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Phase-flip bifurcation induced by time delay

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We present a general bifurcation in the synchronized dynamics of time-delay-coupled nonlinear oscillators. The relative phase between the oscillators jumps from zero to π as a function of the coupling; this phase-flip bifurcation is accompanied by a discontinuous change in the frequency of the synchronized oscillators. This phenomenon is of broad relevance, being observed in regimes of oscillator death as well as in periodic, quasiperiodic, and chaotic dynamics. Time-delay coupling is necessary for the phase-flip bifurcation. We illustrate the phenomenon, and present analytical results for paradigmatic nonlinear systems. Possible applications are discussed.

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The synchronization of coupled nonlinear oscillators can be either the in-phase state, or—as indeed in Huygens' first observation of the phenomenon [1]—the *antiphase* state. Such phase properties of synchronized systems [2] have been the focus of considerable recent interest [3].

In this paper we describe a general bifurcation in nonlinear oscillators when their interaction is time delayed. As the delay is varied, at a critical value of the parameters, the relative phase between the oscillators changes abruptly from zero to π. This bifurcation, which we call the *phase flip*, is observed in the synchronization regime and is a general feature of time-delay-coupled systems. This had been observed in the special case of relaxation oscillations, namely, when the interaction leads to oscillator death [4]. In the present paper, we show that this bifurcation has a broad range of occurrence: it is observed for periodic as well as chaotic oscillators, for identical as well as nonindentical coupled systems, and in a variety of different dynamical regimes.

This bifurcation has broad relevance due to the ubiquity of time-delay coupling. In most natural systems signals are transmitted with finite velocity, which makes delay coupling appropriate in physical, biological, ecological, or social systems. Several studies have explored the manner in which coupling induces complex phenomena (such as synchronization, for example) in nonlinear dynamical systems, although time-delay coupling has not so far been studied widely in this context. The nature of the coupling makes the system infinite dimensional and therefore less analytically tractable; recent studies of coupled time-delay interacting systems [4–7] have begun to address these issues. For phase oscillators with time-delay coupling the phase jump has been observed, but with multiple coexisting attractors [7]. When the coupling is instantaneous, both in-phase and antiphase states are known to occur, as indeed are states of mixed phase [8]. Studies have even noted a situation where there are time-dependent transitions between them [2].

In the synchronized regime there can, in addition to amplitude death [4,9], be regions of chaotic, periodic, or quasiperiodic dynamics. Given the wealth of dynamical behavior that occurs in such systems, therefore, the occurrence of a bifurcation between the in-phase attractor and the out-of-phase attractor of different synchronized dynamics has considerable importance.

We first demonstrate the phase-flip bifurcation in a pair of diffusively coupled Rössler [10] oscillators. This is a paradigmatic system that describe a simple mathematical model of chemical kinetics that incorporates reaction-diffusion. For simplicity we take both oscillators (differentiated by subscript 1 or 2) to be identical, but couple the variables y_1 and y_2 at different times:

$$\dot{x}_{1,2}(t) = - y_{1,2} - z_{1,2},$$
$$\frac{\dot{y}_{1,2}(t)}{dt} = x_{1,2} + a y_{1,2} + \epsilon[y_{2,1}(t-\tau) - y_{1,2}(t)],$$
$$\frac{\dot{z}_{1,2}(t)}{dt} = b + z_{1,2}(x_{1,2} - c). \tag{1}$$

The uncoupled (ϵ=0) subsystems are chaotic [10] for $a=b=0.1$ and $c=14$; we study the coupled system as a function of the coupling ϵ and the time delay τ.

For the case of instantaneous coupling (τ=0) this system has been studied in detail [2,11] and a variety of dynamical phenomena are known to occur. These also obtain for finite coupling; some preliminary results for time-delay-coupled Rössler systems have been presented in [4]. Figure 1(a) schematically depicts the different dynamical states that arise for a range of the parameters ϵ and τ [12]. The shaded region, marked C, corresponds to chaotic states while the white region shows the regions of regular behavior. These include a stable fixed point FP regime (which corresponds to amplitude death [4]) and periodic (P) and quasiperiodic (QP) dynamics.

The phase-flip bifurcation takes place across the bold line in Fig. 1(a) in the whole range of ϵ and for a certain range of τ. The arrows depict the direction of the transition from in-phase to out-of-phase motion [Fig. 1(b)]. The attached numbers label different dynamics before and after this bifurcation: 1 has amplitude death (FP) both before and after the transition, 2 is from periodic to periodic dynamics, 3 and 5 are from periodic to chaotic motion, 4 is from chaotic to chaotic motion, and 6 is from periodic to quasiperiodic dynamics.

The largest few Lyapunov exponents as a function of the time-delay parameter are shown for this system in Fig. 2

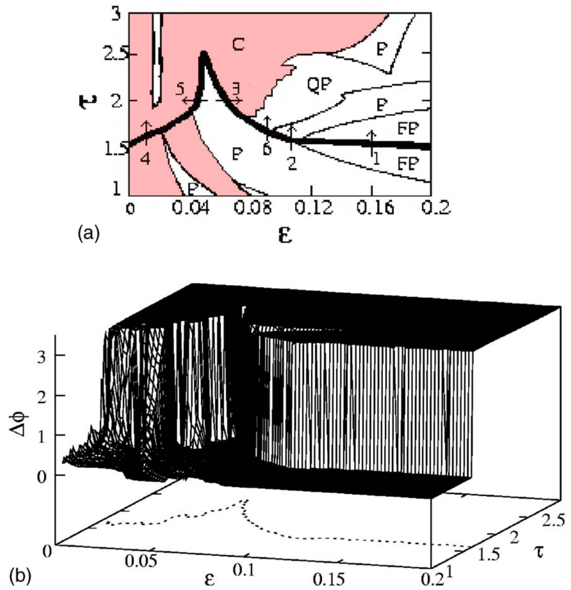

FIG. 1. (Color online) (a) Schematic phase diagram [4] for identical coupled Rössler oscillators, Eq. (1), in the ϵ-τ plane. The thick solid line indicates the locus of the phase-flip bifurcation and the numbered arrows (see the text for details) indicate transitions between different types of motion. (b) Plots of phase difference $\Delta\phi$ in the ϵ-τ plane; contour is drawn for $\Delta\phi=\pi/2$.

across the different settings of the phase-flip bifurcation. Figure 2(a) is for the transition along the arrow marked 1 with $\epsilon=0.16$. Since λ_1 is negative, the dynamics eventually goes to a fixed point attractor [4]. Inset in Fig. 2(a) are representative trajectories at $\tau=1.52$, before the bifurcation, and at $\tau=1.63$, after the bifurcation which can be clearly seen in Fig. 2(b) where the relative phase is shown as a function of τ. In either case the trajectories spiral into the fixed point, but for $\tau=1.52$, which is prior to the bifurcation, the oscillators are in-phase, while for $\tau=1.63$ they are out of phase. Note that eventually the coupling vanishes and the fixed point corresponds to that of the uncoupled systems. The largest Lyapunov exponent for this fixed point solution, the position of which does not change with time delay, is also shown in Fig. 2(a) as λ_{FP}. Since this is negative in the FP region the fixed point is stable [4,13].

The location of the bifurcation can be easily seen in a graph of the largest Lyapunov exponent. Note that there is a discontinuity in the slope at τ_c (marked by the arrow). The phase difference between the oscillators, defined as $\Delta\phi = \langle |\phi_1(t)-\phi_2(t)|\rangle$ where $\langle\cdot\rangle$ denotes the average over time and $\phi_i(t) \simeq \tan^{-1}[y_i(t)/x_i(t)]$ [4,14] is shown as a function of τ in Fig. 2(b). At τ_c this difference abruptly changes from 0 to π: this is the phase flip [see this bifurcation in Fig. 1(b) for other regions of parameter space].

Since the two oscillators are synchronized, their frequencies are identical. At the phase flip, this frequency also changes abruptly. Their oscillation frequency Ω (measured from the peak-to-peak separation [4]) is shown as a function of τ in Fig. 2(c) where the abrupt change at the phase-flip bifurcation is evident. Trajectories before and after the bifurcation for the transitions along the arrows marked 2, 3, and 4

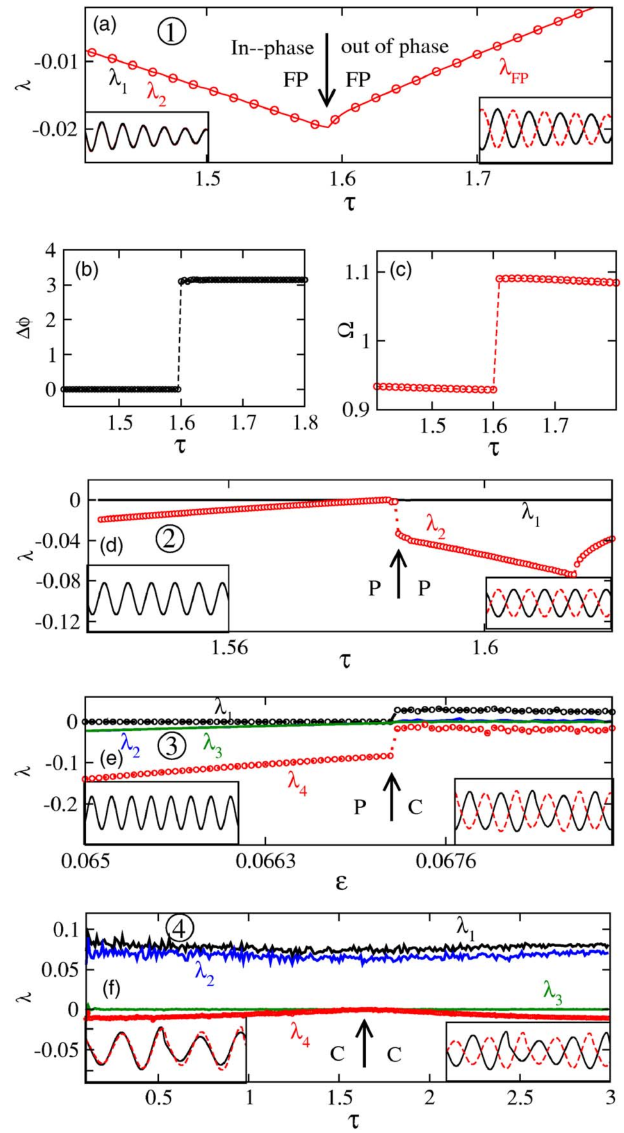

FIG. 2. (Color online) (a) Spectrum of Lyapunov exponents for the Rössler system Eq. (1) as a function of the time delay τ at fixed coupling strength $\epsilon=0.16$. In amplitude death, all Lyapunov exponents are negative. (b) The phase difference between the oscillators: this is 0 before and π after the transition at $\tau_c \sim 1.58$. (c) The frequency of the synchronized oscillators as a function of τ. The largest few Lyapunov exponents (d) as a function of τ for $\epsilon=0.105$, (e) as a function of ϵ for $\tau=2$, and (f) as a function of τ for $\epsilon=0.01$. In all cases, trajectories before and after the transition are shown in the insets on the left and right of the panels, respectively.

are shown in the inset of Figs. 2(d)–2(f). When the transition is from one limit cycle to another, λ_1 remains zero across the transition, but λ_2 shows a discontinuity [Fig. 2(d)]. In the transition from a limit cycle to a chaotic attractor [Fig. 2(e)], λ_1 and λ_4 are discontinuous, while $\lambda_2=\lambda_3$ remain equal to 0, while when the transition is in the chaotic region [Fig. 2(f)] both λ_1 and λ_2 are positive, and λ_4, which is negative, has a *maximum* at the phase flip. In all cases there is a discontinuity in the largest negative Lyapunov exponent or its derivative across this bifurcation: this property can be used to construct an order parameter for the bifurcation.

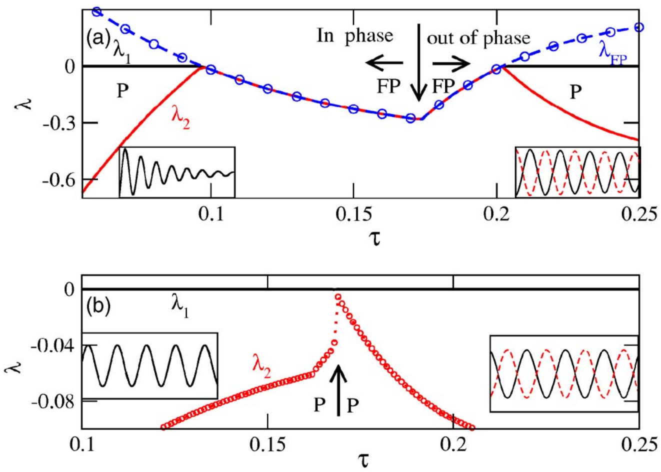

FIG. 3. (Color online) The largest two Lyapunov exponents for the coupled oscillators Eq. (2) as a function of the time delay with fixed coupling. (a) ϵ=9. When all Lyapunov exponents are negative, there is oscillator death (see the inset trajectories, in phase for τ=0.15 and out of phase for τ=0.19). (b) ϵ=40, when the transition is from limit cycle to limit cycle.

In order to gain insight into the nature of the mechanism for this bifurcation, we analyze the somewhat simpler system of identical delay-coupled limit cycle oscillators. This model has been the subject of several earlier studies [6], and is given by the equation

$$\dot{Z}_j(t) = [1 + i\omega - |Z_j(t)|^2]Z_j(t) + \epsilon[Z_k(t-\tau) - Z_j(t)], \quad (2)$$

where $j,k=1,2$ and $k \neq j$. The variables $Z_j(t)$ are the complex amplitudes of the oscillators of frequency ω, and ϵ is the coupling strength.

Numerical results are presented in Fig. 3 for ω=9. The spectrum of Lyapunov exponents along with λ_{FP} the exponent for the fixed point (Z_j=0) is shown in Fig. 3(a). As in the case of the Rössler system, the two oscillators are in phase in the amplitude death region when τ=0.15 and are out of phase when τ=0.19. The phase for each oscillator is $\phi_j=\tan^{-1}[\mathrm{Im}(Z_j)/\mathrm{Re}(Z_j)]$, and the frequency Ω (which differs from ω) now depends on both ϵ and τ. At $\tau=\tau_c$ the phase of one oscillator abruptly flips: the phase difference $\Delta\phi=|\phi_1-\phi_2|$ shown in Fig. 4(a) goes from 0 to π and simultaneously as for the chaotic oscillators, the frequency Ω abruptly increases [Fig. 4(b)]. This type of jump in phase has also been observed in phase oscillators [7] when there are multiple coexisting attractors. In the amplitude death region, however, there is a single stable attractor, and hence the phase jump phenomenon is a bifurcation, independent of initial conditions [15]. The phase flip also occurs in a region of a purely limit cycle behavior: the dynamics before and after are both periodic [Fig. 3(b)]. This occurs for higher coupling (ϵ=40); here λ_1 remains zero across the transition while λ_2 is discontinuous [16].

We now obtain an analytic estimate for the phase-flip bifurcation for the system (2). Consider the characteristic eigenvalue equation [6]

$$\lambda^2 - 2(a+i\omega)\lambda + (a^2 - \omega^2 + i2a\omega) - \epsilon^2 e^{-2\lambda\tau} = 0, \quad (3)$$

where $a=1-\epsilon$. Setting $\lambda=\alpha+i\beta$ in Eq. (3) and separating real and imaginary parts leads to the pair of equations

$$\alpha^2 - \beta^2 - 2\delta_1 + a^2 - \omega^2 - \epsilon^2 e^{-2\alpha\tau}\cos(2\beta\tau) = 0, \quad (4a)$$

$$2\alpha\beta - 2\delta_2 + 2a\omega + \epsilon^2 e^{-2\alpha\tau}\sin(2\beta\tau) = 0, \quad (4b)$$

where $\delta_1=(a\alpha-\beta\omega)$ and $\delta_2=(\alpha\omega+a\beta)$.

The real part of the eigenvalue is the Lyapunov exponent at the fixed point, $\alpha=\lambda_{FP}$, and by using this in Eqs. (4), β can be determined. The roots for the two equations, are shown separately in Fig. 4(b) and these can be seen to coincide with the oscillator frequency, Ω, prior to τ_c along the lower branch, and above τ_c along the upper branch.

Around the fixed point, Eq. (2) can be written in polar coordinates $Z_j(t)=A_j(t)\exp i\phi_j(t) \equiv \exp(\alpha+i\beta t)$. Separating real and imaginary parts, we get

$$\alpha = a - e^{2\alpha t} + \epsilon e^{\alpha\tau}\cos[\phi_j(t-\tau) - \phi_k(t)], \quad (5)$$

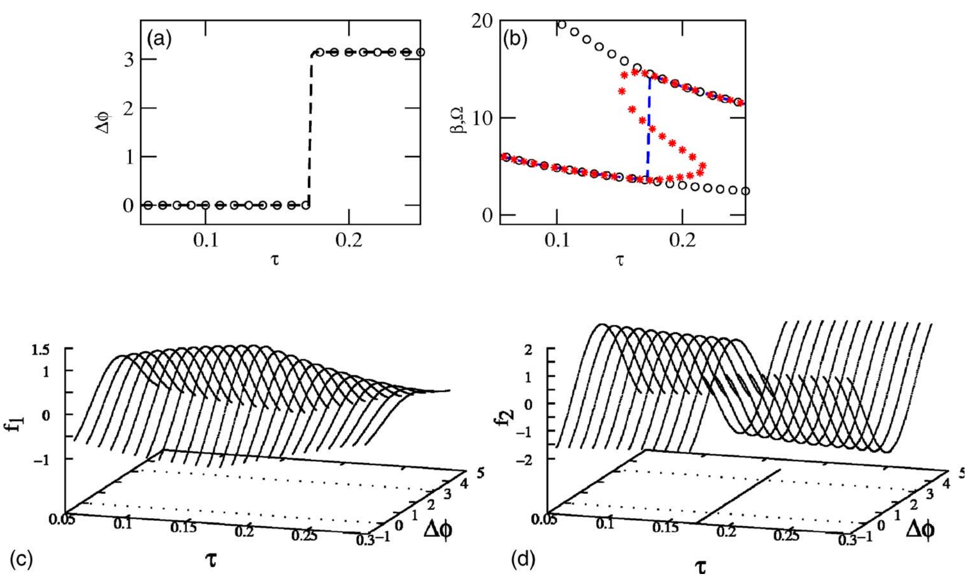

FIG. 4. (Color online) (a) The phase difference between oscillators with time delay τ which is equal to 0 before and π after the transition. (b) Solution of Eqs. (4a) (circles) and (4b) (stars) as a function of the delay parameter τ. The dashed line is the oscillator frequency, Ω which is estimated from the period. The functions (c) f_1 and (d) f_2 in the τ-$\Delta\phi$ plane.

$$\beta = \omega + \epsilon \exp(\alpha\tau)\sin[\phi_j(t-\tau) - \phi_k(t)], \quad (6)$$

where the subscripts take the values $j,k=1,2$ and $j\neq k$ to give two pairs of identities.

The phase difference in the phase- and frequency-locked state is constant; namely, if $\phi_1(t)=\beta t$, then $\phi_2(t)=\beta t+\Delta\phi$ where $\Delta\phi$ is the phase difference. Substituting this in Eqs. (5) and (6) and rearranging, we get a set of conditions

$$f_1(\Delta\phi,\tau) \equiv \cos(\Delta\phi - \beta\tau) - \cos(\Delta\phi + \beta\tau) = 0 \quad (7)$$

and

$$f_2(\Delta\phi,\tau) \equiv \sin(\Delta\phi - \beta\tau) + \sin(\Delta\phi + \beta\tau) = 0. \quad (8)$$

From these equations, it is clear that $\Delta\phi$ can be either 0 or π: the motion is either in phase or out of phase. The functions f_1 and f_2 are shown in Figs. 4(c) and 4(d) in the $\Delta\phi$-τ plane. Since there is no discontinuity in f_1 along the τ axis, both solutions, namely, 0 and π, are allowed. In f_2 there is a discontinuity at τ_c and therefore before and after the bifurcation the solutions of Eqs. (7) and (8) must be distinct: if $\Delta\phi=0$ is a solution for $\tau<\tau_c$ then $\Delta\phi=\pi$ is the solution for $\tau>\tau_c$.

This analysis can be extended to the case of phase flip when the dynamics is on limit cycle attractors, namely, when the fixed point is unstable [see Fig. 3(b)]. A similar estimate of the frequency β agrees with the numerically computed Ω, so that in this case also the eigenvalues of the fixed point determine the characteristics of the bifurcation [15]. Such analysis is not immediately possible for chaotic oscillators.

In summary, we have described a distinctive bifurcation that is characteristic of delay-coupled nonlinear systems [17] in the synchronized state: the two subsystems go from being *in phase* to being *out of phase*. The phase-flip bifurcation, which can occur in different dynamical regimes, always appears to be accompanied by a jump in the frequency of the synchronized oscillators. This bifurcation also occurs when the coupled oscillators are nonidentical, namely, when the parameters are mismatched, as well as when the oscillators are distinct. In this last case the phase difference is only approximately zero or π [4].

We have verified that this bifurcation can be observed in experiment. In amplitude death, since this happens at the point of maximum stability when the Lyapunov exponent goes through a local minimum, the change in dynamics is robust to noise and perturbation. A study of Chua circuits with time-delay coupling has clearly demonstrated the phase flip [15].

The ubiquity of this bifurcation in time-delay-coupled systems is suggestive of its importance, applicability, and utility in a large range of physical situations [18,19]. In coupled laser systems where the time delay can be conveniently varied in experiments, the in-phase regime is one of low frequency, while the high-frequency out-of-phase regime can permit a relatively higher degree of constant output [20]. Ecosystems where the coupling between separated communities naturally involves time delays are another area of application. For instance, it has been observed that synchronization occurs in epidemics: measles infections in different neighboring cities in the United Kingdom are known to be either in phase (Birmingham and Newcastle) or out of phase (Cambridge and Norwich) (see Fig. 1 of Ref. [8]). In this context, analysis of the phase-flip bifurcation in ensembles of oscillators will be of interest, as also the manifestations of this phenomenon in oscillators on networks of complex topology. Studies in these directions are currently under way [15].

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[12] Figure 1(a) is obtained by computing the largest Lyapunov exponent for the system [J. D. Farmer, Physica D **4**, 366 (1982)] and examining regions where it is positive or negative in a 200×200 grid.

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[16] Near the bifurcation, there is a narrow region where multiple attractors, either in or out of phase, coexist; the width of this region increases with τ [15].

[17] We have observed this bifurcation in a wide range of systems such as the multistable Chua and van der Pol oscillators, excitable laser systems, the Fitzhugh-Nagumo neuronal and a food-web ecological model [15].

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