *Slater* qualification are aquivalent. In order to obtain constraint qualification for the linear constraints, it suffices to show that the equality constraints are linearly independent and that there exists a feasible point satisfying all inequalities strictly. Note again that the feasible set does not depend on the relaxation parameter ϵ and hence, the constraint qualification is always uniform with respect to the relaxation.

The linear independence of the equality constraints can be verified by standard reasoning for finite element discretizations and we therefore turn our attention to the inequality constraints. In order to verify constraint qualification, we shall use a natural assumption, namely that the stress and displacement obtained from a design domain completely filled with material satisfy the displacement and stress constraints strictly. This assumption is natural, since one expects the maximal stress and displacement to increase for decreasing mass. Thus, if the constraints are active at maximal mass already, it is quite unlikely to find an optimal design with lower mass anyway, or, in other words, the constraints are too severe to compute a different optimum. In mathematical terms, the assumption can be formulated as follows: Let $\mathbf{R}^1 = \mathbf{1}$ and let $\mathbf{U}^1$ and $\mathbf{S}^1 = boldsymbol\Sigma^1$ be the solutions of the corresponding elasticity problem:

$$(\mathbf{I} - \mathbf{L}_1^T\mathbf{L}_1)\mathbf{B}^T\mathbf{S}^1 = \mathbf{T}, \qquad \mathbf{D\Sigma}^1 - \mathbf{CBU}^1 = \mathbf{0}, \qquad \mathbf{L}_1\mathbf{U}^1 = \mathbf{0},$$

which can be shown to be uniquely defined from standard finite element theory. Then we assume that

$$\boldsymbol{\sigma}^{\min}\mathbf{QPR}^1 < \mathbf{S}^1 < \boldsymbol{\sigma}^{\max}\mathbf{QPR}^1, \qquad \mathbf{u}^{\min} < \mathbf{U}^1 < \mathbf{u}^{\max}, \tag{4.2}$$

where $<$ means strict inequality for each component. Then we obtain the following result:

Theorem 4.1. *Let* (4.2) *be satisfied and let* $\beta > 0$ *be sufficiently small. Then, the constraint set in* (4.1) *with the elimination of variables and constraints as explained above satisfies the linear inequality constraint qualification condition.*

Proof. As noticed above, it suffices to find a feasible point satisfying the constraints strictly. For this sake we choose a density $\mathbf{R}^\eta = \eta\mathbf{1}$ with $\eta < 1$. Then $\|\mathbf{R}^1 - \mathbf{R}^\eta\| = N(1-\eta)$, i.e., the distance to $\mathbf{R}^1$ becomes arbitrarily small as $\eta \to 1$. Because of continuity, we can find $\eta < 1$, and a solution $\mathbf{U}^\eta$, $\mathbf{S}^\eta = \eta\mathbf{\Sigma}^\eta$ of

$$(\mathbf{I} - \mathbf{L}_1^T\mathbf{L}_1)\mathbf{B}^T\mathbf{S}^\eta = \mathbf{T}, \qquad \mathbf{D\Sigma}^\eta - \mathbf{CBU}^\eta = 0, \qquad \mathbf{L}_1\mathbf{U}^\eta = \mathbf{0},$$

such that

$$\boldsymbol{\sigma}^{\min}\mathbf{QPR}^\eta < \mathbf{S}^\eta < \boldsymbol{\sigma}^{\max}\mathbf{QPR}^\eta, \qquad \mathbf{u}^{\min} < \mathbf{U}^\eta < \mathbf{u}^{\max}$$

holds. Moreover, we have

$$\mathbf{0} < \mathbf{R}^\eta = \eta\mathbf{1} < \mathbf{1}$$

and

$$\begin{aligned}-\mathbf{Q}(1-\eta)\mathbf{1} &= -\mathbf{Q}(\mathbf{1} - \mathbf{PR}^\eta) < \beta(\mathbf{\Sigma}^\eta - \mathbf{S}^\eta) \\ &= \beta(1-\eta)\mathbf{\Sigma}^\eta < \mathbf{Q}(\mathbf{1} - \mathbf{PR}^\eta) = \mathbf{Q}(1-\eta)\mathbf{1},\end{aligned}$$

provided β is sufficiently small. Hence, all (reduced) inequality constraints are satisfied strictly by $(\mathbf{R}^\eta, \mathbf{U}^\eta, \mathbf{S}^\eta, \mathbf{\Sigma}^\eta)$, which implies the assertion. □

4.2 First-Order Optimality

If the constraints satisfy constraint qualification conditions, which indeed hold under suitable assumptions as verified above, one can formulate first-order optimality conditions, which must hold for each solution of the optimization problem. We introduce the Lagrangian $\mathcal{L}$ and the Lagrange parameters $\mathbf{\Lambda}_j$ (whose dimension will be clear from their appearence in $\mathcal{L}$) via

$$\begin{aligned}\mathcal{L} \;=\; & \gamma\mathbf{E}^T\mathbf{R} + \tfrac{\epsilon}{2}\mathbf{R}^T\mathbf{KR} + \tfrac{1}{\epsilon}\mathbf{R}^T(\mathbf{E} - \mathbf{MR}) + \\ & \mathbf{\Lambda}_1^T\left((\mathbf{I} - \mathbf{L}_1^T\mathbf{L}_1)\mathbf{B}^T\mathbf{S} - \mathbf{T}\right) + \mathbf{\Lambda}_2^T(\mathbf{D\Sigma} - \mathbf{CBU}) + \mathbf{\Lambda}_3^T\mathbf{L}_1\mathbf{U} + \\ & \mathbf{\Lambda}_4^T(\mathbf{L}_2\mathbf{R} - \mathbf{1}) - \mathbf{\Lambda}_5^T(\mathbf{Q}(\mathbf{1} - \mathbf{PR}) + \beta(\mathbf{\Sigma} - \mathbf{S})) + \\ & \mathbf{\Lambda}_6^T(\beta(\mathbf{\Sigma} - \mathbf{S}) - \mathbf{Q}(\mathbf{1} - \mathbf{PR})) + \mathbf{\Lambda}_7^T\left(\boldsymbol{\sigma}^{\min}\mathbf{QPR} - \mathbf{S}\right) + \\ & \mathbf{\Lambda}_8^T(\mathbf{S} - \boldsymbol{\sigma}^{\max}\mathbf{QPR}) - \mathbf{\Lambda}_9^T\mathbf{R} + \mathbf{\Lambda}_{10}^T(\mathbf{R} - \mathbf{1}) + \\ & \mathbf{\Lambda}_{11}^T\left(\mathbf{u}^{\min} - \mathbf{U}\right) + \mathbf{\Lambda}_{12}^T(\mathbf{U} - \mathbf{u}^{\max})\end{aligned}$$

with $\mathbf{\Lambda}_j \geq 0$ for $j \geq 5$.

By setting all derivatives with respect to primal variables equal to zero as well as using the inequalities and complementarity, we obtain the first-order optimality conditions:

$$
\begin{gathered}
\nabla_{\mathbf{R}}\mathcal{L} = \gamma\mathbf{E} + \epsilon\mathbf{K}\mathbf{R} + \tfrac{1}{\epsilon}\left(\mathbf{E} - 2\mathbf{M}\mathbf{R}\right) + \mathbf{L}_2^T\mathbf{\Lambda}_4 - \mathbf{\Lambda}_9 + \mathbf{\Lambda}_{10} + \\
+\mathbf{P}^T\mathbf{Q}^T\left(\mathbf{\Lambda}_5 + \mathbf{\Lambda}_6 + \boldsymbol{\sigma}^{\min T}\mathbf{\Lambda}_7 - \boldsymbol{\sigma}^{\max T}\mathbf{\Lambda}_8\right) = \mathbf{0}, \\
\nabla_{\mathbf{U}}\mathcal{L} = -\mathbf{B}^T\mathbf{C}^T\mathbf{\Lambda}_2 + \mathbf{L}_1^T\mathbf{\Lambda}_3 - \mathbf{\Lambda}_{11} + \mathbf{\Lambda}_{12} = \mathbf{0}, \\
\nabla_{\mathbf{\Sigma}}\mathcal{L} = \mathbf{D}^T\mathbf{\Lambda}_2 + \beta(\mathbf{\Lambda}_6 - \mathbf{\Lambda}_5) = \mathbf{0}, \\
\nabla_{\mathbf{S}}\mathcal{L} = \mathbf{B}\left(\mathbf{I} - \mathbf{L}_1^T\mathbf{L}_1\right)^T\mathbf{\Lambda}_1 + \beta(\mathbf{\Lambda}_5 - \mathbf{\Lambda}_6) - \mathbf{\Lambda}_7 + \mathbf{\Lambda}_8 = \mathbf{0}, \\
\left(\mathbf{I} - \mathbf{L}_1^T\mathbf{L}_1\right)\mathbf{B}^T\mathbf{S} = \mathbf{T}, \quad \mathbf{D}\mathbf{\Sigma} - \mathbf{C}\mathbf{B}\mathbf{U} = \mathbf{0}, \\
\mathbf{L}_1\mathbf{U} = \mathbf{0}, \quad \mathbf{L}_2\mathbf{R} = \mathbf{1}, \\
-\mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R}) - \beta(\mathbf{\Sigma} - \mathbf{S}) \leq \mathbf{0}, \quad \mathbf{\Lambda}_5^T\left(\mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R}) + \beta(\mathbf{\Sigma} - \mathbf{S})\right) = \mathbf{0}, \quad \mathbf{\Lambda}_5 \geq \mathbf{0}, \\
\beta(\mathbf{\Sigma} - \mathbf{S}) - \mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R}) \leq \mathbf{0}, \quad \mathbf{\Lambda}_6^T\left(\beta(\mathbf{\Sigma} - \mathbf{S}) - \mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R})\right) = \mathbf{0}, \quad \mathbf{\Lambda}_6 \geq \mathbf{0}, \\
\boldsymbol{\sigma}^{\min}\mathbf{Q}\mathbf{P}\mathbf{R} - \mathbf{S} \leq \mathbf{0}, \quad \mathbf{\Lambda}_7^T\left(\boldsymbol{\sigma}^{\min}\mathbf{Q}\mathbf{P}\mathbf{R} - \mathbf{S}\right), \quad \mathbf{\Lambda}_7 \geq \mathbf{0}, \\
\boldsymbol{\sigma}^{\max}\mathbf{Q}\mathbf{P}\mathbf{R} - \mathbf{S} \leq \mathbf{0}, \quad \mathbf{\Lambda}_8^T\left(\boldsymbol{\sigma}^{\max}\mathbf{Q}\mathbf{P}\mathbf{R} - \mathbf{S}\right), \quad \mathbf{\Lambda}_8 \geq \mathbf{0}, \\
-\mathbf{R} \leq \mathbf{0}, \quad \mathbf{\Lambda}_9^T\mathbf{R} = \mathbf{0}, \quad \mathbf{\Lambda}_9 \geq \mathbf{0}, \\
\mathbf{R} - \mathbf{1} \leq \mathbf{0}, \quad \mathbf{\Lambda}_{10}^T(\mathbf{R} - \mathbf{1}), \quad \mathbf{\Lambda}_{10} \geq \mathbf{0}, \\
\mathbf{u}^{\min} - \mathbf{U} \leq \mathbf{0}, \quad \mathbf{\Lambda}_{11}^T\left(\mathbf{u}^{\min} - \mathbf{U}\right) = \mathbf{0}, \quad \mathbf{\Lambda}_{11} \geq \mathbf{0}, \\
\mathbf{U} - \mathbf{u}^{\max} \leq \mathbf{0}, \quad \mathbf{\Lambda}_{12}^T\left(\mathbf{U} - \mathbf{u}^{\max}\right) = \mathbf{0}, \quad \mathbf{\Lambda}_{12} \geq \mathbf{0}.
\end{gathered}
$$

5 Solution of the Discretized Problem

5.1 Continuation in ϵ

As motivated in the introduction and in section 3.1 the problem (4.1) will be solved for a decreasing sequence of ϵ. As $\epsilon \to 0$, in analogy to Γ-convergence of the perimeter funtctional, we expect convergence of the sequence of the minimum solutions to a final solution. Moreover, since the double obstacle term is of leading order in ϵ, such a final solution will have a sharp interface between material ($\{\rho{=}1\}$) and void ($\{\rho{=}0\}$).

To achieve this we will use a continuation method such that we choose a decreasing sequence $\{\epsilon^l\}$ with $\epsilon^l \to 0$ for $l = 0, \ldots, L$, where L describes the total number of continuation levels. The corresponding optimization problems are then solved by an interior-point method, as desribed in the next section. Between the levels ϵ can be reduced, e.g., like $\epsilon^{l+1} := \delta\epsilon^l$ with $0 < \delta < 1$ or like $\epsilon^{l+1} := (\epsilon^0)^l$ if $0 < \epsilon^0 < 1$. If we decrease ϵ too slow, we may expect from theory and observations from numerical test that the final solution is not changed, but we end up with a possibly unnecessary high number of levels L. On the other hand if ϵ is decreased too fast, the optimization process might get stuck in some undesired local minimum, since the objective functional $M(\rho)$ is turned from convex to concave too quickly.

5.2 Interior-Point Methods

In the last two decades interior-point methods have developed to efficient methods for large scale nonlinear programming. A major characteristic of these methods is that all inequality constraints are satisfied strictly, which led to the labelling *interior-point* methods.

For a short introduction we consider the following general optimization problem:

$$\begin{aligned} f(\mathbf{x}) \quad &\rightarrow \quad \min_{\mathbf{x}\in\mathbb{R}^n} \\ \text{s.t.} \quad c_{\mathcal{I}}(\mathbf{x}) &\geq \mathbf{0}, \\ c_{\mathcal{E}}(\mathbf{x}) &= \mathbf{0}, \end{aligned} \tag{5.1}$$

where all appearing functions should be sufficiently differentiable and $c_{\mathcal{I}}(\mathbf{x}) = (c_i(\mathbf{x}))_{i\in\mathcal{I}}$ and $c_{\mathcal{E}}(\mathbf{x}) = (c_i(\mathbf{x}))_{i\in\mathcal{E}}$ denote the inequality and equality constraints with the corresponding index sets $\mathcal{I}$ and $\mathcal{E}$, respectively. This problem is then modified such that the restricting inequality constaints are treated implicitly by adding them to the objective functional using some barrier term. The predominant barrier function is the logarithmic barrier function and so the new objective is now a sum of the original one and a logarithmic interior part:

$$\begin{aligned} f(\mathbf{x}) - \mu \sum_{i\in\mathcal{I}} \ln c_i(\mathbf{x}) \quad &\rightarrow \quad \min_{\mathbf{x}\in\mathbb{R}^n} \\ \text{s.t.} \quad c_{\mathcal{E}}(\mathbf{x}) &= \mathbf{0}, \end{aligned} \tag{5.2}$$

where $\mu > 0$ is called the barrier parameter. Minimization of (5.2) for a decreasing sequence of the barrier parameter $\mu \rightarrow 0$ will result in a sequence of minimizers converging to the minimizer of the original problem (5.1).

In primal-dual methods we treat the primal variables and the dual variables (the Lagrangian multipliers of the problem) independently. Using the following notation we state the first order optimality conditions for (5.2): $C_{\mathcal{I}}(\mathbf{x}) = \operatorname{diag}(c_i(\mathbf{x}), i \in \mathcal{I})$, $\boldsymbol{\lambda}_{\mathcal{E}}$ the vector of Lagrange multipliers for the equality constraints and $\mathbf{e}$ a vector of ones in the appropriate dimension:

$$\begin{aligned} \nabla f(\mathbf{x}) - \mu \nabla c_{\mathcal{I}}(\mathbf{x})^T C_{\mathcal{I}}(\mathbf{x})^{-1}\mathbf{e} - \nabla c_{\mathcal{E}}(\mathbf{x})^T \boldsymbol{\lambda}_{\mathcal{E}} &= \mathbf{0}, \\ c_{\mathcal{E}}(\mathbf{x}) &= \mathbf{0}. \end{aligned} \tag{5.3}$$

Alternatively, if we define new variables $\boldsymbol{\lambda}_{\mathcal{I}} = \mu C_{\mathcal{I}}(\mathbf{x})^{-1}\mathbf{e}$ and consider $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\mathcal{I}}, \boldsymbol{\lambda}_{\mathcal{E}})$ and $c(\mathbf{x}) = (c_{\mathcal{I}}(\mathbf{x}), c_{\mathcal{E}}(\mathbf{x}))$, we can rewrite (5.3) as a system in the primal variables $\mathbf{x}$ and the dual variables $\boldsymbol{\lambda}$:

$$\begin{aligned} \nabla f(\mathbf{x}) - \nabla c(\mathbf{x})^T \boldsymbol{\lambda} &= \mathbf{0}, \\ C_{\mathcal{I}}(\mathbf{x})\boldsymbol{\lambda}_{\mathcal{I}} - \mu\mathbf{e} &= \mathbf{0}, \\ c_{\mathcal{E}}(\mathbf{x}) &= \mathbf{0}. \end{aligned} \tag{5.4}$$

The second equation in (5.4) can be interpreted as the perturbed complementarity condition for the inequality constaints in the KKT conditions for (5.1). The left-hand-side of (5.4) defines a function $F_\mu(\mathbf{x}, \boldsymbol{\lambda})$. Instead of minimizing (5.2) for $\mu \rightarrow 0$, we look for solutions of $F_\mu(\mathbf{x}, \boldsymbol{\lambda}) = 0$ for $\mu \rightarrow 0$. For a fixed μ (5.4) can be solved, e.g., using a Newton-type method, where the Newton direction $(\triangle\mathbf{x}, \triangle\boldsymbol{\lambda})$ is defined as the solution of $F_\mu(\mathbf{x}, \boldsymbol{\lambda})'(\triangle\mathbf{x}, \triangle\boldsymbol{\lambda}) = -F_\mu(\mathbf{x}, \boldsymbol{\lambda})$:

$$\begin{pmatrix} \nabla^2 f & -\nabla c_{\mathcal{I}}^T & -\nabla c_{\mathcal{E}}^T \\ \nabla c_{\mathcal{I}}\Lambda_{\mathcal{I}} & C_{\mathcal{I}} & \mathbf{0} \\ \nabla c_{\mathcal{E}} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \triangle\mathbf{x} \\ \triangle\boldsymbol{\lambda}_{\mathcal{I}} \\ \triangle\boldsymbol{\lambda}_{\mathcal{E}} \end{pmatrix} = - \begin{pmatrix} \nabla f - \nabla c^T \boldsymbol{\lambda} \\ C_{\mathcal{I}}\boldsymbol{\lambda}_{\mathcal{I}} - \mu\mathbf{e} \\ c_{\mathcal{E}} \end{pmatrix}, \qquad (5.5)$$

where $\Lambda_{\mathcal{I}} = \operatorname{diag}(\boldsymbol{\lambda}_i, i \in \mathcal{I})$ and all arguments in (5.5) are omitted.

5.3 Adaption of the Problem to IPOPT

We solve the problem (4.1) with IPOPT, which is a free available optimization code realizing a primal-dual iterior-point method. IPOPT, implemented by A. Wächter and L. T. Biegler, is able to solve problems of the following form:

$$\begin{aligned} f(\mathbf{x}) \quad &\rightarrow \quad \min_{\mathbf{x}\in\mathbb{R}^n} \\ \text{s.t.} \quad c_{\mathcal{E}}(\mathbf{x}) &= \mathbf{0}, \\ \mathbf{x}^{\min} \leq \mathbf{x} &\leq \mathbf{x}^{\max}. \end{aligned}$$

The stopping criterion of IPOPT is defined using the primal-dual equations (5.4):

$$E_\mu(\mathbf{x}, \boldsymbol{\lambda}) := \max\left\{ \frac{\|\nabla f(\mathbf{x}) - \nabla c(\mathbf{x})^T \boldsymbol{\lambda}\|_\infty}{s_d}, \frac{\|C_{\mathcal{I}}(\mathbf{x})\boldsymbol{\lambda}_{\mathcal{I}} - \mu\mathbf{e}\|_\infty}{s_c}, \|c_{\mathcal{E}}(\mathbf{x})\|_\infty \right\},$$

where s_c and s_d are scaling parameters. The optimization process is now stopped if

$$E_0(\mathbf{x}, \boldsymbol{\lambda}) \leq e_0 \qquad (5.6)$$

is fulfilled, where e_0 is a given error tolerance. More information about the implementation of IPOPT can be found in Wächter and Biegler [32].

General nonlinear programming problems with inequality constraints $c_{\mathcal{I}}(\mathbf{x}) \leq 0$ can be written in the above framework using slack variables. So we reformulate (4.1) in the above form by introducing some vector $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) \in \mathbb{R}^{12M}$ of slack variables, leading to:

$$\begin{aligned} \gamma\mathbf{E}^T\mathbf{R} + \frac{\epsilon}{2}\mathbf{R}^T\mathbf{K}\mathbf{R} + \frac{1}{\epsilon}\mathbf{R}^T(\mathbf{E} - \mathbf{M}\mathbf{R}) \quad &\rightarrow \quad \min_{(\mathbf{R},\mathbf{U},\mathbf{S},\mathbf{Z})\in\mathbb{R}^{3N+15M}} \\ \left(\mathbf{I} - \mathbf{L}_1^T\mathbf{L}_1\right)\mathbf{B}^T\mathbf{S} - \mathbf{T} &= \mathbf{0}, \\ -\mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R}) - \beta(\mathbf{C}\mathbf{B}\mathbf{U} - \mathbf{S}) + \mathbf{Z}_1 &= \mathbf{0}, \\ \beta(\mathbf{C}\mathbf{B}\mathbf{U} - \mathbf{S}) - \mathbf{Q}(\mathbf{1} - \mathbf{P}\mathbf{R}) + \mathbf{Z}_2 &= \mathbf{0}, \\ \boldsymbol{\sigma}^{\min}\mathbf{Q}\mathbf{P}\mathbf{R} - \mathbf{S} + \mathbf{Z}_3 &= \mathbf{0}, \\ \mathbf{S} - \boldsymbol{\sigma}^{\max}\mathbf{Q}\mathbf{P}\mathbf{R} + \mathbf{Z}_4 &= \mathbf{0}, \\ \mathbf{0} \leq \mathbf{R} &\leq \mathbf{1}, \\ \mathbf{u}^{\min} \leq \mathbf{U} &\leq \mathbf{u}^{\max}, \\ \mathbf{0} &\leq \mathbf{Z}. \end{aligned} \qquad (5.7)$$

Finally we solve a programming problem with $3N + 15M - 2N_1 - N_2$ unknowns, $2N+12M-2N_1-N_2$ equality constraints and $6N+12M-4N_1-2N_2$ bound constraints. A similar discrete programming problem for the problem with the conservative von Mises stress constraints (3.2) can be deduced in a analogous way.

6 Numerical Results

For the numerical examples we have chosen two simple examples where the global optimal designs are known on very coarse grids (see [30]), which provides some reference for our solutions. For sake of simplicity the Young's modulus and the Poisson's ratio of the given material are $E = 1N/m^2$ and $\nu = 0.3$. We used the plain strain model for the computations and also, for simplicity, all structures have a unit thikness of $1m$ and are loaded with half of the unit load. Reasonable bounds for the displacements $\mathbf{u}$ and the stresses σ are provided by the unique solutions $\overline{\mathbf{u}}$ and $\overline{\sigma}$ of the corresponding elasticity problem when the whole design domain Ω is filled with material $\rho = 1$. Then the displacement bounds are e.g. set to

$$u_i^{\max} = -u_i^{\min} = 2\max\{|\overline{u}_i(\mathbf{x})| \;:\; \mathbf{x} \in \Omega\}, \quad i = 1, 2$$

and the stress bounds to

$$\begin{aligned} \sigma_{11}^{\max} = \sigma_{22}^{\max} &= \max\{|\overline{\sigma}_{11}(\mathbf{x})|, |\overline{\sigma}_{22}(\mathbf{x})| \;:\; \mathbf{x} \in \Omega\}, \\ \sigma_{12}^{\max} = \sigma_{21}^{\max} &= \max\{|\overline{\sigma}_{12}(\mathbf{x})|, |\overline{\sigma}_{21}(\mathbf{x})| \;:\; \mathbf{x} \in \Omega\} \end{aligned}$$

with $\sigma^{\min} = -\sigma^{\max}$. The von Mises stress bound is given by

$$\Phi^{\max} = \max\{\Phi(\sigma(\mathbf{x})) \;:\; \mathbf{x} \in \Omega\}.$$

All numerical examples are performed on a pc using a 2.4 GHz Intel CPU and 2 GB memory. For the mesh generation and the finite element part of the computations the software package NETGEN/NGSolve was used. The optimization part was done using the interior-point code IPOPT. As IPOPT is used as a 'black-box', we did not adjust its linear solver (for systems like (5.5)) and its stopping criterion to our needs. So we stop the optimization process per continuation level if the stopping criterion (5.6) is fulfilled with $e_0 = 10^{-5}$ or a maximum number of 200 iterations is reached. For an approximation of the Hessian of the Lagrange functional a BFGS routine is used. We want to point out that these numerical examples just show the potential of this solution approach. Since there was no emphasis on efficiency so far, the computational times are far from being optimal. A more sophisticated way to decrease ϵ, a proper stopping criterion and a linear solver with optimal complexity for the IPOPT package would decrease the runtimes significantly. So far most of the cpu time is spent in solving the linear systems like (5.5).

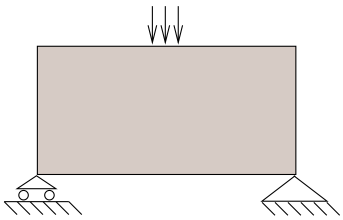


Figure 1: The short beam example.

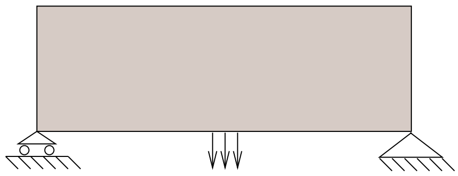


Figure 2: The long beam example.

6.1 A Short Beam Example

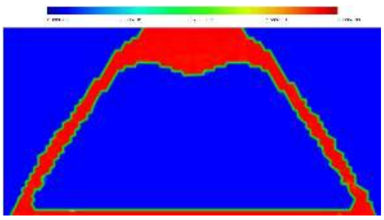


Figure 3: Optimal material distribution.

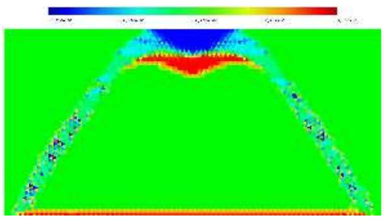


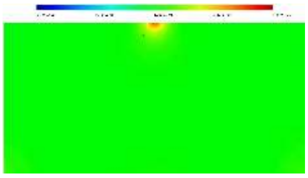
Figure 4: σ_{11} of the optimal design.

As a first example we treat the problem shown in Fig. 1, there the load condition, bearings and geometry are illustrated, with a design domain of dimension $2m \times 1m$. Here we consider stress constraints w.r.t. local stresses. The corresponding bound constraints are:

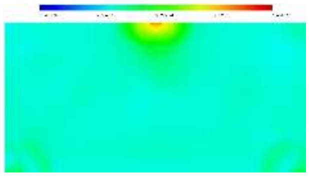
$$\mathbf{u}^{\max} = \begin{pmatrix} 0.51 \\ 0.51 \end{pmatrix} m, \quad \sigma^{\max} = \begin{pmatrix} \sigma_{11}^{\max} \\ \sigma_{22}^{\max} \\ \sigma_{12}^{\max} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.16 \end{pmatrix} \frac{N}{m^2},$$

with $\mathbf{u}^{\min} = -\mathbf{u}^{\min}$ and $\sigma^{\min} = -\sigma^{\max}$. A mesh with 7382 elements is used for the optimization process, so we finally end up with 122185 unknowns. In more details we have 3813, 7626, 22146, and 88584 dofs for the density, displacements, stresses, and slacks respectively. The total number of equality constraints is 96226 and 111462 for the bound constraints. The scaling parameter γ of the mass term in the objective is set to $\gamma = 1.5$ and we start the ϵ-continuation with $\epsilon^0 = 1$. In Fig. 5 we present a sequence of optimal designs, corresponding to the continuation levels $l = 0, \ldots, L$ with $L = 8$. Between the levels ϵ is reduced like $\epsilon^{l+1} = \delta\epsilon^l$ with $\delta = 0.5$. The overall computational time for the 8 levels is about 11 hours and the volume of the final optimal design is 0.34 m^3 (17.2% of $|\Omega|$).

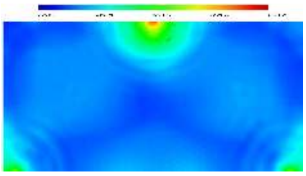
A final design, with the same parameters as above, but after 10 levels can be seen in Fig. 3. The used mesh has 14182 elements, so the optimization



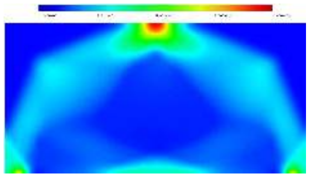
Level 0, $\epsilon = 1$.



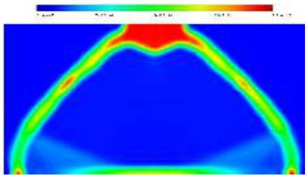
Level 1, $\epsilon = 0.5$.



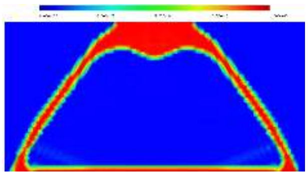
Level 2, $\epsilon = 0.25$



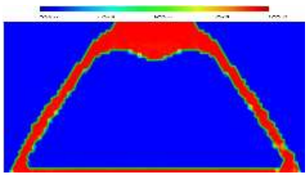
Level 3, $\epsilon = 0.125$.



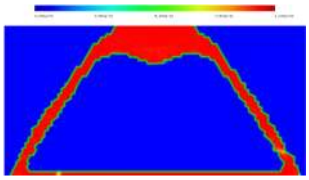
Level 4, $\epsilon = 0.0625$.



Level 5, $\epsilon = 0.03125$.



Level 6, $\epsilon = 0.015625$.



Level 7, $\epsilon = 0.0078125$.

Figure 5: ϵ-continuation over 8 levels.

consists of 234531 unknows (7260, 14520, 42546, and 170184 dofs for the density, displacements, stresses and slacks respectively), 184725 equality constraints, and 213744 bound constraints. For 10 levels the process takes about 17 hours.

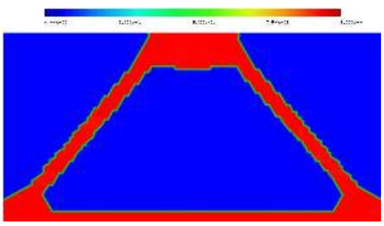


Figure 6: Optimal material distribution for a von Mises constraint.

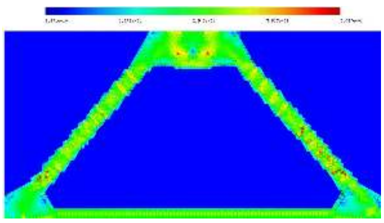


Figure 7: Von Mises stress distribution.

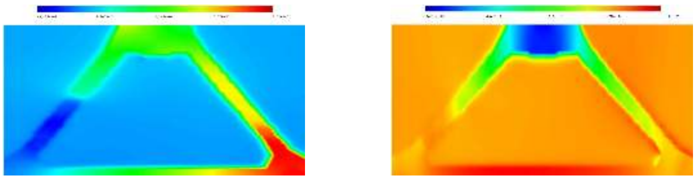


Figure 8: x- and y-displacements of the optimal design.

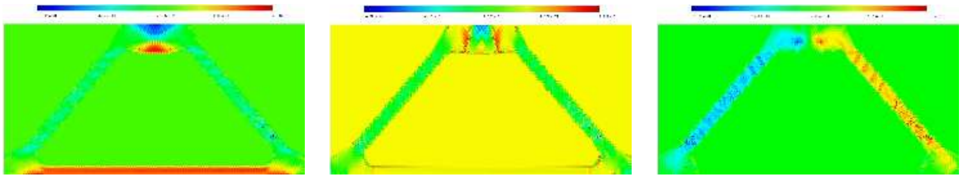


Figure 9: σ_{11}, σ_{22}, and σ_{12} distribution of the optimal design.

Let us again consider the same example, but now with a von Mises stress constraint, $\Phi^{\max} = 0.64$. On a mesh with 14182 elements we end up with 291259 unknowns, due to the additional unknowns representing the conservative von Mises stress approximion. Here 7260, 14520, 42546, 42546, and 184366 dofs are used for the density, displacements, stresses, von Mises approximation, and slacks respectively. The total number of equality constraints is 198907 and 227926 for the bound constraints. With $\gamma = 2$, $\epsilon^0 = 1/3$ and $\epsilon^{l+1} = \epsilon^l/2$ for $L = 10$ levels we end up with a final design as in Fig. 6. Displacements and stresses of the optimal design are shown in Fig. 7 - Fig. 9. In about 13.5 hours the total volume of the design is reduced to 0.4 m^3 (20% of $|\Omega|$).

6.2 A Long Beam Example

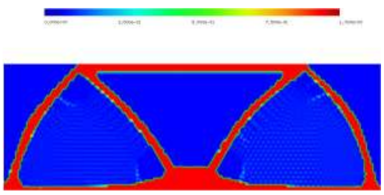


Figure 10: Optimal material distribution.

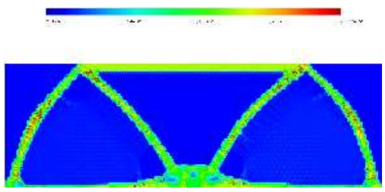


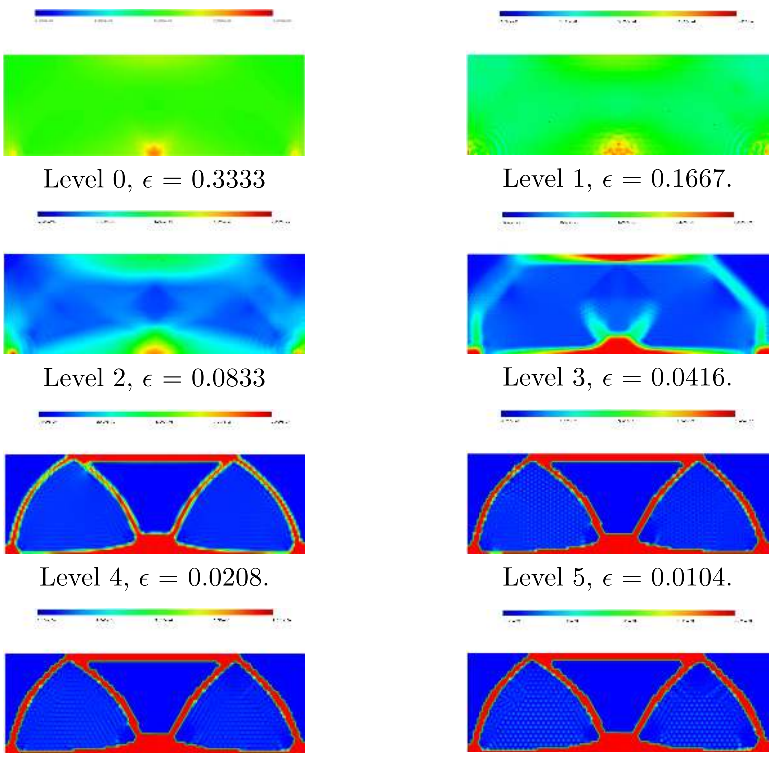
Figure 11: Von Mises stress distribution.

For the second example we consider the load condition, bearings and geometry shown in Fig. 2, where the dimension of the design domain are $3m \times 1m$. Here we choose to calculate an optimal design w.r.t. bounded von

Mises stress and again we list the corresponding bound constraints:

$$\mathbf{u}^{\max} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} m, \quad \Phi^{\max} = 0.71 \frac{N}{m^2},$$

with $\mathbf{u}^{\min} = -\mathbf{u}^{\min}$. For the discretization and the optimization process we use a mesh with 11058 elements, which results in 227191 unknowns, 5691, 11382, 33174, 33174 and 143754 dofs for the density, displacements, stresses, conservative von Mises approximation and slacks, respectively. Here we end up with a total number of 155152 equality constraints and 177900 bound constraints. The two parameters, determining the optimization process, are chosen as follows: $\gamma = 2$, and $\epsilon^0 = 1/3$. Again ϵ is divided by 2 between two levels and the maximal number of levels is set to $L = 8$. As before we show a sequence of optimal designs in Fig. 12. For solving 8 levels it takes about 15 hours and the volume is reduced to 0.75 m^3 (23.5% of $|\Omega|$).



Level 0, $\epsilon = 0.3333$ | Level 1, $\epsilon = 0.1667$.

Level 2, $\epsilon = 0.0833$ | Level 3, $\epsilon = 0.0416$.

Level 4, $\epsilon = 0.0208$. | Level 5, $\epsilon = 0.0104$.

Level 6, $\epsilon = 0.0052$. | Level 7, $\epsilon = 0.0026$.

Figure 12: ϵ-continuation over 8 levels.

In Fig. 10 we see the solution of the same problem, but with 17291 elements and 355097 unkowns (8849, 17698, 51873, 51873, and 224783 dofs for the density, displacements, stresses, convervative von Mises stress approximation, and slacks, respectively. The number of equalilty constraints is 242502 and 277877 for the bound constraints. After about 25 hours we receive the plotted optimal design.

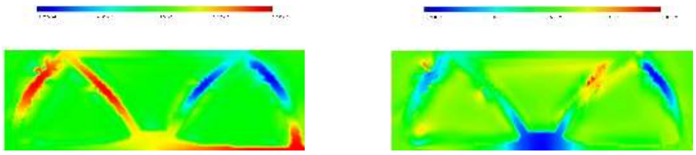


Figure 13: x- and y-displacements of the optimal design.

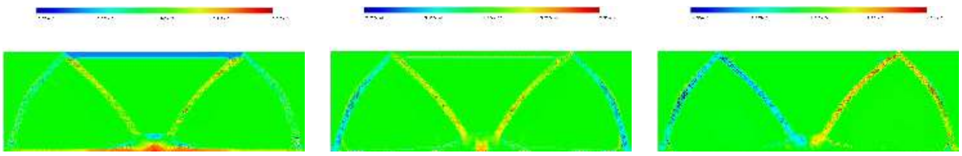


Figure 14: σ_{11}, σ_{22}, and σ_{12} distribution of the optimal design.

7 Conclusions

A new method for solving structural topology optimization problems with stress constraints has been presented. The reformulation of the problem and the phase-field relaxation leads to a parameter-dependent family of large-scale optimization problems satisfying uniform constraint qualification. Using parameter continuaton it is possible to compute optima in a robust way with reasonable effort (e.g. compared to mixed linear programming techniques).

So far no particular emphasis has been laid on the efficient solution of the discretized problems for very large number of unknowns (as appearing e.g. in 3D applications), but there is a lot of potential to speed up the solution techniques, which will be investigated in future research.

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